On Challenges in Coordinate Transformation for Using a High-Gain Multi-Output Nonlinear Observer*

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Abstract— **This paper considers the design of a multi-output high gain observer for a vehicle trajectory tracking application. The high gain observer approach offers the advantages of guaranteed feasibility and global stability with just one constant observer gain for this application. The challenges of transforming the vehicle dynamic model into the required companion form for applying the high gain observer technique are addressed. Transforming a traditional kinematic model to companion form is found to result in an increased number of states. Instead, a coordinate transformation that allows for varying velocity and varying slip angle is shown to be appropriate. The high gain observer methodology for a dynamic system with multiple outputs is presented and the calculation of the Lipschitz constant for the vehicle tracking application is discussed.**

I. Introduction

A. Challenges with High-Gain Observer Design

LMI (Linear Matrix Inequality) based nonlinear observers [1, 2] are powerful tools in state estimation as they come with proof of stability and are relatively easy to implement in real-world applications. The observer gain matrix needs to be found by solving a LMI problem. If found, the LMI problem is said to be feasible and the observer dynamics will then be stable. However, there is no guarantee that a solution to the LMI problem can be found for a specific application. Therefore, one of the main challenges of designing these observers is the unknown feasibility of the LMI problem: the existence of stable observer gains is not guaranteed.

A different kind of estimator is the high-gain observer [3]. For nonlinear systems in (transpose) companion form, stable high-gain observers are guaranteed to exist, if the involved nonlinear functions are Lipschitz [3, 4]. Thus, the high-gain observer always has a feasible solution, while the LMI-based observers in general do not. With all the benefits of using high-gain observers, very few real-world applications of these observers can be found in the literature as the transformation of the nonlinear systems into the required companion form is non-trivial. Also, multi-output applications of high-gain observers have been seldom (if at all) utilized in the literature. In this paper, we show how to design a high gain observer for a vehicle system that is originally not in the companion form and has multiple outputs. The design process can be inspiring for other systems and might help lead to more practical applications of high- gain observers in the future.

B. Vehicle Tracking Problem

Tracking surrounding vehicles is critical in many applications such as collision avoidance and autonomous driving [2]. By tracking the trajectories of other vehicles on the road, essential variables in collision prediction (e. g. time-to-collision) can be calculated using estimates of the vehicles' position, velocity, and orientation [5]. Hence, designing observers to accurately estimate surrounding vehicles' states is valuable. However, vehicle motion typically involves nonlinear dynamic models. Some of the previous works addressed this problem by turning the original nonlinear model into multiple linear models typically including a "straight line driving" and a "constant turn rate driving" model [6, 7]. Using these linear models, they utilize Interacting Multiple Model (IMM) filters (e. g. IMM Kalman Filters) for state estimation [6, 7]. These papers based on linearization lack a proof of global stability and do not cover all possible vehicle maneuvers. Also, implementing IMM filters is more computationally demanding as they require real-time evaluation of each model's probability.

Recently, a few papers have investigated the use of LMI-based nonlinear observers for vehicle state estimation. They are obtained based on a single nonlinear model and include constant observer gains. Thus, they are easy to implement. However, they have some shortcomings including limited stability regions and the use of simplifying assumptions in the model (e.g. assumption of constant velocity). The designed observers are guaranteed to be stable only for a small region of steering angle and a limited range of vehicle direction angle due to the non-monotonic nonlinear functions involved in the model. Therefore, switched gain observers with different gains in different piecewise regions were required to cover the entire operating range. Also, these observers assume constant velocity and are not able to accurately estimate the states of vehicles with variable velocities. Hence, this paper will explore the design of high-gain observers allowing for variable velocity, guaranteed stability, and guaranteed feasibility.

The outline of the paper is as follows. Section II describes the design of a high-gain observer for multi-output applications. Section III discusses two different approaches for transformation of a kinematic vehicle model into the companion form to be used in high-gain observer design.

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Section IV describes the calculation of the observer gain matrix. Section V contains the conclusions.

II. Multi- Output High- Gain Observer Design

While the single output high-gain observer design is well-developed in the literature, the multi-output version is less well-known and is not available in a standard system result format. In this section, a high-gain observer is designed for a multi-output companion form system. Consider the following two-output system which is in companion form:

$$\dot{z} = Fz + Gf(z), \qquad y = Hz \tag{1}$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and $f(z)$ is the nonlinear function in the model. Assume the observer dynamics are:

$$\dot{\hat{z}} = F\hat{z} + Gf(\hat{z}) + L(y - H\hat{z}) \tag{3}$$

where L is the constant observer gain matrix. The observer error dynamics $\tilde{z}$ is derived based on (1) and (3):

$$\begin{aligned} \dot{\tilde{z}} &= \dot{z} - \dot{\hat{z}} \\ &= Fz + Gf(z) - F\hat{z} - Gf(\hat{z}) - L(y - H\hat{z}) \\ &= (F - LH)\tilde{z} + G\tilde{f}(z, \hat{z}) \end{aligned} \tag{4}$$

where $\tilde{z} = z - \hat{z}$ and $\tilde{f} = f(z) - f(\hat{z})$. Here, we assume that the nonlinear process equations are Lipschitz. In other words:

$$\left\|\tilde{f}_1\right\|_2 \le \gamma_1 \|\tilde{z}\|_2, \qquad \left\|\tilde{f}_2\right\|_2 \le \gamma_2 \|\tilde{z}\|_2 \tag{5}$$

Define the following transformation for the error variables:

$$e = T^{-1}(\theta)\tilde{z} \tag{6}$$

and

$$T(\theta) = \begin{bmatrix} \theta & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta^3 \end{bmatrix}, \qquad \theta > 1 \tag{7}$$

Finding the transformed error variable dynamics by implementing the transformation (6) on (4) yields:

$$T(\theta)\dot{e} = (F - LH)Te + G\tilde{f}(z, \hat{z}) \tag{8}$$

or

$$\dot{e} = (T^{-1}FT - KHT)e + T^{-1}G\tilde{f}(z, \hat{z}) \tag{9}$$

where K is the transformed observer gain matrix:

$$K = T^{-1}L \tag{10}$$

Note that:

$$T^{-1}FT = \theta F \tag{11}$$

and

$$HT = \theta H \tag{12}$$

Then, (9) is simplified as:

$$\dot{e} = \theta(F - KH)e + T^{-1}G\tilde{f}(z, \hat{z}) \tag{13}$$

Based on (5), we assume that there exists k_f such that:

$$\left\|T^{-1}G\tilde{f}(z, \hat{z})\right\|_2 \le k_f \|e\|_2 \tag{14}$$

Theorem 1. If there exists $P > 0$, $\lambda > 0$, L, and $\theta > 1$ such that:

$$F^T P + PF - H^T Q - Q^T H < -\lambda I \tag{15}$$

and

$$\theta > \theta_0 = \frac{2k_f \lambda_{max}(P)}{\lambda} \tag{16}$$

in which $\lambda_{max}(.)$ is the maximum eigenvalue, then the estimation error $\tilde{z}$ is exponentially stable by using the observer gain:

$$K = P^{-1}Q^T \tag{17}$$

Proof. Consider the following Lyapunov function candidate:

$$V = e^T P e, \qquad P > 0 \tag{18}$$

and P is symmetric. Taking derivative of this Lyapunov function:

$$\dot{V} = \dot{e}^T P e + e^T P \dot{e} \tag{19}$$

and replacing (13) in (19):

$$\begin{aligned} \dot{V} &= \theta e^T[(F - KH)^T P + P(F - KH)]e \\ &+ \tilde{f}^T(G^T T^{-1} P)e + e^T(PT^{-1}G)\tilde{f} \end{aligned} \tag{20}$$

Exploiting inner product notation, equation (20) can be modified:

$$\begin{aligned} \dot{V} &= \theta e^T[F^T P + PF - H^T K^T P - PKH]e \\ &+ 2\left(T^{-1}G\tilde{f}\right).(Pe) \end{aligned} \tag{21}$$

Using the Cauchy-Schwarz inequality:

$$\begin{aligned} \dot{V} &\le \theta e^T[F^T P + PF - H^T K^T P - PKH]e \\ &+ 2\left\|T^{-1}G\tilde{f}\right\|_2 \|P\|_2 \|e\|_2 \end{aligned} \tag{22}$$

For the positive definite Hermitian matrix P:

$$\|P\|_2 = \sigma_{max}(P) = |\lambda(P)|_{max} = \lambda_{max}(P) \tag{23}$$

Use (14), (17), and (23) in (22):

$$\dot{V} \le e^T\left[\theta(F^T P + PF - H^T Q - Q^T H) + 2k_f \lambda_{max}(P) I\right]e \tag{24}$$

Based on (15), (16), and (24):

$$\begin{aligned} \dot{V} &\le e^T\left[-\lambda\theta I + 2k_f \lambda_{max}(P) I\right]e \\ &= \left(-\lambda\theta + 2k_f \lambda_{max}(P)\right) e^T e < 0 \end{aligned} \tag{25}$$

$\dot{V} < 0$ and the proof is complete. ∎

III. Vehicle Model Transformation

In this section, we investigate the transformation of a vehicle model to the companion form needed for the high-gain observer design. A standard transformation of the original vehicle model comes with significant disadvantages. A modified model, on the other hand, is much more effective for transformation into the companion form.

A. Original Vehicle Model

Fig. 1 shows a vehicle with velocity V, orientation (yaw) angle ψ, slip angle β, and steering angle δ_F. The original vehicle (bicycle) model considered in this paper is [5]:

$$\dot{X} = V\cos(\psi + \beta) \tag{26}$$

$$\dot{Y} = V\sin(\psi + \beta) \tag{27}$$

$$\dot{\psi} = V(\cos\beta)\tan\delta_F / l \tag{28}$$

$$\dot{\delta}_F = 0 \tag{29}$$

where A is the vehicle's acceleration and the parameter l is the wheelbase length of the vehicle:

$$l = l_f + l_r \tag{30}$$

Parameters l_f and l_r are shown in Fig. 1. We use the following relationship between the slip and steering angles [5]:

$$\beta = \tan^{-1}\left(\frac{\ell_r \tan(\delta_f)}{\ell_f + \ell_r}\right) \tag{31}$$

This model was previously used for vehicle tracking in [2].

B. Transformation of the Simplified Vehicle Model

We started by transforming the above simplified model to companion form with the following states and output vectors:

$$x = [X \quad Y \quad \psi \quad \delta_f]^T, y = [X \quad Y]^T \tag{32}$$

The assumptions in the simplified model are:

$$A \approx 0, \qquad V = V_1\ (known), \qquad \dot{\beta} = 0 \tag{33}$$

leading to the following vehicle model (based on equations (26)- (33)):

$$\dot{x} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{\psi} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} V_1\cos(\psi) \\ V_1\sin(\psi) \\ V_1\tan\delta_F / l \\ 0 \end{bmatrix} \tag{34}$$

The companion form requires using the two outputs and their derivatives as the states. To transform the model into companion form, define:

$$w_1 = y_1 = X, \qquad w_2 = y_2 = Y \tag{35}$$

Take derivative and use (34) and (35):

$$\dot{w}_1 = w_3 = \dot{X} = V_1\cos(\psi) \tag{36}$$

$$\dot{w}_2 = w_4 = \dot{Y} = V_1\sin(\psi) \tag{37}$$

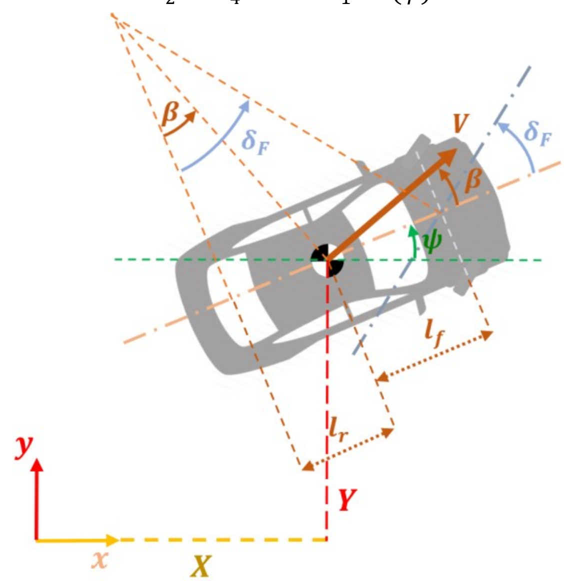

Fig. 1. Motion schematic and model variables for a Vehicle

The dynamic of $\dot{\psi}$ is not captured yet and another derivative is required. Finding derivatives of (36) and (37):

$$\ddot{w}_1 = \dot{w}_3 = w_5 = \ddot{X} = -V_1^2\tan\delta_F\sin(\psi)/l \tag{38}$$

$$\ddot{w}_2 = \dot{w}_4 = w_6 = \ddot{Y} = V_1^2\tan\delta_F\cos(\psi)/l \tag{39}$$

Note that $\ddot{w}_1$ and $\ddot{w}_2$ cannot be written in terms of w_1, w_2, w_3, and w_4. Therefore using (33), (34) and that $\dot{\delta_F} = \dot{\beta} = 0$:

$$\dddot{w}_1 = \dot{w}_5 = \dddot{X} = -V_1^3\tan^2\delta_F\cos(\psi)/l^2 \tag{40}$$

$$\dddot{w}_2 = \dot{w}_6 = \dddot{Y} = -V_1^3\tan^2\delta_F\sin(\psi)/l^2 \tag{41}$$

Considering (36)- (41), $\frac{V_1\tan\delta_F}{l}$ is related to the transformed states:

$$(w_6w_3 - w_5w_4)/V_1^2 = V_1\tan\delta_F / l \tag{42}$$

Replacing (36), (37), and (42) in (40) and (41):

$$\dot{w}_5 = -w_3\left(\frac{w_6w_3 - w_5w_4}{V_1^2}\right)^2 \tag{43}$$

$$\dot{w}_6 = -w_4\left(\frac{w_6w_3 - w_5w_4}{V_1^2}\right)^2 \tag{44}$$

In summary, the transformed model in companion form is:

$$\dot{w} = \begin{bmatrix} \dot{w}_1 \\ \dot{w}_3 \\ \dot{w}_5 \\ \dot{w}_2 \\ \dot{w}_4 \\ \dot{w}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} f(w) \tag{45}$$

and

$$f(w) = \begin{bmatrix} \dot{w}_5 \\ \dot{w}_6 \end{bmatrix} = \begin{bmatrix} -w_3 \left(\dfrac{w_6 w_3 - w_5 w_4}{V_1^2} \right)^2 \\ -w_4 \left(\dfrac{w_6 w_3 - w_5 w_4}{V_1^2} \right)^2 \end{bmatrix} \quad (46)$$

While the model (34) had four states, the transformed system (45) has six states. The redundancy of model (45) can be explained by two constraints.

The first constraint is $V = V_1$ from (33). Note that we have $w_3 = \dot{X}$ and $w_4 = \dot{Y}$ from (36) and (37). The first constraint can be written as:

$$w_3^2 + w_4^2 = V_1^2 \quad (47)$$

The second constraint comes from the constant velocity or $\dot{V} = A \approx 0$ based on (33). From (45) we have $\dot{w}_3 = w_5$ and $\dot{w}_4 = w_6$. Therefore, taking derivative from (47) will give the second constraint:

$$w_3 w_5 + w_4 w_6 = 0 \quad (48)$$

A 4th order system has been translated to a 6th order system plus two constraints. To design a high-gain observer only model (45) must be utilized. The constraints (47) and (48) must be ignored, since algebraic constraints cannot really be a part of the model utilized in the high gain observer.

C. Transformation of a Modified Vehicle Model

By removing some of the assumptions in (33), we improved the previous model and solved the issue with the disadvantages of increase in system order. Instead of constant velocity, we assume constant acceleration so that the observer will have much better performance when the velocity is changing. The states and output vectors for the improved model are:

$$x = [X \quad Y \quad V \quad A \quad \psi \quad \beta]^T, y = [X \quad Y]^T \quad (49)$$

Thus, two new states V and A have been added. Further, the slip angle β is used as the state instead of steering angle. The new assumptions are:

$$\dot{A} \approx 0, \qquad \dot{\delta}_f \approx 0 \quad (50)$$

Note that by rewriting (31) as:

$$\frac{\tan \beta}{l_r} = \frac{\tan(\delta_f)}{l} \quad (51)$$

and replacing it in (28), we obtain:

$$\dot{\psi} = V \sin \beta / l_r \quad (52)$$

Also, from (50) and (51):

$$\dot{\beta} \approx 0 \quad (53)$$

The improved vehicle model based on assumptions (50) is:

$$\dot{x} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{V} \\ \dot{A} \\ \dot{\psi} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} V\cos(\psi + \beta) \\ V\sin(\psi + \beta) \\ A \\ 0 \\ V \sin \beta / l_r \\ 0 \end{bmatrix} \quad (54)$$

The model presented in (54) is already in 6th order form but not in the companion form and it cannot directly be used for high-gain observer design. Now consider the following transformed states and output vectors:

$$z = [X \quad \dot{X} \quad \ddot{X} \quad Y \quad \dot{Y} \quad \ddot{Y}]^T, y = [X \quad Y]^T \quad (55)$$

The transformed vehicle model can be written as:

$$\dot{z} = \begin{bmatrix} \dot{X} \\ \ddot{X} \\ \dddot{X} \\ \dot{Y} \\ \ddot{Y} \\ \dddot{Y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} f(z) \quad (56)$$

We need to calculate $f(z)$ in equation (56):

$$f(z) = \begin{bmatrix} \dot{z}_3 \\ \dot{z}_6 \end{bmatrix} = \begin{bmatrix} \dddot{X} \\ \dddot{Y} \end{bmatrix} = \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} \quad (57)$$

The first step is to calculate the acceleration:

$$\begin{aligned} \ddot{X} &= \frac{d(\dot{X})}{dt} \\ &= \frac{dV}{dt}\cos(\psi + \beta) - \frac{d(\psi + \beta)}{dt} V \sin(\psi + \beta) \end{aligned} \quad (58)$$

$$\begin{aligned} \ddot{Y} &= \frac{d(\dot{Y})}{dt} \\ &= \frac{dV}{dt}\sin(\psi + \beta) + \frac{d(\psi + \beta)}{dt} V \cos(\psi + \beta) \end{aligned} \quad (59)$$

Use (54) in (58) and (59):

$$\ddot{X} = z_3 = A\cos(\psi + \beta) - \frac{V^2 \sin \beta}{l_r} \sin(\psi + \beta) \quad (60)$$

$$\ddot{Y} = z_6 = A\sin(\psi + \beta) + \frac{V^2 \sin \beta}{l_r} \cos(\psi + \beta) \quad (61)$$

or

$$A\cos(\psi + \beta) = z_3 + \frac{V^2 \sin \beta}{l_r} \sin(\psi + \beta) \quad (62)$$

$$A\sin(\psi + \beta) = z_6 - \frac{V^2 \sin \beta}{l_r} \cos(\psi + \beta) \quad (63)$$

Rewriting equations (60) and (61) in the following forms:

$$\left(-\frac{V^2 \sin\beta}{l_r}\sin(\psi+\beta)\right)\sin(\psi+\beta) \\ = (z_3 - A\cos(\psi+\beta))\sin(\psi+\beta) \tag{64}$$

$$\left(\frac{V^2 \sin\beta}{l_r}\cos(\psi+\beta)\right)\cos(\psi+\beta) \\ = (z_6 - A\sin(\psi+\beta))\cos(\psi+\beta) \tag{65}$$

Subtract (64) from (65):

$$\frac{V^2 \sin\beta}{l_r} = z_6\cos(\psi+\beta) - z_3\sin(\psi+\beta) \tag{66}$$

Note that from (54) and (55):

$$\cos(\psi+\beta) = \frac{z_2}{V} \tag{67}$$

$$\sin(\psi+\beta) = \frac{z_5}{V} \tag{68}$$

and velocity is assumed to be non-zero. Using (67) and (68), (66) is written as:

$$\frac{V\sin\beta}{l_r} = \frac{z_6 z_2 - z_3 z_5}{V^2} \tag{69}$$

The second step is to calculate jerk from (60) and (61) by considering (50):

$$\begin{aligned}\dddot{X} = \frac{d(\ddot{X})}{dt} &= \frac{dA}{dt}\cos(\psi+\beta) \\ &-A\sin(\psi+\beta)\frac{d(\psi+\beta)}{dt} \\ &-2V\frac{dV}{dt}\frac{\sin\beta}{l_r}\sin(\psi+\beta) \\ &-\frac{V^2\sin\beta}{l_r}(\cos(\psi+\beta))\frac{d(\psi+\beta)}{dt}\end{aligned} \tag{70}$$

$$\begin{aligned}\dddot{Y} = \frac{d(\ddot{Y})}{dt} &= \frac{dA}{dt}\sin(\psi+\beta) \\ &+A\cos(\psi+\beta)\frac{d(\psi+\beta)}{dt} \\ &+2V\frac{dV}{dt}\frac{\sin\beta}{l_r}\cos(\psi+\beta) \\ &-\frac{V^2\sin\beta}{l_r}(\sin(\psi+\beta))\frac{d(\psi+\beta)}{dt}\end{aligned} \tag{71}$$

Implementing (50) and (54) on (70) and (71):

$$\begin{aligned}\dddot{X} = &-3A\frac{V\sin\beta}{l_r}\sin(\psi+\beta) \\ &-V\left(\frac{V\sin\beta}{l_r}\right)^2\cos(\psi+\beta)\end{aligned} \tag{72}$$

$$\begin{aligned}\dddot{Y} = &\,3A\frac{V\sin\beta}{l_r}\cos(\psi+\beta) \\ &-V\left(\frac{V\sin\beta}{l_r}\right)^2\sin(\psi+\beta)\end{aligned} \tag{73}$$

Use (62) and (63) in (72) and (73):

$$\dddot{X} = -3z_6\frac{V\sin\beta}{l_r} + 2V\left(\frac{V\sin\beta}{l_r}\right)^2\cos(\psi+\beta) \tag{74}$$

$$\dddot{Y} = 3z_3\frac{V\sin\beta}{l_r} + 2V\left(\frac{V\sin\beta}{l_r}\right)^2\sin(\psi+\beta) \tag{75}$$

Use (67), (68), and (69) on (74) and (75):

$$\dddot{X} = -3z_6\frac{z_6z_2 - z_3z_5}{V^2} + 2z_2\left(\frac{z_6z_2 - z_3z_5}{V^2}\right)^2 \tag{76}$$

$$\dddot{Y} = 3z_3\frac{z_6z_2 - z_3z_5}{V^2} + 2z_5\left(\frac{z_6z_2 - z_3z_5}{V^2}\right)^2 \tag{77}$$

From (67) and (68):

$$V^2 = z_2^2 + z_5^2 \tag{78}$$

Implement (78) on (76) and (77):

$$\dddot{X} = -3z_6\frac{z_6z_2 - z_3z_5}{z_2^2 + z_5^2} + 2z_2\left(\frac{z_6z_2 - z_3z_5}{z_2^2 + z_5^2}\right)^2 \tag{79}$$

$$\dddot{Y} = 3z_3\frac{z_6z_2 - z_3z_5}{z_2^2 + z_5^2} + 2z_5\left(\frac{z_6z_2 - z_3z_5}{z_2^2 + z_5^2}\right)^2 \tag{80}$$

Summarizing, the transformed model in companion form can be described as follows:

$$\dot{z} = Fz + Gf(z), \qquad y = Hz \tag{81}$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{82}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$f(z) = \begin{bmatrix} -3z_6\dfrac{z_6z_2 - z_3z_5}{z_2^2 + z_5^2} + 2z_2\left(\dfrac{z_6z_2 - z_3z_5}{z_2^2 + z_5^2}\right)^2 \\ 3z_3\dfrac{z_6z_2 - z_3z_5}{z_2^2 + z_5^2} + 2z_5\left(\dfrac{z_6z_2 - z_3z_5}{z_2^2 + z_5^2}\right)^2 \end{bmatrix} \tag{83}$$

Now that the transformed model in (81 - 83) is in companion form, it can be used for high-gain observer design and is compatible with the design procedure presented in section II.

IV. FINDING THE OBSERVER GAIN FOR VEHICLE MODEL

In this section, the observer gain L is obtained by solving LMI (15) for $\lambda = 10$, using the SEDUMI solver in MATLAB software. We also added the constraint $P < 30\, I_{6\times 6}$ to limit the $\lambda_{max}(P)$ and the resulting θ from (16). The results are:

$$K = \begin{bmatrix} 11.73 & 20.25 & -7.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 11.73 & 20.25 & -7.3 \end{bmatrix} \quad (84)$$

$$\lambda_{max}(P) = 26.5 \quad (85)$$

Parameter k_f in (17) depends on the Lipschitz constants of the complex nonlinear functions of the process dynamics (1). Finding these constants on the $\mathbb{R}^6$ space is challenging. For simplicity, the constants are calculated for $\dot{\psi} = c$ hyperplanes (constant $\dot{\psi}$). From (54), (69), and (78):

$$\dot{\psi} = \frac{z_6 z_2 - z_3 z_5}{z_2^2 + z_5^2} = c \quad (86)$$

Therefore (83) can be simplified as:

$$f(z) \approx \begin{bmatrix} -3cz_6 + 2c^2 z_2 \\ 3cz_3 + 2c^2 z_5 \end{bmatrix} \quad (87)$$

And by assuming $\dot{\psi} = c < 1$ rad/s one can take:

$$\gamma_1 = \gamma_2 \approx 3c \quad (88)$$

By taking $k_f = 1.25$, we are assuming a maximum vehicle yaw angle rate of 0.3 rad/s. The observer gain obtained from these assumptions is:

$$L = \begin{bmatrix} 67.20 & 0 \\ 823.1 & 0 \\ 2818.5 & 0 \\ 0 & 67.20 \\ 0 & 823.1 \\ 0 & 2818.5 \end{bmatrix} \quad (89)$$

V. CONCLUSIONS

In this paper, a multi-output high gain nonlinear observer was designed for a vehicle trajectory tracking application. The high gain observer approach has the advantages of guaranteed feasibility and global stability with one constant observer gain for all ranges of motion. The challenges of transforming the vehicle dynamic model into the companion form needed for applying the high gain observer technique were addressed. A coordinate transformation that allows for varying velocity and varying slip angle was shown to be appropriate. The high gain observer methodology for a dynamic system with multiple outputs was presented. Finally, the calculation of the Lipschitz constant for the vehicle tracking application was discussed, and the observer gain matrix for this application was determined.

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