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Preview Control Barrier Functions for Linear Continuous-Time Systems with Previewable Disturbances

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***Abstract*— Many (semi-)autonomous systems are equipped with mechanisms that provide a window of projecting into the future. Predictions/projections of future exogenous inputs or disturbances, commonly referred to as preview or lookahead, have been widely studied in predictive control systems to yield less conservative controllers that would otherwise have to consider them as worst-case disturbances. However, the incorporation of such preview information has been less studied in the context of safety. Thus, this paper proposes a preview control barrier function (Prev-CBF) that can enforce the controlled invariance of a safe set for a class of linear continuous-time systems with previewable disturbances. Specifically, our approach can leverage future information about external disturbances, e.g., road gradients or predictive trajectories of other agents/vehicles, for a (small) window into the future and can explicitly take input constraints/bounds into consideration to provide strong safety guarantees in a less conservative manner than existing approaches that do not leverage such information.**

I. INTRODUCTION

Autonomous or semi-autonomous cyber-physical systems are often equipped with mechanisms, e.g., forward looking sensors such as cameras or LIDAR, map or forecast information, or more complicated predictive models of external agents, that provide a window of projecting into the future. Through these mechanisms, at run time, the systems have a preview of what lies ahead. Therefore, using this information to enhance system performance while keeping strong safety guarantees holds significant promise, particularly since many cyber-physical systems such as self-driving cars and smart grids are safety-critical.

Although some optimal controllers (e.g., [1], [2]) and model predictive controllers (e.g., [3], [4]) can take advantage of certain forms of preview information, where preview information is naturally incorporated into the state propagation constraints, these approaches in general lack controlled invariance or recursive feasibility properties to provide strong safety guarantees. On the other hand, safety in the sense of staying in a safe region indefinitely (and equivalently, avoiding a user-defined unsafe region) is often achieved via robust controlled invariant set computation (e.g., [5]) or via control barrier functions (CBFs) (e.g., [6]–[10]), that have recently garnered considerable attention in various applications including autonomous vehicles (e.g., [6], [8]), exoskeleton robots (e.g., [11]) and bipedal robots (e.g., [9], [12]). However, most of these safety control approaches do not leverage preview information but instead enforce robust safety by considering the worst-case (future) disturbances.

Some recent exceptions are the studies in [13], [14], where it was shown that there is (much) value in preview information in safety control for discrete-time systems in terms of enlarging the associated maximal controlled invariant sets with increasing preview time. However, to our best knowledge, the same has not been established for systems with continuous-time dynamics. A related but distinct approach for incorporating predictions into safety control for continuous-time systems has been recently proposed in the form of a predictive CBF in [15], where the "predictions" relate to a future reference/desired trajectory that can be proactively modified, somewhat similarly to a reference governor, but it does not directly apply to previewable disturbances we consider in this paper, e.g., road gradients, curvatures or friction coefficients, or predicted/communicated future trajectories of other agents, that can often not be modified or controlled.

Contributions. Inspired by the work in [13], [14] that demonstrated the value of information about previewable disturbances for discrete-time systems, this paper considers its continuous-time (CT) counterpart by introducing a novel *preview control barrier function* (Prev-CBF) that combines the framework of the standard CBFs for CT systems (without preview), e.g., [6], with the predictor feedback idea of using future predictions of the system states/outputs in the literature on time-delay systems, e.g., [16], [17]. Specifically, we propose Prev-CBFs to provide safety guarantees for linear continuous-time systems of relative degree 2 with previewable (but generally immutable/uncontrolled) disturbances, e.g., of road curvatures, gradients, or friction coefficients, or information/prediction of future trajectories of other agents, for a (brief) window into the future that can often be obtained from forward looking sensors, maps or forecasts/predictions.

In contrast to standard CBFs, the proposed Prev-CBFs can leverage available preview information to construct potentially less conservative CBFs in the sense of minimizing the required interventions for guaranteeing safety by rendering the associated Prev-CBF constraints more permissive (i.e., the constraints are more likely than not to be inactive). Further, our approach explicitly takes input constraints into consideration when constructing Prev-CBFs. The effectiveness of the proposed methods is demonstrated on examples of an assistive shoulder robot with interaction torque preview and of vehicle lane keeping with road curvature preview.

This paper is organized as follows. In Section II, we introduce some preliminary material and formulate the problem of interest, while in Section III, we introduce the proposed

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Prev-CBFs. Then, we demonstrate the usefulness of our approach using two illustrative examples in Section IV and also discuss the (empirical) value of preview for continuous-time systems. Lastly, we conclude our paper with a summary of our contributions and some future directions in Section V.

II. Preliminaries and Problem Formulation

A. Notations and Definitions

$\mathbb{R}^n$ and $\mathbb{R}_+$ represent the sets of real numbers of dimension n and non-negative real numbers, respectively. I_n represents an identity matrix of size n and $\mathbf{0}_{m\times n}$ a matrix of zeros of size $m\times n$. Moreover, all vector inequalities are element-wise inequalities, while $\text{sgn}(\cdot)$ and $|\cdot|$ are element-wise signum and absolute value operators, respectively, and for a vector v, $\text{diag}(v)$ is a matrix whose diagonals are elements of v. Further, a function $\alpha : [0,\infty) \to [0,\infty)$ is of class $\mathcal{K}_\infty$, if α is continuous, strictly increasing, unbounded, and $\alpha(0) = 0$.

B. Problem Formulation

Consider a linear continuous-time control system with additive previewable disturbances, Σ, given by

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + B_d d(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

with states $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$, a scalar output $y(t) \in \mathbb{R}$, control inputs $u(t) \in \mathcal{U} \subset \mathbb{R}^m$, and bounded previewable disturbances $d(t) \in \mathcal{D} \subseteq \mathbb{R}^l$. Moreover, we assume that the system Σ in (1) has relative degree 2, i.e., $CB = 0$ and $CAB \neq 0$, and the control input is bounded as $\mathcal{U} \triangleq \{u \mid |u| \leq u_m\}$, with $u_m \in \mathbb{R}^m$. Note that we use the term previewable disturbance in this paper to represent any exogenous inputs, signals or parameters for which preview or predictions of their future values may be available. For instance, the reference signal in a tracking problem or predicted trajectories of other agents can be treated as "previewable disturbances" and so can road curvatures, gradients or friction coefficients in driving scenarios that can be measured by (limited range) sensing and perception modules.

Definition 1 (System with Preview). *We denote a system* Σ with T_p-horizon preview *as Σ_p, where the disturbances within the next time horizon of $T_p(t)$, $\mathbf{d}_p(t) \triangleq \{d(\tau) \in \mathcal{D}, t \leq \tau \leq t + T_p(t)\} \in \mathcal{D}^{[0,T_p]}$, can be measured at each time t, with $\mathcal{D}^{[0,T_p]}$ being the set of all $\mathbf{d}_p(t)$. Thus, the control input $u(t)$ at time t can be determined based on the state $x(0)$ and the disturbances $d(\tau)$ for τ from 0 to $t+T_p(t)$.*

Assumption 1. *For a known (potentially time-varying) preview horizon $T_p(t)$ at given time $t \in \mathbb{R}_+$, the previewable disturbance $\mathbf{d}_p(t) \triangleq \{d(\tau) \in \mathcal{D}, t \leq \tau \leq t+T_p(t)\}$ is known.*

Definition 2 (Safe Sets). *Let $S_x \subseteq \mathbb{R}^n$ be a* safe set *of Σ_p that describes desirable/given safety constraints on the states, and let $S_{x,p} \subseteq \mathbb{R}^n \times \mathcal{D}^{[0,T_p]}$ be the* T_p-augmented safe set *of Σ_p, defined as*

$$S_{x,p} \triangleq \{(x, \mathbf{d}_p) \mid x \in S_x, \mathbf{d}_p \in \mathcal{D}^{[0,T_p]}\}.$$

Definition 3 (Controlled Invariant Set). *A set $\mathcal{C} \subseteq S_x$ is a* controlled invariant set *of Σ_p in a safe set $S_x \subseteq \mathbb{R}^n$ if for all $x(0) \in \mathcal{C}$, there exists some $u(t) \in \mathbb{R}^m$ such that for all $d(t) \in \mathcal{D}$, we have $x(t) \in \mathcal{C} \subseteq S_x$, $\forall t \geq 0$. $\mathcal{C}_{max}$ is the* maximal controlled invariant set *in S_x if $\mathcal{C}_{max}$ contains any controlled invariant set of Σ in S_x.*

Further, a set $\mathcal{C}_p \in S_{x,p}$ is a controlled invariant set *of Σ_p in an augmented safe set $S_{x,p}$ if for all $(x(0), \mathbf{d}_p(0)) \in \mathcal{C}_p$, there exists some $u(t) \in \mathbb{R}^m$ for all $t \geq 0$ such that $(x(t), \mathbf{d}_p(t)) \in \mathcal{C}_p \subseteq S_{x,p}$, $\forall t \geq 0$. $\mathcal{C}_{max,p}$ is the* maximal controlled invariant set *in $S_{x,p}$ if $\mathcal{C}_{max,p}$ contains any controlled invariant set of Σ_p in $S_{x,p}$.*

Further, the above definitions can be viewed as the continuous-time counterpart to the discrete-time definitions in [13]. More importantly, it was proven in [13] that in the discrete-time setting, there is much value in having preview in that the maximal controlled invariant sets for systems without preview is a subset of the maximal controlled invariant sets for systems with preview. Motivated by this work, we hypothesize that there is also value in preview for continuous-time systems.

In particular, instead of finding the maximal controlled invariant set for Σ_p, this paper will focus on control barrier functions with preview for Σ_p that renders some time-varying $\mathcal{C}_{x,t} \subseteq S_x$ controlled invariant by constructing a time-varying *preview safe set $\mathcal{C}_{x,p,t} \subseteq S_{x,p}$* that is controlled invariant and implies the controlled invariance of $\mathcal{C}_{x,t} \subseteq S_x$.

Definition 4 (Preview safe set). *Given a system with preview Σ_p (with known $\mathbf{d}_p$) and a corresponding safe set S_x, a set $\mathcal{C}_{x,p,t}$ defined over a time-varying preview-dependent nonlinear function $h : \mathcal{X} \times \mathcal{D}^{[0,T_p]} \times \mathbb{R}_+ \to \mathbb{R}$:*

$$\mathcal{C}_{x,p,t} \triangleq \{(x, \mathbf{d}_p, t) \mid h(x, \mathbf{d}_p, t) \geq 0\}, \tag{2}$$

$$\partial\,\mathcal{C}_{x,p,t} \triangleq \{(x, \mathbf{d}_p, t) \mid h(x, \mathbf{d}_p, t) = 0\}, \tag{3}$$

$$Int(\mathcal{C}_{x,p,t}) \triangleq \{(x, \mathbf{d}_p, t) \mid h(x, \mathbf{d}_p, t) > 0\}, \tag{4}$$

is a preview safe set for Σ_p if $(x(t), \mathbf{d}_p(t), t) \in \mathcal{C}_{x,p,t}$ for all $t \geq 0$ implies that $x(t) \in S_x$ for all $t \geq 0$.

Then, the problem of interest can be formally written as:

Problem 1 (Safety with Preview). *Given a system with preview Σ_p and a safe set S_x (cf. Definitions 1–2), construct a preview control barrier function corresponding to $\mathcal{C}_{x,p,t}$ in (2) that guarantees controlled invariance of Σ_p in S_x.*

III. Main results

The proposed approach to solve Problem 1 is detailed here.

A. Preview Control Barrier Functions (Prev-CBFs)

First, we expand the conventional definition of control barrier functions (CBFs) and safety to take advantage of the available preview information of disturbances.

Definition 5 (Preview CBF). *Given a system with preview Σ_p satisfying Assumption 1 and a safe set S_x (cf. Definitions 1–2), a continuously differentiable map $h : \mathcal{X} \times \mathcal{D}^{[0,T_p]} \times \mathbb{R}_+ \to \mathbb{R}$ of a preview safe set $\mathcal{C}_{x,p,t}$ in (2) is a preview CBF for Σ_p and S_x, if a class $\mathcal{K}_\infty$ function α exists such that*

$$\sup_{u\in\mathcal{U}} \dot{h}(x, u, \mathbf{d}_p, t) \geq -\alpha(h(x, \mathbf{d}_p, t)), \tag{5}$$

for all $x \in \mathcal{X}$, $t \in \mathbb{R}_+$. Further, for any $(x(t), \mathbf{d}_p(t), t) \in \mathcal{C}_{x,p,t}$, we define the corresponding safe input set:

$$K_{\mathcal{C}_{x,p,t}}(x,t) = \{u \in \mathcal{U} \mid \dot{h}(x,u,\mathbf{d}_p,t) \geq -\alpha(h(x,\mathbf{d}_p,t))\}. \quad (6)$$

Theorem 1 (Safety with Preview)**.** *Let S_x be the safe set for the system with preview Σ_p. If h is a Prev-CBF corresponding to the preview safe set $\mathcal{C}_{x,p,t}$ in* (2)*, then any Lipschitz continuous controller $u(x,t) \in K_{\mathcal{C}_{x,p,t}}(x,t)$ for Σ_p with $x(0) \in S_x$ renders the preview safe set $\mathcal{C}_{x,p,t}$ controlled invariant. Consequently, such a controller $u(x,t)$ also renders some set $\mathcal{C}_{x,t} \subset S_x$ for Σ_p controlled invariant, thus Σ_p is safe with preview, i.e., $x(t) \in S_x$ for all $t \geq 0$.*

Proof. Since h is a Prev-CBF corresponding to the preview safe set $\mathcal{C}_{x,p,t}$, any controller $u(x,t) \in K_{\mathcal{C}_{x,p,t}}(x,t)$ satisfies $\dot{h}(x,u,\mathbf{d}_p,t) \geq -\alpha(h(x,\mathbf{d}_p,t))$ for all $t \geq 0$, thus $\mathcal{C}_{x,p,t}$ is controlled invariant, i.e., $h(x,\mathbf{d}_p,t) \geq 0$ for all $t \geq 0$. Finally, by construction (cf. Definition 4), the controlled invariance of $\mathcal{C}_{x,p,t}$ implies that the system Σ_p is safe with preview for all $t \geq 0$ with respect to S_x. □

B. Closed-Form Candidate Preview CBF

First, we present a framework for deriving a closed-form candidate preview CBF (Prev-CBF) to obtain a preview safe set (cf. Definition 4) for the system with preview Σ_p (cf. Definition 1) from given desired output/state bounds that must be satisfied at all times in the form of

$$|y(t)| = |Cx(t)| \leq y_m, \quad \forall t \geq 0, \quad (7)$$

with inputs that are bounded as $|u(t)| \leq u_m$, where $y_m \in \mathbb{R}$ and $u_m \in \mathbb{R}^m$ are constant vectors[1]. Our framework is motivated by the observation that the design of a control input at the current time t with a future preview of the disturbances for a horizon of T_p is similar in spirit to the design of a control input at time $t - T$ for the current time in a linear time-delay system with input delay T:

$$\dot{x}'(t) = Ax'(t) + Bu'(t - T)$$

with state $x'(t)$ and input $u'(t)$. For such problems, one effective method to control the system is to consider a predictor-based model reduction approach, e.g., [16], [17], where a change of state to the future state

$$x'(t+T) = e^{AT}x'(t) + \int_t^{t+T} e^{A'(t+T-\tau)}Bu'(\tau - T)d\tau$$

is considered.

In our problem with Σ_p (cf. Definition 1), the "predicted" state T_p seconds into the future is given by

$$x(t+T_p) = \epsilon(t,T_p) + \int_t^{t+T_p} e^{A(t+T_p-\tau)}Bu(\tau)d\tau, \quad (8)$$

where $\epsilon(t,T_p) \triangleq e^{AT_p}x(t) + \int_t^{t+T_p} e^{A(t+T_p-\tau)}B_d d(\tau)d\tau$. Then, to ensure the satisfaction of the output/state bounds in (7) at all times, we enforce that that maxima or minima of the output trajectories in the (immediate) future under maximum deceleration or acceleration, respectively, always satisfy the output/state bounds at all times. By enforcing the constraints for the maxima or minima, this also ensures the constraint satisfaction for all times between the current time t to the time corresponding to the maxima or minima, which we call the (minimum) *stopping time*, as defined in the following:

Definition 6 (Stopping Time T_s)**.** *For the system Σ in* (1)*, at any given time $t \geq 0$, we define the (minimum) stopping time $T_s(t)$ as the* minimum *time such that $\dot{y}(t+T_s) = 0$ under maximum deceleration when $\dot{y} \geq 0$ or maximum acceleration when $\dot{y} \leq 0$ (since we consider a relative degree 2 system).*

Then, since at all times, we enforce the satisfaction of the output/state constraints for the moving future time horizon that includes the current time, this is equivalent to ensuring the controlled invariance of a time-varying set that is guaranteed to not violate the output/state constraints. In other words, the satisfaction of output/state bounds in (7) for all times is guaranteed by enforcing that the predicted outputs/states T_s seconds into the future, where T_s is the stopping time, always satisfy the output/state bounds, as follows:

$$|Cx(t+T_s)| \leq y_m, \quad \forall t \geq 0. \quad (9)$$

Note that this idea of utilizing the maxima or minima of a future "predicted" output trajectory is similar to the concept of maximizers/minimizers in [15] for a "predicted" reference/desired trajectory, and the notion of (minimum) stopping time is similarly used in the derivation of the closed-form CBF for a lane keeping example in [6] under the assumption of a constant maximum acceleration/deceleration.

Moreover, we make a simplifying assumption henceforth in this paper for the sake of ease of exposition, as given below. It is possible to have shorter or longer preview horizon than the stopping time with some modifications, which we will further explore in a future work.

Assumption 2. *The stopping time and the preview horizon coincide, i.e., $T_p(t) = T_s(t)$ for all $t \geq 0$.*

Then, under this assumption, we present a candidate Prev-CBF for obtaining a controlled invariant preview safe set. For better readability, the time dependence of T_p and T_s is dropped for the remainder of the paper, except for emphasis.

Lemma 1 (Closed-Form Candidate Preview CBF)**.** *Suppose Assumptions 1–2 hold, Then,*

$$h(x(t),\mathbf{d}_p(t),t) = y_m + D(t)u_m - sgn(\dot{y}(t))C\epsilon(t,T_s) \geq 0, \quad (10)$$

with $D(t) \triangleq (\int_t^{t+T_s} Ce^{A(t+T_s-\tau)}Bd\tau)diag(sgn(CAB))$ and $\epsilon(t,T_s)$ defined below (8) *(with $T_p = T_s$), is a valid candidate Prev-CBF to guarantee the satisfaction of the safety bounds in* (9) *corresponding to $\mathcal{C}_{x,p,t}$ and, in turn, the safety bounds in* (7) *corresponding to S_x, as is required by the construction of $\mathcal{C}_{x,p,t}$ in Definition 4.*

Proof. First, under the assumption of a relative degree 2 system, we can find the output acceleration as

$$\ddot{y}(t) = CA^2x(t) + CABu(t) + CAB_d d(t) + CB_d\dot{d}(t). \quad (11)$$

[1]Note that a simple modification can be carried out to also handle interval bounds on the output/state and the input that are asymmetric by designating the midpoints of the intervals as a known/previewable disturbance and the deviation from the midpoints as symmetrically bounded auxiliary output/states or inputs.

Then, in accordance to the definition of the (minimum) stopping time T_s (cf. Definition 6) that the control input leads to the maximum deceleration when $\dot{y} > 0$ or to the maximum acceleration when $\dot{y} < 0$, it can be shown that this maximum deceleration/acceleration is achieved by $u(\tau) = -\text{sgn}(\dot{y}(t))\text{diag}(\text{sgn}(CAB))u_m$, $\forall \tau \in [t, t+T_s]$, which can also be shown to trivially satisfy $|u(\tau)| \leq u_m$, and we can further find that $CABu(\tau) = -\text{sgn}(\dot{y}(t))|CAB|u_m$.

Then, from (8), the output T_s seconds into the future is

$$y(t+T_s) \quad = C\epsilon(t, T_s) - \text{sgn}(\dot{y}(t))D(t)u_m \tag{12}$$

when $\dot{y}(t) > 0$, with $\epsilon(t, T_s)$ and $D(t)$ defined below (8) and (10), respectively, with $T_p = T_s$. Thus, enforcing $C\epsilon(t, T_s) - D(t)u_m \leq y_m$ trivially implies $y(t+T_s) \leq y_m$. Similarly for $\dot{y}(t) < 0$, we can obtain the lower bound $C\epsilon(t, T_s) + D(t)u_m \geq -y_m$. The two constraints can be combined[2] as

$$y_m + D(t)u_m - \text{sgn}(\dot{y}(t))C\epsilon(t, T_s) \geq 0,$$

the satisfaction of which guarantees the satisfaction of the safety bounds in (9), $T_p = T_s$ seconds into the future. Thus, this is a valid candidate preview control barrier function (Prev-CBF) where there always exists a sequence of inputs $u(\tau)$, specifically $u(\tau) = -\text{sgn}(\dot{y}(t))\text{diag}(\text{sgn}(CAB))u_m$, such that $|u(\tau)| \leq u_m$ for all $\tau \in [t, t+T_s]$ that can satisfy $h(x(t), \mathbf{d}_p(t), t) \geq 0$ for all $t \geq 0$. Finally, by virtue of the constraint in (9) being satisfied at a minima or maxima (i.e., when $\dot{y}(t+T_s) = 0$), the safety bounds is also satisfied for the entire time horizon from t to $t+T_s$; hence, (7) holds. □

C. Stopping Time

As is evident from (10) in Lemma 1, the construction of the candidate Prev-CBF depends on the stopping time $T_s(t)$ and further, we will find its time derivative $\dot{T}_s(t)$ for enforcing the safety constraint in (5).

Lemma 2 (Stopping Time). *The stopping time $T_s(t)$ is given by the (smallest positive) solution to the following equation:*

$$\begin{aligned} &CAx(t) + CEd(t) + \textstyle\int_t^{t+T_s} CA^2[e^{A(\tau-t)}x(t) \\ &+\textstyle\int_t^{\tau} e^{A(\tau-s)}(B_d d(s) - sgn(\dot{y}(t))B diag(sgn(CAB))u_m)ds] \\ &-|CAB|u_m + CAB_d d(t) + CB_d\dot{d}(t)d\tau = 0, \end{aligned} \tag{13}$$

and its time-derivative is given by:

$$\begin{aligned} \dot{T}_s(t) = &-1 - \Delta(t)^{-1}[(CAB + G(t))u(t) \\ &+sgn(\dot{y}(t)(|CAB| + G(t)diag(sgn(CAB)))u_m], \end{aligned} \tag{14}$$

with $G(t)$ and $\Delta(t)$ defined as:

$$\begin{aligned} &G(t) = \textstyle\int_t^{t+T_s(t)} CA^2 e^{A(\tau-t)}B d\tau, \\ &\Delta(t) \\ &= CA^2 e^{AT_s(t)}x(t) - sgn(\dot{y}(t))|CAB|u_m + CB_d\dot{d}(t+T_s(t)) \\ &\quad +CAB_d d(t+T_s(t)) + \textstyle\int_t^{t+T_s(t)} CA^2 e^{A(t+T_s(t)-s)} \\ &\quad\quad (B_d d(s) - sgn(\dot{y}(t))B diag(sgn(CAB))u_m)ds. \end{aligned}$$

Proof. As described in the proof of Lemma 1, $u(\tau) = -\text{sgn}(\dot{y}(t))\text{diag}(\text{sgn}(CAB))u_m$, $\forall \tau \in [t, t+T_s]$ as per the definition of stopping time provided by Definition 6. In this case, the maximal output deceleration/acceleration from (11) corresponding to the input is given by:

$$\ddot{y}(\tau) = CA^2x(\tau) - \text{sgn}(\dot{y}(t))|CAB|u_m + CAB_d d(\tau) + CB_d\dot{d}(\tau).$$

[2]This also applies when $\dot{y}(t) = 0$ since $T_s(t) = 0$ and thus, $D(t) = 0$.

Then, we can derive the stopping time T_s by integrating the above with respect to time and setting the corresponding output velocity to 0, i.e.,

$$\begin{aligned} \dot{y}(t+T_s) = \dot{y}(t) + \textstyle\int_t^{t+T_s}(&CA^2x(\tau) - \text{sgn}(\dot{y}(t))|CAB|u_m \\ &+CAB_d d(\tau) + CB_d\dot{d}(\tau)d\tau) = 0, \end{aligned}$$

with $x(\tau) = e^{A(\tau-t)}x(t) + \int_t^{\tau} e^{A(\tau-s)}B_d d(s) - \text{sgn}(\dot{y}(t))e^{A(\tau-s)}B\text{diag}(\text{sgn}(CAB))u_m ds$ and $\dot{y}(t) = CAx(t) + CEd(t)$, which reduces to (13).

Further, to obtain $\dot{T}_s$, we carry out an implicit differentiation of (13) with respect to time to yield

$$\begin{aligned} &CABu(t) + \Delta(t)(1 + \dot{T}_s) + \text{sgn}(\dot{y}(t))|CAB|u_m \\ &+G(t)(u(t) + \text{sgn}(\dot{y}(t))\text{diag}(\text{sgn}(CAB))u_m) = 0, \end{aligned}$$

with $G(t)$ and $\Delta(t)$ defined below (14), from which $\dot{T}_s(t)$ can be found as in (14). □

D. Closed-Form Preview Control Barrier Function

Equipped with the candidate Prev-CBF in Lemma 1 and the stopping time and its time derivative in Lemma 2, we next show that it is indeed a valid Prev-CBF according to the definition in Definition 5.

Proposition 1 (Closed-Form Preview CBF). *Given a system with preview Σ_p (cf. Definition 1) that satisfies Assumptions 1 and 2 with stopping time $T_s(t)$ and its time-derivative $\dot{T}_s(t)$ in Lemma 2, the mapping $h : \mathbb{R}^n \times \mathcal{D}^{[0,T_p]} \times \mathbb{R}_+ \to \mathbb{R}$ given by* (10) *is a preview control barrier function (Prev-CBF) for Σ_p, if a class $\mathcal{K}_\infty$ function α exists such that* (5) *holds with*

$$\dot{h}(x, u, \mathbf{d}_p, t) = \dot{D}(t)u_m - sgn(\dot{y}(t))C\dot{\epsilon}(t, T_s), \tag{15}$$

where $\dot{D}(t)$ and $\dot{\epsilon}(t, T_s)$ can be computed as

$$\begin{aligned} \dot{D}(t) = &\left[\textstyle\int_t^{t+T_s} CAe^{A(t+T_s-\tau)}Bd\tau(1+\dot{T}_s) - Ce^{AT_s}B\right] \\ &diag(sgn(CAB)), \\ \dot{\epsilon}(t, T_s) = &\ Ae^{AT_s}x(t) + B_d d(t+T_s) + Ce^{AT_s}Bu(t) \\ &+(1+\dot{T}_s)\textstyle\int_t^{t+T_s} Ae^{A(t+T_s-\tau)}B_d d(\tau)d\tau. \end{aligned}$$

Further, (9) *holds and thus, the output/state bounds* (7) *holds.*

Proof. This proposition follows directly from applying Theorem 1 to the closed-form candidate Preview CBF in Lemma 1. In particular, the closed-form condition for (5) can be found by deriving the time derivative of the candidate Prev-CBF that can be obtained as in (15) with $\dot{D}(t)$ and $\dot{\epsilon}(t, T_s(t))$ that can derived as follows:

$$\begin{aligned} \dot{D}(t) &= [\tfrac{d}{dt}(\textstyle\int_t^{t+T_s(t)} Ce^{A(t+T_s(t)-\tau)}Bd\tau)]\text{diag}(\text{sgn}(CAB)) \\ &= [(\textstyle\int_t^{t+T_s(t)} CAe^{A(t+T_s(t)-\tau)}Bd\tau)(1+\dot{T}_s(t)) \\ &\quad -Ce^{AT_s(t)}B]\text{diag}(\text{sgn}(CAB)), \end{aligned}$$

$$\begin{aligned} &\dot{\epsilon}(t, T_s(t)) \\ &= \tfrac{d}{dt}(e^{AT_s(t)}x(t) + \textstyle\int_t^{t+T_s(t)} e^{A(t+T_s(t)-\tau)}B_d d(\tau)d\tau) \\ &= Ae^{AT_s(t)}x(t) + B_d d(t+T_s(t)) + e^{AT_s(t)}Bu(t) \\ &\quad +(1+\dot{T}_s(t))\textstyle\int_t^{t+T_s(t)} Ae^{A(t+T_s(t)-\tau)}B_d d(\tau)d\tau, \end{aligned}$$

by applying chain and Leibniz integral rules, with T_s and $\dot{T}_s$ given in Lemma 2. □

E. Optimization-Based Safety Control

Next, we show that these Prev-CBFs can be coupled with a legacy (stabilizing) controller to stabilize/control the system while guaranteeing safety.

Proposition 2 (Optimization-Based Safety Control). *Given any (legacy) feedback controller $u = k(x,t)$ for the system Σ_p, we can minimally modify this controller to guarantee safety for each time instance t using $u(x, \mathbf{d}_p, t)$ that is obtained from solving the following quadratic program (QP):*

$$\begin{aligned} u(x, \mathbf{d}_p, t) = \arg\min_{u \in \mathcal{U}} \tfrac{1}{2}\|u - k(x,t)\| \\ s.t.\ P(t)u \leq q(t), \end{aligned} \tag{16}$$

with $T_s(t)$, $G(t)$ and $\Delta(t)$ from Lemma 2, $\epsilon(t, T_s(t))$ defined below (8), and any class $\mathcal{K}_\infty$ function α, as well as

$$\begin{aligned} P(t) &\triangleq sgn(\dot{y}(t))Ce^{AT_s(t)}B + \Phi(t)(CAB + G(t)), \\ q(t) &\triangleq \alpha(y_m + D(t)u_m - sgn(\dot{y}(t))C\epsilon(t, T_s(t))) \\ &\quad - sgn(\dot{y}(t))(CAe^{AT_s(t)}x(t) + CB_d d(t + T_s(t))) \\ &\quad - Ce^{AT_s(t)}Bdiag(sgn(CAB))u_m \\ &\quad - sgn(\dot{y}(t))\Phi(t)(|CAB| + G(t)diag(sgn(CAB)))u_m, \\ \Phi(t) &\triangleq \Delta(t)^{-1}\int_t^{t+T_s(t)} CAe^{A(t+T_s(t)-\tau)} \\ &\quad (Bdiag(sgn(CAB))u_m - sgn(\dot{y}(t))B_d d(\tau))d\tau. \end{aligned}$$

Proof. The Prev-CBF constraint in Theorem 1 with h from Lemma 1 and $\dot{h}$ from Proposition 1 can be written as:

$$\begin{aligned} &\dot{D}(t)u_m - \mathrm{sgn}(\dot{y}(t))C\dot{\epsilon}(t, T_s(t)) \\ &\geq -\alpha(y_m + D(t)u_m - \mathrm{sgn}(\dot{y}(t))C\epsilon(t, T_s(t))). \end{aligned}$$

Then, by substituting the definitions of $D(t)$, $\dot{D}(t)$ and $\epsilon(t, T_s(t))$ in Lemma 1, Proposition 1 and below (8) into the above and rearranging, we obtain the constraint $P(t)u \leq q(t)$ in (16). □

Solving (13) analytically to find $T_s(t)$ for the application of Proposition 2 is non-trivial, but it can be found numerically, e.g., using MATLAB functions `fsolve` or `fzero`. Moreover, it was observed in simulations that the computational time of solving the QP in (16) can further be reduced by symbolically solving (13), e.g., with MATLAB Symbolic Math Toolbox (including `vpasolve`), and using the symbolic solution when solving the QP at run time.

IV. Illustrative Examples

A. Assistive Shoulder Exoskeleton Robot

First, we consider the problem of an industrial shoulder exoskeleton robot system [18] with the following dynamics:

$$I_j\ddot{e} + B_j\dot{e} + K_j e = \tau_e + u, \tag{17}$$

where $e(t) = \theta(t) - \theta_d(t)$ is the angular deviation/error from a desired trajectory $\theta_d(t)$, and $\dot{e}$, $\ddot{e}$ are the associated error velocity and acceleration, respectively. $I_j \triangleq I_h + I_r$, $B_j \triangleq K_h + K_r$ and $K_j \triangleq K_h + K_r$ represent inertia, damping coefficient and stiffness of the joint human-exoskeleton system, with I_h, B_h and K_h being the human shoulder inertia, damping coefficient and stiffness, while I_r, B_r and K_r are the robot inertia, damping coefficient and stiffness.

We assume that the shoulder robot and human shoulder joint are matched and that the interaction torque between robot and human, τ_e, is previewable and satisfies Assumptions 1 and 2. The control input u act like a damping or stiffness torque that induces acceleration or deceleration to prevent system states from exceeding predetermined safety limit. In this example, the safety constraint we consider is $|e(t)| \leq \delta$ for all $t \geq 0$.

Putting the system into the state space form Σ in (1) with state $x(t) = \begin{bmatrix} e^T(t) & \dot{e}^T(t) \end{bmatrix}^T$, input $u(t)$ and previewable disturbance $d(t) = \tau_e(t)$, the system matrices are given by

$$A = \begin{bmatrix} 0 & I \\ -I_j^{-1}K_j & -I_j^{-1}B_j \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_j^{-1} \end{bmatrix}, B_d = \begin{bmatrix} 0 \\ I_j^{-1} \end{bmatrix}, \\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

where the system output $y(t) = e(t) = Cx(t)$ satisfy the safety constraint, i.e., $|e(t)| \leq \delta$, that can be expressed in the form of (7) with $y_m = \delta$ and the control input is bounded as $|u(t)| \leq u_m$.

The system parameters/signals used in this example are $I_j = 1\,Nms^2/rad$, $B_j = 2\,Nms/rad$, $K_j = 2\,Nm/rad$, $\tau_e(t) = 0.43\sin(0.2\pi t)\,Nm$ and $\delta = 0.2\,rad$, while u_m is varied between 1.0 to 3.0 in this example to study its impact on both our proposed preview CBF approach in Section III and the standard ("lane keeping") CBF approach in [6], which are described in more detail below. Moreover, for simplicity and ease of exposition, the legacy controller we consider when applying an optimization-based safety controller (including in (16) for the proposed preview CBF approach) is considered to be identically zero, i.e., $k(x,t) = 0$, and as such, the control input $u(t)$ also represents the deviation/intervention that the safety controller must provide to ensure safety, which will also be compared when applying both the preview CBF and standard CBF approaches.

1) Preview CBFs: The proposed Prev-CBF (cf. Section III) enforces the safety/controlled invariance guarantees for the shoulder robot to satisfy $|e(t)| = |Cx(t)| \leq y_m = \delta$ with control input $u(t)$ that is constrained by $|u(t)| \leq u_m$. Specifically, we consider the optimization-based safety controller (16) in Proposition 2 with "zero" legacy control, i.e., $k(x,t) = 0$, and the closed-form Prev-CBF described in Lemmas 1 and 2 and Proposition 1.

2) Standard CBF: For comparison, we also applied the standard CBF approach in [6], specifically the CBF for the lane keeping example in [6, Section V-B], where, in addition to the output constraint:

$$|y(t)| = |e(t)| = |Cx(t)| \leq y_m,$$

it was assumed that the output acceleration is bounded, i.e.,

$$|\ddot{y}(t)| = |\ddot{e}(t)| = |C\ddot{x}(t)| \leq a_{\max},$$

by a constant $a_{\max}$, which in the presence of input constraints $|u| \leq u_m$, is implicitly bounded by the dynamics in (17):

$$u = I_d\ddot{e} + B_d\dot{e} + K_d e - \tau_e.$$

Thus, under the assumption of "preview" information about the bounds $\dot{e}_{\max}$, $e_{\max}$ and $\tau_{e,\max}$ of $|\dot{e}(t)| \leq \dot{e}_{\max}$, $|e(t)| \leq e_{\max}$ and $|\tau_e(t)| \leq \tau_{e,\max}$ for all $t \geq 0$, by triangle inequality,

$$|u| \leq I_d a_{\max} + B_d\dot{e}_{\max} + K_d e_{\max} + \tau_{e,\max} = u_m$$

such that the output acceleration bound $a_{\max}$ must satisfy

$$a_{\max} = I_d^{-1}(u_m - B_d\dot{e}_{\max} - K_d e_{\max} - \tau_{e,\max}). \tag{18}$$

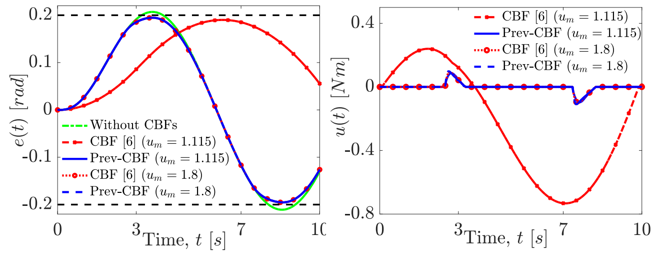

Fig. 1: Angular error (**left**) and input (**right**) trajectories: (i) Without CBFs (exceeds black dashed bounds), (ii) standard CBF [6] with $u_m = 1.115$, (iii) Prev-CBF with $u_m = 1.115$, (iv) standard CBF [6] with $u_m = 1.8$ and (v) Prev-CBF with $u_m = 1.8$.

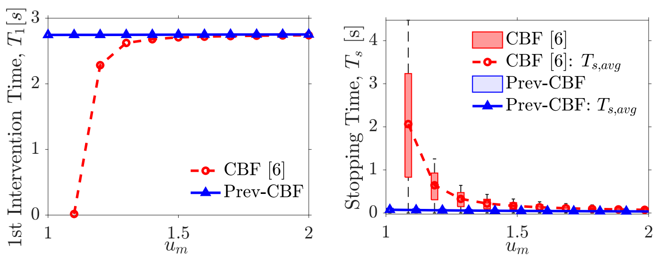

Fig. 2: First intervention times (**left**) and stopping times (**right**; as box plots) when varying u_m between 1 and 2; standard CBF [6] generally intervenes earlier and has longer stopping times than the proposed Prev-CBF with diminishing benefits as u_m is increased.

Moreover, in order to satisfy the output acceleration bound, the control input of the standard CBF approach must satisfy

$$u(t) \in [-I_d a_{\max} + F_0(t), I_d a_{\max} + F_0(t)], \qquad (19)$$

where $F_0(t) \triangleq B_d \dot{e}(t) + K_d e(t) - \tau_e(t)$.

Under the above problem formulation, the standard lane keeping CBF approach in [6, Section V-B] proposed the following CBF:

$$h(x) = (y_m - sgn(\dot{y}(t))y(t)) - \frac{\dot{y}(t)^2}{2a_{\max}}. \qquad (20)$$

In our simulation, we assume the following parameters: $\dot{e}_{\max} = 0.1326$, $e_{\max} = 0.2$ and $\tau_{e,\max} = 0.43$, and by varying the input bounds u_m from 1.0 to 3.0, we correspondingly consider a_{max} from 0.0046 to 1.9046 in order to satisfy (18).

Figure 1 shows the simulation results of the angular error and input trajectories when no safety controller is applied (without CBFs) as well as when the standard CBF in [6] and Prev-CBF are applied with $u_m = 1.115$ and $u_m = 1.8$. We observe that without a safety controller, the safety constraint (black dashed lines) is violated, while the angular errors with both standard CBF and Prev-CBF (i.e., with the safety constraints) are safe. However, when u_m is small, the standard CBF leads to a large deviation from the case without CBFs while Prev-CBF remains relatively close to the (desired) trajectory without CBFs. This is also observed in the input trajectories where the intervention input $u(t)$ is large and starts very early with the standard CBF, whereas Prev-CBF only intervenes close to the safety boundaries. On the other hand, when $u_m = 1.8$, both standard CBF and Prev-CBF do not need to intervene much or early, although the standard CBF still intervenes earlier than the Prev-CBF.

Moreover, we investigated the impact of varying the input bound u_m on the first intervention time T_1, defined as the first time the (intervention) input is non-zero. The closer this time is to 3.2089 s (the time of safety violation without CBFs), then the later the safety intervention needs to be initiated. In other words, smaller T_1 indicates that the safety controller is more permissive and does not modify the legacy controller unnecessarily. Further, we studied the impact of varying the input bound on the (minimum) stopping time T_s, where a larger stopping time indicates that the associated safety controller needs a longer time (and has less control authority) to bring the system to a stop and to avoid a safety violation. As can be observed in Figure 2, Prev-CBFs intervene later than standard CBFs and has shorter stopping times, indicating that Prev-CBFs are able to take advantage of preview information to be more permissive and less conservative than standard CBFs. It is also noteworthy, that when $u_m = 1.0$, $a_{\max}$ in (18) becomes negative, indicating that the input set in (19) becomes empty and as a result, the standard CBF is infeasible, which is the reason why the results of the standard CBF only starts at $u_m = 1.1$, while the Prev-CBF remains feasible and the plot starts at $u_m = 1.0$.

B. Lane Keeping with Road Curvature Preview

Next, we consider the example of centering a vehicle in its lateral direction given a preview of the road curvature. Specifically, we consider the lane-keeping problem in [6, Section V-B], whose dynamics can be written as in (1) with

$$A = \begin{bmatrix} 0 & 1 & v_0 & 0 \\ 0 & -\frac{C_f+C_r}{Mv_0} & 0 & \frac{bC_r-aC_f}{Mv_0} - v_0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{bC_r-aC_f}{I_z v_0} & 0 & -\frac{a^2C_f+b^2C_r}{I_z v_0} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{C_f}{M} \\ 0 \\ a\frac{C_f}{I_z} \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}^\top, C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

where the state is $x := [y, \nu, \psi, r]^\top$, with lateral displacement y, lateral velocity ν, error yaw angle ψ and yaw rate r. The front tire steering angle constitutes the input u to the system. The desired yaw rate $r_d = \frac{v_0}{R}$, where v_0 is the constant longitudinal velocity and R is the road curvature. Moreover, the system parameters/signals are taken from [6, Section V-B]: Vehicle mass $M = 1650\,kg$, moment of inertia about center of mass $I_z = 2315.3\,kgm^2$, distances of front and rear wheels from center of mass $a = 1.11\,m$ and $b = 1.59\,m$, respectively, and front and rear tire stiffness parameters $C_f = 98800\,N/rad$ and $C_r = 133000\,N/rad$, respectively.

Similar to [6], a legacy controller $k(x,t) = -K(x - x_{ff})$, with $x_{ff} = [0\ 0\ 0\ r_d]^\top$, is considered for centering the vehicle in the lane, while safety constitutes satisfaction of the lateral displacement constraint $|y| \leq y_m$ owing to the vehicle not crossing the lane boundaries, where y_m is chosen as 0.6 m in this example. Moreover, the actuation limit $|u| \leq u_m$ is considered with different values of u_m for comparison.

1) Preview CBFs: The lane-keeping Prev-CBF directly enforces the safety/controlled invariance guarantees to satisfy $|y(t)| = |Cx(t)| \leq y_m$, given actuation constraints $|u(t)| \leq u_m$ using the optimization-based safety controller (16) in Proposition 2 with the legacy controller $u = -K(x - x_{ff})$, and the closed-form Prev-CBF described in Lemmas 1 and 2 and Proposition 1.

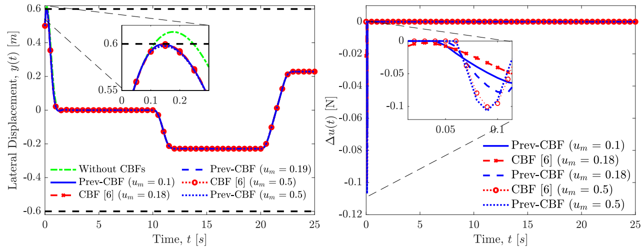

Fig. 3: Lateral displacement trajectories $y(t)$ (**left**) and trajectories of CBF intervention of $u(t)$ (**right**) given by $\Delta u(t) \triangleq u(t) - k(x(t), t)$, where $k(x(t), t)$ is the legacy controller.

2) Standard CBF: For comparison, we consider the standard CBF approach for lane-keeping devised in [6, Section V-B] given in (20), where $y(t)$ and $\dot{y}(t) = \nu + \psi v_0$ are obtained from the system dynamics, y_m represents the lane width from the center and the acceleration limit $a_{\max}$ are obtained from input constraints $|u| \leq u_m$ and the relationship between the acceleration $\ddot{y}$ and input u from the system dynamics:

$$M\ddot{y} = C_f u - F_0,$$

with $F_0 = C_f \frac{\nu + ar}{v_0} + C_r \frac{\nu - br}{v_0} + M v_0 r_d$ that is assumed to satisfy $|F_0| \leq F_{0,\max}$. Specifically, $a_{\max}$ can be found as

$$a_{\max} = \tfrac{1}{M}(C_f u_m - F_{0,\max})$$

and the control input must satisfy:

$$u(t) \in [\tfrac{1}{C_f}(-M a_{\max} + F_0(t)), \tfrac{1}{C_f}(M a_{\max} + F_0(t))]. \quad (21)$$

From Fig. 3 (left), it can be observed that the vehicle crosses the lane boundary (black dashed lines) without any CBFs, while both Prev-CBF and the CBF in [6] guarantee safety with input limits $u_m = 0.18$ and $u_m = 0.5$. If we further decrease the input limits to a very small value of u_m, both Prev-CBF and standard lane-keeping CBF [6] lack sufficient actuation authority to drive the system to safety. However, Prev-CBF is applicable for a broader input range than standard CBF in [6], indicating that the preview information provides an advantage. This can be observed in Fig. 3 (left) for $u_m = 0.1$, where Prev-CBF guarantees safety, while standard CBF failed (and hence, not depicted). Additionally, Fig. 3 (right) demonstrates that for any given input constraint u_m, such as $u_m = 0.18$, the proposed Prev-CBF requires an intervention to modify the legacy controller later than the CBF in [6], while still ensuring safety.

Discussion of Results: In summary, there is value for preview information in both simulation examples in terms of delaying the intervention and decreasing the amount of intervention, but this value of preview decreases as u_m increases. This is as expected since when the input range is larger relative to the worst-case magnitudes of the previewable disturbances, then the stopping time is dominated by the input bounds and the feasible input sets for the standard CBF in (19) and (21) become essentially unbounded. Moreover, this result confirms the findings in [13] for the value of preview in discrete-time systems and shows that there is similarly value in preview information for continuous-time systems.

V. Conclusion

In this paper, we presented a novel preview control barrier function for linear systems with previewable disturbances for a (brief) window in the future, as the continuous-time counterpart of some recent work in [13], [14] for discrete-time systems, and similarly show that there is value in preview information in terms of minimally intervening/modifying the inputs of a legacy/human controller while guaranteeing safety. In comparison with standard CBFs, the preview information enables us to utilize the full range of the control authority and is thus less conservative, which was demonstrated using assistive shoulder robot and lane-keeping examples. Future directions include the extensions of preview CBFs to consider nonlinear continuous-time systems with higher relative degrees as well as the presence of non-previewable uncertainties or insufficiently long preview.

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