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Robust Control Barrier Functions for Control Affine Systems with Time-Varying Parametric Uncertainties

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Abstract: This paper introduces robust control barrier functions for uncertain control affine systems, where the (parametric) uncertainties can be time-varying and nonlinearly affecting the system dynamics and/or safety sets. In particular, we propose two methods based on mixed-monotone decomposition and robust optimization where the controlled invariance condition remains linear in the control inputs despite nonlinear uncertainties. We show that these functions guarantee the robust controlled invariance of a given parameter-dependent safety set while existing adaptive approaches may not. Moreover, we propose alternative robust control Lyapunov functions where the control inputs also appear linearly; thus, these robust control barrier and Lyapunov functions can be coupled and remain a quadratic program that can be solved online. Finally, we demonstrate using two illustrative examples that our approaches have comparable performance with adaptive approaches while guaranteeing robust safety.

Keywords: Safety; Constrained control; Control problems under uncertainties; Nonlinear control and optimization; Robust control

1. INTRODUCTION

Control barrier functions (CBFs) are an extension of barrier certificates that provide a way to impose the forward controlled invariance of safety sets (Ames et al. (2016)). However, since internal/external uncertainties such as noise or disturbances are inevitable in the real world, it is often impossible to obtain precise mathematical models of the (cyber-physical) system dynamics for ensuring safety using standard CBFs. The degradation of safety under model uncertainty was studied in Kolathaya and Ames (2018) and an analysis on the robustness of CBFs to additive perturbations was given in Xu et al. (2015).

Of more interest to this paper are control design approaches for handling model/parametric uncertainties. These methods can be generally categorized into adaptive and robust approaches. In adaptive methods, the parametric uncertainties are often assumed to appear linearly and be constant. Under these assumptions, Taylor and Ames (2020); Lopez et al. (2020) introduced adaptive and robust adaptive CBFs, respectively, that involve adaptation laws to estimate the unknown and constant (linear) parameters. Further, under the additional assumption of persistence of excitation, Black et al. (2021) proposed an approach that guarantees fixed-time convergence of parameter estimates to their true values, thereby making the controller less conservative. Nonetheless, these approaches often do not guarantee safety for the true parameter (but for the estimated parameter) and/or when the parametric uncertainties are time-varying or nonlinear.

On the other hand, robust CBF methods that ensure safety for all possible values of (parametric) uncertainties have been proposed in Jankovic (2018); Breeden and Panagou (2021) under the assumption that the uncertainties are additive and bounded. These approaches also tend to conservatively bound the uncertainties when they appear nonlinearly by assuming a compact/bounded domain. Further, similar to adaptive methods, these approaches do not guarantee robust safety when the safety sets are dependent on unknown parameters. Additionally, Buch et al. (2021) and Seiler et al. (2021) considered input-dependent uncertainties that satisfy specific structures such as sector-boundedness or integral quadratic constraints, respectively, in contrast to polytopic and interval/hyperbox parametric uncertainties considered in this work.

Contributions. In this paper, we propose robust control barrier functions to guarantee robust safety of uncertain control affine systems, where, unlike in most existing adaptive and robust CBF methods, the bounded (parametric) uncertainties can be time-varying and affect the system dynamics and/or safety sets nonlinearly. These uncertainties can stem from dynamic or static sources including process noise or unmodeled dynamics or changing system parameters, e.g., due to wear and tear or change in user/passaenger weight. Our approaches are built upon mixed-monotonicity (e.g., Moisan and Bernard (2010); Coogan and Arcak (2015); Khajenejad and Yong (2023); Yang et al. (2019); Coogan (2020)) for general nonlinear parametric uncertainties and robust optimization/dual linear programming (Bertsimas et al. (2011)) for the affine parametric uncertainty case to obtain controlled

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invariance conditions that are linear in the control inputs. Furthermore, using similar techniques, we propose alternative robust control Lyapunov functions that also lead to conditions where the control inputs appear linearly. Thus, the proposed approaches can be used to design novel quadratic program-based rCLF-rCBF controllers that guarantee that the system satisfies (uncertain) safety conditions at all times and that the system converges to a reference/the origin if the system is sufficiently safe, in presence of time-varying, nonlinear parametric uncertainties. The proposed methods are demonstrated on adaptive cruise control and safe assistive shoulder robot examples.

2. PRELIMINARIES

2.1 Notations and Definitions

$\mathbb{R}^n$ represents the set of real numbers of dimension $n \geq 0$, while I_n represents an identity matrix of size n and $\mathbf{0}_{m\times n}$ represents a matrix of zeros of size $m \times n$. Further, all vector inequalities are element-wise inequalities.

Definition 1 (Locally Lipschitz Functions). *A function* $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathcal{T} \subseteq \mathbb{R}^m$ *is* locally Lipschitz continuous *if for all* $x \in \mathcal{X}$*, there exist an open neighborhood* $\mathcal{N}_x$ *of* x *and a constant* $L_f \geq 0$ *such that* $\|f(x') - f(x'')\|_2 \leq L_f\|x' - x''\|_2$, $\forall x', x'' \in \mathcal{N}_x$.

Definition 2 (Mixed-Monotone Mappings and Decomposition Functions). *(Yang et al., 2019, Definition 4) A mapping* $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathcal{T} \subseteq \mathbb{R}^m$ *is mixed-monotone if there exists a decomposition function* $f_d : \mathcal{X}\times\mathcal{X} \to \mathcal{T}$ *satisfying: (i)* $f_d(x, x) = f(x)$*, (ii)* $x_1 \geq x_2 \Rightarrow f_d(x_1, y) \geq f_d(x_2, y)$*, and (iii)* $y_1 \geq y_2 \Rightarrow f_d(x, y_1) \leq f_d(x, y_2)$.

Note that all locally Lipschitz continuous functions are mixed-monotone (Khajenejad and Yong (2023)) and their decomposition functions are generally not unique and can be computed in a tractable manner, as described in Yang et al. (2019); Khajenejad and Yong (2023); Coogan (2020). All results of this paper apply for *any* of the above methods for computing decomposition functions, although we mainly used the remainder-form variant from Khajenejad and Yong (2023) in our simulations. Further, a function $\alpha : [0, \infty) \to [0, \infty)$ is of class $\mathcal{K}_\infty$, i.e., $\alpha \in \mathcal{K}_\infty$, when α is continuous, strictly increasing, unbounded, and $\alpha(0) = 0$.

2.2 Problem Formulation

Consider the following control affine system with time-varying, nonlinear parametric uncertainties:

$$\dot{x}(t) = f(x(t), \theta^*(t)) + g(x(t), \theta^*(t))u(t), \qquad (1)$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ and $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ are state and input vectors, respectively, while $\theta^*(t) \in \Theta \subset \mathbb{R}^p$ represents an unknown parameter vector that is time-varying with unknown variation $\dot{\theta}^*(t) \in \Theta_d \subset \mathbb{R}^p$, where the sets Θ and Θ_d are known and bounded polytopes. Moreover, $f : \mathcal{X} \times \Theta \to \mathcal{R}^n$ and $g : \mathcal{X} \times \Theta \to \mathcal{R}^{n\times m}$ are locally Lipschitz continuous functions. This system is a generalization of the uncertain control affine systems considered in existing literature, where a fixed uncertain parameter θ^* is assumed to enter the system dynamics linearly. Note that in the rest of this paper, we omit the explicit dependence on time of all signals to improve readability when it is clear from context.

Furthermore, safety is generally defined as forward invariance of a safety set that is only dependent on known states of a system. However, safety of a system can also be directly effected by uncertainties in the system. Thus, we consider uncertainty-dependent safety sets.

Definition 3 (Uncertainty-Dependent Safe Set). $\mathcal{S}_\theta$ *is a superlevel set defined on a continuously differentiable function* $h : \mathcal{X} \times \Theta \to \mathbb{R}$ *that is parameterized by* θ *such that* $\mathcal{S}_\theta \triangleq \{x \in \mathcal{X} \mid h(x, \theta) \geq 0\}$ *and* $\partial\mathcal{S}_\theta \triangleq \{x \in \mathcal{X} \mid h(x, \theta) = 0\}$.

Then, the problem we seek to address in this paper can be formally written as:

Problem 1 (Robust Safety). *Given an uncertain control affine system* (1) *and an uncertainty-dependent safety set* $\mathcal{S}_{\theta^*}$*, construct a* robust *control barrier function that guarantees the* robust *controlled invariance of all possible safety sets, i.e.,* $\mathcal{S}_\theta$ *for all* $\theta \in \Theta$ *and* $\dot{\theta} \in \Theta_d$ *(and thus, for all unknown time-varying* $\theta^*(t)$ *and* $\dot{\theta}^*(t)$, $\forall t \geq 0$*).*

Note that robust safety is a stronger condition that must hold for all time-varying $\theta(t) \in \Theta$ and $\dot{\theta}(t) \in \Theta_d$ in contrast to adaptive safety conditions that are only required to hold for the estimated $\hat{\theta}(t)$ (under the assumption of fixed θ^*). While the definitions in Taylor and Ames (2020); Lopez et al. (2020) are also parameter-dependent, upon closer review, their proposed controllers in (Taylor and Ames, 2020, Theorem 3) and (Lopez et al., 2020, Theorem 2) only enforce safety for the estimated parameters and not the true parameters. In fact, our simulation in Section 4.1 shows that adaptive safety may lead to the violation of the invariance of the true but unknown $\mathcal{S}_{\theta^*}$, since in general, $\hat{\theta}(t) \neq \theta^*$ either during the transient (if the estimate converges) or if $\hat{\theta}(t)$ does not converge to θ^*. It is also noteworthy that since we require robust safety for all possible $\theta(t)$, in some cases, it may also be possible (and beneficial) to directly consider a robust parameter-independent function $h(x) \triangleq \min_{\theta\in\Theta} h(x, \theta)$ if this function remains smooth. Our proposed approaches in this paper also directly apply to this special case. Further, since the robust safety condition in Problem 1 typically involve nonlinearities and require semi-infinite constraints (i.e., an infinite number of constraints for all $\theta(t)$ in an infinite dense set Θ), we also consider the problem of finding sufficient and/or necessary conditions that can be obtained in a computationally tractable manner.

Problem 2 (Tractable Robust Control Barrier Function Conditions). *Given an uncertainty-dependent safety set* $\mathcal{S}_{\theta^*}$*, find sufficient and/or necessary robust control barrier function (rCBF) conditions that are computationally tractable, i.e., without semi-infinite constraints nor optimization subroutines, for several classes of uncertain control affine systems.*

3. MAIN RESULTS

In this section, the definitions and the proposed theories to solve Problems 1 and 2 are described in detail.

3.1 Robust Control Barrier Functions (rCBFs)

To address Problem 1, we extend conventional definitions of control barrier functions to the setting with uncertain parameters that may be time-varying.

Definition 4 (Robust Control Barrier Function (rCBF)). *For an uncertain control affine system (1), a continuously differentiable function $h : \mathcal{X} \times \Theta \to \mathbb{R}$ is a robust control barrier function (rCBF) for the uncertainty-dependent safety set $\mathcal{S}_{\theta^*}$ (cf. Definition 3), if there exists a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that*

$$\sup_{u\in\mathcal{U}} \dot{h}(x,u,\theta,\dot{\theta}) \geq -\alpha(h(x,\theta)), \tag{2}$$

for all $x \in \mathcal{X}$, $\theta \in \Theta$, $\dot{\theta} \in \Theta_d$, and $t \geq 0$, where

$$\dot{h}(x,u,\theta,\dot{\theta}) \triangleq \tfrac{\partial h}{\partial x}(x,\theta)(f(x,\theta)+g(x,\theta)u) + \tfrac{\partial h}{\partial \theta}(x,\theta)\dot{\theta}. \tag{3}$$

Moreover, for any $x \in \mathcal{S}_\Theta \triangleq \bigcap_{\theta\in\Theta} \mathcal{S}_\theta$, we define the corresponding safe input set:

$$K_{\mathcal{S}_\Theta}(x) = \{u \in \mathcal{U} \mid \dot{h}(x,u,\theta,\dot{\theta}) \geq -\alpha(h(x,\theta)), \forall \theta \in \Theta, \dot{\theta} \in \Theta_d\}. \tag{4}$$

Theorem 1. *(*Robust Safety*). Let $\mathcal{S}_{\theta^*}$ be the true uncertainty-dependent safety set defined as the superlevel set of a continuously differentiable function $h : \mathcal{X} \times \Theta \to \mathbb{R}$ and $\mathcal{S}_\Theta \triangleq \bigcap_{\theta\in\Theta} \mathcal{S}_\theta$. If h is a robust control barrier function (rCBF) on $\mathcal{S}_\Theta$ and $\frac{\partial h}{\partial x}(x,\theta) \neq 0$, $\forall x \in \partial\mathcal{S}_\Theta$, then any Lipschitz continuous (with respect to (w.r.t.) the state x) controller $u(x) \in K_{\mathcal{S}_\Theta}(x)$ for the system (1) renders the set $\mathcal{S}_\Theta$ robustly safe, i.e., it also renders $h(x,\theta^*) \geq 0$, $\forall x \in \mathcal{S}_\Theta \subseteq \mathcal{S}_{\theta^*}$. Thus, the system (1) is safe for the true uncertain time-varying parameters $\theta^*(t)$ that are bounded by polytopes or intervals/hyperboxes, $\forall x \in \mathcal{S}_\Theta \subseteq \mathcal{S}_{\theta^*}, t \geq 0$.*

Proof. If h is an rCBF on $\mathcal{S}_\Theta$ for the system (1), then any controller $u(x) \in K_{\mathcal{S}_\Theta}(x)$ satisfies $\dot{h}(x,u,\theta,\dot{\theta}) \geq -\alpha(h(x,\theta))$ for all $\theta \in \Theta$ and $\dot{\theta} \in \Theta_d$, thus $\mathcal{S}_\Theta$ is forward invariant, i.e., $h(x,\theta) \geq 0$ for all $\theta \in \Theta$. Then, since by definition, $\mathcal{S}_\Theta = \bigcap_{\theta\in\Theta} \mathcal{S}_\theta \subseteq \mathcal{S}_{\theta^*}$, this means that the true unknown parameter-dependent safety set $\mathcal{S}_{\theta^*}$ is also rendered controlled invariant for all $x' \in \mathcal{S}_\Theta$, i.e., $h(x,\theta^*) \geq 0$. □

3.2 Tractable rCBF Condition

Note that the robust safety condition in (2) involves semi-infinite (i.e., 'for all') constraints. Hence, this section addresses Problem 2, where we propose two sufficient and/or necessary conditions for satisfying (2) that can be obtained in a computationally tractable manner with only finitely many constraints, under various assumptions about the uncertain parameter and parameter variation sets (Θ, Θ_d) and how $\theta^*, \dot{\theta}^*$ enter (2), with increasing levels of assumptions/restrictions but with possibly decreased conservatism.

3.2.1. Robust CBF via Mixed-Monotone Decompositions (rCBF-MM)

First, we consider the case with general nonlinear parametric uncertainties without any restriction on the nonlinearity besides continuity as described in the following.

Assumption 1. *The parameter and parameter variation sets, i.e., Θ and Θ_d, are bounded sets that are contained within known intervals/hyperboxes $\mathbb{I}\Theta \triangleq [\underline{\theta}, \overline{\theta}]$ and $\mathbb{I}\Theta_d \triangleq [\underline{\theta}_d, \overline{\theta}_d]$, respectively. Moreover, the continuously differentiable safety function $h(x,\theta)$ (with respect to x) and the choice of class κ_∞ function α are such that $\dot{h}(x,u,\theta,\dot{\theta}) + \alpha(h(x,\theta))$ (for a known state x and to-be-designed u, this function is only dependent on θ and $\dot{\theta}$) is a locally Lipschitz continuous function in θ and $\dot{\theta}$ with known bounds for its Jacobian with respect to $[\theta^\top, \dot{\theta}^\top]^\top$, i.e., $J(\theta,\dot{\theta}) \in \mathbb{I}J \triangleq [\underline{J}, \overline{J}]$ for all $\theta \in \mathbb{I}\Theta$ and $\dot{\theta} \in \mathbb{I}\Theta_d$. These bounds can either be state/input-dependent or uniform over x and/or u.*

Under the above assumption, we propose a robust CBF that leverages mixed-monotonicity theory (e.g., Khajenejad and Yong (2023); Coogan (2020)) to bound the terms that are nonlinear in the parametric uncertainties.

Definition 5 (rCBF-MM). *Consider an uncertain control affine system (1), a continuously differentiable function $h : \mathcal{X} \times \Theta \to \mathbb{R}$ for the uncertainty-dependent safety set $\mathcal{S}_{\theta^*}$ and a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ that satisfy Assumption 1, i.e., there exist mixed-monotone decomposition functions h_d, $\tilde{f}_d$, $\tilde{g}_d$, and $\tilde{h}_d$ for $h(x(t),\theta)$, $\tilde{f}(\theta) \triangleq \frac{\partial h}{\partial x} f(x(t),\theta)$, $\tilde{g}(\theta) \triangleq \frac{\partial h}{\partial x} g(x(t),\theta)$, and $\tilde{h}(\theta) \triangleq \frac{\partial h}{\partial \theta}(x(t),\theta)$, respectively (during optimization at each time instance, $x(t)$ is known/measured; hence these functions are (nonlinear) functions of the parameter only, with implicit dependence on the known x). Then, the function h is a robust control barrier function via mixed-monotone bounding (rCBF-MM) if the following holds for all $x \in \mathcal{X}$, $t \geq 0$:*

$$\left\{\begin{array}{l} \sup\limits_{u^+,u^-} \ \tilde{g}_d(\underline{\theta},\overline{\theta})u^+ - \tilde{g}_d(\overline{\theta},\underline{\theta})u^- \\ \quad s.t. \ u^+ - u^- \in \mathcal{U}, \\ \qquad\ \ u^+ \geq 0, u^- \geq 0 \end{array}\right\} \geq \begin{array}{l} -\alpha(h_d(\underline{\theta},\overline{\theta})) \\ -\tilde{f}_d(\underline{\theta},\overline{\theta}) - \Delta, \end{array} \tag{5}$$

with $\Delta \triangleq \min\{\tilde{h}_d(\underline{\theta},\overline{\theta})\underline{\theta}_d, \tilde{h}_d(\underline{\theta},\overline{\theta})\overline{\theta}_d, \tilde{h}_d(\overline{\theta},\underline{\theta})\underline{\theta}_d, \tilde{h}_d(\overline{\theta},\underline{\theta})\overline{\theta}_d\}$, where u^+ and u^- are non-negative auxiliary vector variables from which the safe input can be obtained as $u = u^+ - u^-$. Moreover, for any $x \in \mathcal{S}_\Theta \triangleq \bigcap_{\theta\in\Theta} \mathcal{S}_\theta$, we define the corresponding safe input set:

$$K^{MM}_{\mathcal{S}_\Theta}(x) = \{u = u^+ - u^- \in \mathcal{U} \mid (5) \ holds\}. \tag{6}$$

Theorem 2. *[Sufficient Condition for rCBF-MM]. Let $\mathcal{S}_{\theta^*}$ be the true uncertainty-dependent safety set defined as the superlevel set of a continuously differentiable function function $h : \mathcal{X} \times \Theta \to \mathbb{R}$ that satisfies Assumption 1. If h is a rCBF-MM on $\mathcal{S}_\Theta$ (cf. Definition 5) and $\frac{\partial h}{\partial x}(x,\theta)g(x,\theta) \neq 0$ for all $x \in \partial\mathcal{S}_\Theta$, then any Lipschitz continuous (w.r.t. x) controller $u(x) \in K^{MM}_{\mathcal{S}_\Theta}(x)$ for the system (1) renders the set $\mathcal{S}_\Theta$ robustly safe; thus, the system (1) is safe for the true uncertain time-varying $\theta^*(t)$ for all $x \in \mathcal{S}_\Theta \subseteq \mathcal{S}_{\theta^*}, t \geq 0$.*

Proof. From (3), $\dot{h}(x,u,\theta,\dot{\theta}) = \tilde{f}(x,\theta) + \tilde{g}(x,\theta)u + \tilde{h}(x,\theta)\dot{\theta}$. By mixed-monotonicity (Coogan and Arcak, 2015, Theorem 1), $\tilde{f}(x,\theta) \geq \tilde{f}_d(\underline{\theta},\overline{\theta})$ and $\tilde{g}_d(\overline{\theta},\underline{\theta}) \geq \tilde{g}(x,\theta) \geq \tilde{g}_d(\underline{\theta},\overline{\theta})$. Moreover, we equivalently decompose u as $u = u^+ - u^-$ with $u^+, u^- \geq 0$ (since any vector u can be decomposed into a difference of two element-wise positive vectors u^+ and u^-). Then, since $\tilde{g}(x,\theta)u^+ \geq \tilde{g}_d(\underline{\theta},\overline{\theta})u^+$ and $-\tilde{g}(x,\theta)u^- \geq -\tilde{g}_d(\overline{\theta},\underline{\theta})u^-$,

$$\tilde{g}(x,\theta)u \geq \tilde{g}_d(\underline{\theta},\overline{\theta})u^+ - \tilde{g}_d(\overline{\theta},\underline{\theta})u^-.$$

Further, from interval multiplication in interval arithmetic (Jaulin et al. (2001)), we obtain $\tilde{h}(x,\theta)\dot{\theta} \geq \Delta$, where Δ is defined below (5). Since h being an rCBF-MM for the uncertain control affine system (1) implies that

$\dot{h}(x,u,\theta,\dot{\theta}) \geq \tilde{f}_d(\underline{\theta},\overline{\theta}) + \tilde{g}_d(\underline{\theta},\overline{\theta})u^+ - \tilde{g}_d(\overline{\theta},\underline{\theta})u^- + \Delta \geq -\alpha(h_d(\underline{\theta},\overline{\theta})) \geq -\alpha(h(x,\theta))$, by Theorem 1, the controller renders $\mathcal{S}_\Theta$ robustly safe. □

The above rCBF-MM is only sufficient since the interval bounding using MM decompositions may sometimes be conservative. Moreover, it is noteworthy that when $\tilde{f}(x,\theta)$ is linear in θ, i.e., $\tilde{f}(x,\theta) = A(x)\theta$, the inequality $A(x)^{\oplus}\underline{\theta} - A(x)^{\ominus}\overline{\theta} \leq A(x)\theta$ holds by (Efimov et al., 2013, Lemma 1), where $A(x)^{\oplus} \triangleq \max\{A(x), 0\}$ and $A(x)^{\ominus} \triangleq A(x)^{\oplus} - A(x)$.

3.2.2. Robust CBF via Robust Linear Programs (rCBF-L)

Since bounding with MM decompositions in the previous subsection may be conservative, we also consider the case when the parametric uncertainties appear linearly. In this case, we can also directly consider polytopic parametric uncertainties, as follows.

Assumption 2. *The parameter and parameter variation sets, i.e.,* $\Theta = \{\theta \in \mathbb{R}^p \mid P_\theta\theta \leq q_\theta\}$ *and* $\Theta_d = \{\dot{\theta} \in \mathbb{R}^p \mid P_{\theta_d}\dot{\theta} \leq q_{\theta_d}\}$, *are convex polytopes in their hyperplane representations with known* P_θ, P_{θ_d}, q_θ *and* q_{θ_d}. *Moreover, the continuously differentiable safety function* $h(x,\theta)$ *with respect to x) and the choice of class* κ_∞ *function* α *are such that* $h(x,\theta)$ *and* $\dot{h}(x,u,\theta,\dot{\theta}) + \alpha(h(x,\theta))$ *are linear in* θ *and* $\dot{\theta}$.

The assumption of the 'linear/affine' structure/nature of the parametric uncertainties allows us to leverage robust linear optimization (Bertsimas et al. (2011)) to *tightly* bound the impact of the parametric uncertainties on safety, as described next.

Definition 6 (rCBF-L). *Consider an uncertain control affine system (1), a continuously differentiable function* $h : \mathcal{X} \times \Theta \to \mathbb{R}$ *for the uncertainty-dependent safety set* $\mathcal{S}_{\theta^*}$ *and a class* $\mathcal{K}_\infty$ *function* $\alpha(\cdot)$ *that satisfy Assumption 2, i.e.,* $\dot{h}(x,u,\theta,\dot{\theta}) + \alpha(h(x,\theta)) \in \mathbb{R}$ *is in the following affine form:*

$$(\phi(x) + \rho(x)u)^\top\theta + \psi(x)^\top\dot{\theta} + \sigma(x) + \tau(x)u, \tag{7}$$

where $\dot{h}(x,u,\theta,\dot{\theta})$ *is defined in (3), with* $\phi(\cdot) \in \mathbb{R}^p$, $\rho(\cdot) \in \mathbb{R}^{p\times m}$, $\psi(\cdot) \in \mathbb{R}^p$, $\sigma(\cdot) \in \mathbb{R}$ *and* $\tau(\cdot) \in \mathbb{R}^{1\times m}$. *Then, the function h is a robust control barrier function via robust linear optimization (rCBF-L) if the following holds:*

$$\left\{\begin{array}{l} \sup\limits_{u,\mu} \; -\begin{bmatrix} q_\theta^\top & q_{\theta_d}^\top \end{bmatrix}\mu + \sigma(x) + \tau(x)u \\ s.t. \begin{bmatrix} P_\theta & 0 \\ 0 & P_{\theta_d} \end{bmatrix}^\top \mu = -\begin{bmatrix} \phi(x)+\rho(x)u \\ \psi(x) \end{bmatrix}, \\ \quad u \in \mathcal{U}, \mu \geq 0, \end{array}\right\} \geq 0. \tag{8}$$

for all $x \in \mathcal{X}$, *where* μ *is non-negative auxiliary vector variable. Moreover, for any* $x \in \mathcal{S}_\Theta \triangleq \bigcap_{\theta\in\Theta}\mathcal{S}_\theta$, *we define the corresponding safe input set:*

$$K^L_{\mathcal{S}_\Theta}(x) = \{u \in \mathcal{U} \mid (8) \text{ holds}\}. \tag{9}$$

Theorem 3. *[Necessary and Sufficient Condition for rCBF-L]. Let* $\mathcal{S}_{\theta^*}$ *be the true uncertainty-dependent safety set defined as the superlevel set of a continuously differentiable function function* $h : \mathcal{X} \times \Theta \to \mathbb{R}$ *that satisfies Assumption 2. If h is a rCBF-L on* $\mathcal{S}_\Theta$ *(cf. Definition 6) and* $\frac{\partial h}{\partial x}(x,\theta)g(x,\theta) \neq 0$ *for all* $x \in \partial\mathcal{S}_\Theta$, *then any Lipschitz continuous (w.r.t. x) controller* $u(x) \in K^L_{\mathcal{S}_\Theta}(x)$ *for the system (1) renders the set* $\mathcal{S}_\Theta$ *robustly safe; thus, the system (1) is safe for the true uncertain time-varying* $\theta^*(t)$ *for all* $x \in \mathcal{S}_\Theta \subseteq \mathcal{S}_{\theta^*}, t \geq 0$.

Proof. For any $x \in \mathcal{S}_\Theta$, the satisfaction of (2) with $\dot{h}(x,u,\theta,\dot{\theta}) + \alpha(h(x,\theta))$ in the form of (7) for all $\theta \in \{\theta \in \mathbb{R}^p \mid P_\theta\theta \leq q_\theta\}$ and $\dot{\theta} \in \{\dot{\theta} \in \mathbb{R}^p \mid P_{\theta_d}\dot{\theta} \leq q_{\theta_d}\}$ is equal to:

$$\left\{\begin{array}{l} \sup\limits_{u}\inf\limits_{\theta,\dot{\theta}} (\phi(x)+\rho(x)u)^\top\theta + \psi(x)^\top\dot{\theta} + \sigma(x) + \tau(x)u \\ \quad s.t. \begin{bmatrix} P_\theta & 0 \\ 0 & P_{\theta_d} \end{bmatrix}\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \leq \begin{bmatrix} q_\theta \\ q_{\theta_d} \end{bmatrix}. \end{array}\right\} \geq 0,$$

Then, by robust optimization Bertsimas et al. (2011), we obtain the condition in (8). Thus, $\mathcal{S}_\Theta$ is robustly safe. □

Note that Assumption 1 is generally less restrictive than Assumption 2 since any locally Lipschitz continuous function, including linear functions, is mixed-monotone (Khajenejad and Yong (2023)). However, when both rCBF-MM and rCBF-L apply, i.e., when Assumption 2 is satisfied, rCBF-MM may result in more conservative results than rCBF-L. More importantly, our proposed robust approaches demonstrate that the estimation/adaptation of $\hat{\theta}(t)$ alone is not necessarily beneficial because we would still need to be robust to the worst-case parametric uncertainties, unless bounds on the parameter estimation errors can also be computed, in which case, we only need to be robust to the worst-case estimation errors. Further, while our work mainly focused on systems with relative degree 1 (i.e., $\frac{\partial h}{\partial x}(\cdot)g(\cdot) \neq 0$), similar ideas can also be applied to systems with higher relative degrees to obtain robust exponential CBFs.

3.3 Optimization-Based Control

Next, we show that robust CBFs can be coupled with a safe legacy controller or with robust (and adaptive) control Lyapunov functions, which will be introduced in Section 3.3.2, to stabilize/control the system while guaranteeing safety.

3.3.1. Safe Legacy Controller: Suppose we are given a legacy feedback controller $u = k(x) \in \mathcal{U}$ for the uncertain control affine system (1) and we wish to guarantee safety. To minimally modify this controller while guaranteeing safety, we can consider the following Quadratic Program (QP):

$$u(x) = \arg\min_{u\in\mathcal{U}} \tfrac{1}{2}\|u - k(x)\| \tag{10a}$$

$$s.t. \; (5) \text{ holds}, \tag{10b}$$

for the general case with rCBF-MM. A similar QP can be obtained with rCBF-L by replacing (5) with (8) in (10).

3.3.2. Unifying with Lyapunov: Moreover, robust CBFs can be unified with robust control Lyapunov functions as follows for the uncertain control affine system (1):

Definition 7 (Robust Control Lyapunov Function (rCLF)). *A continuously differentiable function* $V : \mathcal{X} \to \mathbb{R}$ *is a robust control Lyapunov function (rCLF) for the uncertain control affine system (1) if there exist extended class* κ_∞ *functions* α_1, α_2, *and* α_3 *such that:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{11}$$

$$\inf_{u\in\mathcal{U}} \dot{V}(x,u,\theta) \leq -\alpha_3(\|x\|), \; \forall\, \theta \in \Theta, \tag{12}$$

for all $x \in \mathcal{X}$ *and where* $\dot{V}(x,u,\theta)$ *is defined as*

$$\dot{V}(x,u,\theta) \triangleq \frac{\partial V}{\partial x}(x)(f(x,\theta)+g(x,\theta)u). \tag{13}$$

It can be shown that if such rCLF exists, then there exist a controller that robustly asymptotically stabilizes the system to the origin. Moreover, similar to rCBF, the rCLF condition in (12) contains semi-infinite constraints. Thus, corresponding rCLF-MM and rCLF-L functions with finitely many constraints (hence, tractable) can be defined under the assumptions of local Lipschitz continuity (hence, mixed-monotonicity) and linearity with respect to θ in an interval and polytope Θ, respectively, as summarized below. Their robust stability proofs are analog to the proofs for rCBF-MM and rCBF-L, and are omitted for the sake of brevity.

Definition 8 (rCLF-MM). *Consider an uncertain control affine system* (1)*, a parameter set contained within known intervals* $\mathbb{I}\Theta \triangleq [\underline{\theta},\overline{\theta}]$ *and a continuously differentiable function* $V : \mathcal{X} \to \mathbb{R}$ *satisfying* (11) *such that* $\dot{V}(x,u,\theta)$ *(cf.* (13)*) is locally Lipschitz, i.e., there exists mixed-monotone decomposition functions* $(L_f V)_d$ *and* $(L_g V)_d$ *for the Lie derivatives* $L_f V \triangleq \frac{\partial V}{\partial x} f(x,\theta)$ *and* $L_g V \triangleq \frac{\partial V}{\partial x} g(x,\theta)$*, respectively. Then,* V *is a robust control Lyapunov function via mixed-monotone bounding (rCLF-MM) if the following holds:*

$$\left\{\begin{array}{l} \inf\limits_{u^+,u^-} (L_gV)_d(\overline{\theta},\underline{\theta})u^+ - (L_gV)_d(\underline{\theta},\overline{\theta})u^- \\ \quad s.t.\ u^+ - u^- \in \mathcal{U}, u^+ \geq 0, u^- \geq 0, \end{array}\right\} \tag{14}$$
$$\leq -\alpha_3(\|x\|) - (L_fV)_d(\overline{\theta},\underline{\theta}),$$

for some $\mathcal{K}_\infty$ *function* α_3*, where* u^+ *and* u^- *are non-negative auxiliary variables from which the safe input can be obtained as* $u = u^+ - u^-$.

Definition 9 (rCLF-L). *Consider an uncertain control affine system* (1)*, a polytopic parameter set* $\Theta = \{\theta \in \mathbb{R}^p \mid P_\theta \theta \leq q_\theta\}$ *and a continuously differentiable function* $V : \mathcal{X} \to \mathbb{R}$ *satisfying* (11) *such that* $\dot{V}(x,u,\theta)$ *(cf.* (13)*) is in the following affine form:*

$$\dot{V}(x,u,\theta) = (\phi(x)+\rho(x)u)^\top \theta + \sigma(x) + \tau(x)u. \tag{15}$$

Then, the function V *is a robust control Lyapunov function via robust linear optimization (rCLF-L) if the following holds:*

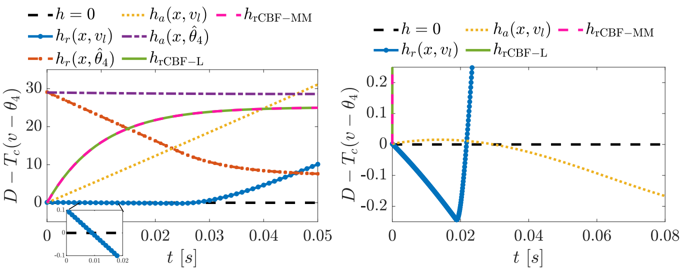
$$\left\{\begin{array}{l} \inf\limits_{u,\mu}\ q_\theta^\top \mu + \sigma(x) + \tau(x)u \\ s.t.\ P_\theta{}^\top \mu = \phi(x) + \rho(x)u, \\ \quad\ u \in \mathcal{U}, \mu \geq 0, \end{array}\right\} \leq -\alpha_3(\|x\|), \tag{16}$$

for some class $\mathcal{K}_\infty$ *function* α_3*, where* μ *is non-negative auxiliary vector variable.*

Having introduced various forms of rCLFs and rCBFs, we can combine robust stabilization and robust safety objectives in a optimization-based control approach. In particular, combining rCLF-MM and rCBF-MM yields the following QP:

$$u(x) = \underset{u\in\mathcal{U},\delta\in\mathbb{R}}{\arg\min}\ \tfrac{1}{2}u^T H(x)u + p\delta^2 \tag{17a}$$
$$s.t.\ (14) \text{ holds} + \delta, \tag{17b}$$
$$(5) \text{ holds}, \tag{17c}$$

where (17b) represents (14) with $-\alpha_3(\|x\|) + \delta$ on the right hand side, $H(x)$ is any positive definite matrix (pointwise in x) and δ is a relaxation variable that ensures solvability of the QP with a penalty weight $p > 0$ (i.e., to guarantee safety while relaxing stability if both cannot be simultaneously satisfied). A QP for combining rCLF-L and rCBF-L can be similarly obtained by replacing (14) with (16) and (5) with (8) in (17).



(a) Time-invariant v_l, i.e., $\dot{v}_l = 0$ (b) Time-varying v_l, i.e., $\dot{v}_l \neq 0$

Fig. 1. ACC-TTC with known mass, where $h_a(\cdot,\cdot)$ represents aCBF (Taylor and Ames (2020)), $h_r(\cdot,\cdot)$ represents RaCBF (Lopez et al. (2020)), $h_{\mathrm{rCBF-L}}$ and $h_{\mathrm{rCBF-MM}}$ represent the proposed robust CBFs.

4. SIMULATION EXAMPLES

4.1 Adaptive Cruise Control with Safe Time-to-Collision

We consider the problem of adaptive cruise control (ACC) similar to Taylor and Ames (2020), but with unknown lead vehicle velocity and with a time-to-collision (TTC) safety constraint instead of a time headway (TH) constraint that leads to an uncertainty-dependent safety set [1] . The system dynamics is given by:

$$\frac{d}{dt}\begin{bmatrix} v \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ -v \end{bmatrix} - \begin{bmatrix} \frac{1}{m} & \frac{v}{m} & \frac{v^2}{m} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ v_l \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} u,$$

where v is the ego/follower vehicle velocity, D is the distance between the ego and lead vehicles, m is the mass of the ego vehicle, and u is the force input. The unknown parameters are composed of unknown rolling friction force parameters f_0, f_1, f_2 and the unknown lead vehicle velocity v_l. Further, the *time-to-collision* (TTC) safety constraint is given as

$$D \geq T_c(v - v_l), \tag{18}$$

where $T_c > 0$ represents the time-to-collision and to ensure safety, we consider the following uncertainty-dependent safety set $\mathcal{S}_{\theta^*}$ (cf. Definition 3) with:

$$h(x,\theta^*) \triangleq D - T_c(v - v_l^*) \geq 0, \tag{19}$$

where v_l^* is one of the true unknown parameters. Further, we consider rCLFs with $V = (v - v_d)^2$ to enable tracking of a desired velocity v_d, when possible. All (known or unknown) system parameters are taken from Ames et al. (2016).

Next, we consider two cases where the vehicle mass m is known or unknown such that $g(x,\theta^*)$ in (1) is independent or dependent on the unknown parameter θ^*, respectively:

Known Mass: Since m is assumed to be known, $g(x,\theta^*)$ in (1) is independent of θ^* and the uncertain parameter vector in this setting is $\theta^* = [f_0^*\ f_1^*\ f_2^*\ v_l^*]^\top$. Moreover, we consider the cases when $\dot{v}_l = 0$ and $\dot{v}_l \neq 0$. In the first case

[1] A detailed discussion on the pros and cons of TTC and TH as safety indicators can be found in Vogel (2003).

with $\dot{v}_l = 0$, the aCBF and RaCBF (with corresponding adaptive CLFs) approaches in Taylor and Ames (2020); Lopez et al. (2020) apply, and the adaptation gains are chosen to satisfy the lower bounds given in Taylor and Ames (2020); Lopez et al. (2020). For a fair comparison, we only consider the RaCBF method in Lopez et al. (2020) and not RaCBF+SMID. However, as shown in Figure 1 (left), when we start close to the safety boundary, these approaches in Taylor and Ames (2020); Lopez et al. (2020) (that only guarantee that $D - T_c(v - \hat{v}_l) \geq 0$) resulted in a violation of the TTC constraint $D - T_c(v - v_l^*) \geq 0$ for RaCBF while aCBF happened to satisfy the TTC constraint, presumably due to its very high level of conservatism. On the other hand, our proposed approach guarantees the satisfaction of the (unknown) TTC constraint $D - T_c(v - v_l^*) \geq 0$. Note further that the fixed-time adaptive approach in Black et al. (2021) is not applicable due to the absence of persistence of excitation.

Moreover, note that (Taylor and Ames, 2020, Theorem 3) and (Lopez et al., 2020, Theorem 2) only guarantee/require safety for $\mathcal{S}_{\hat{\theta}}$ by proving $h(x, \hat{\theta}) \geq \frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta} \geq 0$ for all $t \geq 0$, where $\tilde{\theta} \triangleq \theta^* - \hat{\theta}$ represents the estimation error, θ^* is the true parameter and $\Gamma \succ 0$ represents the adaptive gain used for parameter estimation. Consequently, this does not necessarily prove that $h(x, \theta^*) \geq 0$ (i.e., that $\mathcal{S}_{\theta^*}$ is invariant/safe) since generally, $h(x, \hat{\theta}) \neq h(x, \theta^*)$. If we consider a linear safety function of the form $h(x, \theta) = H_1 x + H_2 \theta$, where H_1 and H_2 are matrices of appropriate dimensions then $h(x, \hat{\theta}) = h(x, \theta^*) - H_2(\theta^* - \hat{\theta}) = h(x, \theta^*) - H_2\tilde{\theta}$. Hence, $h(x, \hat{\theta}) \geq \frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta}$ implies $h(x, \theta^*) \geq \frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta} + H_2\tilde{\theta}$, where the right-hand side can be negative based on the values of Γ, $\tilde{\theta}$ and H_2. Specifically, it is evident that for any fixed/chosen Γ and H_2, $H_2\tilde{\theta}$ may be negative and smaller than $\frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta}$ for small enough magnitudes of $\tilde{\theta}$; thus, $h(x, \theta^*) \geq \frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta} + H_2\tilde{\theta}$ is not necessarily non-negative for all $t \geq 0$.

In the second case with $\dot{v}_l \neq 0$, from Figure 1 (right), the aCBF and RaCBF approaches in Taylor and Ames (2020); Lopez et al. (2020) both fail to guarantee safety, which is expected because the controllers proposed in Taylor and Ames (2020); Lopez et al. (2020) do not take into consideration time-varying parameters. On the other hand, it is evident that both our proposed methods guarantee safety when $\dot{v}_l \neq 0$.

Unknown Mass and Time-Varying Lead Vehicle Velocity: We additionally assume that m is unknown and consider a nonlinear friction term f_0 using the Magic Formula for tires, $f_0 = \mu_s mg(sin(C_s tan^{-1}(S_f\xi - E_s(S_f\xi - tan^{-1}(S_f\xi)))))$ (Pacejka, 2005, Ch.1, Eq.1.6), where we consider the coefficient of friction $\mu_s = 1$, the shape factors $C_s = 2$ and $E_s = 1$, the stiffness factor $S_f = 10$ and the uncertain wheel slip ξ is such that $|\xi| \leq \pi \times 10^{-6}$. Thus, the uncertain parameter vector is $\theta^* = \begin{bmatrix} \xi^* & \frac{f_1}{m}^* & \frac{f_2}{m}^* & v_l^* & \frac{1}{m}^* \end{bmatrix}^\top$. Here we also allow the lead vehicle velocity v_l to be time-varying. In this case, the approaches in Taylor and Ames (2020); Lopez et al. (2020); Black et al. (2021) do not apply because by definition, $\dot{h} = \frac{\partial h}{\partial x}\dot{x} + \frac{\partial h}{\partial \theta}\dot{\theta}$, and the approaches proposed in Taylor and Ames (2020); Lopez et al. (2020); Black et al. (2021) do not take into consideration the second term and because $g(x, \theta^*)$ in (1) is dependent of θ^*. However, with our proposed rCBF-MM, Figure 2 shows that the TTC safety constraint was satisfied and the ego vehicle slows down when it is close to the safety boundary and tends towards the average lead vehicle velocity when sufficiently safe. Further, Figure 2 shows that the TTC safety constraint is also satisfied when using rCBF-L (albeit with $|f_0| \leq 1$ instead of the magic formula due to unmet system/structural assumptions).

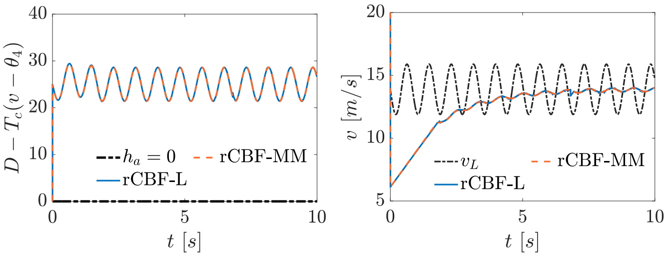


Fig. 2. ACC-TTC with unknown m and time-varying v_l.

In summary, we observed that even when all the preceding assumptions are satisfied for the aCBF and RaCBF methods proposed in Taylor and Ames (2020); Lopez et al. (2020), they may not always guarantee safety (RaCBF) or are very conservative (aCBF), whereas our proposed methods guarantee safety and are less conservative while being applicable to a broader class of systems.

4.2 Safe Assistive Shoulder Robot

In our second example, we consider the problem of rendering the (second-order) impedance control for an assistive shoulder exoskeleton robot given by

$$I_d\ddot{\phi} + B_d\dot{\phi} + K_d(\phi - \phi_e) = \tau_e \tag{20}$$

safe using our proposed robust CBF+CLF approaches, where ϕ, $\dot{\phi}$ and $\ddot{\phi}$ represent the vectors of Euler angles, angular velocities and angular accelerations, respectively. ϕ_e and τ_e are known values. I_d, B_d and K_d represent the inertial matrix, damping matrix and stiffness matrix of human-robot system. Assuming that we wish to control $B_d \triangleq B_h + B_r$ in order to decrease the "effective" damping that is experienced by the human user (to reduce muscle fatigue), where B_h is the unknown human shoulder damping coefficient and B_r is our (robot) control input, the dynamics given in (20) can be rearranged into a state space representation given as:

$$\dot{x} = Fx + \Lambda b_h + \Lambda b_r + E, \tag{21}$$

where $x = [\phi^\top \quad \dot{\phi}^\top]^\top$ represents the state vector of joint angular displacements and angular velocities, the uncertain parameter b_h is a column vector whose entries are the elements of the human damping matrix and b_r represents the control input (column) vector whose entries are the elements of B_r, while the matrices F, Λ and E are:

$$F = \begin{bmatrix} 0 & I \\ -I_d^{-1}K_d & 0 \end{bmatrix}, \Lambda = \begin{bmatrix} 0 \\ -I_d^{-1}\Phi \end{bmatrix}, E = \begin{bmatrix} 0 \\ I_d^{-1}(K_d\theta_e + \tau_e) \end{bmatrix},$$

where Φ is an appropriate matrix-valued function of $\dot{\phi}$. Specifically, from (20), the term $B_d\dot{\phi} = (B_h + B_r)\dot{\phi}$ is rearranged such that $(B_h + B_r)\dot{\phi} = \Phi(b_h + b_r)$, where b_r and b_h are column vectors whose elements are the elements of

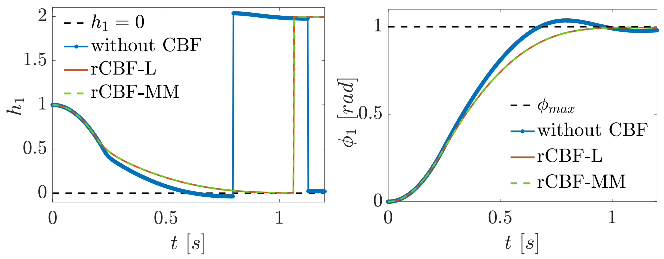


Fig. 3. Safe assistive shoulder robot control.

damping matrices B_r and B_h, respectively. If the damping matrices are assumed to be diagonal, then Φ reduces to a diagonal matrix whose entries are the angular velocities $\dot{\phi}$. Further, the safety conditions of avoiding awkward shoulder orientations and limiting the joint torques that the human experiences can be simplified to bounds on the angular displacements and accelerations, i.e., the safety constraints are $|\phi_i| \leq \phi_{max}$ and $|\ddot{\phi}_i| \leq \ddot{\phi}_{max}$. Inspired by the lane keeping example in Ames et al. (2016), we can encode safety using:

$$h_i = (\phi_{max} - sgn(\dot{\phi}_i)\phi_i) - \frac{1}{2}\frac{\dot{\phi}_i^2}{\ddot{\phi}_{max}} \geq 0 \quad (22)$$

for each joint i. Moreover, we construct rCLFs based on:

$$V = (\phi - \phi_e)^\top(\phi - \phi_e) + \dot{\phi}^\top\dot{\phi} \quad (23)$$

to enforce reference tracking of ϕ_e and consider control input bounds $\|b_r\|_\infty \leq u_{max}$.

In this simulation example, the system parameters are chosen as $I_d = 0.5I$, $B_h = 3I$, $K_d = 50I$, $\phi_{max} = 1$, $\ddot{\phi}_{max} = 10$, $\phi_e = 1$, $\tau_e = 0.1\mathbb{1}$ and $u_{max} = 20$. From Figure 3 (only trajectories of ϕ_1 and h_1 are shown for brevity), we observe that the shoulder robot with rCBF-MM and rCBF-L can achieve the desired reference angle ϕ_e without exceeding ϕ_{max}, while providing a little more margin from ϕ_{max} than when b_h is exactly known, whereas without any robust CBF, the safety conditions are violated (i.e., ϕ_{max} is exceeded).

5. CONCLUSION

We presented two computationally tractable robust control barrier functions that guarantee safety for control affine systems with time-varying, nonlinear parametric uncertainties. We leveraged theories of mixed-monotonicity and robust optimization to formulate novel robust control barrier functions that provide tools to ensure controlled invariance of safe sets, thereby ensuring safety. We extended these ideas to construct robust control Lyapunov functions and coupled them with robust control barrier functions to develop safety controllers that ensure safety and stability. The efficacy of our proposed approaches was demonstrated on two examples, where our proposed approaches have comparable performance to adaptive methods, while ensuring that the system is robustly safe even in cases where the adaptive methods failed to ensure safety. For future work, we will consider parameter estimation as an extension of Lopez et al. (2020) to further improve performance.

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