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Regularity of the value function and quantitative propagation of chaos for mean field control problems

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Abstract. We investigate a mean field optimal control problem obtained in the limit of the optimal control of large particle systems with forcing and terminal data which are not assumed to be convex. We prove that the value function, which is known to be Lipschitz continuous but not of class C^1, in general, without convexity, is actually smooth in an open and dense subset of the space of times and probability measures. As a consequence, we prove a new quantitative propagation of chaos-type result for the optimal solutions of the particle system starting from this open and dense set.

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Introduction

The paper is about the regularity of the value function and quantitative propagation of chaos for mean field control (MFC for short) problems obtained as the limit of optimal control problems for large particle systems with forcing and terminal data which are not assumed to be convex. The value function of MFC problems is known to be a Lipschitz continuous but, in general, not C^1-function in the space of time and probability measures. Our first result is that there exists an open and dense subset of time and probability measures where the value function is actually smooth. The second result is a new quantitative propagation of chaos-type property for the optimal solutions of the particle system starting from this open and dense set.

The background

In order to state the results it is necessary to introduce the general set-up.

We consider the problem of controlling optimally N particles in order to minimize a criterion of the form

$$J^N(t_0,\mathbf{x}_0,\alpha) := \mathbb{E}\left[\frac{1}{N}\int_{t_0}^T \sum_{i=1}^N L(X_t^i,\alpha_t^i)dt + \int_{t_0}^T \mathcal{F}(m^N_{\mathbf{X}_t})dt + \mathcal{G}(m^N_{\mathbf{X}_T})\right]. \tag{0.1}$$

Here $T>0$ is a fixed time horizon, $t_0\in[0,T]$ and $\mathbf{x}_0=(x_0^1,\dots,x_0^N)\in(\mathbb{R}^d)^N$ are respectively the initial time and the initial position of the system at time t_0. The minimization is over the set $\mathcal{A}^N$ of admissible controls $\alpha=(\alpha^k)_{k=1}^N$ in $L^2([0,T]\times\Omega;(\mathbb{R}^d)^N)$ which are adapted to the filtration generated by the independent d-dimensional Brownian motions $(B^i)_{i=1,\dots,N}$, and the trajectories $\mathbf{X}=(X^1,\dots,X^N)$ satisfy, for each $k\in\{1,\dots,N\}$,

$$X_t^k = x_0^k + \int_{t_0}^t \alpha_s^k ds + \sqrt{2}(B_t^k - B_{t_0}^k) \quad \text{for} \quad t\in[t_0,T].$$

In (0.1), $m^N_{\mathbf{X}_t}$ is the empirical measure of the process $\mathbf{X}_t$ given by

$$m^N_{\mathbf{X}_t} := \frac{1}{N}\sum_{k=1}^N \delta_{X_t^k}, \tag{0.2}$$

where δ_x is the Dirac mass at x.

The maps $\mathcal{F}$ and $\mathcal{G}$, which are defined on (suitable subsets of) the set of Borel probability measures on $\mathbb{R}^d$, describe the interactions between the particles X^i. Finally, the cost function $L=L(x,\alpha):\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is convex and grows quadratically in the second variable.

The value function for this optimization problem reads

$$\begin{aligned}\mathcal{V}^N(t_0,\mathbf{x}_0) &:= \inf_{\alpha\in\mathcal{A}^N} J^N(t_0,\mathbf{x}_0,\alpha)\\ &= \inf_{\alpha\in\mathcal{A}^N}\mathbb{E}\left[\int_{t_0}^T\left(\frac{1}{N}\sum_{k=1}^N L(X_t^k,\alpha_t^k)+\mathcal{F}(m^N_{\mathbf{X}_t})\right)dt + \mathcal{G}(m^N_{\mathbf{X}_T})\right].\end{aligned} \tag{0.3}$$

In a more general framework and under slightly different conditions on the data, Lacker [24] proved that the empirical measure $m^N_{\mathbf{X}_t^N}$ associated to the optimal trajectories of (0.3) converges in a suitable sense to the (weak) optimal solution of the mean field control problem (written here in a strong sense) consisting in minimizing the quantity

$$J^\infty(t_0,m_0,\alpha) = \mathbb{E}\left[\int_{t_0}^T L(X_t,\alpha_t)dt + \int_{t_0}^T \mathcal{F}(\mathcal{L}(X_t))dt + \mathcal{G}(\mathcal{L}(X_T))\right], \tag{0.4}$$

where m_0 is an initial distribution of the particles at time t_0, $\alpha\in\mathcal{A}$, the set of admissible controls consisting of square integrable $\mathbb{R}^d$-valued processes adapted to a Brownian motion B and to an initial condition $\overline{X}_0$, which is independent of B and of law m_0, and the process $(X_t)_{t\in[t_0,T]}$ satisfies

$$X_t = \overline{X}_0 + \int_{t_0}^t \alpha_s ds + \sqrt{2}(B_t - B_{t_0}) \quad \text{for} \quad t \in [t_0, T],$$

and $\mathcal{L}(X_t)$ denotes the law of X_t.

The value function $\mathcal{U}$ of the last optimization problem is given (heuristically at this stage) by

$$\begin{aligned} \mathcal{U}(t_0, m_0) :&= \inf_{\alpha\in\mathcal{A}} J^\infty(t_0, m_0, \alpha) \\ &= \inf_{\alpha\in\mathcal{A}} \mathbb{E}[\int_{t_0}^T \big(L(X_t, \alpha_t) + \mathcal{F}(\mathcal{L}(X_t)\big)dt + \mathcal{G}(\mathcal{L}(X_T)]. \end{aligned} \tag{0.5}$$

In addition, [24] points out that there is a propagation of chaos-type property (an easy consequence of Sznitman characterization of propagation of chaos), if the minimization problem (0.5) has a unique weak minimizer. Note, however, that such uniqueness is known only when the maps $(x, \alpha) \to L(x, \alpha)$, $m \to \mathcal{F}(m)$ and $m \to \mathcal{G}(m)$ are globally convex, as in Carmona and Delarue [8]. The conclusions of [24] were extended to problems with interaction through the controls in Djete [17] and for problems with a common noise in Djete, Possamaï and Tan [18]; see also related results for games in Djete [16] and in Laurière and Tangpi [26]. Several other results on the convergence of MFC problems without diffusion were obtained in Cavagnari, Lisini, Orrieri and Savaré [10] and Gangbo, Mayorga and Swiech [20]. A quantitative convergence rate for the value function $\mathcal{V}^N$ to $\mathcal{U}$ was given, for problems on a finite state space, in Kolokoltsov [23] and Cecchin [12] and, for problems on the continuous state space, in Baryaktar and Chakraborty [1] under a certain structural dependence of the data on the measure variable, in Germain, Pham and Warin [21] under the assumption that the limit value is smooth, and in Cardaliaguet, Daudin, Jackson and Souganidis [7] under a decoupling assumption on the Hamiltonian. In addition, a propagation of chaos is proved in [1,12,21] assuming, however, that the limit value function is smooth. The characterization of the value function $\mathcal{U}$ as viscosity solution was recently obtained by Cosso, Gozzi, Kharroubi, Pham, and Rosestolato [13] and, using a completely different approach, by Ceccin and Delarue [11].

The results

In this paper we study nonconvex MFC problems for which the limit value function is not expected to be globally smooth and show that for a large class (a dense and open set) of initial times and measures the value function $\mathcal{U}$ is smooth (Theorem 1.1) and the propagation of chaos holds with a rate (Theorem 1.2).

To write the results in this introduction requires considerable notation. Thus we postpone stating the precise theorems to Sect. 1.

We explain, however, the very general idea of proof. We identify the open and dense in time and space of probabilities set $\mathcal{O}$ where $\mathcal{U}$ is smooth as the set of initial conditions (t_0, m_0) from which starts a unique (in a strong sense) and stable (in a suitable linear sense) minimizer of J^∞. This step, which is reminiscent of ideas from standard optimal control (see, for instance, the book of Cannarsa and Sinestrari [3]), is similar to what was obtained by Briani and Cardaliaguet [2] for MFC problems in a different framework, namely, the state space is the torus and the initial measures have smooth densities. Here, the state space is all of $\mathbb{R}^d$ and the initial conditions are arbitrary probability measures. To show that $\mathcal{U}$ is smooth in $\mathcal{O}$, we adapt ideas used in the construction of a solution to the master equation in mean field games given in Cardaliaguet, Delarue, Lasry and Lions [6]. Here however we argue without the convexity, which translates to monotonicity for general mean field games, assumption used extensively in [6]. Then, using the regularity of $\mathcal{U}$, we derive the propagation of chaos property for the optimal solution of the particle system when starting from the set $\mathcal{O}$. The key argument is the fact that the optimal trajectories of (0.5) that start in $\mathcal{O}$ remain there, while $\mathcal{U}$ is smooth and satisfies (almost) the same Hamilton–Jacobi equation as $\mathcal{V}^N$ in $\mathcal{O}$.

We finally comment about possible extensions to problems with common noise emphasizing once more that we do not assume any monotonicity/convexity. The convergence (with algebraic rate) of $\mathcal{V}^N$ to $\mathcal{U}$ for MFC problems with common noise was established in [7]. However, the generalization of the results of the present paper to such a setting is far from clear. For example, one of the basic tools we use to prove that the set $\mathcal{O}$ is open and dense is a result of Lions-Malgrange-type, which yields the uniqueness of the mean field game system characterizing the optimal solution (see system (1.12) below), but with initial conditions both in u and m: see the proof of Lemma 2.7. At this time, we do not know if there is a counterpart of this argument for problems with a common noise, where the mean field system becomes a system of forward-backward stochastic partial differential equations (see [6]).

Organization of the paper The paper is organized as follows. In Sect. 1, we introduce the standing assumptions, present the main results and recall some preliminary facts that are needed for the rest of the paper. Section 2 is about establishing the smoothness of $\mathcal{U}$ (Theorem 1.1). Section 3 is devoted to showing the propagation of chaos property (Theorem 1.2).

Notations

We work on $\mathbb{R}^d$ and write B_R for the open ball centered at the origin with radius R and I_d for the identity matrix.

We denote by $\mathcal{P}(\mathbb{R}^d)$ the set of Borel probability measures on $\mathbb{R}^d$. Given $m \in \mathcal{P}(\mathbb{R}^d)$ and $p \geq 1$, we write $M_p(m)$ for the p^{th}-moment of m, that is, $M_p(m) = \int_{\mathbb{R}^d} |x|^p dm$. Then $\mathcal{P}_p(\mathbb{R}^d)$ is the set of $m \in \mathcal{P}(\mathbb{R}^d)$ such that $M_p(\mathbb{R}^d) < \infty$. We endow $\mathcal{P}_p(\mathbb{R}^d)$ with the Wasserstein metric $\mathbf{d}_p$, defined by

$$\mathbf{d}_p^p(m, m') := \inf_{\pi \in \Pi(m,m')} \int_{\mathbb{R}^d} |x-y|^p d\pi(x,y),$$

where $\Pi(m, m')$ is the set of all $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ with marginals m and m'. For $p = 1$, we recall the duality formula

$$\mathbf{d}_1(m, m') = \sup_{\phi} \int_{\mathbb{R}^d} \phi d(m - m'),$$

where the supremum is taken over all 1-Lipschitz maps $\phi : \mathbb{R}^d \to \mathbb{R}$.

For $\mathbf{x} = (x^1, ..., x^N) \in (\mathbb{R}^d)^N$, $m^N_{\mathbf{x}} \in \mathcal{P}(\mathbb{R}^d)$ is the empirical measure of $\mathbf{x}$, that is, $m^N_{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$.

Given a map $U : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$, we denote by $\delta U/\delta m$ its flat derivative and by $D_m U$ its Lions-derivative when these derivatives exist, and we use the corresponding notations for second order derivatives. We refer to [6] and the books of Carmona and Delarue [9] for definitions and properties.

We write C^k_{loc} and C^k for the sets of maps on $\mathbb{R}^d$ with continuous and continuous and bounded derivatives up to order k. For $r > 0$, $r \notin \mathbb{N}$, we denote by $C^{r,2r}$ the standard parabolic Hölder spaces and by $C^{r,2r}_c$ the subset of functions of $C^{r,2r}$ with a compact support. The notation $\|u\|_\infty$ stands for the supremum of a map u (or essential supremum, depending on the context) in all variables.

Given a topological vector space, we write by E' for its dual space.

We often need to compare continuous maps defined on different intervals of $[0, T]$. For this, we simply extend the maps continuously on $[0, T]$ by a constant. For instance, if $t_0 \in (0, T]$, E is a topological space and $f : [t_0, T] \to E$ is continuous, we set $f(s) = f(t_0)$ for $s \in [0, t_0]$.

Finally, throughout the proofs C denotes a positive constant, which, unless otherwise said, depends on the data and may change from line to line.

1. The assumptions, the main results and some preliminary facts

The standing assumptions

We state our standing assumptions on the maps $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $\mathcal{F} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ and $\mathcal{G} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$, which constitute the data of our problem. We recall that $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the Legendre transform of H with respect to the second variable:

$$L(x, a) = \sup_{p\in\mathbb{R}^d} \left[-a \cdot p - H(x, p)\right].$$

We assume that

$$\begin{cases} H = H(x,p) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \text{ is of class } C^4_{loc} \text{ and strictly convex with respect to the} \\ \text{second variable, that is, for each } R > 0 \text{ there exists} \\ c_R, C_R > 0 \text{ such that, for all } (x,p) \in \mathbb{R}^d \times \overline{B_R}, \\ \qquad D^2_{pp}H(x,p) \geq c_R I_d \ \text{ and } \ |D^2_{xx}H(x,p)| + |D^2_{xp}H(x,p)| \leq C_R, \end{cases} \tag{1.1}$$

$$\begin{cases} \text{there exists a constant } C > 0 \text{ such that, for all } (x,p) \in \mathbb{R}^d \times \mathbb{R}^d, \\ -C + C^{-1}|p|^2 \leq H(x,p) \leq C(1 + |p|^2) \ \text{ and } \ |D_x H(x,p)| \leq C(|p| + 1), \end{cases} \tag{1.2}$$

$$\begin{aligned} &\mathcal{F} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R} \text{ is of class } C^2 \text{ with } \mathcal{F}, D_m\mathcal{F}, D^2_{ym}\mathcal{F} \text{ and } D^2_{mm}\mathcal{F} \text{ uniformly bounded,} \\ &\text{and, moreover, } x \to \tfrac{\delta \mathcal{F}}{\delta m}(m,x) \text{ is bounded in } C^2 \text{ uniformly in } m, \\ &\text{while } y \to \tfrac{\delta^2 \mathcal{F}}{\delta m^2}(m,x,y) \text{ is bounded in } C^2 \text{ uniformly in} (m,x), \end{aligned} \tag{1.3}$$

$$\mathcal{G} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R} \text{ is of class } C^4 \text{ with all derivatives up to order 4 uniformly bounded.} \tag{1.4}$$

For simplicity, in what follows we put together all the assumptions above in

$$\text{assume that (1.1), (1.2), (1.3) and (1.4) hold.} \tag{1.5}$$

The assumptions on the Hamiltonian H are fairly standard, although a little restrictive, and are used in [7] to obtain, independent of N, Lipschitz estimates on the value function $\mathcal{V}^N$. We recall this estimate in Lemma 1.7 below. An example satisfying (1.5) is a Hamiltonian of the form $H(x,p) = |p|^2 + V(x)\cdot p$ for some smooth and globally Lipschitz continuous vector field $V : \mathbb{R}^d \to \mathbb{R}^d$. The regularity conditions on $\mathcal{F}$ and $\mathcal{G}$ are also important to obtain the estimates of Lemma 1.7 and to prove the regularity of $\mathcal{U}$.

The results

Given $(t_0, \mathbf{x}) \in [0,T) \times (\mathbb{R}^d)^N$, $\mathcal{V}^N(t_0,\mathbf{x})$ is the value function of the optimal control of the N-particle problem given by (0.3).

We now define in a rigorous way the value function $\mathcal{U}$ of the MFC, which was informally introduced in (0.5). For each initial point $(t_0, m_0) \in [0,T) \times \mathcal{P}_1(\mathbb{R}^d)$, we use the set $\mathcal{M}(t_0,m_0)$ of controls given by

$$\mathcal{M}(t_0,m_0) := \left\{ \begin{array}{l} (m,\alpha) \in C^0([t_0,T],\mathcal{P}_1(\mathbb{R}^d)) \times L^0([t_0,T]\times\mathbb{R}^d;\mathbb{R}^d) \ : \\ \int_0^T \int_{\mathbb{R}^d} |\alpha|^2 m < \infty \ \text{ and} \\ \partial_t m - \Delta m + \operatorname{div}(m\alpha) = 0 \ \text{ in } \ (t_0,T)\times\mathbb{R}^d \ \text{ and } \ m(0) = m_0 \text{ in } \mathbb{R}^d \end{array} \right\}, \tag{1.6}$$

with the equation in (1.6) understood in the sense of distributions and where $L^0([t_0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ is the set of Borel measurable maps from $[t_0,T]\times\mathbb{R}^d$ to $\mathbb{R}^d$.

Then the value function $\mathcal{U}(t_0,m_0)$ of the MFC problem is given by

$$\mathcal{U}(t_0,m_0) := \inf_{(m,\alpha)\in\mathcal{M}(t_0,m_0)} \left\{ \int_{t_0}^T \int_{\mathbb{R}^d} L(x,\alpha(t,x))m(t,dx) + \mathcal{F}(m(t))dt + \mathcal{G}(m(T)) \right\}. \tag{1.7}$$

We are going to prove the regularity of $\mathcal{U}$ and the propagation of chaos in the set $\mathcal{O}$ defined by

$$\mathcal{O} := \left\{ \begin{array}{l} (t_0,m_0) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d) \ : \ \text{there exists a unique stable minimizer} \\ \text{in the definition of } \mathcal{U}(t_0,m_0) \end{array} \right\}. \tag{1.8}$$

The notion of stability is defined in terms of the linearized MFG system and is introduced in Sect. 2.

The first main result is stated next.

Theorem 1.1. *Assume (1.5). The value function $\mathcal{U}$ is globally Lipschitz continuous on $[0,T]\times\mathcal{P}_1(\mathbb{R}^d)$ and of class C^1 in the set $\mathcal{O}$, which is open and dense in $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$. Moreover, $\mathcal{U}$ is a classical solution in $\mathcal{O}$ of the master Hamilton–Jacobi equation*

$$-\partial_t \mathcal{U}(t,m) - \int_{\mathbb{R}^d} \operatorname{div}(D_m\mathcal{U}(t,m,y))m(dy) + \int_{\mathbb{R}^d} H(y, D_m\mathcal{U}(t,m,y))m(dy) = \mathcal{F}(m). \tag{1.9}$$

In addition, for any (t_0, m_0) *in* $\mathcal{O}$*, there exists* $\varepsilon > 0$ *and a constant* $C > 0$*, which depends on* (t_0, m_0) *and is such that, for any* $t \in [0,T]$*,* $x, y \in \mathbb{R}^d$ *and* $m^1, m^2 \in \mathcal{P}_2(\mathbb{R}^d)$ *with* $|t - t_0| < \varepsilon$*,* $\mathbf{d_2}(m_0, m^1) < \varepsilon$ *and* $\mathbf{d_2}(m_0, m^2) < \varepsilon$*,*

$$\left|D_m\mathcal{U}(t, m^1, x) - D_m\mathcal{U}(t, m^2, y)\right| \leq C(|x - y| + \mathbf{d}_2(m^1, m^2)). \quad (1.10)$$

By a classical solution of (1.9), we mean that the derivatives of $\mathcal{U}$ involved in the equation exist and are continuous.

Our second main result is a quantitative propagation of chaos property about the optimal trajectories of the underlying N-particle system.

Theorem 1.2. *Assume* (1.5)*. There exists a constant* $\gamma \in (0, 1)$ *depending only on the dimension d such that, for every* $(t_0, m_0) \in \mathcal{O}$ *with* $M_{d+5}(m_0) < +\infty$*, there is* $C = C((t_0, m_0)) > 0$ *such that, if* $\mathbf{Z} = (Z^k)_{k=1,\ldots,N}$ *is a sequence of independent random variables with law* m_0*,* $\mathbf{B}_\cdot = (B^k_\cdot)_{k=1,\ldots,N}$ *is a sequence of independent Brownian motions independent of* $\mathbf{Z}$*, and* $\mathbf{Y}_\cdot^N = (Y_\cdot^{N,k})_{k=1,\ldots,N}$ *is the optimal trajectory for* $\mathcal{V}^N(t_0, (Z^k)_{k=1,\ldots,N})$*, that is, for each* $k = 1, \ldots, N$ *and* $t \in [t_0, T]$*,*

$$Y_t^{N,k} = Z^k - \int_{t_0}^t H_p(Y_s^k, D\mathcal{V}^N(s, \mathbf{Y}_s^N))ds + \sqrt{2}(B_t^k - B_{t_0}^k), \quad (1.11)$$

then

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} \mathbf{d_1}(m^N_{\mathbf{Y}_t^N}, m(t))\right] \leq CN^{-\gamma}.$$

Remark. The value of the constant γ depends on the convergence rate of $\mathcal{V}^N$ to $\mathcal{U}$ in Proposition 1.8 below, which is unfortunately not explicit and poorly understood.

Some preliminary facts

We recall here some well known facts about MFC that we use in the paper.

We begin with some regularity properties of the underlying backward-forward MFG system. Fix $(t_0, m_0) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$. We recall (see, for example, [28,29] for the original statement or [14], in a framework closer to our setting) that there exists at least one minimizer for $\mathcal{U}(t_0, m_0)$ and that, if $(m, \alpha) \in \mathcal{M}(t_0, m_0)$ is a minimizer, then there exists a multiplier $u : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that $\alpha = -H_p(x, Du)$ and the pair (u, m) solves the MFG-system

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = F(x, m(t)) \;\text{ in }\; (t_0, T) \times \mathbb{R}^d, \\ \partial_t m - \Delta m - \operatorname{div}(H_p(x, Du)m) = 0 \;\text{ in }\; (t_0, T) \times \mathbb{R}^d, \\ m(t_0) = m_0, \; u(T, x) = G(x, m(T)) \;\text{ in }\; \mathbb{R}^d, \end{cases} \quad (1.12)$$

where

$$F(x, m) = \frac{\delta \mathcal{F}}{\delta m}(m, x) \;\text{ and }\; G(x, m) = \frac{\delta \mathcal{G}}{\delta m}(m, x).$$

Note that, in view of the assumed strict convexity of H, given (m, α), the relation $\alpha = -H_p(x, Du)$ defines uniquely Du.

Lemma 1.3. *Assume* (1.5) *and let* (u, m) *be a the solution of (1.12). Then, for any* $\delta \in (0,1)$*, there exists* $C > 0$*, which is independent of* (t_0, m_0)*, such that*

$$\|u\|_{C^{(3+\delta)/2,3+\delta}} + \sup_{t\neq t'} \frac{\mathbf{d_2}(m(t), m(t'))}{|t-t'|^{1/2}} \leq C \quad \text{and}$$

$$\sup_{t\in[t_0,T]} \int_{\mathbb{R}^d} |x|^2 m(t,dx) \leq C \int_{\mathbb{R}^d} |x|^2 m_0(dx). \tag{1.13}$$

Remark 1.4. Note that, under our standing assumptions, we do not have, in general, uniqueness of the solution to (1.12). Indeed, the problem defining $\mathcal{U}(t_0, m_0)$ may have several minimizers and/or solutions of (1.12) may not necessarily be associated with a minimizer of $\mathcal{U}(t_0, m_0)$.

Proof of Lemma 1.3. The uniform bound on Du follows as in the proof of Lemma 3.3 in [7], and then the estimate on m are immediate. The $C^{(3+\delta)/2,3+\delta}$-local regularity of u is a consequence of the classical parabolic regularity theory, since the map $x \to G(m, x)$ is bounded C^4 uniformly in m and the map $(t, x) \to F(x, m(t))$ is of class $C^{(1+\delta)/2,1+\delta}$ for any $\delta \in (0, 1)$. For the space regularity, this comes from the boundedness in C^2 of the map $x \to \frac{\delta \mathcal{F}}{\delta m}(m, x)$, uniform in m. For the time regularity, we have, in view of the equation satisfied by m,

$$\frac{d}{dt} F(x, m(t)) = \int_{\mathbb{R}^d} (\Delta_y \frac{\delta F}{\delta m}(m(t), x, y) - H_p(y, Du(t, y))$$
$$\cdot D_y \frac{\delta F}{\delta m}(m(t), x, y))m(t, y)dy,$$

which is bounded in view of the boundedness in C^2, uniform in (m, x), of the map $y \to \frac{\delta^2 \mathcal{F}}{\delta m^2}(m, x, y)$.

The only point is to explain why this regularity holds globally in space. For this, we first note that the uniform boundedness of $D^2_{mm}\mathcal{F}$ and of $D^2_{mm}\mathcal{G}$ yield a $C > 0$ such that, for all $x, y \in \mathbb{R}^d$,

$$\sup_{t\in[t_0,T]} |F(x, m(t)) - F(y, m(t))| + |G(x, m(t)) - G(y, m(t)| \leq C|x - y|.$$

It then follows from the maximum principle that u is, uniformly in (t_0, m_0) and in t, Lipschitz continuous in the space variable. The same argument applied to the equation satisfied by u_{x_i} for each $i = 1, \ldots, d$, implies that Du is also uniformly Lipschitz continuous in the space variable.

The general conclusion can be established similarly, using the maximum principle for the global estimates and the parabolic regularity for the local one. □

In view of the uniform estimates in (1.13), we have the following stability for minimizers in (1.7) when they are unique.

Lemma 1.5. *Assume* (1.5)*, fix* $(t_0, m_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$*, suppose that* $\mathcal{U}(t_0, m_0)$ *has a unique minimizer* (m, α)*, and let* u *be the associated multiplier. If* (t_0^n, m_0^n) *converges to* (t_0, m_0) *in* $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ *and if* (m^n, α^n) *is a minimizer for*

$\mathcal{U}(t_0^n, m_0^n)$ with associated multiplier u^n, then u^n, Du^n and D^2u^n converge respectively to u, Du and D^2u in $C^{\delta/2,\delta}$. In addition, if $t_0^n = t_0$ for all n, the convergence of (u^n) holds in $C^{(2+\delta)/2,2+\delta}$.

Proof. It easily follows from the regularity of u (see (1.13)) that, without loss of generality, we can assume that $t_0^n = t_0$. Moreover, again in view of (1.13) and the continuity of $\mathcal{U}$ (see Lemma 1.7 below), the minimizer (m^n, α^n) converge along subsequences in $C^0([t_0, T] \times \mathcal{P}_1(\mathbb{R}^d)) \times C^{1,2}_{loc}([t_0, T] \times \mathbb{R}^d)$ to minimizers for $\mathcal{U}(t_0, m_0)$. Since the latter is assumed to have a unique minimizer (m, α), the convergence holds along the whole sequence. Arguing as in Lemma 1.3, we can also check that the convergence of the u^n's holds in $C^{(2+\delta)/2,2+\delta}$, because $z^n = u^n - u$ solves a linearized equation of the form

$$-\partial_t z^n - \Delta z^n + V^n \cdot Dz^n = F(x, m^n(t)) - F(x, m(t))$$
$$z^n(T, x) = G(x, m^n(T)) - G(x, m(T)),$$

with $V^n(t, x) = \int_0^1 H_p((1-s)Du^n + sDu, x)ds$, and where, in view of the regularity of $\mathcal{F}$ and $\mathcal{G}$ and the convergence of m^n to m, $F(\cdot, m^n(\cdot)) - F(\cdot, m(\cdot))$ and $G(\cdot, m^n(T)) - G(\cdot, m(T))$ converge as $n \to \infty$ to 0. □

The following second-order optimality condition is used several times in the proofs of the main results.

Lemma 1.6. *Assume (1.5), fix $(t_0, m_0) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^d)$ and let (m, α) be a minimizer for $\mathcal{U}(t_0, m_0)$. Fix $\beta \in C^0([t_0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ or $\beta \in L^\infty([t_0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ with $\beta = 0$ in a neighborhood of t_0 and let $\rho \in C^0([t_0, T], (C^{2+\delta}(\mathbb{R}^d))')$ be the solution in the sense of distributions to*

$$\begin{cases} \partial_t \rho - \Delta \rho + \operatorname{div}(\rho\alpha) + \operatorname{div}(m\beta) = 0 \ \text{ in } \ (t_0, T) \times \mathbb{R}^d, \\ \rho(t_0) = 0 \ \text{ in } \ \mathbb{R}^d. \end{cases} \tag{1.14}$$

Then

$$\begin{aligned} \int_{t_0}^T \Big(\int_{\mathbb{R}^d} & L_{\alpha,\alpha}(x, \alpha(t,x))\beta(t,x) \cdot \beta(t,x) m(t,dx) \\ & + \langle \frac{\delta F}{\delta m}(\cdot, m(t), \cdot), \rho(t) \otimes \rho(t) \rangle \Big) dt \\ & + \langle \frac{\delta G}{\delta m}(\cdot, m(T), \cdot), \rho(T, \cdot) \otimes \rho(T, \cdot) \rangle \geq 0. \end{aligned} \tag{1.15}$$

This statement is an adaptation of an analogous result in [2]. The existence of the solution to (1.14) and the proof of (1.15) are given in the Appendix.

It is well-known that the map $\mathcal{V}^N$ defined in (0.3) solves the uniformly parabolic Hamilton–Jacobi–Bellman (HJB for short) equation

$$\begin{cases} -\partial_t \mathcal{V}^N(t, \mathbf{x}) - \sum_{j=1}^N \Delta_{x^j} \mathcal{V}^N(t, \mathbf{x}) + \frac{1}{N} \sum_{j=1}^N H(x^j, ND_{x^j} \mathcal{V}^N(t, \mathbf{x})) = \mathcal{F}(m_{\mathbf{x}}^N) \ \text{ in } \ (0, T) \times (\mathbb{R}^d)^N, \\ \mathcal{V}^N(T, \mathbf{x}) = \mathcal{G}(m_{\mathbf{x}}^N) \ \text{ in } \ (\mathbb{R}^d)^N, \end{cases}$$

and, therefore, $\mathcal{V}^N$ is smooth for any N. This is in contrast with the limit $\mathcal{U}$, which might not be C^1.

The following result, proved in [7], states however that both maps are uniformly Lipschitz continuous.

Lemma 1.7. *(Regularity of $\mathcal{V}^N$ and of $\mathcal{U}$) Assume* (1.5)*. There exists constant $C > 0$ depending on the data such that*

$$\|\mathcal{V}^N\|_\infty + N \sup_{j=1,\dots,N} \|D_{x^j}\mathcal{V}^N\|_\infty \le C,$$

and, for all $(t,m),(t',m') \in [0,T]\times\mathcal{P}_1(\mathbb{R}^d)$,

$$|\mathcal{U}(t,m) - \mathcal{U}(t',m')| \le C(|t-t'| + \mathbf{d}_1(m,m')).$$

The following convergence rate is the main result of [7].

Proposition 1.8. (Quantified convergence of $\mathcal{V}^N$ to $\mathcal{U}$) *Assume (1.5). There exists $\gamma \in (0,1)$ depending on dimension only and $C > 0$ depending on the smoothness of the data such that, for any $(t,\mathbf{x}) \in [0,T]\times(\mathbb{R}^d)^N$,*

$$\left|\mathcal{V}^N(t,\mathbf{x}) - \mathcal{U}(t,m^N_{\mathbf{x}})\right| \le C\frac{1}{N^\gamma}\left(1 + M_2(m^N_{\mathbf{x}})\right).$$

2. The regularity of $\mathcal{U}$

We prove here Theorem 1.1. A crucial step is the analysis of a linearized system, which is reminiscent of a linearized system studied in [6] and [2] for MFG problems. The main and important difference from [6] is that here we do not assume that $\mathcal{F}$ and $\mathcal{G}$ are convex, while [2] deals with problems on the torus. We go around the lack of monotonicity by using the notions of stability and strong stability of a solution, which are introduced next using the linearized system. Finally, stability is also used to define and analyze the open and dense set $\mathcal{O}$ on which the map $\mathcal{U}$ will eventually be smooth.

The linearized system

We fix $t_0 \in [0,T)$, a constant C_0, and, for $m_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $V : [t_0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ with $\|V\|_{C^{1,3}} \le C_0$, let m be the solution to

$$\begin{cases} \partial_t m - \Delta m - \mathrm{div}(Vm) = 0 \;\text{ in }\; (t_0,T)\times\mathbb{R}^d, \\ m(t_0) = m_0 \;\text{ in }\; \mathbb{R}^d. \end{cases} \tag{2.1}$$

We analyze the inhomogeneous linearized system

$$\begin{cases} -\partial_t z - \Delta z + V(t,x)\cdot Dz = \dfrac{\delta F}{\delta m}(x,m(t))(\rho(t)) + R^1(t,x) \;\text{ in }\; (t_0,T)\times\mathbb{R}^d, \\ \partial_t\rho - \Delta\rho - \mathrm{div}(V\rho) - \sigma\mathrm{div}(m\Gamma Dz) = \mathrm{div}(R^2) \;\text{ in }\; (t_0,T)\times\mathbb{R}^d, \\ \rho(t_0) = \xi \;\text{ and }\; z(T,x) = \dfrac{\delta G}{\delta m}(x,m(T))(\rho(T)) + R^3 \;\text{ in }\; \mathbb{R}^d, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} &\sigma \in [0,1] \;\text{ and }\; \delta \in (0,1), \\ &\Gamma \in C^0([0,T]\times\mathbb{R}^d;\mathbb{R}^{d\times d}) \quad\text{with}\quad \|\Gamma\|_\infty \le C_0, \\ &R^1 \in C^{\delta/2,\delta},\quad R^2 \in L^\infty([t_0,T],(W^{1,\infty})'(\mathbb{R}^d,\mathbb{R}^d)),\quad R^3 \in C^{2+\delta} \quad\text{and}\quad \xi \in (W^{1,\infty})'. \end{aligned} \tag{2.3}$$

The pair $(z,\rho) \in C^{0,1}([t_0,T]\times\mathbb{R}^d)\times C^0([0,T],(C^{2+\delta})')$ is a solution to (2.2) if z and ρ satisfy respectively the first and second equation in the sense of distributions.

Note that, because of the regularity of ρ and the assumptions on $\mathcal{F}$ and $\mathcal{G}$, the maps $(t,x) \to \frac{\delta F}{\delta m}(x, m(t))(\rho(t))$ and $x \to \frac{\delta G}{\delta m}(x, m(T))(\rho(T))$ are continuous and bounded.

We will often use system (2.2) in which $V(t,x) = H_p(x, Du(t,x))$ and $\Gamma(t,x) = H_{pp}(x, Du(t,x))$, where (u,m) is a classical solution to (1.12). In this case, V, Γ and m satisfy the conditions above.

Next we introduce the notion of strong stability for the homogeneous version of (2.2), that is the system

$$\begin{cases} -\partial_t z - \Delta z + V(t,x)\cdot Dz = \frac{\delta F}{\delta m}(x, m(t))(\rho(t)) \ \text{ in } \ (t_0,T)\times\mathbb{R}^d, \\ \partial_t \rho - \Delta\rho - \operatorname{div}(V\rho) - \sigma \operatorname{div}(m\Gamma Dz) = 0 \ \text{ in } \ (t_0,T)\times\mathbb{R}^d, \\ \rho(t_0) = 0 \ \text{ and } \ z(T,x) = \frac{\delta G}{\delta m}(x, m(T))(\rho(T)) \ \text{ in } \ \mathbb{R}^d. \end{cases} \tag{2.4}$$

We say that

$$\text{the system (2.4) is strongly stable if, for any } \sigma \in [0,1], \\ \text{its unique solution is } (z,\rho) = (0,0). \tag{2.5}$$

When dealing with the optimal control system, we will use a weaker notion of stability for (2.4), which only requires that the unique solution to (2.4) *with* $\sigma = 1$ is $(z,\rho) = (0,0)$. We need however the notion of strong stability for Proposition 2.4 below in order to prove the existence of a solution to (2.2) by a continuation method.

The main result of the subsection is a uniqueness and regularity result for the solution to (2.2).

Lemma 2.1. *Assume* (1.5) *and* (2.5). *There exist a neighborhood* $\mathcal{V}$ *of* (V,Γ) *in the topology of locally uniform convergence, and* $\eta, C > 0$ *such that, for any* $(V', t_0', m_0', \Gamma', R^{1,'}, R^{2,'}, R^{3,'}, \xi', \sigma')$ *with*

$$(V',\Gamma') \in \mathcal{V}, \quad |t_0' - t_0| + \mathbf{d}_2(m_0', m_0) \le \eta, \quad \|V'\|_{C^{1,3}} + \|\Gamma'\|_\infty \le 2C_0, \quad \sigma' \in [0,1], \\ R^{1,'} \in C^{\delta/2,\delta},\ R^{2,'} \in L^\infty([t_0,T], (W^{1,\infty})'(\mathbb{R}^d,\mathbb{R}^d)),\ R^{3,'} \in C^{2+\delta},\ \xi' \in (W^{1,\infty})', \tag{2.6}$$

any solution (z', ρ') *to* (2.2) *associated with these data on* $[t_0', T]$ *and* m' *the solution to* (2.1) *with drift* V' *and initial condition* m_0' *at time* t_0' *satisfies*

$$\|z'\|_{C^{(2+\delta)/2,2+\delta}} + \sup_{t\in[t_0',T]} \|\rho'(t,\cdot)\|_{(C^{2+\delta})'} + \sup_{t'\neq t} \frac{\|\rho'(t',\cdot) - \rho'(t,\cdot)\|_{(C^{2+\delta})'}}{|t'-t|^{1/2}} \le CM', \tag{2.7}$$

where

$$M' := \|\xi'\|_{(W^{1,\infty})'} + \|R^{1,'}\|_{C^{\delta/2,\delta}} + \|R^{3,'}\|_{C^{2+\delta}} + \sup_{t\in[t_0',T]} \|R^{2,'}(t)\|_{(W^{1,\infty})'}. \tag{2.8}$$

An immediate consequence is the following corollary.

Corollary 2.2. *Assume* (1.5) *and* (2.5). *Then, for any* (V', m_0', Γ') *satisfying* (2.6)*, the corresponding homogeneous linearized system is strongly stable.*

The proof of Lemma 2.1 follows some of the ideas of [6], where a similar system is studied. The main differences are that, here, we use the stability condition instead of the monotonicity assumption of [6] and work in an unbounded space.

In what follows, we need a preliminary result which we state and prove next. The difference between the estimate below and the one of Lemma 2.1 is the right hand side of the former which depends on the solution itself.

Lemma 2.3. *Assume* (1.5) *and let* (z,ρ) *be a solution to* (2.2). *There is a constant* $C>0$, *depending only on the regularity of* $\mathcal{F}$, $\mathcal{G}$ *and on* $\|V\|_{C^{1,3}}+\|\Gamma\|_\infty$, *such that*

$$\|z\|_{C^{(2+\delta)/2,2+\delta}}+\sup_{t\in[t_0,T]}\sup_{t'\neq t}\frac{\|\rho(t',\cdot)-\rho(t,\cdot)\|_{(C^{2+\delta})'}}{|t'-t|^{\delta/2}}\leq C\left(M+\sup_{t\in[t_0,T]}\|\rho(t)\|_{(C^{2+\delta})'}\right),$$

where

$$M:=\|\xi\|_{(W^{1,\infty})'}+\|R^1\|_{C^{\delta/2,\delta}}+\sup_{t\in[t_0,T]}\|R^2(t)\|_{(W^{1,\infty})'}+\|R^3\|_{C^{2+\delta}}.$$

Proof. Throughout the proof, C denotes a constant that depends only on the data and may change from line to line.

Set $R=\sup_{t\in[t_0,T]}\|\rho(t)\|_{(C^{2+\delta})'}<\infty$. It follows that the maps $(t,x)\to\frac{\delta F}{\delta m}(x,m(t))(\rho(t))$ and $x\to\frac{\delta G}{\delta m}(x,m(T))(\rho(T))$ are bounded by CR, the latter in $C^{2+\delta}$.

Then, standard parabolic regularity gives that z is bounded in $C^{(1+\delta)/2,1+\delta}$ by CR.

The main step of the proof is to show that

$$\sup_{t\in[t_0,T]}\sup_{t'\neq t}\frac{\|\rho(t')-\rho(t)\|_{(C^{2+\delta})'}}{|t'-t|^{\delta/2}}\leq C(M+R).$$

Arguing by duality, we fix $t_0\leq t_1<t_2\leq T$ and, with $\overline{\psi}\in C^{2+\delta}$, we consider the solution ψ^i for $i=1,2$ to

$$\begin{cases}-\partial_t\psi^i-\Delta\psi^i+V(t,x)\cdot D\psi^i=0 \ \text{ on } \ (t_0,t_i)\times\mathbb{R}^d,\\ \psi(t_i)=\overline{\psi} \ \text{ in } \ \mathbb{R}^d,\end{cases}\tag{2.9}$$

which, in view of the assumption on V and parabolic regularity, satisfies, for $i=1,2$, the bound

$$\|\psi^i\|_{C^{(2+\delta)/2,2+\delta}}\leq C\|\overline{\psi}\|_{C^{2+\delta}}.$$

It is immediate that, for $i=1,2$,

$$\langle\rho(t_i),\overline{\psi}\rangle=\langle\xi,\psi^i(t_0)\rangle-\sigma\int_{t_0}^{t_i}\int_{\mathbb{R}^d}\Gamma Dz\cdot D\psi^i m\,dxdt-\int_{t_0}^{t_i}\langle R^2(t),D\psi^i(t)\rangle dt.$$

Thus

$$\langle \rho(t_2)-\rho(t_1), \overline{\psi}\rangle = \langle \xi, \psi^2(t_0)-\psi^1(t_0)\rangle - \sigma\int_{t_0}^{t_1}\int_{\mathbb{R}^d} \Gamma Dz\cdot(D\psi^2-D\psi^1)m$$
$$-\sigma\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\Gamma Dz\cdot D\psi^2 m - \int_{t_0}^{t_1}\langle R^2(t), (D\psi^2-D\psi^1)(t)\rangle dt \tag{2.10}$$
$$-\int_{t_1}^{t_2}\langle R^2(t), D\psi^2(t)\rangle dt.$$

Note that $\psi^2-\psi^1$ solves (2.9) on (t_0,t_1) with a terminal condition at t_1 given by $(\psi^2-\psi^2)(t_1,\cdot)=\psi^2(t_1,\cdot)-\psi^2(t_2,\cdot)$, which, in view of the regularity of ψ^2, is bounded in $W^{2,\infty}$ by $C(t_2-t_1)^{\delta/2}\|\overline{\psi}\|_{C^{2+\delta}}$.

It then follows from the maximum principle that $(\psi^2-\psi^1)(t,\cdot)$ is bounded in $W^{2,\infty}$ by $C(t_2-t_1)^{\delta/2}\|\overline{\psi}\|_{C^{2+\delta}}$ for any t, and, hence,

$$\left|\langle \xi, \psi^2(t_0,\cdot)-\psi^1(t_0,\cdot)\rangle\right| \le \|\xi\|_{(W^{1,\infty})'}\|\psi^2(t_0,\cdot)-\psi^1(t_0,\cdot)\|_{W^{1,\infty}}$$
$$\le C(t_2-t_1)^{\delta/2}\|\overline{\psi}\|_{C^{2+\delta}}\|\xi\|_{(W^{1,\infty})'} \le C(t_2-t_1)^{\delta/2}\|\overline{\psi}\|_{C^{2+\delta}}M,$$
$$\left|\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\Gamma Dz\cdot(D\psi^2-D\psi^1)m dx dt\right| \le C\|Dz\|_\infty (t_2-t_1)^{\delta/2}\|\overline{\psi}\|_{C^{2+\delta}}$$
$$\le C(t_2-t_1)^{\delta/2}\|\overline{\psi}\|_{C^{2+\delta}}R$$

and

$$\left|\int_{t_0}^{t_1}\langle R^2(t),(D\psi^2-D\psi^1)(t)\rangle dt\right| \le \sup_t \|R^2(t)\|_{(W^{1,\infty})'}\sup_t\|D\psi^2(t)-D\psi^1(t)\|_{W^{1,\infty}}$$
$$\le C(t_2-t_1)^{\delta/2}\|\overline{\psi}\|_{C^{2+\delta}}M.$$

Note that the regularity of ψ^2 yields

$$\left|\sigma\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\Gamma Dz\cdot D\psi^2 m dx dt\right| \le C(t_2-t_1)\|Dz\|_\infty\|D\psi^2\|_\infty \le C(t_2-t_1)^{1/2}\|\overline{\psi}\|_{C^{2+\delta}}R$$

and

$$\left|\int_{t_1}^{t_2}\langle R^2(t), D\psi^2(t)\rangle dt\right| \le (t_2-t_1)\sup_t\|R^2(t)\|_{(W^{1,\infty})'}\sup_t\|D\psi^2(t)\|_{W^{1,\infty}}$$
$$\le C(t_2-t_1)\|\overline{\psi}\|_{C^{2+\delta}}M.$$

Since $\overline{\psi}$ is arbitrary, it follows from (2.10) that

$$\|\rho(t_2)-\rho(t_1)\|_{(C^{2+\delta})'} \le C(t_2-t_1)^{\delta/2}(M+R).$$

This regularity of ρ implies that the maps $(t,x)\to \frac{\delta F}{\delta m}(x,m(t))(\rho(t))$ and $x\to\frac{\delta G}{\delta m}(x,m(T))(\rho(T))$ are bounded in $C^{\delta/2,\delta}$ and $C^{2+\delta}$ respectively by $C(M+R)$. Thus z is also bounded in $C^{(2+\delta)/2,2+\delta}$ by $C(M+R)$. $\square$

Proof of Lemma 2.1. The first (and main) part of the proof consists in showing the existence of $\mathcal{V}$, $\eta>0$ and $C>0$ such that, for any $(V', t_0', m_0', \Gamma', R^{1,'}, R^{2,'},$

$R^{3,'}, \xi', \sigma'$) as in (2.6), any solution (z', ρ') to (2.2) associated with these data on $[t_0', T]$ satisfies, with M' is defined by (2.8),

$$\sup_{t\in[t_0',T]} \|\rho'(t)\|_{(C^{2+\delta})'} \le CM'.$$

We prove this claim by contradiction, assuming the existence of sequences $(V^n)_{n\in\mathbb{N}}$, $(t_0^n, m_0^n)_{n\in\mathbb{N}}$, $(m^n)_{n\in\mathbb{N}}$, $(\Gamma^n)_{n\in\mathbb{N}}$, $(R^{i,n})_{n\in\mathbb{N}}$, $(\xi^n)_{n\in\mathbb{N}}$, and $(\sigma^n)_{n\in\mathbb{N}}$ such that (V^n, Γ^n) converges locally uniformly to (V, Γ),

$$|t_0^n - t_0| + \mathbf{d}_2(m_0^n, m_0) \le 1/n, \quad \|V^n\|_{C^{1,3}} + \|\Gamma^n\|_\infty \le 2C_0, \quad \sigma^n \in [0,1],$$

and

$$\|\xi^n\|_{(W^{1,\infty})'} + \|R^{3,n}\|_{C^{2+\delta}} + \|R^{1,n}\|_{C^{\delta/2,\delta}} + \sup_{t\in[t_0,T]} \|R^{2,n}(t)\|_{(W^{1,\infty})'} \le 1,$$

and, for each n, a solution (z^n, ρ^n) to (2.2) associated with the data above, such that

$$\lambda^n = \sup_{t\in[t_0^n,T]} \|\rho^n(t)\|_{(C^{2+\delta})'} \ge n.$$

It follows that $(\widetilde{z}^n, \widetilde{\rho}^n) = \frac{1}{\lambda^n}(z^n, \rho^n)$ solves the system

$$\begin{cases} -\partial_t \widetilde{z}^n - \Delta \widetilde{z}^n + V^n(t,x)\cdot D\widetilde{z}^n = \dfrac{\delta F}{\delta m}(x, m^n(t))(\widetilde{\rho}^n(t)) + \dfrac{1}{\lambda^n} R_1^n(t,x) & \text{in } (t_0^n, T)\times\mathbb{R}^d, \\ \partial_t \widetilde{\rho}^n - \Delta\widetilde{\rho}^n - \mathrm{div}(V^n\widetilde{\rho}^n) - \sigma^n \mathrm{div}(m\Gamma^n D\widetilde{z}^n) = \dfrac{1}{\lambda^n}\mathrm{div}(R_2^n) & \text{in } (t_0^n, T)\times\mathbb{R}^d, \\ \widetilde{\rho}^n(t_0^n) = \dfrac{1}{\lambda^n}\xi^n \ \text{ and } \ \widetilde{z}^n(T,x) = \dfrac{\delta G}{\delta m}(x, m^n(T))(\widetilde{\rho}^n(T)) + \dfrac{1}{\lambda^n} R_3^n & \text{in } \mathbb{R}^d. \end{cases} \tag{2.11}$$

Since, by definition, $\sup_{t\in[t_0^n,T]} \|\widetilde{\rho}^n(t)\|_{(C^{2+\delta})'} = 1$, Lemma 2.3 implies that the $(\widetilde{z}^n)$'s and $\widetilde{\rho}^n(t,\cdot)$'s are bounded in $C^{(2+\delta)/2, 2+\delta}$ and $C^{\delta/2}([0,T], (C^{2+\delta})')$ respectively.

Hence, we may assume that, up to a subsequence, the sequences $(\sigma^n)_{n\in\mathbb{N}}$, $(\widetilde{z}^n)_{n\in\mathbb{N}}$, $(\widetilde{\rho}^n)_{n\in\mathbb{N}}$ and $(V^n)_{n\in\mathbb{N}}$ converge respectively to some $\sigma\in[0,1]$, $\widetilde{z} \in C^{1,2}_{loc}$, $\widetilde{\rho} \in C^0([t_0,T], (C_c^{2+\delta})')$, where $(C_c^{2+\delta})'$ is endowed with the weak$-*$ topology, and $V \in C^0([t_0,T], C^{2+\delta}_{loc})$.

The goal is to show that $(\widetilde{z}, \widetilde{\rho})$ is a nonzero solution to the homogenous equation (2.4), which will contradict the strong stability of the system.

There are two difficulties that need to be addressed both caused by the above claimed weak convergence of the $\widetilde{\rho}^n$'s to $\widetilde{\rho}$.

The first is to prove that, as $n\to\infty$,

$$\frac{\delta F}{\delta m}(x, m^n(t))(\widetilde{\rho}^n(t)) \to \frac{\delta F}{\delta m}(x, m(t))(\widetilde{\rho}(t)) \text{ and } \frac{\delta G}{\delta m}(x, m^n(T))(\widetilde{\rho}^n(T)) \to \frac{\delta G}{\delta m}(x, m(T))(\widetilde{\rho}(T)),$$

and the second is to show that, since $\sup_t \|\widetilde{\rho}^n(t)\|_{(C^{2+\delta})'} = 1$, we must have $(\widetilde{z}, \widetilde{\rho})$ is nonzero.

To overcome these two issues it is necessary to upgrade the convergence of the $\widetilde{\rho}^n$'s to $\widetilde{\rho}$ in $C^0([t_0,T], (C^{2+\delta})')$ from weak to strong.

We first note that $\widetilde{\rho}^n = \widetilde{\rho}^{n,1} + \widetilde{\rho}^{n,2}$ with $\widetilde{\rho}^{n,1}$ and $\widetilde{\rho}^{n,2}$ solving respectively

$$\partial_t \widetilde{\rho}^{n,1} - \Delta \widetilde{\rho}^{n,1} - \mathrm{div}(V^n \widetilde{\rho}^{n,1}) = \frac{1}{\lambda^n}\mathrm{div}(R_2^n) \ \text{ and } \ \rho^{n,1}(t_0) = \frac{1}{\lambda^n}\xi^n,$$

and

$$\partial_t \widetilde{\rho}^{n,2} - \Delta \widetilde{\rho}^{n,2} - \mathrm{div}(V^n \widetilde{\rho}^{n,2}) - \sigma^n \mathrm{div}(\Gamma^n D\widetilde{z}^n m^n) = 0 \ \text{ and } \ \widetilde{\rho}^{n,2}(t_0) = 0. \tag{2.12}$$

We show next that $\sup_t \|\widetilde{\rho}^{n,1}(t)\|_{(C^{2+\delta})'} \to 0$. Indeed, for fixed $\bar{t} \in (t_0^n, T]$ and $\overline{\psi} \in C^{2+\delta}$, let ψ^n be the solution to the dual problem

$$\begin{cases} -\partial_t \psi^n - \Delta \psi^n + V^n(t,x) \cdot D\psi^n = 0 \ \text{ in } \ (t_0^n, \bar{t}) \times \mathbb{R}^d, \\ \psi^n(\bar{t}) = \overline{\psi} \ \text{ in } \ \mathbb{R}^d. \end{cases} \tag{2.13}$$

It follows from the standard parabolic regularity that ψ^n is bounded in $C^{(2+\delta)/2,2+\delta}$ by $C\|\overline{\psi}\|_{C^{2+\delta}}$ with C is independent of n and $\bar{t}$, and since, in view of the duality, we have

$$\langle \widetilde{\rho}^{n,1}(\bar{t}), \overline{\psi}\rangle = \frac{1}{\lambda^n}\big(\langle \xi^n, \psi^n(t_0^n)\rangle - \int_{t_0^n}^{\bar{t}} \langle R^{2,n}, D\psi^n\rangle dt\big),$$

we obtain

$$\sup_{t\in[t_0^n,T]} \|\widetilde{\rho}^{n,1}(t)\|_{(C^{2+\delta})'} \le \frac{C}{\lambda^n}\big(\|\xi^n\|_{(W^{1,\infty})'} + \sup_{t\in[t_0^n,T]} \|R^{2,n}(t)\|_{(W^{1,\infty})'}\big).$$

Hence, $\sup_{t\in[t_0^n,T]} \|\widetilde{\rho}^{n,1}(t)\|_{(C^{2+\delta})'} \to 0$ as $\dfrac{1}{\lambda^n}(\|\xi^n\|_{(W^{1,\infty})'} + \sup_{t\in[t_0^n,T]} \|R^{2,n}(t)\|_{(W^{1,\infty})'}) \to 0$.

It follows from (2.12) that, for any $\bar{t}^n \in [t_0^n, T]$ and $\overline{\psi}^n \in C^{2+\delta}$, if ψ^n is the solution to (2.13) on $[t_0^n, \bar{t}^n]$ with terminal condition $\psi^n(\bar{t}^n) = \overline{\psi}^n$, then

$$\langle \widetilde{\rho}^{n,2}(\bar{t}^n), \overline{\psi}^n\rangle = -\sigma^n \int_{t_0^n}^{\bar{t}^n} \int_{\mathbb{R}^d} \Gamma^n D\widetilde{z}^n \cdot D\psi^n m^n. \tag{2.14}$$

In order to prove the uniform convergence of the $\widetilde{\rho}^{n,2}$'s in $(C^{2+\delta})'$, we assume that the $\|\overline{\psi}^n\|_{C^{2+\delta}}$'s are bounded and, without loss of generality, that the $\bar{t}^n$'s and $\overline{\psi}^n$'s converge respectively to $\bar{t} \in [t_0, T]$ and $\overline{\psi} \in C^{2+\delta}$, the last convergence being in $C^{2+\delta_1}_{loc}$ for any $\delta_1 \in (0, \delta)$. We need to prove that the $(\langle \widetilde{\rho}^{n,2}(\bar{t}^n), \overline{\psi}^n\rangle)$'s converge to $\langle \widetilde{\rho}^2(\bar{t}), \overline{\psi}\rangle$, where $\widetilde{\rho}^2$ is the solution to

$$\partial_t \widetilde{\rho}^2 - \Delta \widetilde{\rho}^2 - \mathrm{div}(V \widetilde{\rho}^2) - \sigma \mathrm{div}(\Gamma D\widetilde{z} m) = 0 \ \text{ and } \ \widetilde{\rho}^2(t_0) = 0.$$

Note that

$$\langle \widetilde{\rho}^2(\bar{t}), \overline{\psi}\rangle = -\sigma \int_{t_0}^{\bar{t}} \int_{\mathbb{R}^d} \Gamma D\widetilde{z} \cdot D\psi m,$$

where ψ is the solution to

$$-\partial_t \psi - \Delta \psi + V(t,x) \cdot D\psi = 0 \ \text{ in } \ (t_0, \bar{t}) \times \mathbb{R}^d \ \text{ and } \ \psi(\bar{t}) = \overline{\psi} \ \text{ in } \ \mathbb{R}^d,$$

and recall that the Γ^n's and $D\widetilde{z}^n$'s are bounded and converge locally uniformly to Γ and to $D\widetilde{z}$ respectively.

Similarly, due to the parabolic regularity, the ψ^n's are bounded in $C^{(2+\delta)/2,2+\delta}$ and the $D\psi^n$'s converge locally uniformly to $D\psi$.

Moreover, since m^n is the solution to

$$\partial_t m^n - \Delta m^n + \mathrm{div}(V^n m^n) = 0 \ \text{ in } \ (t_0^n, T] \ \text{ and } \ m^n(t_0^n, \cdot) = m_0^n \ \text{ in } \mathbb{R}^d,$$

with V^n uniformly bounded, we know that the m^n's converge uniformly to m in $\mathcal{P}_1(\mathbb{R}^d)$, and we have the second-order moment estimate

$$\sup_{t,n} \int_{\mathbb{R}^d} |x|^2 m^n(t, dx) \le C \sup_n \int_{\mathbb{R}^d} |x|^2 m_0^n(dx) \le C.$$

In addition, using that $\Gamma D\widetilde{z} \cdot D\psi$ is globally Lipschitz, for any $R \ge 1$ we find

$$\begin{aligned}
\left| \langle \widetilde{\rho}^{n,2}(\bar{t}^n), \overline{\psi}^n \rangle - \langle \widetilde{\rho}^2(\bar{t}), \overline{\psi} \rangle \right| &\le C(|\sigma^n - \sigma| + |\bar{t}^n - \bar{t}| + |t_0^n - t_0|) \\
&\quad + \sigma \left| \int_{t_0 \vee t_0^n}^{\bar{t} \wedge \bar{t}^n} \int_{\mathbb{R}^d} (\Gamma D\widetilde{z} \cdot D\psi m - \Gamma^n D\widetilde{z}^n \cdot D\psi^n m^n) \right| \\
&\le C(|\sigma^n - \sigma| + |\bar{t}^n - \bar{t}| + |t_0^n - t_0| + \sup_t \mathbf{d}_1(m^n(t), m(t)) + R^{-2}) \\
&\quad + \sigma \left| \int_{t_0 \vee t_0^n}^{\bar{t} \wedge \bar{t}^n} \int_{B_R} (\Gamma D\widetilde{z} \cdot D\psi - \Gamma^n D\widetilde{z}^n \cdot D\psi^n) m^n \right|.
\end{aligned}$$

Letting $n \to \infty$ and then $R \to \infty$ proves the convergence of the $\langle \widetilde{\rho}^{n,2}(\bar{t}^n), \overline{\psi}^n \rangle$'s to $\langle \widetilde{\rho}^2(\bar{t}), \overline{\psi} \rangle$. It follows that the sequence $(\widetilde{\rho}^n)_{n \in \mathbb{N}}$ converges to $\widetilde{\rho} = \widetilde{\rho}^2$ strongly in $C^0([t_0, T], (C^{2+\delta})')$.

To summarize the above, we know that the sequences $(\sigma^n)_{n\in\mathbb{N}}$, $(\widetilde{z}^n)_{n\in\mathbb{N}}$, $(\widetilde{\rho}^n)_{n\in\mathbb{N}}$ and $(V^n)_{n\in\mathbb{N}}$ converge respectively to $\sigma \in [0,1]$, $\widetilde{z}$ in $C^{1,2}_{loc}$, $\widetilde{\rho}$ in $C^0([0,T], (C^{2+\delta})')$ and V in $C^0([t_0,T], C^{2+\delta}_{loc})$.

Passing to the limit in (2.11) we infer that $(\widetilde{z}, \widetilde{\rho})$ is a solution to the homogenous equation (2.4).

Since $\sup_{t\in[t_0^n,T]} \|\rho^n(t)\|_{(C^{2+\delta})'} = 1$ for any n, it follows that $\sup_{t\in[t_0,T]} \|\rho(t)\|_{(C^{2+\delta})'} = 1$. Thus $(\widetilde{z}, \widetilde{\rho})$ is a nonzero solution to the homogenous equation (2.4) which contradicts the strong stability assumption (2.5).

The second part of the proof consists in upscaling the regularity obtained in the first part. For this, we let $(V', t_0', m_0', \Gamma', R^{1,'}, R^{2,'}, R^{3,'}, \xi', \sigma')$ be such that (2.6) holds and (z', ρ') be a solution to (2.2) associated with these data, where m' is the solution to (2.1) with drift V' and initial condition m_0' at time t_0'.

We have already established that, for the M' in (2.8),

$$\sup_{t\in[t_0',T]} \|\rho'(t)\|_{(C^{2+\delta})'} \le CM'.$$

It then follows from Lemma 2.3 that

$$\begin{aligned}
&\|z'\|_{C^{(2+\delta)/2,2+\delta}} + \sup_{t'\neq t} \frac{\|\rho'(t',\cdot) - \rho'(t,\cdot)\|_{C^{2+\delta}}}{|t'-t|^{\delta/2}} \\
&\quad \le C(M' + \sup_{t\in[t_0',T]} \|\rho'(t)\|_{(C^{2+\delta})'}) \le CM'.
\end{aligned}$$

□

We complete the section with an existence result for system (2.2) given a solution m to (2.1).

Proposition 2.4. *Assume* (1.5) *and* (2.5). *Then, for any* ξ, δ, Γ, R^1, R^2, R^3 *as in* (2.3), *there exists a unique solution to the linearized system* (2.2) *with* $\sigma = 1$.

Proof. We use a continuation method. Let Σ be the set of σ's for which (2.2) has a solution for any data ξ, R^1, R^2 and R^3 satisfying (2.3). We have to check that Σ is nonempty, open and closed in $[0, 1]$.

If $\sigma = 0$, then the equation for ρ (which does not involve z) has a unique solution; then one can solve the equation for z in a standard way. Hence $0 \in \Sigma$ and Σ is nonempty.

We now check that Σ is closed. Let $\sigma^n \to \sigma \in [0, 1]$ and (z^n, ρ^n) be the associated solution to (2.2) given some ξ, R^1, R^2 and R^3. In view of Lemma 2.1, we have

$$\|z^n\|_{C^{(2+\delta)/2,2+\delta}} + \sup_{t\in[t_0,T]} \|\rho^n(t,\cdot)\|_{(C^{2+\delta})'} + \sup_{t'\neq t} \frac{\|\rho^n(t',\cdot) - \rho'(t,\cdot)\|_{(C^{2+\delta})'}}{|t'-t|^{1/2}} \le C.$$

Then it is easy to pass to the limit in system (2.2) to find a solution to (2.2) for σ as a limit (up to subsequence) of the (z^n, ρ^n)'s. This solution is unique thanks to Lemma 2.1. So Σ is closed.

Finally, we check that Σ is open. We fix ξ, R^1, R^2 and R^3 and $\sigma \in \Sigma$ and, for $\sigma' \in [0, 1]$ close to σ, we build the solution of (2.2) for σ' by a Banach fixed point argument. Indeed, given (z', ρ') in $C^{(2+\delta)/2,2+\delta} \times C^0([t_0, T], (C^{2+\delta})')$, let (z'', ρ'') be the solution to

$$\begin{cases} -\partial_t z'' - \Delta z'' + V(t,x)\cdot Dz'' = \dfrac{\delta F}{\delta m}(x, m(t))(\rho''(t)) + R^1 \ \text{ in } \ (t_0,T)\times\mathbb{R}^d, \\ \partial_t \rho'' - \Delta\rho'' - \operatorname{div}(V\rho'') - \sigma\operatorname{div}(m\Gamma Dz'') = \operatorname{div}(R^2 + (\sigma'-\sigma)m\Gamma Dz') \ \text{ in } \ (t_0,T)\times\mathbb{R}^d, \\ \rho''(t_0) = \xi \ \text{ and } \ z''(T,x) = \dfrac{\delta G}{\delta m}(x, m(T))(\rho''(T)) + R^3 \ \text{ in } \ \mathbb{R}^d, \end{cases}$$

which is uniquely solvable since $\sigma \in \Sigma$. Moreover, a fixed point of the map $(z', \rho') \to (z'', \rho'')$ is a solution to (2.2) for σ'. We prove next that this map is a contraction. Let (z'_1, ρ'_1) and (z'_2, ρ'_2) be two data, with associated solutions (z''_1, ρ''_1) and (z''_2, ρ''_2) respectively. The difference $(z''_2 - z''_1, \rho''_2 - \rho''_1)$ satisfies an equation of the form (2.2) with σ and with $R^1 = R^3 = \xi = 0$ and $R^2 = (\sigma' - \sigma)m\Gamma D(z'_2 - z'_1)$. By Lemma 2.1, we have therefore

$$\begin{aligned} &\|z''_2 - z''_1\|_{C^{(2+\delta)/2,2+\delta}} + \sup_{t\in[t_0,T]} \|(\rho''_2 - \rho''_1)(t,\cdot)\|_{(C^{2+\delta})'} \\ &\quad + \sup_{t'\neq t} \frac{\|(\rho''_2 - \rho''_1)(t',\cdot) - \rho'(t,\cdot)\|_{(C^{2+\delta})'}}{|t'-t|^{1/2}} \\ &\le C|\sigma'-\sigma| \sup_{t\in[t_0,T]} \|(m\Gamma D(z'_1 - z'_2))(t,\cdot)\|_{(W^{1,\infty})'} \le C|\sigma'-\sigma| \|z'_2 - z'_1\|_{C^{(2+\delta)/2,2+\delta}}. \end{aligned}$$

This shows that the map $(z', \rho') \to (z'', \rho'')$ is a contraction if $|\sigma' - \sigma|$ is small enough. Therefore Σ is open, which completes the proof of the proposition. □

The stability property

We discuss here the notion of stability of a solution (u, m) of the MFG-system arising in MFC.

Let $(t_0, m_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ and (m, α) be a minimizer for $\mathcal{U}(t_0, m_0)$ with associated multiplier u, that is, the pair (u, m) solves (1.12) and $\alpha(t, x) = -H_p(x, Du(t, x))$.

Definition 2.5. The solution (u, m) is strongly stable (resp. stable), if for all $\sigma \in [0, 1]$ (resp. $\sigma = 1$) the only solution $(z, \mu) \in C^{(1+\delta)/2,1+\delta} \times C^0([t_0, T]; (C^{2+\delta}(\mathbb{R}^d))')$ to the linearized system

$$\begin{cases} -\partial_t z - \Delta z + H_p(x, Du) \cdot Dz = \dfrac{\delta F}{\delta m}(x, m(t))(\mu(t)) \text{ in } (t_0, T) \times \mathbb{R}^d, \\ \partial_t \mu - \Delta\mu - \mathrm{div}(H_p(x, Du)\mu) - \sigma \mathrm{div}(H_{pp}(x, Du)Dzm) = 0 \text{ in } (t_0, T) \times \mathbb{R}^d, \\ \mu(t_0) = 0 \text{ and } z(T, x) = \dfrac{\delta G}{\delta m}(x, m(T))(\mu(T)) \text{ in } \mathbb{R}^d, \end{cases} \tag{2.15}$$

Since, given a minimizer (m, α), the relation $\alpha = -H_p(x, Du)$ defines Du uniquely, the stability condition depends on (m, α) only. We say that the minimizer (m, α) is strongly stable (resp. stable) if (u, m) is strongly stable (resp. stable).

The above makes also clear the definition of the regularity set $\mathcal{O}$ in (1.8). We remark that at this point we do not know whether $\mathcal{O}$ is a nonempty set. This will follow from Lemma 2.8 below.

We also note that (2.15) is the linearized system studied in the previous subsection for the particular choice of vector field $V(t, x) = H_p(x, Du(t, x))$ and matrix $\Gamma(t, x) = H_{pp}(x, Du(t, x))$. To emphasize that we are working with this particular system and also be consistent with other references, heretofore we use the notation (z, μ) instead of (z, ρ).

The following lemma asserts that the minimizers starting from an initial condition in $\mathcal{O}$ are actually strongly stable.

Lemma 2.6. *Assume (1.5), fix $(t_0, m_0) \in \mathcal{O}$ and let (m, α) be the unique stable minimizer associated to $\mathcal{U}(t_0, m_0)$. Then (m, α) is strongly stable.*

Proof. Let (z, ρ) be a solution of (2.4). If $\sigma = 1$, the fact that $(z, \rho) = (0, 0)$ is just the assumed stability of (m, α). If $\sigma = 0$, then the equation of ρ does not depend on z and thus $\rho = 0$, which in turn implies that $z = 0$. So in what follows we assume that

$$\sigma \in (0, 1).$$

Lemma 2.1 gives that $z \in C^{(2+\delta)/2,2+\delta}$, while using the duality, we have, for any $t \in [t_0, T]$,

$$\langle z(t, \cdot), \mu(t)\rangle = -\int_{t_0}^{t} (\int_{\mathbb{R}^d} \Big(\sigma H_{pp}(x, Du)Dz \cdot dzmdx) + \langle \frac{\delta F}{\delta m}(\cdot, m(t))(\mu(t)), \mu(t)\rangle\Big)dt,$$

and, in particular, for $t = T$, we get

$$\int_{t_0}^{T} \Big(\int_{\mathbb{R}^d} \sigma H_{pp}(x, Du) Dz \cdot Dz m dx + \langle \frac{\delta F}{\delta m}(\cdot, m(t))(\mu(t)), \mu(t) \rangle \Big) dt \\ + \langle \frac{\delta G}{\delta m}(\cdot, m(T))(\mu(T)), \mu(T) \rangle = 0. \tag{2.16}$$

We now use that (m, α) is a minimizer as well as the second-order condition (1.15) with $(\rho, \beta) = (\mu, \sigma H_{pp}(x, Du)Dz)$. Recalling that $L_{\alpha,\alpha}(x, \alpha(t, x))$ $H_{pp}(x, Du(t, x)) = Id$, we get

$$\int_{t_0}^{T} \int_{\mathbb{R}^d} \Big(\sigma^2 H_{pp}(x, Du) Dz \cdot Dz m \\ + \int_{\mathbb{R}^d} \frac{\delta F}{\delta m}(x, m(t), y) \mu(t, x) \mu(t, y) dy \Big) dx dt \\ + \int_{\mathbb{R}^{2d}} \frac{\delta G}{\delta m}(x, m(T), y) \mu(T, x) \mu(T, y) dy dx \geq 0,$$

and, in view of (2.16),

$$(\sigma - \sigma^2) \int_{t_0}^{T} \int_{\mathbb{R}^d} H_{pp}(x, Du) Dz \cdot Dz \; m \leq 0.$$

Since $H_{pp} > 0$ and $\sigma - \sigma^2 > 0$, the last inequality yields $Dz \; m = 0$, from which we easily conclude, going back to the equations satisfied by μ and by z, that $(z, \mu) = (0, 0)$. □

We turn next to $\mathcal{O}$. The next lemma establishes an important property together with the fact $\mathcal{O}$ is not empty. A similar statement is proved in [2] when the state space is the torus. The adaptation to the whole space is given here for the sake of completeness.

Lemma 2.7. *Assume (1.5). Fix $(t_0, m_0) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^d)$ and let (m, α) be a minimizer for $\mathcal{U}(t_0, m_0)$. Then $(t, m(t))$ belongs to $\mathcal{O}$ for any $t \in (t_0, T)$.*

Proof. Fix $(t_0, m_0) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^d)$, and let (m, α) be a minimizer for $\mathcal{U}(t_0, m_0)$ and u its associated multiplier.

For $t_1 \in (t_0, T)$, set $m_1 = m(t_1)$ and let $(\widetilde{m}, \widetilde{\alpha})$ be an optimal solution for $\mathcal{U}(t_1, m_1)$ with associated multiplier $\widetilde{u}$. Since, in view of the dynamic programming principle,

$$(\widehat{m}, \widehat{\alpha}) = \begin{cases} (m, \alpha) \text{ on } [t_0, t_1) \times \mathbb{R}^d, \\ (\widetilde{m}, \widetilde{\alpha}) \text{ on } [t_1, T] \times \mathbb{R}^d, \end{cases}$$

is optimal for $\mathcal{U}(t_0, m_0)$, we know from Lemma 1.3 that $\widehat{\alpha} \in C^{(1+\delta)/2, 1+\delta}$. It follows that $\alpha(t_1, \cdot) = \widetilde{\alpha}(t_1, \cdot)$ and thus that $Du(t_1, \cdot) = D\widetilde{u}(t_1, \cdot)$. Thus, the pair $((z^k)_{k=1,\dots,d}, \mu) = ((\partial_{x_k}(u - \widetilde{u}))_{k=1,\dots,d}, m - \widetilde{m})$ solves the system

$$\begin{aligned} & -\partial z^k - \Delta z^k + g^k(t, x) = 0 \;\text{ in }\; (t_1, T) \times \mathbb{R}^d, \\ & \partial \mu - \Delta \mu + \text{div}(h) = 0 \;\text{ in }\; (t_1, T) \times \mathbb{R}^d, \\ & \mu(t_1) = 0 \;\; z^k(t_1, \cdot) = 0 \;\text{ in }\; \mathbb{R}^d, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned}g^k(t,x) &= H_{x_k}(x,Du) - H_{x_k}(x,D\widetilde{u}) + H_p(x,Du)\cdot D(\partial_{x_k}u) - H_p(x,D\widetilde{u})\cdot D(\partial_{x_k}\widetilde{u})\\ &\quad - F_{x_k}(x,m(t)) + F_{x_k}(x,m^1(t)),\\ h &= H_p(Du)m - H_p(D\widetilde{u})\widetilde{m}.\end{aligned}$$

In order to estimate g^k and h, we note that, since $t_1 > t_0$, $m, \widetilde{m} \in C^{1,2}([t_1,T]\times\mathbb{R}^d)$ and $m(t,\cdot)$ and $\widetilde{m}(t,\cdot)$ are bounded in L^2. It follows that

$$\begin{aligned}&\sum_{k=1}^d |g^k(t,x)|^2 \le C(|z(t,x)|^2 + |Dz(t,x)|^2 + \|\mu(t)\|_{L^2}^2),\\ &|h(t,x)|^2 \le C(|z(t,x)|^2 + |\mu(t,x)|^2),\\ &|\mathrm{div}(h(t,x))|^2 \le C(|z(t,x)|^2 + |Dz(t,x)|^2 + |\mu(t,x)|^2 + |D\mu(t,x)|^2).\end{aligned}\tag{2.18}$$

Then a Lions-Malgrange-type argument shows that $z_k = \mu = 0$, and, hence, the solution starting from (t_1, m_1) is unique. We refer to Lions and Malgrange [27] for the original argument and Cannarsa and Tessitore [4] and [2] for its adaptation to forward-backward equations.

Next we check that this solution is stable. Let (z,μ) be a solution to (2.15) in $[t_1,T]\times\mathbb{R}^d$ with $\sigma = 1$, which by the standard parabolic regularity is actually classical. An elementary calculation yields

$$\begin{aligned}&\int_{t_1}^T \left(\int_{\mathbb{R}^d} H_{pp}(x,Du(t,x))Dz\cdot Dz\; mdx + \left\langle \frac{\delta F}{\delta m}(\cdot,m(t)),\mu(t),\mu(t)\right\rangle\right)dt\\ &\qquad + \left\langle \frac{\delta G}{\delta m}(\cdot,m(t))(\mu(t)),\mu(t)\right\rangle = 0.\end{aligned}\tag{2.19}$$

Using Lemma 1.6, we know that, for any $\beta \in L^\infty([t_0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ vanishing near $t = t_0$, if ρ is the solution in the sense of distributions to (1.14) in $[t_0,T]\times\mathbb{R}^d$, then

$$\begin{aligned}\widetilde{J}(\beta) = \int_{t_0}^T \Big(\int_{\mathbb{R}^d} & L_{\alpha,\alpha}(x,\alpha(t,x))\beta(t,x)\cdot\beta(t,x)m(t,dx)\\ &+ \langle \frac{\delta F}{\delta m}(\cdot,m(t),\cdot),\rho(t)\otimes\rho(t)\rangle\Big)dt\\ &+ \langle \frac{\delta G}{\delta m}(\cdot,m(T),\cdot),\rho(T,\cdot)\otimes\rho(T,\cdot)\rangle \ge 0.\end{aligned}$$

The solution $\overline{\rho}$ to (1.14) associated to the map $\overline{\beta}$ defined by $\overline{\beta} = 0$ on $[t_0,t_1)$ and $\overline{\beta} = -H_{pp}(x,Du)Dz$ on $[t_1,T]$ is given by $\overline{\rho}(t) = 0$ on $[t_0,t_1)\times\mathbb{R}^d$ and $\overline{\rho}(t) = \mu(t)$ on $[t_1,T]\times\mathbb{R}^d$.

It then follows from (2.19) that $\widetilde{J}(\overline{\beta}) = 0$, and, hence, $\overline{\beta}$ is a minimizer for $\widetilde{J}$, a fact that, by standard arguments (see, for example, [2])), implies that β is a continuous map. Thus $Dz(t_1,\cdot) = 0$.

We differentiate with respect to space variable the first equation in (2.15) and obtain that $((\partial_{x_k}z)_{k=1,\dots,d},\mu)$ solves a system of the form (2.17) with zero initial condition and data g and h satisfying (2.18). This implies, as before, that $((\partial_{x_k} z)_{k=1,\dots,d}, \mu) = (0,0)$. Coming back to (2.15), we obtain $z = \mu = 0$. Therefore the solution is stable. □

The next theorem establishes the key properties of $\mathcal{O}$.

Theorem 2.8. *Assume (1.5). The set $\mathcal{O}$ is open and dense in $[0,T) \times \mathcal{P}_2(\mathbb{R}^d)$.*

Proof. Lemma 2.7 implies that the set $\mathcal{O}$ is a nonempty, dense subset of $[0,T)\times\mathcal{P}_2(\mathbb{R}^d)$.

Next we show that $\mathcal{O}$ is open arguing by contradiction. For this, we fix $(t_0, m_0) \in \mathcal{O}$ and assume that there are initial positions $(t^n, m_0^n) \notin \mathcal{O}$ which converge to (t_0, m_0) in $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$. Let (m,α) be the unique and stable minimizer for (t_0, m_0) and u be the associated multiplier, that is, $\alpha = -H_p(x, Du)$ and the pair (u, m) solves (1.12).

Since $(t^n, m_0^n) \notin \mathcal{O}$, there are two cases (up to subsequences): either, for all n, there exist several minimizers for $\mathcal{U}(t^n, m_0^n)$ or, for all n, there exists a unique minimizer which is not stable. This latter case is ruled out by Lemma 2.1 and the strong stability of (m,α).

It remains to consider the first case and we argue as follows. Let $(m^{n,1}, \alpha^{n,1})$ and $(m^{n,2}, \alpha^{n,2})$ be two distinct minimizers starting from (t^n, m_0^n) with associated multipliers $u^{n,1}$ and $u^{n,2}$ respectively.

Since the problem with initial condition (t_0, m_0) has a unique minimizer, it follows from Lemma 1.5 that, for $i = 1, 2$, the $(m^{n,i}, \alpha^{n,i})$'s converge to (m,α) in $C^0([0,T];\mathcal{P}_1(\mathbb{R}^d)) \times C^{\delta/2\delta}$ while the $u^{n,i}$'s, $Du^{n,i}$'s and $D^2u^{n,i}$'s converge to u, Du and D^2u respectively in $C^{\delta/2,\delta}$.

Set

$$\theta^n = \|Du^{n,1} - Du^{n,2}\|_{C^{\delta/2,\delta}} + \sup_{t\in[t_0,T]} \mathbf{d}_1(m^{n,1}(t), m^{n,2}(t))$$

and note that, since $(m^{n,1}, \alpha^{n,1})$ and $(m^{n,2}, \alpha^{n,2})$ are distinct, $\theta^n > 0$ and, in view of the previous discussion, $\theta_n \to 0$, and, finally,

$$\theta^n \le C\|Du^{n,1} - Du^{n,2}\|_{C^{\delta/2,\delta}}. \tag{2.20}$$

This last estimate follows from the fact that again the uniform parabolicity implies that the $D^2u^{n,i}$'s are uniformly bounded. Applying Gronwall's inequality to the stochastic differential equations associated with the Kolmogorov equations satisfied by $m^{n,1}$ and $m^{n,2}$, we find the distance $\sup_{[t_n,T]} \mathbf{d}_1(m^{n,1}(t), m^{n,2}(t))$ is controlled by $C\|Du^{n,1} - Du^{n,2}\|_\infty$ and thus by $C\|Du^{n,1} - Du^{n,2}\|_{C^{\delta/2,\delta}}$.

Next we introduce the differences $z^n = (u^{n,1} - u^{n,2})/\theta^n$, $\mu^n = (m^{n,1} - m^{n,2})/\theta^n$ and observe that

$$\begin{cases} -\partial_t z^n - \Delta z^n + H_p(x, Du^{n,1})\cdot Dz^n = \dfrac{\delta F}{\delta m}(x, m^{n,1}(t))(\mu^n(t)) + R^{n,1}, \\ \partial_t \mu^n - \Delta\mu^n - \mathrm{div}(H_p(x, Du^{n,1})\mu^n) - \mathrm{div}(H_{pp}(x, Du^{n,1})Dz^n m^{n,1}) = \mathrm{div}(R^{2,n}), \\ \mu^n(t_0) = 0,\ z^n(T,x) = \dfrac{\delta G}{\delta m}(x, m^{n,1}(T))(\mu^n(T)) + R^{3,n}, \end{cases}$$

with

$$R^{n,1} = (\theta^n)^{-1}\Big[H(x,Du^{n,2}) - H(x,Du^{n,1}) - H_p(x,Du^{n,1})\cdot(Du^{n,2}-Du^{n,1})$$
$$- \Big(F(x,m^{n,2}(t)) - F(x,m^{n,1}(t)) - \frac{\delta F}{\delta m}(x,m^{n,1}(t))(m^{n,2}(t)-m^{n,1}(t))\Big)\Big],$$
$$R^{n,2} = -(\theta^n)^{-1}\Big[H_p(x,Du^{n,2})m^{n,2} - H_p(x,Du^{n,1})m^{n,1} - H_p(x,Du^{n,1})(m^{n,2}-m^{n,1})$$
$$- H_{pp}(x,Du^{n,1})(Du^{n,2}-Du^{n,1})m^{n,1}\Big],$$

and

$$R^{n,3} = -(\theta^n)^{-1}\Big[G(x,m^{n,2}(T)) - G(x,m^{n,1}(T))$$
$$-\frac{\delta G}{\delta m}(x,m^{n,1}(T))(m^{n,2}(T)-m^{n,1}(T))\Big].$$

It follows from the regularity of F, G and H and the definition of θ^n that

$$\|R^{n,1}\|_{C^{\delta/2,\delta}} + \|R^{n,3}\|_{C^{2+\delta}} + \sup_{t\in[t_0,T]} \|R^{n,2}(t)\|_{(W^{1,\infty})'} \le C\theta^n. \qquad (2.21)$$

However, Lemma 2.1 yields that the sequence $(z^n)_{n\in\mathbb{N}}$ tends to 0 in $C^{(1+\delta)/2,1+\delta}$, a contradiction with (2.20). □

The smoothness of $\mathcal{U}$ in $\mathcal{O}$

We prove here Theorem 1.1.

Before we present the arguments, we state below as lemma a preliminary fact that is needed to establish the regularity of $\mathcal{U}$. It is about a stability property in the appropriate norms for the multipliers associated with minimizers starting in $\mathcal{O}$. In turn, this will allow us to compute the derivative of $\mathcal{U}$ with respect to m. Its proof is presented at the end of this subsection.

Lemma 2.9. *Assume* (1.5) *and fix* $(t_0,m_0)\in\mathcal{O}$. *There exists* $\delta, C>0$ *such that, for any* t_0', m_0^1, m_0^2 *satisfying* $(t_0', m_0^i)\in\mathcal{O}$, $|t_0'-t_0|<\delta$, $\mathbf{d}_2(m_0,m_0^i)<\delta$, *if* (m^i,α^i) *is the unique minimizer starting from* (t_0', m_0^i) *with associated multiplier* u^i *for* $i=1$ *and* $i=2$, *then*

$$\|u^1-u^2\|_{C^{(2+\delta)/2,2+\delta}} + \sup_{t\in[t_0',T]} \mathbf{d}_2(m^1(t),m^2(t)) \le C\mathbf{d}_2(m_0^1,m_0^2).$$

We continue with the proof of Theorem 1.1 which consists of three parts. In the first, we establish the regularity of $\mathcal{U}$ in $\mathcal{O}$. In the second, we show that the infinite dimensional Hamilton–Jacobi equation (1.9) is satisfied in $\mathcal{O}$. Finally, the third part is about (1.10).

Proof of Theorem 1.1. The proof consists of three parts, namely the regularity of $\mathcal{U}$, the fact that the Hamilton–Jacobi equation is satisfied, and the regularity of $D_m\mathcal{U}$.

Part 1: The regularity of $\mathcal{U}$ Lemma 1.7 yields the Lipschitz continuity of $\mathcal{U}$.

We establish that $\mathcal{U}$ is differentiable at any $(t_0,m_0)\in\mathcal{O}$. We fix such a (t_0,m_0). Let (m,α) be the unique stable minimizer for $\mathcal{U}(t_0,m_0)$ and u its associated multiplier. We check that $D_m\mathcal{U}(t_0,m_0,\cdot)$ exists and is given by $Du(t_0,\cdot)$.

Let $\delta > 0$ and $\delta' \in (0, \delta)$ be such that the δ-neighborhood of the $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$-compact set $\{(t, m(t)) : t \in [t_0, T]\}$ is contained in $\mathcal{O}$, and, for any $m_0^1 \in B(m_0, \delta')$, $\sup_{t\in[t_0,T]} \mathbf{d_1}(m(t), m^1(t)) < \delta$, where (m^1, α^1) is the minimizer for $\mathcal{U}(t_0, m_0^1)$.

Fix $m_0^1 \in B(m_0, \delta')$. Let (m^1, α^1) be the minimizer for $\mathcal{U}(t_0, m_0^1)$, u^1 its associated multiplier, (z, μ) the solution of the linearized system (2.15) with initial condition $\mu(0) = m_0^1 - m_0$ (which exists thanks to Proposition 2.4), set

$$(w, \rho) = (u^1 - u - z, m^1 - m - \mu)$$

and note that (w, ρ) satisfies the linearized system (2.2) with $\xi = 0$,

$$\begin{aligned} R^1(t, x) =& - (H(x, Du^1) - H(x, Du) - H_p(x, Du) \cdot (Du^1 - Du)) \\ & + F(x, m^1) - F(x, m) - \frac{\delta F}{\delta m}(x, m(t))(m^1(t) - m(t)), \\ R^2(t, x) =& H_p(x, Du^1)m^1 - H_p(x, Du)m - H_p(x, Du)(m^1 - m) \\ & - H_{pp}(x, Du) \cdot (Du^1 - Du)m, \end{aligned}$$

and

$$R^3(x) = G(x, m^1(T)) - G(x, m(T)) - \frac{\delta G}{\delta m}(x, m(T))(m^1(T) - m(T)).$$

Then, using Lemma 2.9, we get

$$\|R^1\|_{C^{\delta/2,\delta}} + \|R^3\|_{C^{2+\delta}} + \sup_{t\in[t_0,T]} \|R^2(t)\|_{(W^{1,\infty})'} \leq C\mathbf{d}_2^2(m_0^1, m_0)$$

and, in view of Lemma 2.1,

$$\|u^1 - u - w\|_{C^{(2+\delta)/2,2+\delta}} + \sup_{t\in[t_0,T]} \|m^1(t) - m(t) - \mu(t)\|_{(C^{2+\delta})'} \leq C\mathbf{d}_2^2(m_0^1, m_0).$$

Recall that $\alpha^1 = -H_p(x, Du^1)$. Thus

$$\alpha^1 = \alpha - H_{pp}(x, Du).Dw + o(\mathbf{d}_1(m_0^1, m_0)).$$

where $o(\cdot)$ is small in uniform norm.

It follows that

$$\begin{aligned} \mathcal{U}(t_0, m_0^1) =& \int_{t_0}^T \Big(\int_{\mathbb{R}^d} L(x, \alpha^1)m^1 + \mathcal{F}(m^1)\Big)dt + \mathcal{G}(m^1(T)) \\ =& \mathcal{U}(t_0, m_0) + \int_{t_0}^T \int_{\mathbb{R}^d} \Big(D_\alpha L(x, \alpha) \cdot (-H_{pp}(x, Du)Dwm + L(x, \alpha)\mu(t, x) \\ & + F(x, m(t))\mu(t, x)\Big)dxdt \\ & + \int_{\mathbb{R}^d} G(x, m(T))\mu(T, x)dx + o(\mathbf{d}_2(m_0^1, m_0)). \end{aligned}$$

On the other hand, recalling the equations satisfied by u and μ and using duality we find

$$\begin{aligned}&\int_{\mathbb{R}^d} G(x,m(T))\mu(T,x)dx - \int_{\mathbb{R}^d} u(t_0,x)(m_0^1-m_0)(dx)\\ &\quad = \int_{t_0}^T\int_{\mathbb{R}^d}\Big((H(x,Du)-F(x,m(t))-H_p(x,Du)\cdot Du)\mu\\ &\qquad - H_{pp}(x,Du)Du\cdot Dwm\Big).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{U}(t_0,m_0^1) &= \mathcal{U}(t_0,m_0) + \int_{\mathbb{R}^d} u(t_0,x)(m_0^1-m_0)(dx)\\ &\quad + \int_{t_0}^T\int_{\mathbb{R}^d}\Big((H(x,Du)-H_p(x,Du)\cdot Du + L(x,\alpha))\mu\\ &\quad - (H_{pp}(x,Du)Du\cdot Dw + D_\alpha L(x,\alpha)\cdot(H_{pp}(x,Du)Dw))m\Big)dxdt\\ &\quad + o(\mathbf{d}_2(m_0^1,m_0))\end{aligned}$$

In view of the relationship (convex duality) between H and L and the fact that $\alpha = -H_p(x,Du)$, we have $D_\alpha L(x,\alpha) = -Du$ and, therefore,

$$H(x,Du) - H_p(x,Du)\cdot Du + L(x,\alpha) = 0$$

and

$$H_{pp}(x,Du)Du\cdot Dw + D_\alpha L(x,\alpha)\cdot(H_{pp}(x,Du)Dw) = 0.$$

Thus,

$$\mathcal{U}(t_0,m_0^1) = \mathcal{U}(t_0,m_0) + \int_{\mathbb{R}^d} u(t_0,x)(m_0^1-m_0)(dx) + o(\mathbf{d}_2(m_0^1,m_0)).$$

Applying the above equality to $m_0^1 = (1-s)m_0 + s\delta_y$ (for some $y\in\mathbb{R}^d$ and $s\in(0,1)$), we infer that the limit

$$\lim_{s\to 0^+}\frac{1}{s}(\mathcal{U}(t_0,(1-s)m_0+s\delta_y) - \mathcal{U}(t_0,m_0)) = u(t_0,y)$$

exists.

Recalling the stability of the map $m_0 \to (u,m)$ proved in Lemma 1.5, this limit depends in a continuous way of (m_0,y). It follows from Lemma B.1 in [5] that $\mathcal{U}(t_0,\cdot)$ has a linear derivative in a neighborhood of m_0 given by $u(t_0,\cdot)$. Hence

$$D_m U(t_0,m_0,x) = Du(t_0,x).$$

Using again the stability of the map $m_0\to(u,m)$, we actually have that $(t_0,m_0)\to D_mU(t_0,m_0,\cdot)$ is continuous in $\mathcal{O}$ with respect to the $\mathbf{d}_2$-distance for the measure variable into C^2.

Part 2: The Hamilton–Jacobi equation Next we show that $\mathcal{U}$ is a classical solution to (1.9).

Using the notation of Part 1 and the dynamic programming principle with $h > 0$ small, we find

$$\mathcal{U}(t_0, m_0) = \int_{t_0}^{t_0+h} \Big(\int_{\mathbb{R}^d} L(x, \alpha(t,x)) m(t,x) dx + \mathcal{F}(m(t)) \Big) dt + \mathcal{U}(t_0 + h, m(t_0 + h)),$$

and, in view of the C^1-regularity of $\mathcal{U}$,

$$\mathcal{U}(t_0 + h, m(t_0 + h)) - \mathcal{U}(t_0 + h, m_0) = \int_{t_0}^{t_0+h} \int_{\mathbb{R}^d} \big(\mathrm{Tr} D^2_{ym} \mathcal{U}(t_0 + h, m(t), y)$$
$$+ D_m \mathcal{U}(t_0 + h, m(t), y) \cdot H_p(t, Du(t, y)) \big) \, m(t, dy) dy dt.$$

It follows that $\partial_t \mathcal{U}(t_0, m_0)$ exists and is given by

$$\partial_t \mathcal{U}(t_0, m_0) = - \int_{\mathbb{R}^d} L(x, \alpha(t_0, x)) m_0(dx) - \mathcal{F}(m_0)$$
$$- \int_{\mathbb{R}^d} \Big(\mathrm{Tr} D^2_{ym} \mathcal{U}(t_0, m_0, y) + D_m \mathcal{U}(t_0, m_0, y) \cdot H_p(t_0, Du(t_0, y)) \Big) m_0(dy).$$

Since $Du(t_0, x) = D_m \mathcal{U}(t_0, m_0, x)$ and $\alpha(t_0, x) = -H_p(x, Du(t_0, x))$, (1.9) is then satisfied.

Part 3: The regularity of $D_m \mathcal{U}$ We prove that (1.10) holds.

Let $\delta, C > 0$ be given by Lemma 2.9. For any $t \in (t_0 - \delta, t_0 + \delta)$ and $m_0^2, m_0^2 \in \mathcal{P}_2(\mathbb{R}^d)$ with $\mathbf{d}_2(m_0, m_0^1) < \delta$, $\mathbf{d}_2(m_0, m_0^2) < \delta$, $(t, m_0^1) \in \mathcal{O}$ and $(t, m_0^2) \in \mathcal{O}$, and for any $x^1, x^2 \in \mathbb{R}^d$, let u^1 (respectively u^2) be the multiplier associated with the unique minimizer (m^1, α^1) for $\mathcal{U}(t, m_0^1)$ (respectively with the unique minimizer (m^2, α^2) for $\mathcal{U}(t, m_0^2)$).

Since, as already established, $D_m \mathcal{U}(t, m_0^1, x^1) = Du^1(t, x^1)$ and $D_m \mathcal{U}$ $(t, m_0^2, x^2) = Du^2(t, x^2)$, we find, using Lemmas 1.3 and 2.9, that

$$\big| D_m \mathcal{U}(t, m_0^1, x^1) - D_m \mathcal{U}(t, m_0^2, x^2) \big| \leq \big| Du^1(t, x^2) - Du^2(t, x^2) \big|$$
$$+ \| D^2 u^1 \|_\infty |x^1 - x^2|$$
$$\leq C(\mathbf{d}_2(m_0^1, m_0^2) + |x^1 - x^2|).$$

□

We conclude with the remaining proof.

Proof of Lemma 2.9. Let (m, α) be the unique stable minimizer starting from (t_0, m_0) with multiplier u. It follows from Lemma 2.6 that the associated linear system (2.15) is strongly stable.

We set $V = -H_p(x, Du)$ and $\Gamma = -H_{pp}(x, Du)$, consider the neighborhood $\mathcal{V}$ (in the local uniform convergence) of (V, Γ) given in Lemma 2.1, and choose $\delta > 0$ so that, for any m_0^1 such that $\mathbf{d}_2(m_0, m_0^1) < \delta$, we have that

$$(V^1, \Gamma^1) \in \mathcal{V},$$

with $V^1 = -H_p(x, Du^1)$ and $\Gamma_1 = H_{pp}(x, Du^1)$, u^1 being the multiplier associated with the optimal solution (m^1, α^1) for $\mathcal{U}(t_0, m_0^1)$. The above is possible since, if δ is small, then, in view of Lemma 1.5, u^1 is close to u in $C^{\delta/2, \delta}$.

Furthermore, choosing, if necessary, δ even smaller, we have that, for some $\eta > 0$ to be chosen below, and, for any t_0', m_0^i, (m^i, α^i) and u^i as above such that $|t_0' - t_0| < \delta$ and $\mathbf{d}_2(m_0, m_0^i) < \delta$ for $i = 1$ and $i = 2$,

$$\|u^1 - u^2\|_{C^{(2+\delta)/2, 2+\delta}} + \sup_{t \in [t_0', T]} \mathbf{d}_2(m^1(t), m^2(t)) \le \eta. \tag{2.22}$$

Classical estimates on the Kolmogorov equation (see, for instance, [6]) yield that, for t_0', m_0^i, (m^i, α^i) and u^i as above, we have

$$\sup_{t \in [t_0', T]} \mathbf{d}_2(m^1(t), m^2(t)) \le C(\mathbf{d}_2(m_0^1, m_0^2) + \|Du^1 - Du^2\|_\infty), \tag{2.23}$$

for a constant C depending on T, H and $\|D^2 u^i\|_\infty$, which is uniformly bounded by Lemma 2.3.

Then the pair

$$(z, \mu) = (u^1 - u^2, m^1 - m^2)$$

satisfies the linearized system (2.2) with

$$\begin{aligned}
&V(t,x) = -H_p(x, Du^2), \quad \Gamma = -H_{pp}(x, Du^2), \quad \xi = m_0^1 - m_0^2,\\
&R^1 = -(H(x, Du^1) - H(x, Du^2) - H_p(x, Du^2) \cdot (Du^1 - Du^2))\\
&\qquad + F(x, m^1(t)) - F(x, m^2(t))\\
&\qquad - \frac{\delta F}{\delta m}(x, m^2(t))(m^1(t) - m^2(t)), R^2 = H_p(x, Du^1) m^1 - H_p(x, Du^2) m^2\\
&\qquad - H_p(x, Du^2)(m^1 - m^2) - H_{pp}(x, Du^2) \cdot (Du^1 - Du^2) m^2\\
&\quad = (H_p(x, Du^1) - H_p(x, Du^2))(m^1 - m^2)\\
&\qquad + (H_p(x, Du^1) - H_p(x, Du^2) - H_{pp}(x, Du^2) \cdot (Du^1 - Du^2)) m^2,
\end{aligned}$$

and

$$R^3 = G(x, m^1) - G(x, m^2) - \frac{\delta G}{\delta m}(x, m^2(t))(m^1(t) - m^2(t)).$$

Note that

$$\begin{aligned}
R^2 =& (H_p(x, Du^1) - H_p(x, Du^2))(m^1 - m^2)\\
&\quad + (H_p(x, Du^1) - H_p(x, Du^2) - H_{pp}(x, Du^2) \cdot (Du^1 - Du^2)) m^2.
\end{aligned}$$

Thus

$$\begin{aligned}
M &= \|\xi\|_{(W^{1,\infty})'} + \|R^1\|_{C^{\delta/2,\delta}} + \|R^3\|_{C^{2+\delta}} + \sup_{t \in [t_0', T]} \|R^2(t)\|_{(W^{1,\infty})'}\\
&\le \mathbf{d}_1(m_0^1, m_0^2) + C\{\|Du^1 - Du^2\|^2_{C^{\delta/2,\delta}} + \sup_t \mathbf{d}_2^2(m^1(t), m^2(t))\}.
\end{aligned}$$

It follows from Lemma 2.1 that

$$\begin{aligned}
&\|u^1 - u^2\|_{C^{(2+\delta)/2, 2+\delta}} \le C\{\mathbf{d}_1(m_0^1, m_0^2)\\
&+ \|Du^1 - Du^2\|^2_{C^{\delta/2,\delta}} + \sup_t \mathbf{d}_2^2(m^1(t), m^2(t))\}.
\end{aligned}$$

Hence, choosing $\eta > 0$ small enough in (2.22), we find

$$\|u^1 - u^2\|_{C^{(2+\delta)/2, 2+\delta}} \le C\{\mathbf{d}_1(m_0^1, m_0^2) + \sup_t \mathbf{d}_2^2(m^1(t), m^2(t))\},$$

and inserting the last inequality in (2.23) we obtain

$$\sup_t \mathbf{d}_2(m^1(t), m^2(t)) \le C\{\mathbf{d}_2(m_0^1, m_0^2) + \sup_t \mathbf{d}_2^2(m^1(t), m^2(t))\},$$

which yields, for $\eta > 0$ small enough,

$$\sup_t \mathbf{d}_2(m^1(t), m^2(t)) \le C\mathbf{d}_2(m_0^1, m_0^2).$$

Going back to the previous inequality on $\|u^2 - u^2\|_{C^{(2+\delta)/2,2+\delta}}$ completes the proof. □

3. The propagation of chaos

We present the proof of Theorem 1.2, which consists of several steps each of which is stated below as separate lemma. Throughout this section, $\mathbf{Z} = (Z^k)_{k=1,\dots,N}$ is a sequence of independent random variables with law m_0, $\mathbf{B}_\cdot = (B_\cdot^k)_{k=1,\dots,N}$ is a sequence of independent Brownian motions independent of $\mathbf{Z}$, and $\mathbf{Y}_\cdot^N = (Y^{N,k})_{k=1,\dots,N}$ is the optimal trajectory for $\mathcal{V}^N(t_0, (Z^k)_{k=1,\dots,N})$, that is, for each $k = 1, \dots, N$ and $t \in [t_0, T]$, a solution to (1.11).

In preparation, we fix $(t_0, m_0) \in \mathcal{O}$ with $M_{d+5}(m_0) < +\infty$ and let (m, α) be the unique minimizer for $\mathcal{U}(t_0, m_0)$.

It follows from Theorem 1.1 and the compactness of the curve $\{(t, m(t)) : t \in [t_0, T]\}$ that there exists $\delta, C > 0$ such that, for any $t_1 \in [t_0, T]$, $t \in (t_1 - \delta, t_1 + \delta)$ and $m_0^1, m_0^2 \in \mathcal{P}_2(\mathbb{R}^d)$ with $\mathbf{d}_2(m(t_1), m_0^1) < \delta$, $\mathbf{d}_2(m(t_1), m_0^2) < \delta$, $(t_1, m_0^1) \in \mathcal{O}$ and $(t_1, m_0^2) \in \mathcal{O}$, and $x^1, x^2 \in \mathbb{R}^d$,

$$|D_m\mathcal{U}(t_1, m_0^1, x^1) - D_m\mathcal{U}(t_1, m_0^2, x^2)| \le C(|x^1 - x^2| + \mathbf{d}_2(m_0^1, m_0^2)). \quad (3.1)$$

For $\sigma \in (0, \delta)$, set

$$V_\sigma = \{(t, m') \in [t_0, T] \times \mathcal{P}_2(\mathbb{R}^d) : \mathbf{d}_2(m', m(t)) < \sigma\}$$

and

$$V_\sigma^N = \{(t, \mathbf{x}) \in [0, T] \times (\mathbb{R}^d)^N : (t, m_{\mathbf{x}}^N) \in V_\sigma\}.$$

We consider the solution $(\mathbf{X}_t^N)_{t \in [t_0, T]} = (X_t^{N,1}, \dots, X_t^{N,N})_{t \in [t_0, T]}$ to

$$dX_t^{N,j} = Z^j - \int_{t_0}^t H_p(X_s^{N,j}, D_m\mathcal{U}(s, m_{\mathbf{X_s^N}}^N, X_s^{N,j}))ds + \sqrt{2}(B_s^j - B_{t_0}^j), \quad (3.2)$$

on the time interval $[t_0, \tau^N]$, where the stopping time τ^N is defined by

$$\tau^N = \inf\left\{t \in [t_0, T],\ (t, \mathbf{X}_t^N) \notin V_{\delta/2}^N\right\} \text{ or } \tau^N = T, \text{ if there is no such a } t.$$

Note that, in view of (3.1), $\mathbf{X}^N$ is uniquely defined.

Set

$$\widetilde{\tau}^N = \inf\{t \in [t_0, \tau^N],\ (t, \mathbf{Y}_t^N) \notin V_\delta^N\} \text{ or } \widetilde{\tau}^N = \tau^N \text{ is there is no such a} t. \quad (3.3)$$

Let us stress that $\widetilde{\tau}^N$ can be equal to t_0 if $(Z^j)_{j=1,\dots,N} \notin V_\delta^N$.

Lemma 3.1. *Assume* (1.5). *There is a constant* $C > 0$ *depending on* (t_0, m_0) *such that*

$$\mathbb{E}\left[\sup_{t\in[t_0,\tau^N]} \mathbf{d}_1(m^N_{\mathbf{X}^N_t}, m(t))\right] \le CN^{-1/(d+8)} \tag{3.4}$$

and

$$\mathbb{P}\left[\tau^N < T\right] \le CN^{-1/(d+8)}.$$

Proof. The proof is standard and relies on propagation of chaos estimates; see, for instance, the proof of Theorem 5.6 in [15].

Let $\widetilde{X}^{N,i}$ be the i.i.d. solutions to

$$d\widetilde{X}^{N,i}_t = -H_p(\widetilde{X}^{N,j}_t, D_m\mathcal{U}(t, m(t), \widetilde{X}^{N,i}_t))dt + \sqrt{2}dB^j_t \quad \widetilde{X}^{N,i}_{t_0} = Z^i,$$

which exist on $[t_0, T]$, in view of the global Lipschitz property of $D_m\mathcal{U}(t, m(t), \cdot)$.

Then (3.4) is an easy consequence of the inequality

$$\mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{d}^2_2(m^N_{\widetilde{\mathbf{X}}^N_t}, m(t))\right] \le CN^{-2/(d+8)},$$

which follows from Theorem 3.1 in Horowitz and Karandikar [22], for some C depending only on (t_0, m_0) through the regularity of $D_m\mathcal{U}$ in V_δ.

Then

$$\mathbb{P}\left[\tau^N < T\right] \le \mathbb{P}\left[\sup_{t\in[t_0,\tau^N]} \mathbf{d}_1(m^N_{\mathbf{X}^N_t}, m(t)) \ge \delta/2\right] \le C\delta^{-1}N^{-1/(d+8)}.$$

□

Next, for $(t, \mathbf{x}) \in V^N_\delta$, we set

$$\mathcal{U}^N(t, \mathbf{x}) = \mathcal{U}(t, m^N_{\mathbf{x}}).$$

Recalling the regularity of $\mathcal{U}$ given by Theorem 1.1, it is immediate that, on V^N_δ, $\mathcal{U}^N$ is C^1 in time-space with $D_{x^j}\mathcal{U}^N$ Lipschitz continuous in space and (see, for example, [6])

$$D_{x^j}\mathcal{U}^N(t, \mathbf{x}) = \frac{1}{N}D_m\mathcal{U}(t, m^N_{\mathbf{x}}, x^j) \text{ and } |D^2_{x^jx^j}\mathcal{U}^N(t, \mathbf{x}) - \frac{1}{N}D^2_{ym}\mathcal{U}(t, m^N_{\mathbf{x}}, x^j)| \le \frac{C}{N^2} \text{ a.e. in } V^N_\delta.$$

Finally, $\mathcal{U}^N$ satisfies

$$\begin{cases} -\partial_t\mathcal{U}^N(t, \mathbf{x}) - \displaystyle\sum_{j=1}^N \Delta_{x^j}\mathcal{U}^N(t, \mathbf{x}) + O^N(t, \mathbf{x}) \\ \qquad + \frac{1}{N}\sum_{j=1}^N H(x^j, ND_{x^j}\mathcal{U}^N(t, \mathbf{x})) = \mathcal{F}(m^N_{\mathbf{x}}) \text{ a.e. in } V^N_\delta, \\ \mathcal{U}^N(T, \mathbf{x}) = \mathcal{G}(m^N_{\mathbf{x}}) \text{ on } (\mathbb{R}^d)^N, \end{cases} \tag{3.5}$$

where

$$|O^N(t, \mathbf{x})| \le CN^{-1} \qquad \text{a.e. in } V^N_\delta. \tag{3.6}$$

Lemma 3.2. *Let* $\mathbf{Y}^N = (Y^{N,i})_{i=1,\dots,N}$ *and* $\widetilde{\tau}^N$ *be defined by* (1.11) *and* (3.3) *respectively. Then,*

$$\mathbb{E}\left[\int_{t_0}^{\widetilde{\tau}^N} N^{-1}\sum_j |H_p(Y_t^{N,j}, ND_{x^j}\mathcal{U}^N) - H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)|^2 dt\right] \leq C(N^{-1} + R^N),$$

where $R^N = \|\mathcal{U}^N - \mathcal{V}^N\|_\infty$ *and* $\mathcal{U}^N$, $\mathcal{V}^N$ *and their derivatives are evaluated at* $(t, \mathbf{Y}_t^N)$.

Proof. For $t \in [t_0, \widetilde{\tau}^N]$ and, in view of (3.6), we have

$$\begin{aligned}
d\mathcal{U}^N(t, \mathbf{Y}_t^N) &= (\partial_t \mathcal{U}^N + \sum_j \Delta_{x^j}\mathcal{U}^N - \sum_j H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)\cdot D_{x^j}\mathcal{U}^N)dt \\
&\quad + \sqrt{2}\sum_j D_{x^j}\mathcal{U}^N \cdot dB_t^j \\
&= (\frac{1}{N}\sum_j H(Y^{N,j}, ND_{x^j}\mathcal{U}^N(t, \mathbf{Y}_t^N)) - \sum_j H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)\cdot D_{x^j}\mathcal{U}^N \\
&\quad + O^N - \mathcal{F}(m^N_{\mathbf{Y}_t^N}))dt \\
&\quad + \sqrt{2}\sum_j D_{x^j}\mathcal{U}^N \cdot dB_t^j \\
&\geq (\frac{1}{N}\sum_j(-L(Y^{N,j}, -H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)) + C^{-1}|H_p(Y_t^{N,j}, ND_{x^j}\mathcal{U}^N) \\
&\quad - H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)|^2) \\
&\quad - CN^{-1} - \mathcal{F}(m^N_{\mathbf{Y}_t^N}))dt + \sqrt{2}\sum_j D_{x^j}\mathcal{U}^N \cdot dB_t^j,
\end{aligned}$$

the inequality following from the uniform convexity of H in bounded sets.

We take expectations and integrate between t_0 and $\widetilde{\tau}^N$ above to get

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{U}^N(\widetilde{\tau}^N, \mathbf{Y}^N_{\widetilde{\tau}^N})\right] - \mathbb{E}\left[\mathcal{U}^N(t_0, \mathbf{Z}^N)\right] \\
&\quad \geq \mathbb{E}\left[\int_{t_0}^{\widetilde{\tau}^N} (\frac{1}{N}\sum_j\Big(-L(Y^{N,j}, -H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)) + C^{-1}|H_p(Y_t^{N,j}, ND_{x^j}\mathcal{U}^N)\right. \\
&\qquad \left. - H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)|^2\Big) - CN^{-1} - \mathcal{F}(m^N_{\mathbf{Y}_t^N}))dt\right].
\end{aligned}$$

Rearranging, using the definition of R^N, the dynamic programming principle and the optimality of $\mathbf{Y}^N$ for $\mathcal{V}^N(t_0, \mathbf{Z}^N)$ we find

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{U}^N(t_0, \mathbf{Z}^N)\right] + \mathbb{E}\left[\int_{t_0}^{\widetilde{\tau}^N} \frac{1}{CN} \sum_j |H_p(Y_t^{N,j}, ND_{x^j}\mathcal{U}^N) - H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)|^2 dt\right] \\
&\leq \mathbb{E}\left[\int_{t_0}^{\widetilde{\tau}^N} \left(\frac{1}{N}\sum_j L(Y^{N,j}, -H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)\right)\right. \\
&\quad \left. + CN^{-1} + \mathcal{F}(m^N_{\mathbf{Y}_t^N}))dt + \mathcal{V}^N(\widetilde{\tau}^N, \mathbf{Y}^N_{\widetilde{\tau}^N})\right] + R^N \\
&\leq \mathbb{E}\left[\mathcal{V}^N(t_0, \mathbf{Z}^N)\right] + CN^{-1} + R^N,
\end{aligned}$$

and, using once more the definition of R^N, we get

$$\mathbb{E}\left[\int_{t_0}^{\widetilde{\tau}^N} \frac{1}{CN} \sum_j |H_p(Y_t^{N,j}, ND_{x^j}\mathcal{U}^N) - H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)|^2 dt\right] \leq CN^{-1} + 2R^N.$$

□

Lemma 3.3. (Convergence of optimal trajectories) *For* $\mathbf{X}^N = (X^{N,i})$ *and* $\mathbf{Y}^N = (Y^{N,i})$ *defined by* (3.2) *and* (1.11) *respectively, we have*

$$\mathbb{E}\left[\sup_{s\in[t_0,\widetilde{\tau}^N]} N^{-1}\sum_j |X_s^{N,j} - Y_s^{N,j}|\right] \leq C(N^{-1} + R^N)^{1/2}$$

and

$$\mathbb{P}\left[\widetilde{\tau}^N < T\right] \leq C(N^{-1/(d+8)} + (R^N)^{1/2}).$$

Proof. Lemma 3.2, the regularity of $\mathcal{U}^N$ in (3.1) and an application of Gronwall's inequality give the first inequality since, for any $t \in [t_0, T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t_0,t\wedge\widetilde{\tau}^N]} N^{-1}\sum_j |X_s^{N,j} - Y_s^{N,j}|\right] \\
&\leq \mathbb{E}\left[\int_{t_0}^{t\wedge\widetilde{\tau}^N} N^{-1}\sum_j |H_p(Y_t^{N,j}, ND_{x^j}\mathcal{U}^N(t, \mathbf{Y}_t^N))\right. \\
&\quad \left. - H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N(t, \mathbf{Y}_t^N))|dt\right] \\
&\quad + \mathbb{E}\left[\int_{t_0}^{t\wedge\widetilde{\tau}^N} N^{-1}\sum_j |H_p(Y_t^{N,j}, ND_{x^j}\mathcal{U}^N(t, \mathbf{Y}_t^N))\right. \\
&\quad \left. - H_p(X_t^{N,j}, ND_{x^j}\mathcal{U}^N(t, \mathbf{X}_t^N))|dt\right] \\
&\leq C(N^{-1} + R^N)^{1/2} + CN^{-1}\sum_j \mathbb{E}\left[\int_{t_0}^{t\wedge\widetilde{\tau}^N} |X_s^{N,j} - Y_s^{N,j}|ds\right].
\end{aligned}$$

Then,

$$\mathbb{P}\left[\widetilde{\tau}^N < T\right] \le \mathbb{P}\left[\tau^N < T\right] + \mathbb{P}\left[\sup_{s\in[t_0,\widetilde{\tau}^N]} N^{-1}\sum_j |X_s^{N,j} - Y_s^{N,j}| > \delta/2\right]$$
$$\le CN^{-1/(d+8)} + C\delta^{-1}(N^{-1} + R^N)^{1/2}.$$

□

We can proceed now with the proof of the propagation of chaos property.

Proof of Theorem 1.2. It is immediate that

$$\mathbb{E}\left[\sup_{s\in[t_0,\widetilde{\tau}^N]} \mathbf{d}_1(m^N_{\mathbf{Y}^N_s}, m(s))\right] \le \mathbb{E}\left[\sup_{s\in[t_0,\widetilde{\tau}^N]} \mathbf{d}_1(m^N_{\mathbf{Y}^N_s}, m^N_{\mathbf{X}^N_s})\right]$$
$$+ \mathbb{E}\left[\sup_{s\in[t_0,\widetilde{\tau}^N]} \mathbf{d}_1(m^N_{\mathbf{X}^N_s}, m(s))\right] \tag{3.7}$$
$$\le \mathbb{E}\left[\sup_{s\in[t_0,\widetilde{\tau}^N]} N^{-1}\sum_j |X_s^{N,j} - Y_s^{N,j}|\right] + \mathbb{E}\left[\sup_{s\in[t_0,\widetilde{\tau}^N]} \mathbf{d}_1(m^N_{\mathbf{X}^N_s}, m(s))\right].$$

Lemma 3.3 gives that the first term in the right-hand side of (3.7) is not larger than $C(N^{-1} + R^N)^{1/2}$, where R^N can be estimated by Proposition 1.8, while Lemma 3.1 implies that the second term in the right-hand side of (3.7) is not larger than $CN^{-1/(d+8)}$.

Thus, for some γ' depending on d only and C depending on the initial condition (t_0, m_0),

$$\mathbb{E}\left[\sup_{s\in[t_0,\widetilde{\tau}^N]} \mathbf{d}_1(m^N_{\mathbf{Y}^N_s}, m(s))\right] \le CN^{-\gamma'}.$$

Finally, we have

$$\mathbb{E}\left[\sup_{s\in[t_0,T]} \mathbf{d}_1(m^N_{\mathbf{Y}^N_s}, m(s))\right]$$
$$\le \mathbb{E}\left[\sup_{s\in[t_0,\widetilde{\tau}^N]} \mathbf{d}_1(m^N_{\mathbf{Y}^N_s}, m(s))\right] + \mathbb{E}\left[\sup_{s\in[t_0,T]} \mathbf{d}_1^2(m^N_{\mathbf{Y}^N_s}, m(s))\right]^{1/2} \mathbb{P}\left[\widetilde{\tau}^N < T\right]^{1/2}$$
$$\le CN^{-\gamma'} + C(N^{-1/(d+8)} + (R^N)^{1/2})^{1/2},$$

the last inequality coming from Lemma 3.3 and the facts that, since, in view of Lemma 1.3, $m(s)$ has a uniformly bounded second-order moment and, by Lemma 1.7, the drift of the process $\mathbf{Y}^N$ is also uniformly bounded,

$$\mathbb{E}\left[\sup_{s\in[t_0,T]} \mathbf{d}_1^2(m^N_{\mathbf{Y}^N_s}, m(s))\right] \le C$$

Using once more Proposition 1.8, we get that, for a new constant $\gamma'' \in (0,1)$ depending on d only and a new constant C' depending on the initial condition (t_0, m_0),

$$\mathbb{E}\left[\sup_{s\in[t_0,T]} \mathbf{d}_1(m^N_{\mathbf{Y}^N_s}, m(s))\right] \leq C'N^{-\gamma''}.$$

□

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Appendix A: The proof of Lemma 1.6

Proof. A fact similar to Lemma 1.6 was given in [2] for the torus and for smooth initial data. Here, we extend the argument for the whole space and general initial conditions and slightly simplify it.

We begin with the existence of a solution to (1.14), the uniqueness being obvious in view of the regularity of α.

Fix $\beta \in C^\infty_c((t_0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ and note that the product $m\beta$ is smooth, because the only singularity of m is at time t_0. Thus, there exists a unique classical solution to (1.14).

In order to prove its regularity, fix $t_0 < t_1 < t_2$, $\xi \in C^{2+\delta}(\mathbb{R}^d)$, let w be the solution to

$$-\partial_t w - \Delta w - \alpha(t, x) \cdot Dw = 0 \ \text{ in } \ (t_0, T) \times \mathbb{R}^d \quad w(t_2) = \xi \ \text{ in } \ \mathbb{R}^d,$$

and note that, for a constant C depending only on the data of the problem, since the regularity of α depends only on the data of the problem,

$$\|w\|_\infty + \|Dw\|_\infty \leq C\|\xi\|_{W^{1,\infty}} \ \text{ and } \ \|w\|_{C^{\delta/2,\delta}} + \|Dw\|_{C^{\delta/2,\delta}} \leq C\|\xi\|_{C^{2+\delta}}. \tag{A.1}$$

Then,

$$\int_{\mathbb{R}^d} \rho(t_2,x)\xi(x)dx = \int_{\mathbb{R}^d} w(t_1,x)\rho(t_1,x)dx - \int_{t_1}^{t_2}\int_{\mathbb{R}^d} \beta(t,x)\cdot Dw(t,x)m(t,dx)dx,$$

and, choosing $t_1 = t_0$ and t_2 arbitrary in $[t_0,T]$, we get

$$\sup_{t\in[t_0,T]} \|\rho(t)\|_{(W^{1,\infty})'} \le \|\beta(t,\cdot)\|_{L^1_m([0,T]\times\mathbb{R}^d)}. \tag{A.2}$$

In addition, since, thanks to (A.1),

$$\|w(t_1,\cdot)-w(t_2,\cdot)\|_{W^{1,\infty}} \le C(t_2-t_1)^{\delta/2}\|\xi\|_{C^{2+\delta}}$$

using (A.2) we find

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\rho(t_2,x)-\rho(t_1,x))\xi(x)dx\\
&\quad = \int_{\mathbb{R}^d} (w(t_1,x)-w(t_2,x))\rho(t_1,x)dx\\
&\qquad - \int_{t_1}^{t_2}\int_{\mathbb{R}^d} \beta(t,x)\cdot Dw(t,x)m(t,dx)dx\\
&\quad \le \|w(t_1,\cdot)-w(t_2,\cdot)\|_{W^{1,\infty}}\|\rho(t_1,\cdot)\|_{(W^{1,\infty})'}\\
&\qquad + C(t_2-t_1)^{1/2}\|\beta\|_{L^2_m([0,T]\times\mathbb{R}^d)}\|Dw\|_\infty\\
&\quad \le C(t_2-t_1)^{\delta/2}\|\beta(t,\cdot)\|_{L^1_m([0,T]\times\mathbb{R}^d)}\ \|\xi\|_{C^{2+\delta}}\\
&\qquad + C(t_2-t_1)^{1/2}\|\beta\|_{L^2_m([0,T]\times\mathbb{R}^d)}\|\xi\|_{W^{1,\infty}}.
\end{aligned}$$

The last estimates proves the existence of a solution ρ for $\beta \in C^0([t_0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ or for $\beta\in L^\infty$ vanishing near $t=t_0$ by approximation.

Next, let

$$J(m',\alpha') = \int_{t_0}^T \left(\int_{\mathbb{R}^d} L(x,\alpha'(t,x))m'(t,dx) + \mathcal{F}(m'(t))\right)dt + \mathcal{G}(m'(T)).$$

The quantity $J(m',\alpha')$ is defined, for instance, for $m'\in C^0([t_0,T],\mathcal{P}_1(\mathbb{R}^d))$ and $\alpha'\in C^0([t_0,T]\times\mathbb{R}^d;\mathbb{R}^d)$. Let $\beta\in C^\infty_c((t_0,T]\times\mathbb{R}^d)$ and ρ be the classical solution to (1.14), and, for $h>0$ small, let $m_h\in C^0([t_0,T],\mathcal{P}_1(\mathbb{R}^d))$ be the solution to

$$\partial_t m_h - \Delta m_h + \mathrm{div}(m_h(\alpha+h\beta)) = 0 \qquad \text{in } (t_0,T)\times\mathbb{R}^d \ \text{ and } \ m_h(t_0)=m_0 \ \text{ in } \ \mathbb{R}^d.$$

Then $m_h = m + h\rho + h^2\xi_h$, where ξ_h solves in the sense of distribution

$$\partial_t\xi_h - \Delta\xi_h + \mathrm{div}(\xi_h(\alpha+h\beta)) + \mathrm{div}(\beta\rho) = 0 \ \text{ in } \ (t_0,T)\times\mathbb{R}^d \ \text{ and } \ \xi_h(t_0)=0 \ \text{ in } \ \mathbb{R}^d.$$

The regularity of α, β and ρ imply that $\|\xi_h\|_\infty \le C$, with C depending on β, and, as $h\to 0$, the (ξ_h)s converges weakly in L^∞-weak-$*$ to the solution ξ of the same equation with $h=0$.

Then

$$J(m_h, \alpha + h\beta) = \int_{t_0}^{T} \Big(\int_{\mathbb{R}^d} L(x, \alpha + h\beta) m_h(t, dx) + \mathcal{F}(m_h(t)) \Big) dt + \mathcal{G}(m_h(T))$$

$$= J(m, \alpha) + h\Big\{ \int_{t_0}^{T} \Big(\int_{\mathbb{R}^d} D_\alpha L(x, \alpha) \cdot \beta(t, x) m(t, dx) + \int_{\mathbb{R}^d} L(x, \alpha) \rho(t, x) dx$$
$$+ \frac{\delta \mathcal{F}}{\delta m}(m(t))(\rho(t)) \Big) dt + \frac{\delta \mathcal{G}}{\delta m}(m(T))(\rho(T)) \Big\}$$
$$+ \frac{h^2}{2} \Big\{ \int_{t_0}^{T} \Big(\int_{\mathbb{R}^d} D_{\alpha\alpha} L(x, \alpha) \beta(t, x) \cdot \beta(t, x) m(t, dx)$$
$$+ 2 \int_{\mathbb{R}^d} D_\alpha L(x, \alpha) \cdot \beta(t, x) \rho(t, x) dx$$
$$+ \int_{\mathbb{R}^d} 2L(x, \alpha) \xi_h(t, x) dx + 2 \frac{\delta \mathcal{F}}{\delta m}(m(t))(\xi_h(t)) + \frac{\delta^2 \mathcal{F}^2}{\delta m}(m(t))(\rho(t), \rho(t)) \Big) dt$$
$$+ 2 \frac{\delta \mathcal{G}}{\delta m}(m(T))(\xi_h(T)) + \frac{\delta^2 \mathcal{G}}{\delta m^2}(m(T))(\rho(T), \rho(T)) \Big\} + o(h^2).$$

The first-order necessary optimality condition implies that the factor of h above vanishes and, therefore, the limit as h vanishes of the term in h^2 is nonnegative.

Thus

$$\int_{t_0}^{T} \Big(\int_{\mathbb{R}^d} D_{\alpha\alpha} L(x, \alpha) \beta(t, x) \cdot \beta(t, x) m(t, dx) + 2 \int_{\mathbb{R}^d} D_\alpha L(x, \alpha) \cdot \beta(t, x) \rho(t, x) dx$$
$$+ \int_{\mathbb{R}^d} 2L(x, \alpha) \xi(t, x) dx + 2 \frac{\delta \mathcal{F}}{\delta m}(m(t))(\xi(t)) + \frac{\delta^2 \mathcal{F}^2}{\delta m}(m(t))(\rho(t), \rho(t)) \Big) dt$$
$$+ 2 \frac{\delta \mathcal{G}}{\delta m}(m(T))(\xi(T)) + \frac{\delta^2 \mathcal{G}}{\delta m^2}(m(T))(\rho(T), \rho(T)) \ \geq \ 0.$$

Using the equation satisfied by the multiplier u and the equation satisfied by ξ we find

$$\int_{t_0}^{T} \int_{\mathbb{R}^d} (L(x, \alpha) \xi(t, x) dx + \frac{\delta \mathcal{F}}{\delta m}(m(t))(\xi(t))) dt + \frac{\delta \mathcal{G}}{\delta m}(m(T))(\xi(T))$$
$$= \int_{t_0}^{T} \int_{\mathbb{R}^d} ((-H(x, Du) - \alpha \cdot Du) \xi(t, x) dx$$
$$+ \frac{\delta \mathcal{F}}{\delta m}(m(t))(\xi(t))) dt + \frac{\delta \mathcal{G}}{\delta m}(m(T))(\xi(T))$$
$$= - \int_{t_0}^{T} \int_{\mathbb{R}^d} Du(t, x) \cdot \beta(t, x) \rho(t, x) dx dt$$
$$= - \int_{t_0}^{T} \int_{\mathbb{R}^d} D_\alpha L(x, \alpha) \cdot \beta(t, x) \rho(t, x) dx dt.$$

Inserting the last equality in the previous inequality yields the second-order optimality condition when β is smooth. The general case is obtained by approximation using the estimates in the first part of the proof. □

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