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C*-algebras of a Cantor system with finitely many minimal subsets: structures, K-theories, and the index map

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Abstract. We study homeomorphisms of a Cantor set with k ($k < +\infty$) minimal invariant closed (but not open) subsets; we also study crossed product C*-algebras associated to these Cantor systems and certain of their orbit-cut sub-C*-algebras. In the case where $k \geq 2$, the crossed product C*-algebra is stably finite, has stable rank 2, and has real rank 0 if in addition (X, σ) is aperiodic. The image of the index map is connected to certain directed graphs arising from the Bratteli–Vershik–Kakutani model of the Cantor system. Using this, it is shown that the ideal of the Bratteli diagram (of the Bratteli–Vershik–Kakutani model) must have at least k vertices at each level, and the image of the index map must consist of infinitesimals.

Key words: operator algebras, topological dynamics, K-theory, Cantor systems, index map
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Contents

1. *Introduction*

The paper is devoted to the study of interactions between topological dynamical systems on a Cantor set and C*-algebras. There are several constructions of operator algebras arising from dynamical systems in the frameworks of measurable or topological dynamics, and the most important one is the concept of crossed product operator algebras which originated in the work of von Neumann. It is fascinating to see how dynamical properties reveal themselves in the corresponding operator algebras. On the other hand, ideas and methods used in operator algebras can lead to new approaches and information for the study of dynamical systems. We mention here that Bratteli diagrams (one of the central objects of our paper) are proved to be an effective tool in dynamics.

A topological space is called *Cantor* if it is compact, metrizable, totally disconnected, and has no isolated points. Any two Cantor sets are homeomorphic. A *Cantor system* (X, σ) consists of a Cantor set X and a homeomorphism $\sigma : X \to X$. A closed subset $Y \subseteq X$ is said to be invariant if $Y = \sigma(Y)$. It follows from Zorn's lemma that any system (X, σ) has minimal non-empty closed invariant subsets (called *minimal components*). If X is the only non-empty closed invariant subset, the system (X, σ) is said to be *minimal*. This means, in other words, that, for every $x \in X$, the σ-orbit of x is dense in X. In this paper, we will focus on *non-minimal homeomorphisms* of a Cantor set. Our primary interest is the case where a homeomorphism σ has finitely many minimal components $Y_1, Y_2, \dots, Y_k$. These subsets are necessarily disjoint. There are several natural classes of homeomorphisms which have this property, such as interval exchange transformations (treated as Cantor dynamical systems) and aperiodic substitutions; see [**4**, **5**, **9**].

If some of the minimal component, say Y_i, is open, then both Y_i and $X \setminus Y_i$ are again Cantor sets (they are compact, metrizable, totally disconnected, and have no isolated points). Since Y_i and $X \setminus Y_i$ are invariant under σ, the whole system (X, σ) can be decomposed into the Cantor systems (Y_i, σ) and $(X \setminus Y_i, \sigma)$, where (Y_i, σ) is minimal, and $(X \setminus Y_i, \sigma)$ has $k - 1$ minimal components.

Thus, in this paper, we will assume that (X, σ) is *indecomposable*, that is, none of the Y_i are open. Our aim is to study C*-algebras associated to (X, σ). For this, we use the existence of a *Bratteli–Vershik–Kakutani model* of (X, σ) constructed by a sequence of Kakutani–Rokhlin partitions. Our main interest is in the properties of the C*-algebras associated to a Cantor system (X, σ) with finitely many minimal sets. In this paper, we apply the methods developed in K-theory and Bratteli diagrams. Note that the case of minimal homeomorphisms has been extensively studied in earlier papers (see, in particular, [**10**, **15**, **18**, **26–29**] where various aspects of the theory of C*-algebras corresponding to minimal homeomorphisms are discussed).

It is well known that if (X, σ) is minimal, then the crossed product C*-algebra $C(X) \rtimes \mathbb{Z}$ is isomorphic to an inductive limit of circle algebras (A$\mathbb{T}$ algebra); see [**28**].

This is still true if there is only one minimal component. However, once there are at least two minimal components, the C*-algebra $C(X) \rtimes \mathbb{Z}$ is no longer an $A\mathbb{T}$ algebra. In fact, in this case, the C*-algebra $C(X) \rtimes \mathbb{Z}$ is a *stably finite C*-algebra with stable rank* 2; moreover, it has *real rank* 0 if (X, σ) is aperiodic. (See Corollary 7.11; see also [**26**].)

In the case of minimal dynamical systems, it is also well known that each minimal Cantor system (X, σ) has a Bratteli–Vershik–Kakutani model; see [**18**]. Moreover, the (unordered) Bratteli diagram appears in the Bratteli–Vershik–Kakutani model must be simple, and any (non-elementary) simple Bratteli diagram can be ordered to have a continuous Bratteli–Vershik map.

In the case of Cantor systems with k minimal components, it turns out that any such system still has a Bratteli–Vershik–Kakutani model (see §6); and very naturally, the (unordered) Bratteli diagram that appears in the Bratteli–Vershik–Kakutani model is k-simple (Definition 5.1)—or, roughly speaking, it has k sub-diagrams which are simple, and each of them corresponds to one of the minimal components.

However, unlike the simple case, to assign an order to a given (unordered) k-simple Bratteli diagram with $k \geq 2$ so that the Bratteli–Vershik map is continuous is in some sense restrictive (see Definition 5.3), and is even impossible for some (k-simple) Bratteli diagrams. Therefore, not all k-simple Bratteli diagrams can arise from a Cantor system. For more details on orders on Bratteli diagrams, see [**6**, **7**, **15**, **18**, **21**, **23**].

It turns out that these constrains on the Bratteli diagrams can be connected to the crossed product algebra $C(X) \rtimes \mathbb{Z}$ and the associated C*-algebra extension

$$0 \longrightarrow C_0\Big(X \Big\backslash \bigcup_i Y_i\Big) \rtimes_\sigma \mathbb{Z} \longrightarrow C(X) \rtimes_\sigma \mathbb{Z} \longrightarrow \bigoplus_i C(Y_i) \rtimes_\sigma \mathbb{Z} \longrightarrow 0.$$

The index map of the extension above,

$$K_1\Big(\bigoplus_i C(Y_i) \rtimes_\sigma \mathbb{Z}\Big) \to K_0\Big(C_0\Big(X \Big\backslash \bigcup_{i=1}^{k} Y_k\Big) \rtimes \mathbb{Z}\Big),$$

turns out to have an image isomorphic to $\mathbb{Z}^{k-1}$ (in particular, the index map is non-zero if $k \geq 2$).

To connect the index map to Bratteli diagrams, we introduce a sequence of finite directed graphs (transition graphs; see Definition 7.1) for the ordered Bratteli diagram which represents the given Cantor system (X, σ). Then the index map can be recovered from these transition graphs (Theorem 7.5). Using this connection as a bridge, we show that elements in the image of the index map must be *infinitesimals* (Corollary 7.9). The transition graphs are also used in §8 to characterize the (unordered) Bratteli diagrams which appear in the Bratteli–Vershik–Kakutani construction (Theorems 8.1 and 8.2). Relative results on graphs associated to Bratteli diagrams also can be found in [**6**, **7**].

The paper is organized as follows. In §2 we include several definitions and facts from K-theory and Cantor dynamics that are used in the main part of the paper. In §3 we discuss basic properties of the crossed product C*-algebras, generated by homeomorphisms σ with finitely many minimal components. Using the Pimsner–Voiculescu six-term exact sequence, we show (in §3.3) that the index map is non-zero if the Cantor system (X, σ)

contains more than one minimal component, and hence in this case the crossed product C*-algebra cannot be A$\mathbb{T}$, in contrast to the case of minimal Cantor systems. The K-theory of the Putnam's orbit-cutting algebra A_Y, where $Y \subseteq X$ is a closed set with non-empty intersection with each Y_i, $i = 1, \ldots, k$, is considered in §4 (for instance, Y can be $\bigcup_i Y_i$). A dynamical system description of $\mathrm{K}_0(A_Y)$ is given in Theorem 4.4.

In §5 we introduce k-simple ordered Bratteli diagrams. These are Bratteli diagrams with k simple quotients, but with more compatibility conditions on the order structure (see Definition 5.3). With these conditions, the Bratteli–Vershik map is then continuous, and the induced Cantor system is indecomposable and has k minimal components. Then, in §6, we also show that these k-simple ordered Bratteli diagrams (with the Vershik maps) are exactly the Bratteli–Vershik–Kakutani models for arbitrary indecomposable Cantor systems with k minimal components.

The compatibility conditions on the k-simple ordered Bratteli diagram actually are very rigid on the underlining (unordered) Bratteli diagram, and they have deep connections to the index map of the C*-algebra extension associated with the Cantor system. In §7 we build this bridge by means of *transition graphs* (Definition 7.1). It is a sequence of finite directed graphs produced from the given k-simple ordered Bratteli diagram, and it actually provides a combinatorial description of the index map (Theorem 7.5). As consequences, when $k \geq 2$, the K_0-group of the canonical approximately finite (AF) ideal of $\mathrm{C}(X) \rtimes \mathbb{Z}$ must contain infinitesimal elements, it has rational rank at least k, and the C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}$ is stably finite. Together with [**26**, Corollary 2.6], this provides natural examples of stably finite C*-algebra with stable rank 2 and real rank 0.

We use transition graphs again in §8 to describe which unordered Bratteli diagram can carry a Cantor system with k minimal components. With the help of the Euler walk, this question can be answered in a special case (Theorem 8.2).

In §9 we consider Cantor systems with only one minimal component. In this case, the crossed product C*-algebra is an A$\mathbb{T}$ algebra, and hence the K_0-group is a (not necessarily simple) dimension group. We study the connection between its order structure and the boundedness of the invariant measures concentrated on the complement of the minimal components.

In §10 we focus on topological properties of Cantor dynamical systems with finitely many minimal components. We discuss the notions of *chain-transitive* and *moving* homeomorphisms (the latter was introduced in [**1**]). We give a necessary and sufficient condition for a non-minimal homeomorphism to be chain transitive; see Theorem 10.15. In particular, we show that any k-minimal homeomorphism is chain transitive (Corollary 10.16).

2. *Preliminaries and notation*

2.1. *Cantor systems.* By a Cantor dynamical system (X, σ), we mean a Cantor set X together with a homeomorphism $\sigma : X \to X$. A closed set $Y \subseteq X$ is said to be invariant if $\sigma(Y) = Y$, and a closed invariant set is said to be *minimal* if it is non-empty and does not contain any closed invariant subsets other than itself and $\emptyset$. By Zorn's lemma and the compactness of X, minimal invariant subsets always exist; let us call them *minimal components.* In this paper we only consider Cantor systems with finitely many minimal components, unless other properties of σ are explicitly specified. A Cantor system is said to be *aperiodic* if X does not contain any periodic points of σ.

2.2. *Ordered Bratteli diagrams.* Let us recall some definitions and notation on Bratteli diagrams which are crucial in Cantor dynamics for constructing models for given transformations. For more details, we refer to [**3**, **11**, **12**] where various combinatorial and dynamical properties of simple and non-simple Bratteli diagrams are discussed.

Definition 2.1. A *Bratteli diagram* is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{n\geq 0} V^n$ and the edge set $E = \bigcup_{n\geq 0} E^n$ are partitioned into disjoint subsets V^n and E^n where:

(i) $V^0 = \{v_0\}$ is a single point;

(ii) V^n and E^n are finite sets, for all $n \geq 0$;

(iii) there exist $r : E \to V$ (range map r) and $s : E \to V$ (source map s), both from E to V, such that $r(E^n) = V^{n+1}$, $s(E^n) = V^n$ (in particular, $s^{-1}(v) \neq \emptyset$ and $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$).

The structure of every Bratteli diagram B is completely determined by the sequence of incidence matrices (F_n). By definition, the *incidence matrix* F_n has entries

$$f^{(n)}_{v,w} = |\{e \in E_n : s(e) = w, r(e) = v\}|, \quad v \in V_{n+1},\ w \in V_n.$$

A Bratteli diagram B is called *stationary* if $F_n = F_1$ for all n, and B is of *finite rank* if there exist $d \in \mathbb{N}$ such that $|V_n| \leq d$ for all n.

In what follows, we will constantly use the telescoping procedure. A telescoped Bratteli diagram preserves all properties of the initial diagram so that it does not change its dynamical properties.

Definition 2.2. Let B be a Bratteli diagram, and $n_0 = 0 < n_1 < n_2 < \cdots$ be a strictly increasing sequence of integers. The *telescoping of B to (n_k)* is the Bratteli diagram B', with V^{n_i} being the i-level vertex set $(V')^i$ and the edges between $(V')^i$ and $(V')^{i+1}$ being finite paths of the Bratteli diagram B between n_i-level vertices and n_{i+1}-level vertices.

For a Bratteli diagram B, the *tail (cofinal) equivalence* relation $\mathcal{E}$ on the path space X_B is defined as follows: $x\mathcal{E}y$ if there exists $m \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq m$, where $x = (x_n)$, $y = (y_n)$.

A Bratteli diagram B is called *aperiodic* if every $\mathcal{E}$-orbit is countably infinite.

LEMMA 2.3. *Every aperiodic Bratteli diagram B can be telescoped to a diagram B' with the property $|r^{-1}(v)| \geq 2$, $v \in V \setminus V^0$ and $|s^{-1}(v)| \geq 2$, $v \in V \setminus V^0$.*

Remark 2.4. Given an aperiodic dynamical system (X, σ), a Bratteli diagram is constructed by a sequence of refining Kakutani–Rokhlin partitions generated by (X, σ) (for details, see [**18**, **23**]). The nth level of the diagram, (V^n, E^n), corresponds to the nth Kakutani–Rokhlin partition, and the cardinality of the set $E(v_0, v)$ of all finite paths between the top v_0 and a vertex $v \in V^n$ is the height of the σ-tower labeled by the symbol v from that partition. We will give more details of this construction in §6.

Definition 2.5. A Bratteli diagram $B = (V^*, E)$ is called *ordered* if there is a linear order '$>$' on every set $r^{-1}(v)$, $v \in \bigcup_{n\geq 1} V_n$. We also use $>$ to denote the corresponding partial order on E and write $(B, >)$ when we consider B with the order $>$.

Every order $>$ defines the *lexicographic* ordering on the set $E(k, l)$ of finite paths between vertices of levels V^k and V^l: we say that

$$(e_{k+1}, \ldots, e_l) > (f_{k+1}, \ldots, f_l)$$

if and only if there is i with $k + 1 \leq i \leq l$ such that $e_j = f_j$ for $i < j \leq l$, and $e_i > f_i$. It follows that, given the order $>$, any two paths from $E(v_0, v)$ are comparable with respect to the lexicographic ordering generated by $>$. If two infinite paths are tail equivalent, and agree from the vertex v onwards, then we can compare them by comparing their initial segments in $E(v_0, v)$. Thus $>$ defines a partial order on X_B, where two infinite paths are comparable if and only if they are tail equivalent.

With every Bratteli diagram B, one can associate the *dimension group* K_B. This is defined as the direct limit of groups $\mathbb{Z}^{|V_n|}$ with a sequence of positive homomorphisms generated by incidence matrices F_n:

$$\mathrm{K}_B = \lim_{n\to\infty} (\mathbb{Z}^{|V_n|} \xrightarrow{F_n} \mathbb{Z}^{|V_{n+1}|}).$$

Then K_B is an abelian partially ordered group with the distinguished order unit corresponding to V^0. These groups play a prominent role in the classification of AF algebras and Bratteli diagrams; see, for example, [**8**, **14**, **15**].

In particular, it is well known that, for Bratteli diagrams B_1 and B_2, if $\mathrm{K}_{B_1} \cong \mathrm{K}_{B_2}$ (as order-unit groups), then there is a Bratteli diagram B such that both B_1 and B_2 can be obtained from B by the telescoping operations.

2.3. *Crossed product C*-algebras and K-theory.* Consider a compact Hausdorff space X and consider a homeomorphism $\sigma : X \to X$. Let $\mathrm{C}(X)$ denote the C*-algebra of complex-valued continuous functions on X. Then the homeomorphism σ induces an automorphism of $\mathrm{C}(X)$ by $f \mapsto f \circ \sigma^{-1} = \sigma(f)$, which is also denoted by σ. The *crossed product* C*-algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is defined to be the universal C*-algebra generated by $\mathrm{C}(X)$ and a canonical unitary u satisfying

$$u^* f u = \sigma(f), \quad f \in \mathrm{C}(X).$$

In other words, the crossed product C*-algebra is defined by

$$\mathrm{C}(X) \rtimes_\sigma \mathbb{Z} := \mathrm{C}^*\{f, u;\ f \in \mathrm{C}(X),\ uu^* = u^*u = 1,\ u^* f u = f \circ \sigma^{-1}\}.$$

This concept is defined and studied in many books on operator algebras; see, for instance, [**10**, Ch. VIII].

Applying the Pimsner–Voiculescu six-term exact sequence to $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ yields

$$\begin{array}{ccccc} \mathrm{K}_0(\mathrm{C}(X)) & \xrightarrow{1-[\sigma]_0} & \mathrm{K}_0(\mathrm{C}(X)) & \longrightarrow & \mathrm{K}_0(\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}) & \longleftarrow & \mathrm{K}_1(\mathrm{C}(X)) & \xleftarrow[1-[\sigma]_1]{} & \mathrm{K}_1(\mathrm{C}(X)), \end{array} \tag{2.1}$$

where $K_0(A)$ and $K_1(A)$ are respectively the K_0-group and K_1-group of a C*-algebra A. In (2.1), $[\sigma]_0$ and $[\sigma]_1$ denote the maps between the K_0-groups and K_1-groups induced by σ, respectively.

In the case where X is a Cantor set, we have $K_1(C(X)) = \{0\}$; hence, it follows from (2.1) that

$$K_0(C(X) \rtimes_\sigma \mathbb{Z}) = K_0(C(X))/(1 - [\sigma]_0)(K_0(C(X))), \tag{2.2}$$

and the group

$$K_1(C(X) \rtimes_\sigma \mathbb{Z}) = \ker(1 - [\sigma]_0) \tag{2.3}$$

consists of σ-invariant functions.

A closed two-sided *ideal* of a C*-algebra A is a sub-C*-algebra $I \subseteq A$ such that $IA \subseteq I$ and $AI \subseteq I$. If $Y \subseteq X$ is a closed subset, then $C_0(X \setminus Y)$ is an ideal of $C(X)$. If, moreover, Y is invariant (i.e., $\sigma(Y) = Y$), then $C_0(X \setminus Y)$ is a σ-invariant ideal of $C(X)$, and therefore $C_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$ is an ideal of $C(X) \rtimes_\sigma \mathbb{Z}$ with quotient canonically isomorphic to $C(Y) \rtimes_\sigma \mathbb{Z}$. That is, we have a short exact sequence of C*-algebras:

$$0 \longrightarrow C_0(X \setminus Y) \rtimes_\sigma \mathbb{Z} \longrightarrow C(X) \rtimes_\sigma \mathbb{Z} \longrightarrow C(Y) \rtimes_\sigma \mathbb{Z} \longrightarrow 0.$$

In general, for any C*-algebra extension

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0,$$

we have the six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

The map $K_1(A/I) \to K_0(I)$ is called the *index map*, and the map $K_0(A/I) \to K_1(I)$ is called the *exponential map*.

For more information about the K-theory of a C*-algebra, see, for instance, [**31**].

3. *Cantor system with finitely many minimal subsets*

3.1. *Crossed product C*-algebra associated to σ.* Recall that, for a topological dynamical system (X, σ) (X is not necessarily a Cantor set), a closed subset Y is minimal if Y is a closed invariant non-empty subset and Y is minimal among these subsets. Minimal subsets always exist, and each pair of them is disjoint.

Let (X, σ) be a topological dynamical system with only k minimal subsets $Y_1, \ldots, Y_k$. We then have the short exact sequence

$$0 \longrightarrow C_0\Big(X \Big\backslash \bigcup_i Y_i\Big) \rtimes_\sigma \mathbb{Z} \longrightarrow C(X) \rtimes_\sigma \mathbb{Z} \longrightarrow \bigoplus_i C(Y_i) \rtimes_\sigma \mathbb{Z} \longrightarrow 0. \tag{3.1}$$

LEMMA 3.1. *Let $Y_1, \ldots, Y_k$ be all the minimal subsets of (X, σ). Suppose that $U \subseteq X$ is an open set such that $U \cap Y_i \neq \emptyset$, $i = 1, 2, \ldots, k$. Then $\bigcup_{j=-\infty}^{\infty} \sigma^j(U) = X$.*

Proof. We first note that if an open set V intersects a minimal set Y, then the orbit of V contains Y. Consider the open invariant subset

$$Z := \bigcup_{j=-\infty}^{\infty} \sigma^j(U).$$

If Z were a proper subset of X, then $X \setminus Z$ would be a non-empty invariant closed subset of X, and hence it must contain a minimal subset Y. But this minimal subset is disjoint with every set $Y_1, \ldots, Y_k$. This contradicts the assumption that $Y_1, Y_2, \ldots, Y_k$ are all the minimal components. Therefore, $Z = X$, as desired. □

Remark 3.2. It follows from Lemma 3.1 that, for any open subset $U \supseteq \bigcup_i Y_i$, we have that $\bigcup_{j=-\infty}^{\infty} \sigma^j(U) = X$.

LEMMA 3.3. *Under the assumption that all proper minimal components for (X, σ) are not clopen sets, every minimal component Y has empty interior.*

Proof. Suppose Y has a subset U which is open in X. Since Y is minimal and compact, there exists N such that $Y = \bigcup_{j=-N}^{N} \sigma^j(U)$. Hence Y is open in X, which contradicts the assumption. □

COROLLARY 3.4. *Let $Y_1, \ldots, Y_k$ be minimal subsets for (X, σ). If none of the Y_i is clopen, then the ideal $\mathrm{C}_0(X \setminus \bigcup_i Y_i)$ is essential in $\mathrm{C}(X)$.*

PROPOSITION 3.5. *Let α be an action of $\mathbb{Z}$ on a C*-algebra A, and let I be an invariant ideal. If I is essential in A, then $I \rtimes_\alpha \mathbb{Z}$ is essential in $A \rtimes_\alpha \mathbb{Z}$.*

Proof. Consider the conditional expectation

$$\mathbb{E} : A \rtimes_\alpha \mathbb{Z} \ni a \mapsto \int_{\mathbb{T}} \alpha_t^*(a)\, dt \in A,$$

where α_t^* is the dual action of α. Note that $\mathbb{E}$ is faithful.

Let a be a positive element in $A \rtimes_\alpha \mathbb{Z}$ with $a(I \rtimes_\alpha \mathbb{Z}) = \{0\}$. Then, for any $b \in I$, we have that $ab = 0$. Since $b \in I \subseteq A$, we also have that $\mathbb{E}(ab) = \mathbb{E}(a)b = 0$. Since $\mathbb{E}(a) \in A$ and b is arbitrary, we have that $\mathbb{E}(a) = 0$. Since $\mathbb{E}$ is faithful, we have that $a = 0$, as desired. □

COROLLARY 3.6. *For (X, σ) as above, if none of $Y_1, \ldots, Y_k$ is clopen, then the ideal $\mathrm{C}_0(X \setminus \bigcup_i Y_i) \rtimes_\sigma \mathbb{Z}$ is essential in $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$.*

3.2. *AF sub-C*-algebras.* We return to the case where the compact topological space X is a Cantor set. Then each minimal subset Y_i, $i = 1, \ldots, k$, is a Cantor set or consists of a periodic orbit. Indeed, to see this, suppose that a minimal set, say Y_1, contains an isolated point x. Then Orbit(x) is an open set (relative to Y_1), and therefore $\overline{\mathrm{Orbit}(x)} \setminus \mathrm{Orbit}(x)$ is an invariant closed subset of Y_1. Since Y_1 is minimal, we have that $Y_1 = \mathrm{Orbit}(x)$, and then a standard compactness argument shows that Y_1 consists of a periodic orbit.

Let $Y \subseteq X$ be a closed subset with $Y \cap Y_i \neq \emptyset$, $i = 1, \ldots, k$. By Lemma 3.1, we have that $\bigcup_{j=-\infty}^{\infty} \sigma^j(U) = X$ for any clopen set $U \supseteq Y$. Then, by [**27**, Lemma 2.3], the sub-C*-algebra

$$A_Y := \mathrm{C}^*\{g,\ fu;\ g,\ f \in \mathrm{C}(X),\ f|_Y = 0\} \subseteq \mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$$

is AF-dimensional.

In particular, fix $y_i \in Y_i$, $i = 1, 2, \ldots, k$, and define

$$A_{y_1,\ldots,y_k} := \mathrm{C}^*\{g,\ fu;\ g,\ f \in \mathrm{C}(X),\ f(y_i) = 0,\ i = 1, \ldots, k\} \subseteq \mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$$

The C*-algebra $A_{y_1,\ldots,y_k}$ is AF, and hence the ideal $\mathrm{C}_0(X \setminus \bigcup_i Y_i) \rtimes_\sigma \mathbb{Z}$ is also AF; see [**13**, Theorem 3.1].

3.3. *Index maps.* The six-term exact sequence associated to (3.1) is

$$\begin{array}{ccccc}
\mathrm{K}_0\Big(\mathrm{C}_0\Big(X\setminus\bigcup_i Y_i\Big) \rtimes_\sigma \mathbb{Z}\Big) & \longrightarrow & \mathrm{K}_0(\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}) & \longrightarrow & \bigoplus_i \mathrm{K}_0(\mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z}) \\
{\scriptstyle \mathrm{Ind}}\uparrow & & & & \downarrow \\
\bigoplus_i \mathrm{K}_1(\mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z}) & \longleftarrow & \mathrm{K}_1(\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}) & \longleftarrow & \mathrm{K}_1\Big(\mathrm{C}_0\Big(X\setminus\bigcup_i Y_i\Big) \rtimes_\sigma \mathbb{Z}\Big) \cong \{0\}.
\end{array} \tag{3.2}$$

Since the restriction of σ to each Y_i is minimal, it follows from (2.3) that

$$\mathrm{K}_1(\mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z}) = \mathbb{Z},$$

which is generated by the K_1-class of the canonical unitary of $\mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z}$. Also note that (X, σ) is indecomposable, that is, the only clopen σ-invariant subsets are X and $\emptyset$. Then the only invariant $\mathbb{Z}$-valued continuous functions on X are constant functions, and then, by (2.3) again, we obtain that

$$\mathrm{K}_1(\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}) = \mathbb{Z},$$

which is generated by the K_1-class of the canonical unitary of $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$.

THEOREM 3.7. *If a Cantor system (X, σ) is indecomposable and has k minimal components, then the image of the index map* Ind *is isomorphic to $\mathbb{Z}^{k-1}$; in particular, the index map is non-zero if $k \geq 2$.*

Proof. Note that the canonical unitary of $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is sent to the canonical unitaries of $\mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z}$, $i = 1, 2, \ldots, k$. Therefore, the index map $\mathrm{K}_1(\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}) \to \bigoplus_{i=1}^k \mathrm{K}_1(\mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z})$ in (3.2) is given by

$$\mathbb{Z} \ni 1 \mapsto (1, \ldots, 1) \in \mathbb{Z}^k,$$

and hence

$$\mathrm{Image}(\mathrm{Ind}) \cong \left(\bigoplus_{i=1}^{k} \mathrm{K}_1(\mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z})\right) \Big/ \mathrm{K}_1(\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}) \cong \mathbb{Z}^k/\mathbb{Z}(1, 1, \ldots, 1), \tag{3.3}$$

which is isomorphic to $\mathbb{Z}^{k-1}$ and is non-zero if $k \geq 2$. □

An abelian group G is said to have $\mathbb{Q}$-*rank* r if the vector space $G \otimes \mathbb{Q}$ has dimension r over $\mathbb{Q}$. Then another consequence of equation (3.3) is the following theorem.

THEOREM 3.8. *The $\mathbb{Q}$-rank of the group $\mathrm{K}_0(\mathrm{C}_0(X \setminus \bigcup_i Y_i) \rtimes_\sigma \mathbb{Z})$ is at least $k-1$.*

Remark 3.9. The lower bound of the $\mathbb{Q}$-rank of $\mathrm{K}_0(\mathrm{C}_0(X \setminus \bigcup_i Y_i) \rtimes_\sigma \mathbb{Z})$ will be improved to k in Corollary 7.10.

For a clopen set $U \subset X$, denote by χ_U the characteristic function of the set U; note that $\chi_U \in \mathrm{C}(X)$. Also note that any continuous integer-valued function on X which vanishes on $\bigcup_{i=1}^k Y_i$ induces a K_0-class of the C*-algebra $\mathrm{C}_0(X \setminus \bigcup_{i=1}^k Y_i)$ and hence induces a K_0-class of the C*-algebra $\mathrm{C}_0(X \setminus \bigcup_{i=1}^k Y_i) \rtimes_\sigma \mathbb{Z}$ by the embedding of $\mathrm{C}_0(X \setminus \bigcup_{i=1}^k Y_i)$.

THEOREM 3.10. *Consider the minimal components Y_i, $i = 1, \ldots, k$, and choose pairwise disjoint clopen sets $U_i \supseteq Y_i$. Define*

$$d_i := [\chi_{U_i} - \chi_{U_i} \circ \sigma^{-1}]_0 \in \mathrm{K}_0\left(\mathrm{C}_0\left(X \Big\backslash \bigcup_i Y_i\right) \rtimes_\sigma \mathbb{Z}\right).$$

(Note that each integer-valued continuous function $\chi_{U_i} - \chi_{U_i} \circ \sigma^{-1}$ vanishes on $\bigcup_{i=1}^k Y_i$.) Then d_i is independent of the choice of U_i. Moreover, $d_1 = 0$ if $k = 1$. If $k \geq 2$, then

$$\sum_{i=1}^k d_i = 0,$$

and the sum of any proper subset of $\{d_1, d_2, \ldots, d_k\}$ is non-zero.

Proof. Consider

$$v_i := \chi_{U_i} u + \sum_{j \neq i} \chi_{U_j} \in \mathrm{C}(X) \rtimes_\sigma \mathbb{Z},$$

where u is the canonical unitary of $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$. Then it is a lifting of the canonical unitary of $\mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z}$. Since $v_i v_i^* = \sum_{j=1}^k \chi_{U_j}$ (so v_i is a partial isometry) and

$$v_i^* v_i = u^* \chi_{U_i} u + \sum_{j \neq i} \chi_{U_j} = \chi_u \circ \sigma^{-1} + \sum_{j \neq i} \chi_{U_j},$$

the index of the canonical unitary of $\mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z}$ is given by

$$\begin{aligned}
[1 - v_i^* v_i]_0 - [1 - v_i v_i^*]_0 &= \left[1 - \left(\chi_{U_i} \circ \sigma^{-1} + \sum_{j \neq i} \chi_{U_j}\right)\right]_0 - \left[1 - \left(\sum_{j=1}^k \chi_{U_j}\right)\right]_0 \\
&= [\chi_{U_i} - \chi_{U_i} \circ \sigma^{-1}]_0 \\
&= d_i.
\end{aligned}$$

Therefore, d_i is the image of the K_1-class of the canonical unitary of $\mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z}$ under the index map. The theorem then follows from the six-term sequence (3.2) and (3.3). □

COROLLARY 3.11. *The C*-algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is an $\mathrm{A}\mathbb{T}$-algebra (i.e., it is the inductive limit of $F \otimes \mathrm{C}(\mathbb{T})$, where F is a finite-dimensional C*-algebra and $\mathbb{T}$ is the unit circle) if and only if $k = 1$.*

Proof. By [**22**, Theorem 5], an extension of $\mathrm{A}\mathbb{T}$ algebras is $\mathrm{A}\mathbb{T}$ if and only if the index map is zero, and by Theorem 3.7, this holds if and only if $k = 1$. □

Remark 3.12. For the case $k \geq 2$, we will show later (see Corollary 7.11) that the C*-algebra $C(X) \rtimes_\sigma \mathbb{Z}$ is stably finite, has stable rank 2, and has real rank 0 if the Cantor system is aperiodic.

4. *The K-theory of A_Y*

In this section we study a general Cantor system (X, σ). Let $Y \subseteq X$ be a closed subset of X which satisfies the property that, for any open subset U containing Y,

$$\bigcup_{n \in \mathbb{Z}} \sigma^n(U) = X.$$

Such sets are called *basic* in [**23**]. In particular, in the case where X has k minimal subsets $Y_1, \ldots, Y_k$, the set Y can be any closed subset with $Y \cap Y_i \neq \emptyset$, $i = 1, \ldots, k$. By [**27**, Theorem 2.3], A_Y is an AF-algebra. We will calculate the K_0-group of A_Y in this section, and show that it can be identified with a certain ordered group related to the dynamical system (X, σ).

First, let us recall the following result.

PROPOSITION 4.1. ([**26**, Proposition 3.3], [**28**, Theorem 4.1]) *For Y chosen as above, there is an exact sequence*

$$0 \longrightarrow C^\sigma(X, \mathbb{Z}) \xrightarrow{\alpha} C(Y, \mathbb{Z}) \xrightarrow{\beta} K_0(A_Y) \xrightarrow{\iota_*} K_0(C(X) \rtimes_\sigma \mathbb{Z}) \longrightarrow 0, \tag{4.1}$$

where $C^\sigma(X, \mathbb{Z})$ is the group of σ-invariant integer-valued continuous functions on X, α is the restriction map, and

$$\beta(f) = [g - g \circ \sigma^{-1}]$$

for some $g \in C(X, \mathbb{Z})$ with $g|_Y = f$. In particular, if (X, σ) is indecomposable, that is, any σ-invariant clopen subset is trivial, then $C^\sigma(X, \mathbb{Z}) \cong \mathbb{Z}$, and (4.1) is transformed to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} C(Y, \mathbb{Z}) \xrightarrow{\beta} K_0(A_Y) \xrightarrow{\iota_*} K_0(C(X) \rtimes_\sigma \mathbb{Z}) \longrightarrow 0. \tag{4.2}$$

Let us define an ordered group from the dynamical system setting.

Definition 4.2. Let (X, σ) be a Cantor system, and let $Y \subseteq X$ be a closed subset. Define

$$K^0_Y(X, \sigma) = C(X, \mathbb{Z}) / \{f - f \circ \sigma^{-1};\ f \in C(X, \mathbb{Z}),\ f|_Y = 0\},$$

and set

$$K^0(X, \sigma)^+ = \{\bar{f};\ f \geq 0,\ f \in C(X, \mathbb{Z})\} \subseteq K^0_Y(X, \sigma).$$

Note that $C(X, \mathbb{Z}) \cong K_0(C(X))$ as ordered groups (the positive cone of $C(X, \mathbb{Z})$ consisting of positive functions). Then the embedding $C(X) \subseteq A_Y$ induces a map $\theta : C(X, \mathbb{Z}) \to K_0(A_Y)$. Moreover, a direct calculation shows that if $f \in C(X, \mathbb{Z})$ with $f|_Y = 0$, then

$$\theta(f - f \circ \sigma^{-1}) = 0.$$

Therefore, θ induces a map from $K^0_Y(X, \sigma)$ to $K_0(A_Y)$ which sends $K^0(X, \sigma)^+$ to the positive cone of $K_0(A_Y)$.

By (2.2), we have

$$0 \longrightarrow \{f - f \circ \sigma^{-1};\ f \in C(X, \mathbb{Z})\} \longrightarrow C(X, \mathbb{Z}) \xrightarrow{\iota_*} K_0(C(X) \rtimes_\sigma \mathbb{Z}) \longrightarrow 0,$$

and therefore, there is an exact sequence

$$0 \longrightarrow H \longrightarrow \mathrm{K}^0_Y(X,\sigma) \xrightarrow{\iota_*} \mathrm{K}_0(\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}) \longrightarrow 0, \tag{4.3}$$

where

$$H := \frac{\{f - f\circ\sigma^{-1};\ f \in \mathrm{C}(X,\mathbb{Z})\}}{\{f - f\circ\sigma^{-1};\ f \in \mathrm{C}(X,\mathbb{Z}),\ f|_Y = 0\}}.$$

For any $f \in \mathrm{C}(X,\mathbb{Z})$, define $\eta(f - f\circ\sigma^{-1})$ to be the restriction of f to Y. This induces an isomorphism from H to $\mathrm{C}(Y,\mathbb{Z})/\alpha(\mathrm{C}^\sigma(X,\mathbb{Z}))$, which is also denoted by η. Indeed, if

$$f - f\circ\sigma^{-1} = g - g\circ\sigma^{-1},$$

then

$$f - g = (f-g)\sigma^{-1}.$$

That is, $f - g \in \mathrm{C}^\sigma(X,\mathbb{Z})$, and hence θ is well defined. It is also clear that η is a bijection, and thus an isomorphism.

LEMMA 4.3. *With notation as above, the following diagram commutes:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & \mathrm{K}^0_Y(X,\sigma) & \xrightarrow{\iota_*} & \mathrm{K}_0(\mathrm{C}(X)\rtimes_\sigma\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\theta} & & \| & & \\
0 & \longrightarrow & \mathrm{C}(Y,\mathbb{Z})/\alpha(\mathrm{C}^\sigma(X,\mathbb{Z})) & \xrightarrow{\beta} & \mathrm{K}_0(A_Y) & \xrightarrow{\iota_*} & \mathrm{K}_0(\mathrm{C}(X)\rtimes_\sigma\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

In particular, the map θ is an isomorphism.

Proof. To prove this, we only have to verify the commutativity for the first square. Pick any $\overline{f - f\circ\sigma^{-1}} \in H$. Then we have

$$\beta(\eta(\overline{f - f\circ\sigma^{-1}})) = \beta(\overline{f|_Y}) = [f - f\circ\sigma^{-1}]$$

and

$$\theta(\overline{f - f\circ\sigma^{-1}}) = [f - f\circ\sigma^{-1}],$$

as desired. Since the map η is an isomorphism, by the five-lemma, the map θ is also an isomorphism. □

Moreover, the map θ is in fact an order isomorphism.

THEOREM 4.4. *The map θ induces an order isomorphism*

$$(\mathrm{K}^0_Y(X,\sigma), \mathrm{K}^0_Y(X,\sigma)^+, \tilde{1}) \cong (\mathrm{K}_0(A_Y), \mathrm{K}^+_0(A_Y), [1_{A_Y}]).$$

Proof. We need to show that $\theta(\mathrm{K}^0_Y(X,\sigma)^+) = \mathrm{K}^+_0(A_Y)$. It is clear that the image of $\mathrm{K}^0_Y(X,\sigma)^+$ is in $\mathrm{K}^+_0(A_Y)$. On the other hand, using the AF-decomposition of A_Y, it is clear that any positive element in $\mathrm{K}_0(A_Y)$ comes from a positive integer function on X. □

Assume that (X,σ) is indecomposable. Let $W_1 \subseteq W_2$ be closed subsets of X. Then we have

$$0 \longrightarrow K \longrightarrow \mathrm{K}^0_{W_2}(X,\sigma) \longrightarrow \mathrm{K}^0_{W_1}(X,\sigma) \longrightarrow 0, \tag{4.4}$$

where

$$K = \frac{\{f - f \circ \sigma^{-1};\ f \in \mathrm{C}(X, \mathbb{Z}),\ f|_{W_1} = 0\}}{\{f - f \circ \sigma^{-1};\ f \in \mathrm{C}(X, \mathbb{Z}),\ f|_{W_2} = 0\}}.$$

For any $f \in \mathrm{C}(X, \mathbb{Z})$, define

$$\eta(f - f \circ \sigma^{-1}) = f|_{W_2}.$$

Note that this map is well defined. We also define

$$\mathrm{C}(W_2, W_1, \mathbb{Z}) := \{f \in \mathrm{C}(W_2, \mathbb{Z}),\ f|_{W_1} = 0\}.$$

Then we can prove the following statement.

LEMMA 4.5. *If (X, σ) is indecomposable and $W_1 \neq \emptyset$, the map η induces an isomorphism*

$$\eta : K \to \mathrm{C}(W_2, W_1, \mathbb{Z}).$$

Proof. Let us first check that the map η is well defined. Indeed, if

$$(f - f \circ \sigma^{-1}) - (g - g \circ \sigma^{-1}) = h - h \circ \sigma^{-1}$$

for some $f, g, h \in \mathrm{C}(X, \mathbb{Z})$ with $f|_{W_1} = g|_{W_1} = 0$ and $h|_{W_2} = 0$, then

$$(f - g - h) = (f - g - h) \circ \sigma^{-1}.$$

Since (X, σ) is indecomposable, there is no invariant clopen subset of X, and hence $f - g - h$ is a constant function. Since $W_1 \neq \emptyset$ and the restrictions of f, g, h to W_1 are zero, we have that

$$f = g + h.$$

The condition $h|_{W_2} = 0$ implies that

$$f|_{W_2} = g|_{W_2};$$

that is, the map η is well defined.

It is clear that η is surjective. If $\eta(f - f \circ \sigma^{-1}) = 0$, then $f|_{W_2} = 0$; that is, the map η is also injective, and hence it is an isomorphism. □

Thus, the exact sequence (4.4) can be written as

$$0 \longrightarrow \mathrm{C}(W_2, W_1, \mathbb{Z}) \xrightarrow{\eta^{-1}} \mathrm{K}^0_{W_2}(X, \sigma) \longrightarrow \mathrm{K}^0_{W_1}(X, \sigma) \longrightarrow 0, \tag{4.5}$$

and. applying Theorem 4.4, we obtain the following statement.

THEOREM 4.6. *For any non-empty closed subsets $W_1 \subseteq W_2$ which are basic, we have*

$$0 \longrightarrow \mathrm{C}(W_2, W_1, \mathbb{Z}) \xrightarrow{\beta} \mathrm{K}_0(A_{W_2}) \xrightarrow{\iota_*} \mathrm{K}_0(A_{W_1}) \longrightarrow 0, \tag{4.6}$$

where $\beta(f) = g - g \circ \sigma^{-1}$ for some $g \in \mathrm{C}(X, \mathbb{Z})$ with $g|_{W_2} = f$.

5. *k-simple Bratteli diagrams and Bratteli–Vershik models*

In this section we shall introduce certain ordered Bratteli diagrams which will be used to model Cantor systems with finitely many minimal subsets.

5.1. *k-simple Bratteli diagrams.* Let us now introduce a special class of Bratteli diagrams. This class of Bratteli diagrams will serve as models for Cantor systems with finitely many minimal components.

Definition 5.1. Let $k \in \mathbb{N}$. A Bratteli diagram $B = (V, E)$ is said to be *k-simple* if, for each $n \geq 1$, there are pairwise disjoint subsets $V_1^n, \dots, V_k^n$ of V^n such that:

(1) for any $1 \leq i \leq k$ and any $v \in V_i^{n+1}$, we have that $s(r^{-1}(v)) \subseteq V_i^n$;

(2) for any $1 \leq i \leq k$ and any level n, there is $m > n$ such that each vertex of V_i^m is connected to all vertices of V_i^n.

Moreover, denote $V_o^n = V^n \setminus (V_1^n \cup \dots \cup V_k^n)$ for $n \geq 1$.

(1) The diagram B is said to be *strongly k-simple* if, for any level n, there is $m > n$ such that if a vertex $v \in V_o^m$ is connected to some vertex of V_o^n, then v is connected to all vertices of V_o^n.

(2) The diagram B is said to be *non-elementary* if, for any V_o^n, there is $m > n$ such that the multiplicity of the edges between V_o^n and V_o^m is either 0 or at least 2.

A dimension group G is said to be (strongly) k-simple if $G \cong \mathrm{K}_B$ for some (strongly) k-simple Bratteli diagram.

Remark 5.2.

(1) To clarify the meaning of Definition 5.1, we make the following observations.

  (a) The diagram B consists of k many *simple* sub-diagrams B_i, $i = 1, \dots, k$, constructed on the sequence of vertices (V_i^n); there are no edges connecting different sub-diagrams. The part of the diagram whose infinite paths eventually go through vertices of V_o^n for sufficiently large n constitutes an open invariant set which does not contain minimal subsets.

  (b) Without loss of generality, we can assume that $V_o^n \neq \emptyset$; otherwise the corresponding Bratteli–Vershik system would be decomposable.

  (c) The part of the diagram B defined by (V_o^n) induces an ideal of the corresponding AF-algebra. Strongly k-simple diagrams correspond to the case where the ideal is simple.

  (d) A k-simple Bratteli diagram B is non-elementary if and only if the infinite-path space does not have isolated points (so it is a Cantor set). To guarantee that a Bratteli diagram is non-elementary it suffices to require that for every infinite path $x = (x_i)$ there are infinitely many edges x_m such that $|s^{-1}(r(x_m))| > 1$.

(2) Let B be a k-simple Bratteli diagram. Then the sub-diagram restricted to the vertices in V_o^n, denoted by I_B, induces an ordered ideal $\mathrm{K}_{I_B} \subseteq \mathrm{K}_B$ such that $\mathrm{K}_B/\mathrm{K}_{I_B} \cong \bigoplus_{i=1}^k G_i$, where G_i are simple dimension groups induced by the restriction of B to the vertices in V_i^n. Moreover, the diagram B is strongly k-simple if and only if K_{I_B} is a simple dimension group; it is non-elementary if and only if K_{I_B} has no quotient which is isomorphic to $\mathbb{Z}$, and if and only if the space of infinite paths through the sets V_o^n is a locally compact Cantor set.

An ordered Bratteli diagram $B = (V, E, >)$ is a Bratteli diagram with (partial) order $>$ on E so that two edges e and e' are comparable if and only if $r(e) = r(e')$; see Definition 2.5. Denote by $E_{\max}$ and $E_{\min}$ the sets of maximal edges and minimal edges, respectively. This partial order induces a lexicographical partial order on paths (infinite or finite). Denote by $X_{\max}$ and $X_{\min}$ the set of maximal infinite paths and the set of minimal infinite paths, respectively. Also note that if B' is the Bratteli diagram obtained by telescoping on B, the lexicographic order on B' makes it into an ordered Bratteli diagram canonically.

Definition 5.3. An ordered Bratteli diagram $B = (V, E, >)$ is called *k-simple* (with a slight abuse of notation) if it satisfies the following conditions.

(1) The unordered Bratteli diagram $B = (V, E)$ is k-simple in the sense of Definition 5.1.

(2) There are infinite paths $z_{1,\max}, \ldots, z_{k,\max}$ and $z_{1,\min}, \ldots, z_{k,\min}$ such that, for any level n and $1 \leq i \leq k$,

$$\{z^n_{i,\min}, z^n_{i,\max}\} \subset V^n_i$$

and

$$X_{\max} = \{z_{1,\max}, \ldots, z_{k,\max}\},$$
$$X_{\min} = \{z_{1,\min}, \ldots, z_{k,\min}\}.$$

By this condition and Lemma 5.5 below, there is L such that, for all $n \geq L$ and any $v \in V^n_o$, the maximal edge (or minimal edge) starting with v backwards to V^1 will end up in V^1_i for some $1 \leq i \leq k$. Denote $m_+(v) = i$ (or $m_-(v) = i$).

(3) For any $v \in V^n_o$, we have

(a) if e is an edge with $s(e) = v$, then

$$m_-(s(e+1)) = m_+(v)$$

(if $e \in E_{\max}$, the vertex $s(e+1)$ is understood as $s(e'+1)$ with e' a non-maximal edge starting with e and ending at some level $m > n$—such an edge exists and $m_-(s(e+1))$ is well defined, by Condition (2)); and

(b) if e is an edge with $e \notin E_{\max}$, $r(e) = v$ and $s(e) \in V^{n-1}_i$ with $n \geq 3$, then

$$m_-(s(e+1)) = i.$$

If, in addition, the unordered Bratteli diagram (V, E) is strongly k-simple, then B is said to be a strongly k-simple ordered Bratteli diagram.

Remark 5.4. Note that if $k = 1$, then condition (3) is redundant. Moreover, condition (3) is preserved under telescoping.

LEMMA 5.5. *Any ordered Bratteli diagram satisfying condition (2) of Definition 5.3 can be telescoped to an ordered Bratteli diagram satisfying the following condition: if e and e' are in $E_{\max}$ (or $E_{\min}$) with $r(e) = s(e')$, then e is in $X_{\max}$ (or $X_{\min}$).*

Proof. The proof is similar to that of [**18**, Proposition 2.8]. Let T denote the graph obtained from $E_{\max}$ by deleting $z_{1,\max}, \ldots, z_{k,\max}$. By Condition (2), each connected component of T is finite.

Let $n_0 = 0$. Having defined n_k, choose n_{k+1} so that no vertex in V_{n_k} is connected to all vertices of $V_{n_{k+1}}$. Contract the diagram to the subsequence $\{n_k;\ k \geq 0\}$. Then this diagram satisfies the lemma. □

Recall that two ordered Bratteli diagrams B_1 and B_2 are *equivalent* if there is an ordered Bratteli diagram B such that B_1 and B_2 can be obtained by telescoping on B.

LEMMA 5.6. *Let $B = (V, E, >)$ be a k-simple ordered Bratteli diagram. Then it is equivalent to a k-simple Bratteli diagram $B' = (V', E', >')$ satisfying the following conditions:*

(1) *if $e \in E'_{\max}$ with $r(e) \in V'_o$, then $s(e) \notin V'_o$ (so the edge e in condition (3a) cannot be a maximal edge in B');*

(2) *for any $1 \leq i \leq k$ and any $n \geq 1$, each vertex $v \in (V')_i^{n+1}$ is connected to all vertices of $w \in (V')_i^n$; and*

(3) *if B is strongly k-simple, then, for any $n \geq 1$, each vertex of $v \in (V')_o^{n+1}$ is connected to all vertices of $(V')_o^n$ (and hence to all vertices of $(V')^n$).*

Moreover, if B is an unordered k-simple Bratteli diagram, then it is equivalent to an unordered k-simple Bratteli diagram B' which satisfies conditions (2) and (3).

Proof. Condition (1) follows from Lemma 5.5. Since B is k-simple, by (2) of Definition 5.1, condition (2) can also be obtained by a telescoping of B.

If B is strongly k-simple. Then B can be telescoped further so that if a vertex $v \in V_o^{n+1}$ is connected to a vertex of V_o^n, then it is connected to all vertices of V_o^n. For condition (3) we need to find an equivalent Bratteli diagram B' so that each vertex $v \in (V')_o^{n+1}$ is connected to all vertices of $(V')_o^n$.

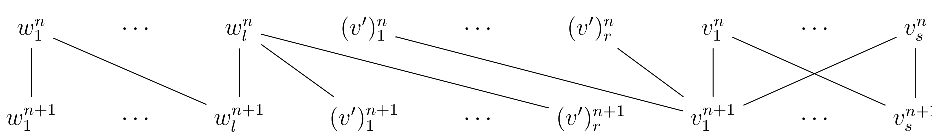
For each $n \geq 1$, write

$$V_o^{n+1} = \{(v')_1^{n+1}, \ldots, (v')_{r_{n+1}}^{n+1}, v_1^{n+1}, \ldots, v_{t_{n+1}}^{n+1}\}$$

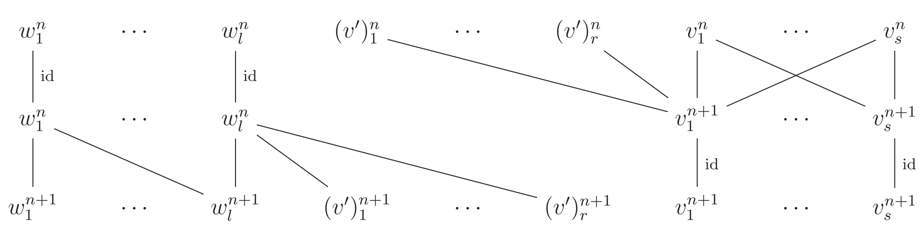
where the vertices v'_i are not connected to V_o^n. Denote by $w_1^n, \ldots, w_l^n$ the vertices from V^n which are not in V_o^n. Interpolate levels n and $n+1$ of B as follows. Consider the vertices

$$\tilde{V}^n : \; w_1^n, \ldots, w_l^n, v_1^{n+1}, \ldots, v_{t_{n+1}}^{n+1}.$$

The map from $V^n \to \tilde{V}^n$ is defined as the identity if restricted to $w_1^n, \ldots, w_l^n$ (in V^n), and the original map (defined by edges of B) if restricted to $v_1^{n+1}, \ldots, v_{t_{n+1}}^{n+1}$. Define the map from $\tilde{V}_n \to V^{n+1}$ as the original map if restricted to $(v')_1^{n+1}, \ldots, (v')_{r_{n+1}}^{n+1}$, and the identity if restricted to $v_1^{n+1}, \ldots, v_{t_{n+1}}^{n+1}$ (in V_{n+1}). This can be illustrated by the following diagrams. The original maps



are interpolated into

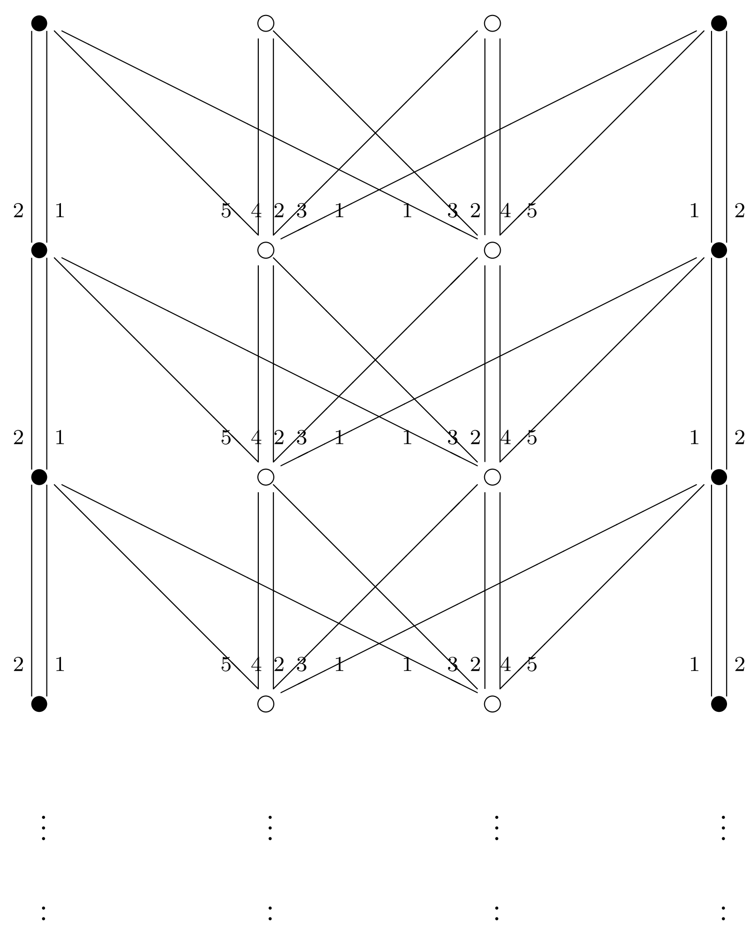


Put an order on this enlarged Bratteli diagram as follows. Consider a vertex of level $\tilde{V}^n$. If it is one of the w_i^n, $i = 1, \ldots, l$, then there is only one edge connecting it backwards, and so just put the trivial order; if it is one of the v_i^{n+1}, $s = 1, \ldots, s$, then the edges backwards are exactly the same edges backwards as in the original Bratteli diagram, and so just put the same order as in the original Bratteli diagram. Put the order similarly for vertices of level V^n. Then it is straightforward to check that its telescoping into the levels V^n is exactly the original ordered Bratteli diagram B.

Denote by B' the telescoping of this diagram into the levels $\tilde{V}^n$. Note that the vertices of $(V')_0^n$ only consist of $v_1^{n+1}, \ldots, v_{t_{n+1}}^{n+1}$, and hence B' is a k-simple ordered Bratteli diagram satisfying condition (3). It follows from Lemma 5.5 that conditions (1) and (2) can also be satisfied by a further telescoping. □

Remark 5.7. In the rest of the paper, we always assume that strong k-simple Bratteli diagrams satisfy conditions (1), (2), and (3) of Lemma 5.6.

Example 5.8. The following is an example of a strongly 2-simple ordered Bratteli diagram (with level 0 omitted):



5.2. *Bratteli–Vershik map.* Let B be a non-elementary ordered k-simple Bratteli diagram, and denote by X_B the space of infinite paths of B. For each finite path ξ, we will denote by χ_ξ the cylinder set consisting of all paths starting with ξ. Then X_B is a Cantor set with topology generated by these cylinder sets because the cylinder sets are clopen X_B.

Let us adapt the well-known construction of the Vershik map $\sigma : X_B \to X_B$ of the simple case to the case of k-simple Bratteli diagrams. Let $\xi = (\xi^1, \xi^2, \dots) \in X_B$ with each $\xi^i \in E$. If $\xi = z_{i,\max}$ for some $1 \le i \le k$, then define

$$\sigma(\xi) = \sigma(z_{i,\max}) = z_{i,\min}.$$

Otherwise, set

$$d(\xi) = \max\{m;\ (\xi^1, \dots, \xi^{m-1}) \in E_{\max}\},$$

and for any vertex v at level n, set $r_{\min}(v) \in E^{1,n}_{\min}$ the minimal edge with range v. Then define

$$\sigma(\xi)(n) = \begin{cases} r_{\min}(s(\xi^{d(\xi)} + 1)) & \text{if } n < d(\xi), \\ \xi^n + 1 & \text{if } n = d(\xi), \\ \xi^n & \text{if } n \ge d(\xi) + 1. \end{cases}$$

LEMMA 5.9. *The map $\sigma : X_B \to X_B$ is a homeomorphism.*

Proof. Define a map $\tau : X_B \to X_B$ as follows. Let $\xi = (\xi^1, \xi^2, \dots) \in X_B$ with each $\xi^i \in E$. If $\xi = z_{i,\min}$ for some $1 \le i \le k$, then define

$$\sigma(\xi) = \sigma(z_{i,\min}) = z_{i,\max}.$$

Otherwise, set

$$c(\xi) = \max\{m;\ (\xi^1, \dots, \xi^{m-1}) \in E_{\min}\},$$

and for any vertex v at level n, set $r_{\max}(v) \in E^{1,n}_{\max}$ the maximal edge with range v. Then define

$$\tau(\xi)(n) = \begin{cases} r_{\max}(s(\xi^{c(\xi)} + 1)) & \text{if } n < d(\xi), \\ \xi^n - 1 & \text{if } n = d(\xi), \\ \xi^n & \text{if } n \ge d(\xi) + 1. \end{cases}$$

Then it is straightforward to calculate that $\sigma \circ \tau = \tau \circ \sigma = \mathrm{id}$, and thus the map σ is one-to-one and onto.

Since X_B is a compact metrizable space, to show that σ is a homeomorphism, it is enough to show that σ is continuous. It is clear that σ is continuous at each $\xi \in X_B \setminus X_{\max}$. Consider $z_{i,\max}$ and a sequence $\xi_j \to z_{i,\max}$ with $\xi_j \notin X_{\max}$. Let $N \in \mathbb{N}$. Then there is J such that, for any $j > J$,

$$\xi_j(n) = z_{i,\max}(n) \quad \text{for all } 1 \le n \le N.$$

Pick an arbitrary ξ_j with $j > N$, and put

$$M = \max\{n;\ \xi(n') \in E_{\max},\ \forall n' \le n\}.$$

Note that $M \ge N$. If $r(\xi_j(M+1)) \in V_i^{M+2}$, then

$$\sigma(\xi_j)(n) = z_{i,\min}(n) = \sigma(z_{i,\max})(n) \quad \text{for all } 1 \le n \le M - 2.$$

If $r(\xi_j(M+1)) \in V^{M+2} \setminus \bigcup_i V_i^{M+2}$ and $\xi_j(M+1) \in E_{\max}$ then, by condition (3a),

$$\sigma(\xi_j)(n) = z_{i,\min}(n) = \sigma(z_{i,\max})(n) \quad \text{for all } 1 \leq n \leq M-1.$$

If $r(\xi_j(M+1)) \in V^{M+2} \setminus \bigcup_i V_i^{M+2}$ and $\xi_j(M+1) \notin E_{\max}$ then, by condition (3b), we still have that

$$\sigma(\xi_j)(n) = z_{i,\min}(n) = \sigma(z_{i,\max})(n) \quad \text{for all } 1 \leq n \leq M-1.$$

Since $M \geq N$, we have that $\sigma(\xi_j)$ is in the $(N-1)$-neighborhood of $z_{i,\min}$ for any $j \geq J$, and hence the map σ is continuous at $z_{i,\max}$, as desired. □

THEOREM 5.10. *The Bratteli–Vershik system (X_B, σ) has k minimal subsets.*

Proof. For each $1 \leq i \leq k$, denote by Y_i the closed subset corresponding to the paths $\{z; z^n \in V_i^n\}$. It follows from condition (1) that $Y_1, \dots, Y_k$ are closed invariant subsets. By condition (2), the sets $Y_1, \dots, Y_k$ are minimal. Note that $z_{i,\max} \in Y_i$ for each i, and the orbit of $z_{i,\max}$ is dense in Y_i.

Let U be any minimal invariant closed non-empty subset of X_B.

Pick any point $x = (x_1, x_2, \dots) \in U$, and fix $n \in \mathbb{N}$ such that the n-neighborhoods of each $z_{i,\max}$ are pairwise disjoint. The finite path $(x_1, \dots, x_{n+1})$ has finitely many successors in $P_{0,n+1}$. In particular, its ith successor is maximal in $P_{0,n+1}$ for some i, and therefore $\sigma^i(x) = (f_1, \dots, f_{n+1})$ with $(f_1, \dots, f_{n+1})$ maximal. Since the diagram satisfies Lemma 5.5, the finite path $(f_1, \dots, f_n)$ is in $X_{\max}$, and hence in the n-neighborhood of $z_{i,\max}$ for some i.

Therefore, at least one of $\{z_{1,\max}, \dots, z_{k,\max}\}$ is in the closure of the orbit of x, and hence U has to contain one of $\{Y_1, \dots, Y_k\}$. Since U is also minimal, it has to be one of $\{Y_1, \dots, Y_k\}$, as desired. □

Consider the C*-algebra $A_{y_1,\dots,y_k}$, and write $V^n = \{v_1, \dots, v_{|V^n|}\}$. For each v_i, denote by $f_1^i < \cdots < f_{l_i}^i$ the finite paths ending at v_i (they form a totally ordered set). Consider

$$F_n := \bigoplus_{i=1}^{|V^n|} \mathrm{C}^*\{\chi_{f_1^i}, \chi_{f_1^i} u, \dots, \chi_{f_l^i} u^{l_i - 1}\} \subseteq A_{y_1,\dots,y_k}.$$

LEMMA 5.11. *The sub-C*-algebras $\{F_n\}$ have the following properties:*

(1) $F_n \cong \bigoplus_{i=1}^{|V^n|} \mathrm{M}_{l_i}(\mathbb{C})$;

(2) $F_1 \subseteq \cdots \subseteq F_n \subseteq F_{n+1} \subseteq \cdots$;

(3) $\bigcup_n F_n$ *is dense in* $A_{y_1,\dots,y_k}$;

(4) *the* K_0*-map induced by the inclusion* $F_n \subseteq F_{n+1}$ *is the same as the multiplicities between* V^n *and* V^{n+1}.

Proof. Property (1) is clear and property (3) is standard (see, for example, [**27**]). Let us show properties (2) and (4).

Let f be a minimal finite path with end point v. Assume that v is sent to $v_1, \dots, v_l$ with multiplicity $m_1, \dots, m_l$ at level $n+1$. Consider $\chi_f u^l$ with l strict smaller than the number of paths ending at v, and denote the edges between v and v_i by $f_1^i < \cdots < f_{m_i}^i$. Then

$$\begin{aligned}\chi_f u^l &= \bigg(\sum_{i=1}^{l}\sum_{j=1}^{m_i}\chi_{ff_j^i}\bigg)u^l \\ &= \sum_{i=1}^{l}\sum_{j=1}^{m_i}(u^*)^{j-1}\chi_{ff_1^i}u^{j-1}u^l \\ &= \sum_{i=1}^{l}\sum_{j=1}^{m_i}((u^*)^{j-1}\chi_{ff_1^i})(\chi_{ff_1^i}u^{l+j-1}) \in F_{n+1}.\end{aligned}$$

Thus, $F_n \subseteq F_{n+1}$.

Let us calculate the K_0-map. Applying the equation above for $l = 0$, we have

$$[\chi_f] = \sum_{i=1}^{l}\sum_{j=1}^{m_i}[(u^*)^{j-1}\chi_{ff_1^i}\chi_{ff_1^i}u^{j-1}] = \sum_{i=1}^{l} m_i[\chi_{ff_1^i}].$$

Since the standard generators of $\mathrm{K}_0(F_n)$ are $[\chi_f]$, the K_0-map agrees with the multiplicity map between V^n and V^{n+1}, as desired. □

As a straightforward corollary, we have the following result.

THEOREM 5.12. *Denote by* K_B *the dimension group associated with* B. *We then have*

$$\mathrm{K}_0(A_{y_1,\ldots,y_k}) \cong \mathrm{K}_B$$

as ordered groups.

6. *From Cantor systems to Bratteli–Vershik models*

In this section we shall show that any Cantor system (X, σ) with finitely many minimal components can be modeled by the Bratteli–Vershik map on an ordered k-simple Bratteli diagram as introduced in the previous section (see Theorems 6.5 and 6.7 below). Results of this kind, based on sequences of Kakutani–Rokhlin partitions, have been discussed in a number of papers; see, for example, [**4**, **18**, **23**]. Nevertheless, it will be useful for the reader to see a complete proof where all details are clarified.

We first show that if, for a given Cantor system (X, σ), a sequence of Kakutani–Rokhlin partitions satisfies certain conditions, then it determines an ordered Bratteli diagram B. Then we prove that if (X, σ) has k minimal subsets, then B is a k-simple Bratteli diagram, described in §5.

Definition 6.1. A *Kakutani–Rokhlin partition* of (X, σ) consists of pairwise disjoint clopen σ-towers

$$\xi_l := \{Z(l, j);\ 1 \le j \le J(l)\}, \quad 1 \le l \le L,$$

of height $J(l)$ such that:

(1) $Z(l, j) \cap Z(l, j') = \emptyset$, $j \neq j'$;

(2) $\bigcup_{l,j} Z_{l,j} = X$; and

(3) $\sigma(Z(l, j)) = Z(l, j + 1)$ for any $1 \le j < J(l)$.

Remark 6.2. Denote by $Z = \bigcup_{l=1}^{L} Z(l,\, J(l))$. Then

$$\bigcup_{l=1}^{L} Z(l,\, 1) = \sigma(Z).$$

LEMMA 6.3. [**18**, Lemma 4.1] *Let Z be a clopen subset such that $y_i \in Z$ for any $1 \le i \le k$, and let $\mathscr{P}$ be a partition of X into clopen sets. Then there is a Kakutani–Rokhlin partition*

$$\{Z(l,\, j);\ 1 \le l \le L,\ 1 \le j \le J(l)\}$$

of X which is finer than $\mathscr{P}$ and $Z = \bigcup_{l=1}^{L} Z(l,\, J(l))$.

Proof. By Lemma 3.1, we have that $X = \bigcup_{i=-N}^{N} \sigma^i(Z)$ for some N. Applying σ^{-N} on both sides, we have that

$$X = \bigcup_{i=0}^{2N} \sigma^{-i}(Z). \tag{6.1}$$

For each $x \in Z$, define

$$r(x) = \min\{i \ge 1;\ \sigma^i(x) \in Z\}.$$

By (6.1), the map $r : Z \to \mathbb{N}$ is well defined. Moreover, the map r is continuous. Write

$$r(Z) = \{J(1),\, J(2),\, \ldots,\, J(L)\},$$

and define

$$Z(l,\, j) = \sigma^j(r^{-1}(J(l))).$$

It is clear that the sets $\{Z(l,\, j)\}$ are clopen and satisfy condition (3). We show that they form a partition of X. If $x \in Z(l_1,\, j_1) \cap Z(l_2,\, j_2)$, then there are $y_1 \in r^{-1}(J(l_1))$ and $y_2 \in r^{-1}(J(l_2))$ such that

$$\sigma^{j_1}(y_1) = \sigma^{j_2}(y_2).$$

If $j_1 > j_2$, then

$$\sigma^{j_1 - j_2}(y_1) = y_2 \in r^{-1}(J(l_2)) \subseteq Z,$$

which contradicts $y_1 \in r^{-1}(J(l_1))$. If $j_1 < j_2$, the same argument leads to a contradiction. If $j_1 = j_2$, then we have that $y_1 = y_2$ and hence $l_1 = l_2$. Thus, the collection $\{Z(l,\, j)\}$ consists of pairwise disjoint sets.

For any $x \in X \setminus Z$, consider

$$n = \min\{i \ge 0;\ \sigma^{-i}(x) \in Z\} \quad \text{and} \quad m = \min\{i \ge 0;\ \sigma^i(x) \in Z\}.$$

By Lemma 3.1, such n and m exist. Note that $m,\, n \ge 1$ and

$$\sigma^{-n}(x) \in r^{-1}(m + n + 1).$$

Therefore, $x \in Z(l,\, n)$ with $J(l) = m + n + 1$.

For any $x \in Z$, consider

$$n = \min\{i \ge 1;\ \sigma^{-i}(x) \in Z\}.$$

Then we have that $r(\sigma^{-n}(x)) = n$ and

$$x \in Z(l,\, J(l)) \tag{6.2}$$

for l with $J(l) = n$. Hence $\{Z(l, j)\}$ is a partition of X, and it actually forms a Kakutani–Rokhlin partition of X with respect to σ.

It is clear that $\bigcup_l Z(l, J(l)) \subseteq Z$ for any $1 \le l \le L$ by the construction. On the other hand, it follows from (6.2) that $\bigcup_l Z(l, J(l)) \supseteq Z$, and hence we have

$$\bigcup_l Z(l, J(l)) = Z.$$

Once we have a Kakutani–Rokhlin partition of X with respect to σ, a similar argument to that of [**28**, Lemma 3.1] shows that $\{Z(l, j)\}$ can always be modified further so that $\{Z(l, j)\}$ is finer than the given partition $\mathscr{P}$. □

THEOREM 6.4. [**18**, Theorem 4.2] *There are Kakutani–Rokhlin partitions of X,*

$$\mathscr{P}_n = \{Z(n, l, j);\ 1 \le l \le L(n),\ 1 \le j \le J(n, l)\},$$

such that:
(1) *the sequence* $(Z_n := \bigcup_{l=1}^{L(n)} Z(n, l, J(n, l)))$ *is a decreasing sequence of clopen sets with intersection* $\{y_1, y_2, \dots, y_k\}$*, where the points* $\{y_1, y_2, \dots, y_k\}$ *are chosen in minimal components* $Y_1, \dots, Y_k$*, respectively;*
(2) *the partition* $\mathscr{P}_{n+1}$ *is finer than the partition* $\mathscr{P}_n$*; and*
(3) $\bigcup_n \mathscr{P}_n$ *generates the topology of X.*

Proof. Choose a sequence of clopen sets $Z_1 \supseteq Z_2 \supseteq \cdots$ with $\bigcap_n Z_n = \{y_1, y_2, \dots, y_k\}$, and a sequence of finite partitions $(\mathscr{P}'_n)$ such that $\cup\mathscr{P}'_n$ generates the topology of X. By Lemma 6.3, there is a Kakutani–Rokhlin partition $\mathscr{P}_1 := \{Z(1, l, j);\ 1 \le l \le L(1),\ 1 \le j \le J(1, l)\}$ such that $\bigcup_l Z(1, l, J(1, l)) = Z_1$ and $\mathscr{P}_1$ is finer than $\mathscr{P}'_1$.

Assume that the Kakutani–Rokhlin partitions $\mathscr{P}_1, \dots, \mathscr{P}_{n-1}$ are constructed. Then, by Lemma 6.3, there is a Kakutani–Rokhlin partition $\mathscr{P}_n := \{Z(n, l, j);\ 1 \le l \le L(n),\ 1 \le j \le J(n, l)\}$ such that $\bigcup_l Z(n, l, J(n, l)) = Z_n$ and $\mathscr{P}_n$ is finer than $\mathscr{P}'_n \vee \mathscr{P}_{n-1}$. Then $(\mathscr{P}_n)$ is the desired sequence of Kakutani–Rokhlin partitions. □

Based on the sequence of Kakutani–Rokhlin partitions, e cab construct an ordered Bratteli diagram $B = (V, E, >)$ following the procedure described in [**18**, §4].

For convenience, we may assume that $L(0) = 1$, $J(0, 1) = 1$, and $Z(0, 1, 1) = X$. The set of vertices V^n of the diagram B is formed by the towers in the Kakutani–Rokhlin partition $\mathscr{P}_n$. To define the set of edges, we say that there is an edge between a tower (vertex) ξ_i^{n-1} at level $n-1$ and a tower (vertex) ξ_j^n at level n if ξ_j^n pass through ξ_i^{n-1}. More precisely, for each n, we have

$$V^n = \{(n, 1), (n, 2), \dots, (n, L(n))\}$$

and

$$E^n = \{(n, l, l', j') | Z(n, l', j' + j) \subseteq Z(n-1, l, j)\ \forall j = 1, \dots, J(n-1, l)\}.$$

Then the source and range maps are

$$s((n, l, l', j')) = (n-1, l)$$

and

$$r((n, l, l', j')) = (n, l').$$

The order on the edges comes from the natural order on each tower $\{Z(n, l, j);\ j = 1, \dots, J(n, l)\}$. That is,

$$(n, l_1, l', j_1') > (n, l_2, l', j_2') \quad \text{if and only if } j_1' > j_2'.$$

THEOREM 6.5. *Let (X, σ) be a Cantor dynamical system with k minimal sets $Y_1, \dots, Y_k$. The ordered Bratteli diagram $B = (V, E, >)$, constructed as above, is non-elementary, and satisfies conditions (1)–(3) of Definition 5.3, that is, B is a non-elementary k-simple ordered Bratteli diagram.*

Proof. For each n and $1 \le i \le k$, set

$$V_i^n := \{(n, l);\ Y_i \cap Z(n, l, j) \neq \emptyset \text{ for some } 1 \le j \le J(n, l)\}.$$

Note that since Y_i is invariant,

$$V_i^n = \{(n, l);\ \ Y_i \cap Z(n, l, j) \neq \emptyset\ \forall 1 \le j \le J(n, l)\}.$$

Since Y_i are pairwise disjoint, by choosing n sufficiently large, we have that $\{V_1^n, V_2^n, \dots, V_k^n\}$ are pairwise disjoint.

Consider an edge (n, l, l', j') with its range $(n, l') \in V_i^n$. Then $Z(n, l', j) \cap Y_i \neq \emptyset$ for all $1 \le j \le J(n, l')$, and thus

$$Z(n-1, l, j) \cap Y_i \neq \emptyset$$

for all $1 \le j \le J(n-1, l)$. That is, the vertex $(n-1, l)$ is in V_i^{n-1}. Hence the Bratteli diagram satisfies Condition (1) of Definition 5.1.

Note that the restriction of σ to Y_i is minimal. Thus, Condition (2) of Definition 5.1 also holds. That is, the unordered diagram (V, E) is k-simple, and this verified condition (1) of 5.3.

Let $((n, l_n, l_n', j_n'))$ be an infinite path in $E_{\min}$. Since it is an infinite path, we have that $l_{n-1}' = l_n$. Since (n, l_n, l_n', j_n') is minimal, we have that $j_n' = 0$. Hence $Z(n, l_n, 1) \subseteq Z(n-1, l_{n-1}, 1)$ for any n, and thus

$$\bigcap_{n=1}^{\infty} Z(n, l_n, 1) \subseteq \bigcap_{n=1}^{\infty}\left(\bigcup_{l=1}^{L(n)} Z(n, l, 1)\right) = \{\sigma(y_1), \sigma(y_2), \dots, \sigma(y_k)\}.$$

By the construction of $Z(n, l, j)$, this is finer than any given partition $\mathscr{P}'$. Therefore the intersection $\bigcap_{n=1}^{\infty} Z(n, l_n, 1)$ is a single point, and there is $1 \le i \le k$ such that

$$\bigcap_{n=1}^{\infty} Z(n, l_n, 1) = \sigma(y_i).$$

This uniquely determines l_n, and thus there are at most k infinite minimal paths. On the other hand, for each y_i, the infinite sequence $((Z(n, l_n, 1)))$ with $\sigma(y_i) \in Z(n, l_n, 1)$ clearly forms a minimal path. Thus, there are k minimal paths.

A similar argument shows that there are also k maximum paths. Therefore, the ordered Bratteli diagram $(V, E, >)$ satisfies condition (2) of Definition 5.3.

For each $1 \le i \le k$ and each $n \ge 1$, define

$$N_{n,i} = \bigcup_{(n,l)\in V_i^n} \bigcup_{j=1}^{J(n,l)} Z(n, l, j).$$

Then each $N_{n,i}$ is a clopen neighborhood of Y_i, and $N_{n,1}, \dots, N_{n,k}$ are pairwise disjoint.

Consider a vertex (a tower) $(n, l) \in V_n \setminus (V_1^n \cup \cdots \cup V_k^n)$. Note that if $n \ge 2$, by condition (1) of Theorem 6.4, there is one and only one $1 \le i \le k$ and one and only one $1 \le j \le k$ such that

$$Z(n, l, 1) \subseteq N_{n-1,i} \quad \text{and} \quad Z(n, l, J(n, l)) \subseteq N_{n-1,j}.$$

Then, by the definition,

$$m_-((n, l)) = i \quad \text{and} \quad m_+((n, l)) = j.$$

Let $(n + 1, l, l', j')$ be an edge such that

$$Z(n + 1, l', j' + j) \subseteq Z(n, l, j), \quad j = 1, \dots, J(n, l).$$

Then

$$s((n + 1, l, l', j')) = (n, l).$$

Note that there is $Z(n, l'')$ such that

$$Z(n + 1, l', j' + J(n, l) + j) \subseteq Z(n, l'', j), \quad j = 1, \dots, J(n, l').$$

Then

$$(n + 1, l, l', j') + 1 = (n + 1, l'', l, j' + J(n, l))$$

and

$$s((n + 1, l, l', j') + 1) = (n, l'').$$

If

$$Z(n, l, J(n, l)) \subseteq N_{n-1,s} \quad \text{and} \quad Z(n, l, J(n, l) + 1) \subseteq N_{n-1,t},$$

for some $1 \le s, t \le k$, then

$$Z(n + 1, l', j' + J(n + 1)) \subseteq N_{n-1,s} \quad \text{and} \quad Z(n + 1, l', j' + J(n + 1) + 1) \subseteq N_{n-1,t}.$$

But

$$Z(n + 1, l', j' + J(n + 1) + 1) = \sigma(Z(n + 1, l', j' + J(n + 1) + 1)),$$

and this forces $s = t$, that is,

$$m_-(s((n + 1, l, l', j') + 1)) = m_-((n, l'')) = t = s = m_+((n, l)).$$

This shows condition (3a) of Definition 5.3.

Condition (3b) of Definition 5.3 can also be verified in a similar way. □

Applying the same argument as that of [**18**, Theorem 4.4], we can prove the following statement.

THEOREM 6.6. *Given (X, σ) and the points $y_1 \in Y_1, \dots, y_k \in Y_k$, the equivalent class of the ordered Bratteli diagram B constructed in Theorem 6.5 does not depend on the choice of Kakutani–Rohklin partitions.*

THEOREM 6.7. *There is a one-to-one correspondence between the equivalence classes of non-elementary k-simple ordered Bratteli diagrams and the pointed topological conjugacy classes of Cantor systems with k minimal invariant subsets.*

7. *Transition graphs of a k-simple ordered Bratteli diagram and the index maps*

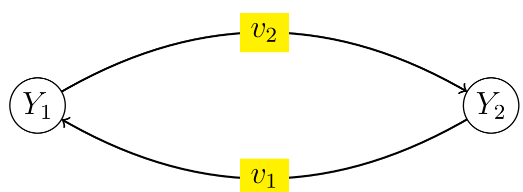
7.1. *Transition graphs associated to Bratteli diagrams.* Consider a k-simple ordered Bratteli diagram $B = (V, E, >)$. Condition (3) of Definition 5.3 is necessary for the continuity of the associated Bratteli–Vershik map. Based on this continuity condition, we will introduce a sequence of directed graphs for B, and it turns out that these graphs are closely related to the index map of the short exact sequence (3.1) associated to the Bratteli–Vershik system (X_B, σ) (see Theorem 7.5). Moreover, we will be able to obtain certain combinatorial properties of these graphs and the Bratteli diagram B from the information on the index map.

In the next definition, we use the word 'vertex' for a Bratteli diagram and for a transition graph. It should be clear from the context which vertex is considered.

Definition 7.1. Let $B = (V, E, >)$ be a k-simple ordered Bratteli diagram. For each level $n \geq 2$, define the *transition graph* L_n to be the following directed graph: the vertices of L_n correspond to the minimal sets $Y_1, \ldots, Y_k$, and the edges are labeled by the vertices from the set V_o^n. For each $v \in V_o^n$, the edge v starts from Y_i and ends at Y_j if and only if

$$m_-(v) = i \quad \text{and} \quad m_+(v) = j.$$

Example 7.2. For the diagram considered in Example 5.8, we obtain the following transition graph at level n:



LEMMA 7.3. *Let $v \in V_o^n = V^n \setminus \bigcup_{i=1}^k V_i^n$ and let $e \in E_{\min}$ with $r(e) = v$ and $s(e) \in V_i^{n-1}$. Take $v' \in V^{n+1}$ and $e' \in E$ with $r(e') = v'$ and $s(e') = v$. Then, for any $e'' \in E$ with $r(e'') = v'$ and $e' < e''$, there is a path $(v_1, \ldots, v_l)$ in L_n with $v_1 = v$ which starts at Y_i and ends at $Y_{m_-(s(e''))}$.*

Proof. Since $e' < e''$, there are

$$\{\underbrace{\{e_1^{(1)}, \ldots, e_{m_1}^{(1)}\}}_{G_1}, \underbrace{\{e_1^{(2)}, \ldots, e_{m_2}^{(2)}\}}_{G_2}, \ldots, \underbrace{\{e_1^{(d)}, \ldots, e_{m_d}^{(d)}\}}_{G_d}, e'''\}$$

with each $e_j^{(s)}$ an edge between level $n-1$ and level n such that:

(1) each group G_s consists of all edges with range $v_s \in V^n$;

(2) inside each group G_s, we have

$$e_1^{(s)} < e_2^{(s)} (= e_1^{(s)} + 1) < \cdots < e_{m_s}^{(s)} (= e_{m_s - 1}^{(s)} + 1);$$

(3) $e_1^{(1)} = e$, $e''' \in E_{\min}$ and $r(e''') = s(e'')$;

(4) there are edges $g_1, \ldots, g_s$ connecting v_s to v' respectively such that

$$e' = g_1 < g_2 (= g_1 + 1) < g_3 (= g_2 + 1) < \cdots < g_d (= g_{d-1} + 1) < g_d + 1 = e''.$$

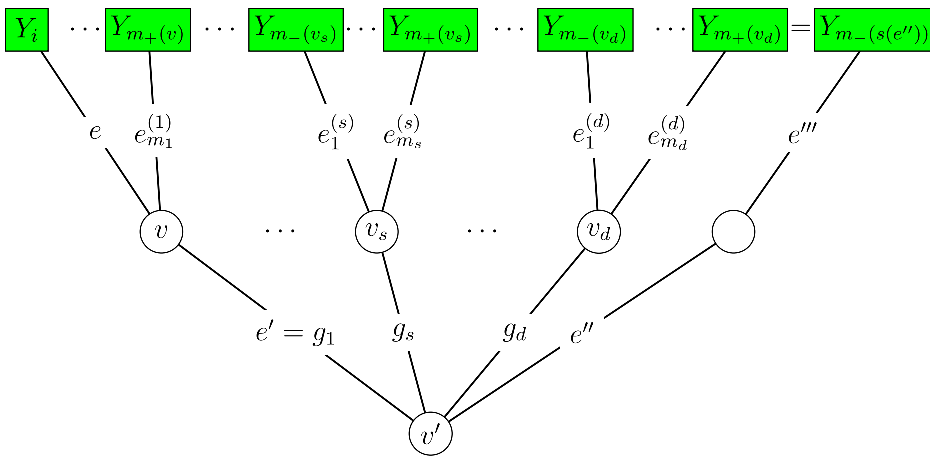
Then each group G_s represents an edge in L_n if $v_s \in V^n \setminus \bigcup_{j=1}^k V_j^n$. Moreover, if $v_s \in V_j^n$ for some j, by condition (3b) of Definition 5.3, then $m_-(v_{s+1}) = j$. Thus, if one deletes these $v_s \in V_j^n$ for some j, the remaining vertices induce a path in L_n.

It is clear that the path starts with Y_i. The end point of the path is $Y_{m_+(v_d)}$. Then it follows from condition (3a) of Definition 5.3 and condition (4) above that

$$Y_{m_+(v_d)} = Y_{m_-(s(g_d+1))} = Y_{m_-(s(e''))},$$

as desired.

The argument can be illustrated by the following diagram:



□

COROLLARY 7.4. *Let $B = (V, E, >)$ be a k-simple non-elementary ordered Bratteli diagram with $k \geq 2$, and let L_n denote the transition graph of B at level n. If there is an edge v_1 that has the vertex Y_i as the source point, then there is a closed walk $(v_1, \ldots, v_l)$ in L_n (so the range point of v_l is Y_i).*

Proof. Without loss of generality, assume that the vertex Y_1 is a source point. Then there are $v \in V^n \setminus \bigcup_{i=1}^k V_i^n$ and $e \in E_{\min}$ such that $r(e) = v$ and $s(e) \in V_1^{n-1}$. Since the diagram is non-elementary, by condition (2) of Definition 5.1, there is $v' \in V^{n+1}$ such that

$$|E_{v,v'} := \{e' \in E;\, s(e') = v,\, r(e') = v'\}| \geq 2.$$

Pick $e', e'' \in E_{v,v'}$ with $e' < e''$. Then, by Lemma 7.3, there is a path in L_n staring at Y_1 and ending at $Y_{m_-(v)} = Y_1$, and it is the desired loop. □

7.2. *Index maps and transition graphs.* In this subsection we shall give a description of the images of the index maps (see Theorem 3.10) using the transition graphs.

Consider a k-simple non-elementary ordered Bratteli diagram $B = (V, E, >)$, and consider the associated Cantor system (X_B, σ). Note that the ideal $C_0(X_B \setminus \bigcup_{i=1}^k Y_i) \rtimes \mathbb{Z}$ has an AF-structure arising naturally from the sub-diagram of B with vertices $\{V_o^n : n = 1, 2, \ldots\}$. In particular, its K_0-group is naturally isomorphic to the dimension group of the sub-diagram of B with vertices $\{V_o^n : n = 1, 2, \ldots\}$. With this identification, we have the following theorem.

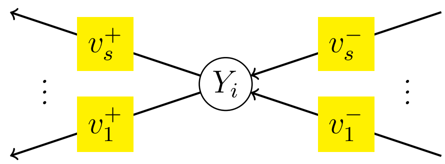
THEOREM 7.5. *Let Y_i be a minimal component of (X_B, σ), and let L_n be the transition graph of B at level n. Denote by $E_+(Y_i)$ the set of edges of L_n which have Y_i as source, and denote by $E_-(Y_i)$ the set of edges of L_n which have Y_i as range. Write*

$$E_+(Y_i) = \{v_1^+, \dots, v_s^+\}$$

and

$$E_-(Y_i) = \{v_1^-, \dots, v_t^-\}.$$

That is,



Let d_i be the element of Theorem 3.10 associated to Y_i. Then d_i is given by

$$(e_{v_1^+} + \cdots + e_{v_s^+}) - (e_{v_1^-} + \cdots + e_{v_t^-}),$$

where e_v stands for $(0, \dots, 0, 1, 0, \dots, 0) \in \bigoplus_{V_o^n} \mathbb{Z}$ with entry 1 at the position v.

Proof. Consider the set $U \subseteq X_B$ consisting of all infinite paths which are in Y_i up to level $n-1$. It is clear that U is a clopen set containing Y_i. By Theorem 3.10, the element d_i is given by

$$[\chi_U - \chi_U \circ \sigma^{-1}]_0 = [\chi_U - \chi_{\sigma(U)}]_0.$$

Denote by $v_1, \dots, v_h$ the vertices in V_o^n connecting to Y_i; and for each $1 \le l \le h$, define

$$E_l := \{e \in E;\ r(e) = v_l,\ s(e) \in V_i^{n-1}\}$$

and

$$\{f_1^{(l)}, \dots, f_{m_l}^{(l)}\} := \{e \in E_l;\ e \notin E_{\max},\ s(e+1) \notin V_i^{n-1}\},$$
$$\{g_1^{(l)}, \dots, g_{r_l}^{(l)}\} := \{e \in E_l;\ e \notin E_{\min},\ s(e-1) \notin V_i^{n-1}\}.$$

Since the number of edges jumping into E_l is the same as the number of edges being pushed out of E_l, we have

$$\begin{cases} m_l = r_l & \text{if } |(E_l \cap E_{\max}) \cup (E_l \cap E_{\min})| \in \{0, 2\}, \\ m_l + 1 = r_l & \text{if } (E_l \cap E_{\max}) \neq \emptyset \text{ but } E_l \cap E_{\min} = \emptyset, \\ m_l = r_l + 1 & \text{if } (E_l \cap E_{\max}) = \emptyset \text{ but } E_l \cap E_{\min} \neq \emptyset. \end{cases} \tag{7.1}$$

For each $e \in E_l$, define $\chi_-(e)$ to be the cylinder set consisting all infinite paths starting with we with w the minimal finite path ending at $s(e)$. Note that, for any $e_1, e_2 \in E_l$,

$$e_{v_l} := [\chi_-(e_1)]_0 = [\chi_-(e_2)]_0 = (0, \dots, 0, 1, 0 \dots, 0) \in \bigoplus_{V_o^n} \mathbb{Z} \subset \mathrm{K}_0(I), \tag{7.2}$$

where the 1 is in position v_l.

We then assert that

$$\chi_U - \chi_{\sigma(U)} = \sum_{l=1}^{h}(\chi_-(g_1^{(l)}) + \cdots + \chi_-(g_{r_l}^{(l)})) - \sum_{l=1}^{h}(\chi_-(f_1^{(l)}+1) + \cdots + \chi_-(f_{m_l}^{(l)}+1)). \tag{7.3}$$

In fact,

$$\chi_U - \chi_{\sigma(U)} = \chi_{U\setminus\sigma(U)} - \chi_{\sigma(U)\setminus U}.$$

Then the assertion follows from the equations

$$\chi_{U\setminus\sigma(U)} = \sum_{l=1}^{h}(\chi_-(g_1^{(l)}) + \cdots + \chi_-(g_{r_l}^{(l)}))$$

and

$$\chi_{\sigma(U)\setminus U} = \sum_{l=1}^{h}(\chi_-(f_1^{(l)}+1) + \cdots + \chi_-(f_{m_l}^{(l)}+1)),$$

which can be verified straightforwardly.

Then, by equations (7.1) and (7.2), we have

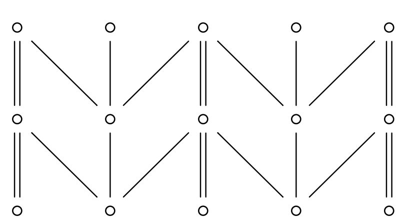
$$\begin{aligned}
&[\chi_U - \chi_{\sigma(U)}]_0 \\
&= \sum_{l=1}^{h}[\chi_-(g_1^{(l)}) + \cdots + \chi_-(g_{r_l}^{(l)})]_0 - \sum_{l=1}^{h}[\chi_-(f_1^{(l)}+1) + \cdots + \chi_-(f_{m_l}^{(l)}+1)]_0 \\
&= \sum_{l=1}^{h}([\chi_-(g_1^{(l)})]_0 + \cdots + [\chi_-(g_{r_l}^{(l)})]_0) - ([\chi_-(f_1^{(l)})]_0 + \cdots + [\chi_-(f_{m_l}^{(l)})]_0) \\
&= \sum_{l=1}^{h}(r_l - m_l)e_{v_l} = (e_{v_1^+} + \cdots + e_{v_s^+}) - (e_{v_1^-} + \cdots + e_{v_t^-}),
\end{aligned}$$

as desired. □

COROLLARY 7.6. *Let $B = (V, E, >)$ be a k-simple ordered Bratteli diagram with $k \geq 2$. Then each transition graph L_n is connected. In particular, L_n has at least $k-1$ edges.*

Proof. If there were a proper connected component of L_n, say, consisting of $\{Y_{n_1}, \ldots, Y_{n_s}\}$, then, by Theorem 7.5, we would have $d_{n_1} + \cdots + d_{n_s} = 0$. But this contradicts the conclusion of Theorem 3.10. □

In general, $k-1$ can be attained. For example, consider the stationary diagram



This diagram can be easily ordered so that it becomes an ordered 3-simple Bratteli diagram. Its transition graph at each level has two edges.

However, if the Bratteli diagram is non-elementary, that is, if the path space is a Cantor set, then L_n has at least k edges.

COROLLARY 7.7. *Let $B = (V, E, >)$ be a non-elementary k-simple ordered Bratteli diagram with $k \geq 2$. The transition graph L_n has at least k edges. In particular, we have that*

$$|V_o^n| = \left| V_n \setminus \bigcup_{i=1}^{k} V_i^n \right| \geq k,$$

for all n.

Proof. It follows from Corollary 7.6 that the transition graph L_n is connected, and it follows from Corollary 7.4 that L_n contains loops. Since L_n has k vertices, it must have at least k edges, as desired. □

Definition 7.8. Let G be a dimension group with a given inductive limit decomposition $G = \varinjlim \mathbb{Z}^{n_i}$. Define $\mathrm{D}(G)$ to be the subgroup consisting of the elements g such that, with $g = (g_i)$, where $g_i \in \mathbb{Z}^{n_i}$, there is $m \in \mathbb{N}$ such that

$$\|g_i\|_\infty \leq m, \quad i = 1, 2, \ldots,$$

where $\|\cdot\|_\infty$ is the standard ℓ^∞-norm of $\mathbb{Z}^{n_i}$.

It is straightforward to show that $\mathrm{D}(G) \cap G^+ = \{0\}$ if the dimension group G has no quotient isomorphic to $\mathbb{Z}$. Also note that if G is a non-cyclic simple dimension group, then any element of $\mathrm{D}(G)$ is an infinitesimal (an element g of a simple dimension group G is said to be an infinitesimal if $-h < mg < h$ for all $m \in \mathbb{N}$ and $h \in G^+ \setminus \{0\}$).

Recall that K_{I_B} is the dimension group of the sub-diagram of B restricted to the vertices V_o^n, $n = 1, 2, \ldots$; hence it is isomorphic to $\mathrm{K}_0(\mathrm{C}_0(X \setminus \bigcup_{i=1}^k Y_i) \rtimes \mathbb{Z})$.

COROLLARY 7.9. *If B is a non-elementary ordered Bratteli diagram, then*

$$\mathrm{Image}(\mathrm{Ind}) \subseteq \mathrm{D}(\mathrm{K}_{I_B})$$

with respect to the inductive limit decomposition of K_{I_B} given by B. In particular,

$$\mathrm{Image}(\mathrm{Ind}) \cap (\mathrm{K}_{I_B})^+ = \{0\}.$$

Moreover, if B is assumed to be strongly k-simple (so the ideal K_{I_B} is simple), then the image of the index map is in the subgroup of K_{I_B} of infinitesimals.

Proof. Since the image of the index map is generated by $\{d_1, \ldots, d_k\}$, it is enough to show that

$$d_i \in \mathrm{D}(\mathrm{K}_{I_B}), \quad i = 1, 2, \ldots, k.$$

But this follows directly from Theorem 7.5, which states that each entry of d_i must be 0, -1, or 1, at any level, as desired. □

The following result can be regarded as a strengthened version of Corollary 7.7.

COROLLARY 7.10. *Denote by r the $\mathbb{Q}$-rank of K_{I_B}. Then $r \geq k$.*

Proof. Denote by $H \subseteq \mathrm{K}_{I_B}$ the image of the index map. By equation (3.3), we have that $\dim_{\mathbb{Q}}(H \otimes \mathbb{Q}) = k - 1$. Note that the dimension group K_{I_B} must contain non-zero positive elements; so pick $p \in \mathrm{K}_{I_B}$ positive and non-zero. We assert that $p \otimes 1_{\mathbb{Q}} \notin H \otimes \mathbb{Q}$. If this were not true, there would be natural numbers m, n and $h \in H$ such that $mp = nh$. Since p is positive, it follows from Corollary 7.9 that $mp = 0$. Since the dimension group K_{I_B} is torsion-free, we have that $p = 0$, which contradicts the choice of p. Therefore $\mathrm{K}_{I_B} \otimes \mathbb{Q} \supsetneq H \otimes \mathbb{Q}$, and hence $r = \dim_{\mathbb{Q}}(\mathrm{K}_{I_B} \otimes \mathbb{Q}) \geq k - 1 + 1 = k$. □

The authors thank the referee for suggesting Pimsner's dynamical criterion for the stable finiteness in the following corollary.

COROLLARY 7.11. *Let (X, σ) be a indecomposable Cantor system with k minimal subsets. Then the C*-algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is stably finite. It has stable rank 2 if $k \geq 2$, and stable rank 1 if $k = 1$. Moreover, if (X, σ) is aperiodic, the C*-algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ has real rank 0.*

Proof. Let us first consider the real rank of $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$. Note that there is a short exact sequence

$$0 \longrightarrow \mathrm{C}_0\Big(X \Big\backslash \bigcup_i Y_i\Big) \rtimes_\sigma \mathbb{Z} \longrightarrow \mathrm{C}(X) \rtimes_\sigma \mathbb{Z} \longrightarrow \bigoplus_i \mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z} \longrightarrow 0. \tag{7.4}$$

Since (X, σ) is assumed to be aperiodic, each minimal component Y_i is homeomorphic to a Cantor set, and therefore the quotient algebra $\bigoplus_i \mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z}$ is an A$\mathbb{T}$ algebra with real rank 0. On the other hand, the ideal $\mathrm{C}_0(X \setminus \bigcup_i Y_i) \rtimes_\sigma \mathbb{Z}$ is AF, so it also has real rank 0. Since $\mathrm{K}_1(\mathrm{C}_0(X \setminus \bigcup_i Y_i) \rtimes_\sigma \mathbb{Z}) = \{0\}$, we have that the exponential map

$$\mathrm{K}_0\Big(\bigoplus_i \mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z}\Big) \to \mathrm{K}_1\Big(\mathrm{C}_0\Big(X \Big\backslash \bigcup_i Y_i\Big) \rtimes_\sigma \mathbb{Z}\Big)$$

of the extension above must be zero. Hence, as an extension of two real rank-0 C*-algebras with zero exponential map, the algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ has real rank 0 (see, for example, [**22**, Proposition 4(i)]).

For the stable rank of $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$, if $k = 1$, it follows from Corollary 3.11 that $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is an A$\mathbb{T}$ algebra, and, in particular, it has stable rank 1. If $k \geq 2$, it follows from [**26**, Corollary 2.6] that $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ has stable rank 2. (Indeed, since both the ideal and the quotient algebras in the extension (7.4) have stable rank 1, it follows from [**30**, Corollary 4.12] that the stable rank of $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is either 1 or 2. Since the index map is non-zero when $k \geq 2$, the stable rank of $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ cannot be 1; see, for instance, [**22**, Proposition 4(ii)].)

Let us show that $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is always stably finite, and let us prove the following general statement instead. Consider an extension of C*-algebras

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\ \pi\ } D \longrightarrow 0$$

with A and D unital. Assume that D is stably finite, I has the property that $[p]_0 \neq 0 \in \mathrm{K}_0(I)$ for any non-zero projection $p \in I$, and

$$\mathrm{Ind}(\mathrm{K}_1(D)) \cap \mathrm{K}_0^+(I) = \{0\}. \tag{7.5}$$

Then A is stably finite.

We only have to show that A is finite (for matrix algebras over A, one can tensor the extension above with a matrix algebra, and proceed with the same argument). Let v be an isometry in A. Since D is finite, the image $\pi(v)$ has to be an unitary. Then

$$\mathrm{Ind}(-[\pi(v)]_1) = [1_A - vv^*]_0 - [1_A - v^*v]_0 = [1_A - vv^*]_0 \in \mathrm{K}_0^+(I).$$

Therefore $[1 - vv^*]_0 = 0$ and hence $1 - vv^* = 0$ (since $1 - vv^*$ is a projection on A). So v must be a unitary, and A is finite.

Consider the extension (7.4). Note that, by Theorem 6.7, one may assume that the Cantor system arises from a non-elementary k-simple ordered Bratteli diagram. It follows from Corollary 7.9 that equation (7.5) always holds, and I is an AF-algebra. Hence the statement follows.

Alternatively, one also can use Pimsner's dynamical criterion for the stable finiteness. Note that the C*-algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is $\mathrm{A}\mathbb{T}$ and hence stably finite if $k = 1$ (Corollary 3.11). So, let us assume that $k \geq 2$, and let us show that every point of X is pseudoperiodic, that is, for any $x_0 \in X$ and any $\varepsilon > 0$, there exist $x_1, x_2, \ldots, x_{n-1}$ such that

$$\mathrm{dist}(x_{i+1}, \sigma(x_i)) < \varepsilon, \quad i = 0, 1, \ldots, n-1,$$

where dist is a compatible metric on X, and x_n is understood as x_0. Then the stable finiteness follows from [**25**, Theorem 9].

The pseudoperiodicity indeed follows from Corollary 7.6: if $x_0 \in Y_i$ for sone $i = 1, \ldots, k$, then the pseudoperiodicity follows from the minimality of Y_i; therefore one may assume that $x_0 \in X \setminus \bigcup_{i=1}^k Y_i$. With the Bratteli–Vershik model, let x_0 be represented by the infinite path $[e_1, e_1, \ldots, e_d, e_{d+1}, \ldots]$, and let d be sufficiently large such that any two paths with same first d segments actually have distance at most ε.

Consider the vertex $r(e_d)$, which is at level $d+1$. Since $x_0 \in X \setminus \bigcup_{i=1}^k Y_i$, one may assume that d is sufficiently large that $r(e_d) \in V_o^{d+1}$. Consider the minimal edge starting with $r(e_d)$ backwards, and denote it by $[e_1', e_2', \ldots, e_d']$. It is clear that x_0 is in the forward orbit of the infinite path $[e_1', \ldots, e_d', e_{d+1}, \ldots]$. By Lemma 5.5, one may assume that $s(e_d')$ is in the sub-diagram $B_{m_-(r(e_d))}$ (and so the finite minimal path $[e_1', e_2', \ldots, e_{d-1}']$ is in the sub-diagram $B_{m_-(r(e_d))}$). Pick an arbitrary infinite path y in the sub-diagram $B_{m_-(r(e_d))}$ which starts with $[e_1', e_2', \ldots, e_{d-1}']$. Note that

$$\mathrm{dist}(y, [e_1', \ldots, e_{d-1}', e_d', e_{d+1}, \ldots]) < \varepsilon.$$

Consider the minimal set $Y_{m_-(r(e_d))}$. By Corollary 7.6, there is a closed walk $(r(e_d), v_2, \ldots, v_l)$ in the transition graph L_{d+1}, where $v_i \in V_o^{d+1}$, $i = 2, \ldots, l$, and $m_+(v_l) = m_-(r(e_d))$. Then this loop provides a partial orbit $x_1, x_2, \ldots, x_n$, where each x_i is an infinite path of the Bratteli diagram such that $x_{i+1} = \sigma(x_i)$, $i = 0, \ldots, n-1$, and $\sigma(x_n) = (e_1'', \ldots, e_{d-1}'', e_d'', \ldots)$, where $e_1'', e_2'', \ldots, e_{d-1}''$ are minimal edges in the

Bratteli sub-diagram $B_{m_-(r(e_d))}$. Pick an arbitrary infinite path z_0 in the Bratteli sub-diagram $B_{m_-(r(e_d))}$ which starts with $[e''_1, e''_2, \dots, e''_{d-1}]$, and note that

$$\mathrm{dist}(z_0, \sigma(x_n)) = \mathrm{dist}(z, [e''_1, \dots, e''_{d-1}, e''_d, \dots]) < \varepsilon.$$

Since the set $Y_{m_-(r(e_d))}$ is minimal (so the forward orbit is dense), there are $z_1, z_2, \dots, z_{n_1} \in Y_{m_-(r(e_d))}$ such that $z_{i+1} = \sigma(z_i)$, $i = 0, \dots, n_1 - 1$, and $\mathrm{dist}(\sigma(z_{n_1}, y)) < \varepsilon$. Therefore, the finite sequence

$$x_0, x_1, x_2, \dots, x_n, z_0, z_1, \dots, z_{n_1}, y, [e'_1, \dots, e'_{d-1}, e'_d, e_{d+1}, \dots], \dots, x_0$$

is the desired pseudoperiodic orbit. □

8. *Realizability of a Bratteli diagram*

Let $B = (V, E)$ be an unordered k-simple Bratteli diagram. In this section let us consider the question when there is an order $>$ on B so that $(V, E, >)$ is an ordered k-simple Bratteli diagram.

Suppose that there is an order $>$ on B so that $(V, E, >)$ is k-simple. Without loss of generality, let us also assume that $(V, E, >)$ satisfies the conditions of Lemma 5.6.

Denote by L_n the transition graphs of $(V, E, >)$ at level n. For any edge v of the transition graph L_n, denote by $Y_{\min(v)}$ the source point of v and denote by $Y_{\max(v)}$ the range point of v.

THEOREM 8.1. *Consider the k-simple ordered Bratteli diagram $(V, E, >)$. The transition graphs $\{L_n; n = 2, \dots\}$ are compatible with the unordered Bratteli diagram (V, E) in the following sense. For any edge w of L_{n+1}, there is a path $(v_1, v_2, \dots, v_l)$ in L_n such that:*

(1) *the edge w and the path $(v_1, \dots, v_l)$ have the same range and source;*

(2) *for any $v \in V_o^n$, the number of times v (as an edge of L_n) appears in $(v_1, \dots, v_l)$ is the same as the multiplicity of the edges in the Bratteli diagram (V, E) between v and w (as vertices of (V, E));*

(3) *if w (as a vertex in V_o^{n+1}) is connected to some vertex in V_i^n for some $1 \le i \le k$, then $(v_1, v_2, \dots, v_l)$ passes through Y_i; and*

(4) *for any edge v of L_n, the vertex v (as a vertex in the Bratteli diagram) is connected to some vertex in $V^{n-1}_{\min(v)}$ and is also connected to some vertex in $V^{n-1}_{\max(v)}$.*

Conversely, if there is a sequence of directed graphs $\{L_n; n = 2, 3, \dots\}$ such that the vertices of each L_n are $\{Y_1, \dots, Y_k\}$, the edges of each L_n are labeled by the vertices in V_o^n, and (L_n) are compatible with (V, E) in the sense above, then there is an order on (V, E) so that it is a k-simple ordered Bratteli diagram.

Proof. Let w be any edge of L_{n+1}. With a slight abuse of notation, let w also denote the vertex in V_o^{n+1} which corresponds to this directed edge of L_{n+1}. Since the edges $r^{-1}(w) \in E_n$ of the Bratteli diagram are totally ordered, write them as

$$e'_1 < e'_2 < \dots < e'_m.$$

Remove all the edges with the source points not in V_o^n, and write the remaining edges as

$$e_1 < e_2 < \dots < e_l.$$

Put $v_i = s(e_i)$, $i = 1, \ldots, l$. Then direct calculation shows that $(v_1, v_2, \ldots, v_l)$ is a path in L_n and satisfies conditions (1), (2), (3) and (4). We leave it to the reader.

Now assume that there are directed graphs $\{L_n; n = 2, 3, \ldots\}$ and a k-simple (unordered) Bratteli diagram (V, E) which are compatible. Let us show that there is an order $>$ on (V, E) so that $(V, E, >)$ is a k-simple ordered Bratteli diagram.

For each $1 \leq i \leq k$ and $n \geq 1$, choose a pair of vertices $(z^n_{i,\min}, z^n_{i,\max})$ in V^n_i. (The infinite paths $(z^1_{i,\min}, z^2_{i,\min}, \ldots)$ and $(z^1_{i,\max}, z^2_{i,\max}, \ldots)$ will be the minimal path and maximal path of the final ordered Bratteli diagram, respectively.)

Then, for each $v \in V^n_i$, put an arbitrary total order on $r^{-1}(v)$ such that $(z^{n-1}_{i,\min}, v)$ is minimal and $(z^{n-1}_{i,\max}, v)$ is maximal.

Now let us consider how to order the edges $r^{-1}(w)$ for some $w \in V^n_o$. On the first level, put an arbitrary total order on $r^{-1}(w)$ if $w \in V^1_o$.

For any $w \in V^n_o$ with $n \geq 2$, define the order the edges $r^{-1}(w)$ as follows. Pick any edge between $V^{n-1}_{\min(w)}$ and w to be the minimal edge and pick any edge between $V^{n-1}_{\max(w)}$ and w to be the maximal edge. (The existence of such edges is ensured by condition (4).)

If $n = 2$, then order the edges $r^{-1}(w)$ with an arbitrary total order with the given minimal element and maximal element.

If $n > 2$, then consider the corresponding path $(v_1, v_2, \ldots, v_l)$ in the transition graph L_{n-1}. For each v_i, $1 \leq i \leq l$, pick an arbitrary edge in E_n connecting v_i (as a vertex in V^{n-1}_o) to w, and denote it by $e(v_i)$. By condition (2), such an edge exists and the collection

$$\{e(v_1), e(v_2), \ldots, e(v_l)\}$$

exhausts all the edges between V^{n-1}_o and w. Set

$$e(v_1) < e(v_2) < \cdots < e(v_l).$$

For each $1 \leq i \leq k$, and edges between V^{n-1}_i and w, by condition (3), there is $1 \leq j \leq l$ such that the vertex Y_i of L_{n-1} is the range of v_j. Then, filling between $e(v_j)$ and $e(v_{j+1})$ by all the edges between V^{n-1}_i and w with an arbitrary order, we obtain a total order on $r^{-1}(w)$.

Note that it follows from the construction above that

$$m_-(w) = Y_{\min(w)} \quad \text{and} \quad m_+(w) = Y_{\max(w)}.$$

Let us verify that the resulting ordered Bratteli diagram $(V, E, >)$ is k-simple.

Since (V, E) is a k-simple unordered Bratteli diagram, condition (1) of Definition 5.3 is automatic.

By the choices of the ordering, it is easy to see that $(z^1_{i,\min}, z^2_{i,\min}, \ldots)$ and $(z^1_{i,\max}, z^2_{i,\max}, \ldots)$, $1 \leq i \leq k$, are the only maximal infinite paths and minimal infinite paths. Thus condition (2) of Definition 5.3 is also satisfied.

Let us verify condition (3) of Definition 5.3. Fix any $v \in V^n_o$. Let e be any edge with $s(e) = v$. Denote $w = r(e)$. Write the path in the transition graph L_n corresponding to w (as an edge in the transition graph L_{n+1}) as $(v_1, v_2, \ldots, v_l)$. Then there is $1 \leq i \leq l$ such that $v = v_i$. By the construction of the transition graph, we have that $m_+(v)$ is the range of v_i in L_n, that is,

$$m_+(v) = Y_{\max(v_i)}. \tag{8.1}$$

Consider the edge $e+1$. By the construction of the order on $r^{-1}(w)$, the vertex $s(e+1)$ is either v_{i+1} or in $V^n_{\max(v_i)}$. If $s(e+1) = v_{i+1}$, then

$$m_-(s(e+1)) = m_-(v_{i+1}) = Y_{\min(v_{i+1})} = Y_{\max(v_i)};$$

and if $s(e+1) \in V^n_{\max(v_i)}$, then $m_-(s(e+1)) = Y_{\max(v_i)}$. Therefore, by (8.1), we always have

$$m_-(s(e+1)) = Y_{\max(v_i)} = m_+(v),$$

which verifies condition (3a) of Definition 5.3.

Now, let e be an edge with $e \notin E_{\max}$, $r(e) = v$ and $s(e) \in V_i^{n-1}$ with $n \geq 3$. Write the path in L_{n-1} corresponding to v (as an edge in L_n) as $(u_1, u_2, \ldots, u_s)$. By condition (3), there is $1 \leq j \leq s$ such that $Y_{\max(u_j)} = Y_i$.

If $j < s$, then we have that either $s(e+1) = u_{j+1}$, in which case

$$m_-(s(e+1)) = Y_{\min(u_{j+1})} = Y_{\max(u_j)} = Y_i,$$

or $s(e+1) \in V_i^{n-1}$. So, in both cases, $m_-(s(e+1)) = Y_i$.

If $j = s$, since $e \notin E_{\max}$, we have that $s(e+1) \in V_i^{n-1}$, and therefore $m_-(s(e+1)) = Y_i$.

Thus, the order satisfies condition (3b) of Definition 5.3, and hence $(V, E, >)$ is a k-simple ordered Bratteli diagram. □

Let $B = (V, E, >)$ be an ordered k-simple Bratteli diagram, and denote by $L_2, L_3, \ldots$ the corresponding transition graphs. Consider the Cantor system (X_B, σ), and denote by $Y_1, \ldots, Y_k$ the minimal subsets. For each $1 \leq i \leq k$, recall that d_i is the image of $\chi_{Y_i} u$ under the index map. By Theorem 7.5, each entry of $d_i \in \bigoplus_{V_o^n} \mathbb{Z}$ has to be ± 1 or 0. Moreover, we also have that, for each $v \in V_o^n$,

$$|\{1 \leq i \leq k;\, d_i(v) \neq 0\}| = 0 \text{ or } 2,$$

and if $d_{i_1}(v) \neq 0$ and $d_{i_2}(v) \neq 0$ for some $i_1 \neq i_2$, then $d_{i_1}(v) + d_{i_2}(v) = 0$; that is, if there is a non-zero pair, it must be either $(+1, -1)$ or $(-1, +1)$.

Thus, the unordered Bratteli diagram (V, E) has the following property. There are elements $d_1, \ldots, d_k$ in K_{I_B} such that:

(a) $c_1 d_1 + \cdots + c_n d_n = 0$ if and only if $c_1 = c_2 = \cdots = c_n$;

(b) for each level n and each $v \in V_o^n$, we have that $d_i(v) \in \{0, \pm 1\}$, $1 \leq i \leq k$;

(c) for each $v \in V_o^n$, we have that $|\{1 \leq i \leq k;\, d_i(v) \neq 0\}| = 0$ or 2, and if

$$\{1 \leq i \leq k;\, d_i(v) \neq 0\} = \{i_1, i_2\},$$

then $(d_{i_1}(v), d_{i_2}(v))$ is either $(+1, -1)$ or $(-1, +1)$.

It turns out that these conditions are also sufficient for the existence of a k-simple order on (V, E) if (V, E) is strongly k-simple.

THEOREM 8.2. *Let $B = (V, E)$ be an unordered strongly k-simple Bratteli diagram satisfying condition (3) of Lemma 5.6 (i.e., each vertex in V_o^{n+1} is connected to all vertices in V^n). Suppose that there are element $d_1, \ldots, d_k \in \mathrm{K}_{I_B} \subseteq \mathrm{K}_B$ satisfying conditions (a), (b), and (c) above. Then there is an order $>$ such that $(V, E, >)$ is an ordered (strongly) k-simple Bratteli diagram.*

Before we prove the theorem, let us recall several facts from graph theory.

Definition 8.3. Let $G = (V, E)$ be a directed graph (there might be multiple edges between two vertices, and loops are also allowed). Let v be a vertex of G. The indegree and outdegree of v, denoted by $\deg^-(v)$ and $\deg^+(v)$ respectively, are the numbers of directed edges leading into and leading away from v, respectively. The degree of v is defined by

$$\deg(v) = \deg^+(v) - \deg^-(v).$$

A (directed) Euler walk in G is a walk (in the directed sense) in G that covers each directed edge exactly once.

We have the following criterion for the existence of an Euler walk in a directed graph.

THEOREM 8.4. *A directed (multi)graph has an Euler walk if and only if it is connected, and* $\deg(v) = 0$ *for every vertex with the possible exception of two vertices* v_0 *and* v_1 *such that* $\deg(v_0) = 1$ *and* $\deg(v_1) = -1$. *In this case,* v_0 *and* v_1 *are the starting point and end point of the Euler walk, respectively.*

Proof of Theorem 8.2. By Theorem 8.1, one only has to construct a sequence of directed graphs $L_2, L_3, \ldots$ which are compatible to $B = (V, E)$.

Note that the vertices of the proposed directed graphs are always $Y_1, Y_2, \ldots, Y_k$. To get L_n, one only has to assign each $v \in V_o^n$ to be a suitable directed edge of L_n.

Fix $n \geq 2$. For each $v \in V_o^n$, if $d_i(v) = 0$ for all i, then choose any Y_i such that there is an edge between V_i^{n-1} and v, and then assign v to be a loop with the base point Y_i.

Otherwise, by condition (c), there are $1 \leq \min(v) \neq \max(v) \leq k$ such that

$$d_{\min(v)}(v) = -1 \quad \text{and} \quad d_{\max(v)}(v) = +1.$$

Then assign v to be an edge from $Y_{\min(v)}$ to $Y_{\max(v)}$.

Denote the resulting directed graph by L_n. We assert that $\{L_2, L_3, \ldots\}$ are compatible with (V, E) in the sense of Theorem 8.1.

First, note that, for each $n \geq 2$ and $v \in V_o^n$, by the construction of L_n, we have

$$d_i(v) = \begin{cases} -1 & \text{if } Y_i \text{ is the source point but not the range point of } v \text{ in } L_n, \\ +1 & \text{if } Y_i \text{ is the range point but not the source point of } v \text{ in } L_n, \\ 0 & \text{otherwise.} \end{cases} \tag{8.2}$$

That is, the elements $d_1, \ldots, d_k$ are induced by the diagram as in Theorem 7.5.

Then it follows from condition (a) that the underlying undirected graphs of $L_2, L_3, \ldots$ are connected. Indeed, if there were a proper connected component of L_n, say with vertices $Y_{n_1}, \ldots, Y_{n_t}$, it would then follow from (8.2) that $d_{n_1} + \cdots + d_{n_t} = 0$, which is in contradiction to condition (a).

Now let w be any edge of L_{n+1}. Consider w as a vertex in V_o^{n+1} and write

$$w = \sum_{j=1}^{l} c_j v_j + \sum_{j=1}^{l'} c'_j v'_j$$

in the Bratteli diagram (V, E), where $c_j, c'_j \in \mathbb{N}$, $v_j \in V_o^n$, and $v'_j \in V^n \setminus V_o^n$.

Since the vertex w is assumed to be connected to all vertices at level n, we have that $\{v_1, \dots, v_l\} = V_o^n$. Let us construct an auxiliary directed graph L_n^w to be the directed (multi)graph obtained by multiple each edge v_j of L_n into c_j edges. Since L_n is connected, it is clear that L_n^w is also connected.

Then, in order to find a path in L_n satisfying conditions (1) and (2) of Theorem 8.1, it is enough to find an Euler walk (i.e., a walk which covers each edge exactly once) in L_n^w which has the starting point $Y_{\min(w)}$ and the ending point $Y_{\max(w)}$. Moreover, since the edges of the graph L_n^w exhaust all vertices in V_o^n,conditions (3) and (4) of Theorem 8.1 are satisfied automatically.

Since L_n^w is connected, by Theorem 8.4, it is enough to show that

$$\deg_{L_n^w}(Y_{\min(w)}) = +1, \quad \deg_{L_n^w}(Y_{\max(w)}) = -1$$

and

$$\deg_{L_n^w}(Y_i) = 0 \quad \text{for all other vertices } Y_i.$$

Consider any vertex Y_i of L_n^w. Then

$$\deg^+_{L_n^w}(Y_i) = \sum_{d_i(v_i)=-1} c_i \quad \text{and} \quad \deg^-_{L_n^w}(Y_i) = \sum_{d_i(v_i)=1} c_i,$$

and therefore

$$\begin{aligned} \deg_{L_n^w}(Y_i) &= \deg^+_{L_n^w}(Y_i) - \deg^-_{L_n^w}(Y_i) \\ &= \sum_{d_i(v_j)=-1} c_j - \sum_{d_i(v_j)=1} c_j \\ &= -d_i(w). \end{aligned}$$

Hence,

$$\begin{cases} \deg_{L_n^w}(Y_i) = -d_i(w) = 0 \quad \text{if } i \notin \{\min(w), \max(w)\}, \\ \deg_{L_n^w}(Y_{\min(w)}) = -d_{\min(w)}(w) = 1, \\ \deg_{L_n^w}(Y_{\max(w)}) = -d_{\max(w)}(w) = -1. \end{cases}$$

Therefore, there is an Euler walk in L_n^w with starting point $Y_{\min(w)}$ and ending point $Y_{\max(w)}$. That is, there is a path in L_n that satisfies conditions (1)–(4) of Theorem 8.1, as desired. □

Remark 8.5. In general, it would be interesting to have an abstract characterization of the dimension groups which can be realized by k-simple ordered Bratteli diagrams.

9. *Cantor system with one minimal subset*

Let us study the ordered K_0-group of $C(X) \rtimes_\sigma \mathbb{Z}$ for a Cantor system with only one minimal component Y, and explore its connection to the boundedness of invariant measures on the open set $X \setminus Y$. By Theorem 3.10, the index map is zero. Also note that the C*-algebra $C(Y) \rtimes_\sigma \mathbb{Z}$ is always an A$\mathbb{T}$-algebra, and has real rank 0 if (X, σ) is aperiodic. Hence, by [**22**, Theorem 5], we have the following structure theorem.

THEOREM 9.1. *If $k = 1$, the C*-algebra $C(X) \rtimes_\sigma \mathbb{Z}$ is an A$\mathbb{T}$-algebra. It has real rank 0 if (X, σ) is aperiodic.*

Denote by $\mathcal{K}$ the algebra of compact operators acting on a separable infinite-dimensional Hilbert space. Then a C*-algebra A is said to be stable if $A \cong A \otimes \mathcal{K}$, and the positive cone of the K_0-group, denoted by $K_0^+(A)$, is defined by

$$K_0^+(A) = \{[p]_0 : p \text{ is a projection of } A \otimes \mathcal{K}\} \subseteq K_0(A).$$

Note that if A is stably finite, $(K_0(A), K_0^+(A))$ is always an ordered group; furthermore, if A is an A$\mathbb{T}$-algebra, $(K_0(A), K_0^+(A))$ is always a dimension group.

As in §3, we have the exact sequence

$$0 \longrightarrow K_0(C_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}) \xrightarrow{\iota} K_0(C(X) \rtimes_\sigma \mathbb{Z}) \xrightarrow{\pi} K_0(C(Y) \rtimes_\sigma \mathbb{Z}) \longrightarrow 0,$$

and therefore $K_0(C(X) \rtimes_\sigma \mathbb{Z})$ is an extension of the dimension group. Denote by $u \in K_0^+(C(X) \rtimes_\sigma \mathbb{Z})$ and $v \in K_0^+(C(Y) \rtimes_\sigma \mathbb{Z})$ the standard order units induced by the constant function 1. Then it is clear that $\pi(u) = v$. If, moreover,

$$\iota(K_0^+(C_0(X \setminus Y) \rtimes_\sigma \mathbb{Z})) \subseteq [0, u], \tag{9.1}$$

then the extension above is an extension of the dimension group with order units in the sense of [**17**, p. 295]. Since $K_0(C(Y) \rtimes_\sigma \mathbb{Z})$ is simple, it follows from [**17**, Theorem 17.9] that the extension is lexicographic, namely,

$$K_0^+(C(X) \rtimes_\sigma \mathbb{Z}) = \iota(K_0^+(C_0(X \setminus Y) \rtimes_\sigma \mathbb{Z})) \cup \pi^{-1}(K_0^+(C(Y) \rtimes_\sigma \mathbb{Z}) \setminus \{0\}).$$

In general, relation (9.1) does not always hold. As we shall see in this section, (9.1) holds if and only if there is no finite σ-invariant measure on the open set $X \setminus Y$, and hence the extension is lexicographic in this case.

Example 9.2. The order-unit group $K_0(C(Y) \rtimes_\sigma \mathbb{Z})$ is always simple, but $K_0(C_0(X \setminus Y) \rtimes_\sigma \mathbb{Z})$ is not necessarily a simple ordered group. For example, let (X_1, σ_1) and (X_2, σ_2) be two almost simple Cantor system with fixed point x_1 and x_2. Attaching X_1 to X_2 by identifying x_1 and x_2, we have a new Cantor set X and an action σ on it with a fixed point $\{x\}$. It is clear that the only non-trivial closed invariant subset is $\{x\}$ (although the system is not almost simple, which requires that every orbit other than $\{x\}$ is dense). However, $K_0(C_0(X \setminus \{x\}) \rtimes_\sigma \mathbb{Z})$ is not simple. One can expand the fixed point to an odometer system to get an aperiodic example. For instance, consider the stationary Bratteli diagram with incidence matrix

$$F = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{pmatrix}.$$

The dimension group associated with the ideal is $\mathbb{Z}[1/2] \oplus \mathbb{Z}[1/3]$, with the usual order.

Let A be a stably finite C*-algebra. Denote

$$D(A) = \{[p]_0;\ p \in A\} \subseteq K_0^+(A).$$

LEMMA 9.3. *Let A be a C*-algebra with an approximate unit consisting of projections. Assume that A has stable rank* 1. *Then A is stable if and only if $D(A) = K_0^+(A)$.*

Proof. If A is stable, then it is clear that $D(A) = \mathrm{K}_0^+(A)$.

Assume that $D(A) = \mathrm{K}_0^+(A)$, and let us show that A is stable. Since A has an approximate unit consisting of projections, by [**19**, Theorem 3.1], it is enough to show that, for any projection $p \in A$, there is a projection $q \in A$ such that q is Murray–von Neumann equivalent to p and $q \perp p$. Indeed, it follows from the stable rank 1 that A has cancelation of projections. Together with $D(A) = \mathrm{K}_0^+(A)$, we have that, for the given projection p, there is a projection $s \in A$ such that s is Murray–von Neumann equivalent to $p \oplus p$, which implies that there is a subprojection $q' \leq s$ which is Murray–von Neumann equivalent to p. Using the cancelation of projections again, we have that the complementary projection $s - q'$ is also Murray–von Neumann equivalent to p. Since A has stable rank 1, there is a unitary $u \in \tilde{A}$ such that $u^*q'u = p$ (i.e., q' and p are unitarily equivalent). Then $q = u^*(s - q')u$ is the desired projection. □

The following result deals with the case of AF-algebras.

LEMMA 9.4. *Let A be an AF-algebra. Then A is stable if and only if any non-zero trace on A is unbounded.*

Proof. If A is stable, then any non-zero trace is unbounded.

If A is not stable, we construct a non-zero bounded trace on A. Write

$$A = \varinjlim(A_n, \phi_n),$$

where each $A_n = \bigoplus_{i=1}^{l_n} \mathrm{M}_{m_{n,i}}(\mathbb{C})$. Then there is a projection $p \in A$ (one may assume that $p \in A_1$) such that, for any n, there exists $1 \leq i \leq l_n$, such that

$$2 \cdot \mathrm{rank}(\pi_{n,i} \circ \phi_{1,n}(p)) > m_{n,i};$$

as otherwise, for any projection p, we can find a projection q such that $p \perp q$ and $p \sim q$. Then it follows from [**19**, Theorem 3.1] that A is stable, which contradicts the assumption.

Set

$$\tau_n = \frac{1}{m_{n,i}} \mathrm{Tr} \circ \pi_{n,i} : A_n \to \mathbb{C}.$$

It is clear that τ_n is a tracial state on A_n, and $\tau_n(p) > 1/2$. Extend τ_n to a linear functional on A with norm 1, and denote it also by τ_n. Pick an accumulation point of $\{\tau_n; n = 1, \ldots, \infty\}$, and denote it by τ. It is clear that τ is a trace on A with norm at most 1. Moreover, since $\tau_n(p) > 1/2$, we have that $\tau(p) \geq 1/2$, and thus τ is non-zero. Therefore A has a non-zero bounded trace, as desired. □

THEOREM 9.5. *The restriction of σ on $X \setminus Y$ has no finite invariant non-zero measure if and only if $\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$ is stable.*

Proof. If $(X \setminus Y, \sigma)$ has a finite invariant measure, then it induces a finite trace τ on $\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$, and hence it cannot be stable.

On the other hand, if $\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$ is not stable, then by Lemma 9.4, there is a bounded non-zero trace on $\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$. The restriction of this trace to $\mathrm{C}_0(X \setminus Y)$ induces a finite non-zero invariant measure. □

Denote by u the standard order unit of $\mathrm{K}_0(\mathrm{C}(X) \rtimes_\sigma \mathbb{Z})$, and consider the generating set

$$\iota^{-1}[0, u] = \{\kappa \in \mathrm{K}_0^+(\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}) : \iota(\kappa) < u\} \subseteq \mathrm{K}_0^+(\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}).$$

The set $\iota^{-1}[0, u]$ is then a *generating interval* in sense that it is a convex upward-directed subset which generates the whole ordered group; see [**17**, Lemma 17.8].

We have the following theorem.

THEOREM 9.6. *The ideal* $\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$ *is stable if and only if* $\iota^{-1}[0, u] = \mathrm{K}_0^+(\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z})$.

Proof. Assume that $\iota^{-1}[0, u] = \mathrm{K}_0^+(\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z})$; that is, for any positive element $a \in \mathrm{K}_0^+(\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z})$, we have that $a < u$ in $\mathrm{K}_0^+(\mathrm{C}(X) \rtimes_\sigma \mathbb{Z})$. Then, for any projection p in a matrix algebra of $\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$, there is a partial isometry v in a matrix algebra of $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ such that $vv^* = p$ and $v^*v \leq 1$. In particular, $v^*pv \in \mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$. Since $\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$ is an ideal, we have that $v^*pv \in \mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$. Hence $[p] \in D(\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z})$. By Lemma 9.3, the ideal $\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$ is stable.

If the ideal $\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$ is stable, then, for any $a \in \mathrm{K}_0^+(\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z})$, there is a projection $p \in \mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$ such that $[p]_0 = a$. It is clear that $p < 1$ in $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$, and therefore $a = [p]_0 < [1]_0 = u$. Hence, $\iota^{-1}[0, u] = \mathrm{K}_0^+(\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z})$. □

COROLLARY 9.7. *The restriction of* σ *on* $X \setminus Y$ *has no non-zero finite invariant measure if and only if* $\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$ *is stable, if and only if* $\iota^{-1}[0, u] = \mathrm{K}_0^+(\mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z})$, *and if and only if the extension is lexicographic.*

10. *Chain transitivity*

10.1. *Topologies on the group of homeomorphisms.* Let X be a Cantor set, and let $H(X)$ denote the group of all homeomorphisms of a Cantor set X. Since all Cantor sets are homeomorphic, we do not need to specify a particular Cantor set while studying the group $H(X)$. In particular, the Cantor set can be represented as the path space of a non-simple Bratteli diagram.

We recall that by an *aperiodic Cantor dynamical system* (X, σ), we mean a homeomorphism σ of a Cantor set X such that, for any $x \in X$, the orbit $\mathrm{Orbit}_\sigma(x) = \{\sigma^i(x) : i \in \mathbb{Z}\}$ is infinite.

The set $\mathcal{A}p$ of aperiodic homeomorphism was studied in [**1**, **2**, **23**, **24**] from various points of view. We recall here a few results that will be used below.

Fix a metric d on X compatible with the clopen topology on X. There are several natural topologies defined on $H(X)$; see [**1**, **16**, **20**]. The most popular one is the topology of uniform convergence, τ_w, that turns $H(X)$ into a Polish group. This topology can be defined in several equivalent ways: for instance, by the metric

$$D(\psi_1, \psi_2) = \sup_{x \in X} d(\psi_1(x), \psi_2(x)) + \sup_{x \in X} d(\psi_1^{-1}(x), \psi_2^{-1}(x)), \quad \psi_1, \psi_2 \in H(X). \tag{10.1}$$

Equivalently, the topology τ_w is generated by the base of neighborhoods $\mathcal{W} = \{W(\psi; E_1, \ldots, E_n)\}$ where

$$W(\psi; E_1, \ldots, E_n) = \{f \in H(X) : f(E_1) = \psi(E_1), \ldots, f(E_n) = \psi(E_n)\}.$$

Here $\psi \in H(X)$, and $E_1, \ldots, E_n$ are any clopen sets. Without loss of generality, we can assume that $(E_1, \ldots, E_n)$ forms a clopen partition of X.

If $D(\sigma_n, \sigma) \to 0$, we say that σ is approximated by a sequence of homeomorphisms (σ_n). We first remark that any homeomorphism of X is approximated by aperiodic homeomorphism.

LEMMA 10.1. [**1**] *The set $\mathcal{A}p$ is dense in $(H(X), \tau_w)$.*

We introduce notation for some classes of homeomorphisms of a Cantor set X: $\mathcal{M}in$ denotes the set of all minimal homeomorphisms (a homeomorphism σ is minimal if every σ-orbit is dense in X); $\mathcal{T}t$ denotes the set of all topologically transitive homeomorphisms (σ is *topologically transitive* if there exists a dense orbit); $\mathcal{M}ov$ is the set of all moving homeomorphisms (a homeomorphism $\sigma \in H(X)$ is called *moving* if, for any non-trivial clopen set $E \subset X$, we have $\sigma(E) \setminus E \neq \emptyset$ and $E \setminus \sigma(E) \neq \emptyset$). The notion of moving homeomorphisms was defined in [**1**].

LEMMA 10.2. [**1**] *The set of moving homeomorphisms is τ_w-closed. An aperiodic homeomorphism σ is moving if and only if σ can be approximated by a sequence of minimal homeomorphisms (or topologically transitive homeomorphisms), that is, $\mathcal{M}ov = \overline{\mathcal{M}in}^{\tau_w} = \overline{\mathcal{T}t}^{\tau_w}$.*

10.2. *Chain-transitive homeomorphisms.* Let (X, σ) be an aperiodic Cantor dynamical system. We now recall several notions related to *chains* in X defined by σ. A finite set $\{x_0, x_1, \ldots, x_n\}$, where $\sigma(x_i) = x_{i+1}$, $i = 0, \ldots, n-1$, is called a *σ-chain* (or simply a *chain*). Given $\varepsilon > 0$ and $x, y \in X$, an *ε-chain* from x to y is a finite sequence $\{x_0, x_1, \ldots, x_n\}$ such that $x_0 = x$, $x_n = y$ and $d(\sigma(x_i), x_{i+1}) < \varepsilon$, $i = 0, \ldots, n-1$. In symbols, an ε-chain from x to y will be denoted by $x \overset{\varepsilon}{\rightsquigarrow} y$.

Given an aperiodic Cantor system (X, σ), it is said that σ is *chain transitive* if, for any two points $x, y \in X$, there exists an ε-chain from x to y.

The following result is a new characterization of moving homeomorphisms as chain-transitive ones.

THEOREM 10.3. *An aperiodic homeomorphism σ of a Cantor set X is chain transitive if and only if σ is moving.*

Proof. ($\Longrightarrow$) If σ is not moving then there is a non-trivial clopen set E such that $\sigma(E) \subset E$, and $F = E \setminus \sigma(E)$ is a non-empty clopen subset (the case where $E \subset \sigma(E)$ is considered similarly). Let x_0, y_0 be any distinct points from F with $d(x_0, y_0) = \delta > 0$. We note that $\sigma^i(x) \in \sigma^i(E) \subset \sigma(E)$, $i > 0$, for any $x \in E$. Take $0 < \varepsilon < \min\{\delta, d(y_0, \sigma(E)), d(x_0, \sigma(E))\}$ (we denote the distance between closed subsets of X by the same letter d; it will be clear from the context which metric space is meant). Then there is no ε-chain from x_0 to y_0 and from y_0 to x_0 as well. Hence, σ is not chain transitive.

($\Longleftarrow$) Conversely, let σ be a moving homeomorphism of a Cantor set X, and let ε be a positive number. Take a partition of X into a finite collection of clopen sets $C(i)$ such that $\mathrm{diam}(C(i)) < \varepsilon$ for any $i = 1, \ldots, N$. We will show that, for any $x, y \in X$, there is a finite ε-chain for σ from x to y. Suppose $x \in C(i_0)$ and $\sigma(x) \in C(i_1)$. Then any z from $C(i_1)$

can be considered as the target of ε-chain $\{x, z\}$ of length 1. So, if $y \in C(i_1)$ we are done. If not, we consider $X \setminus B(1)$, where $B(1) = C(i_0)$. Because σ is moving, the set $\sigma(B(1))$ intersects $X \setminus B(1)$. Let

$$B(2) = \bigcup_{j \in I_1} C(j),$$

where $j \in I_1$ if $C(j) \cap \sigma(B(1)) \neq \emptyset$. If z is a point from $B(2)$, then there exists an ε-chain from x to z of length 2. Indeed, if $z \in C(j)$, $j \in I_1$, take the ε-chain $\{x, x_1, z\}$ where $x_1 \in B(1)$ such that $\sigma(x_1) \in C(j)$. If $y \in B(2)$, we are done. Otherwise, we apply the same argument to the set $X \setminus (B(1) \cup B(2))$. In view of compactness of X, this procedure terminates in a finite number of steps. This means there exists $B(k)$ such that $y \in B(k)$, that is, the point y is the final point of an ε-chain of length k. □

Remark 10.4. We observe that the proof of Theorem 10.3 uses essentially the topological structure of Cantor sets, namely, the existence of partitions of Cantor sets into clopen sets of arbitrary small diameter. Below we give another proof of the implication moving $\Longrightarrow$ chain transitivity based on Lemma 10.2.

Proof. Suppose $\sigma \in H(X)$ is moving. By Lemma 10.2, for any fixed $\varepsilon > 0$, there exists a minimal homeomorphism $f = f_\varepsilon$ such that $D(\sigma, f) < \varepsilon$. Take any two points $x, y \in X$. By minimality of f, find the smallest positive k such that $d(f^k x, y) < \varepsilon$. Consider the set $I = \{x = x_0, x_1 = f(x), x_2 = f^2(x), \ldots, x_k = f^k(x)\}$. We claim that I is an ε-chain for σ from x to y. Indeed,

$$d(\sigma(x_i), x_{i+1}) = d(\sigma(f^i(x)), f(f^i(x))) < \sup_{z \in X} d(\sigma(z), f(z)) < \varepsilon, \qquad i = 0, 1, \ldots, k-1,$$

and $d(x_k, y) < \varepsilon$. Thus, σ is chain transitive. □

Let $\mathcal{C}hT$ be the set of all chain-transitive homeomorphisms. The next corollary follows from Theorem 10.3.

COROLLARY 10.5. *The set $\mathcal{C}hT$ of chain-transitive homeomorphisms of a Cantor set is closed in the topology τ_w of uniform convergence.*

Since the proof of the fact that any moving homeomorphism is chain transitive, given in Remark 10.4, does not explicitly use the fact that X is a Cantor set, we immediately deduce the following corollary.

COROLLARY 10.6. *Suppose σ is a homeomorphism of a compact metric space (Ω, d) which is approximated by minimal homeomorphisms, that is,* $\lim_i D(\sigma, f_i) = 0$ *where each f_i is minimal. Then σ is chain transitive.*

10.3. *Homeomorphisms with finite number of minimal sets.* In this subsection we consider the case where an aperiodic homeomorphism $\sigma \in H(X)$ has a finite number of minimal sets, say $Y_1, \ldots, Y_k$, that is, each Y_i is a closed σ-invariant set such that the orbit $\mathrm{Orbit}_\sigma(z)$ is dense in Y_i for any $z \in Y_i$. For simplicity, we will call such homeomorphisms *k-minimal*.

It worth recalling that we are dealing with indecomposable homeomorphisms σ, that is, every minimal set Y for a k-minimal homeomorphism σ is not open and has empty interior.

Given a closed subset C of X, we say that an open set V is an *ε-neighborhood of C* if $V \supset C$ and $d(C, x) < \varepsilon$ for any $x \in V$.

LEMMA 10.7. *Given an aperiodic Cantor system (X, σ), let $\{Y_\alpha\}$ be the collection of all minimal subsets for σ. Let V be any clopen subset of X such that $V \supset \bigcup_\alpha Y_\alpha$. Then there exist positive integers K_+ and K_- such that*

$$\bigcup_{n=0}^{K_+} \sigma^n(V) = X \quad \text{and} \quad \bigcup_{n=0}^{K_-} \sigma^{-n}(V) = X. \tag{10.2}$$

Moreover, the same result is true when the above condition $V \supset \bigcup_\alpha Y_\alpha$ is replaced by the condition $V \cap Y_\alpha \neq \emptyset$ for any α.

Proof. Let (X, σ) be a Cantor aperiodic dynamical system. Consider the σ-invariant open set $Y = \bigcup_{n\in\mathbb{Z}} \sigma^n(V)$. If the closed set $X \setminus Y$ were non-empty, then it would contain a minimal set for σ; this is impossible because all minimal sets are subsets of Y. Thus, $X = \bigcup_{n\in\mathbb{Z}} \sigma^n(V)$; hence, by compactness, the latter is a finite union. Applying appropriate powers of σ to the above relation, we find some $K_+ \in \mathbb{N}$ and $K_- \in \mathbb{N}$ such that (10.2) holds. The other statement of the lemma is proved similarly. □

In the case of finitely many minimal subsets, we can reformulate Lemma 10.7 in a more appropriate form. For this purpose, we first observe the following useful fact. Let $V \supset Z$ be any neighborhood of a minimal set Z for a homeomorphism σ. It is easily seen that $\sigma^n(V) \supset Z$ $(n \in \mathbb{Z})$ because Z is σ-invariant. Therefore, the following result can be straightforwardly deduced from Lemma 10.7.

COROLLARY 10.8. *Let (X, σ) be an aperiodic Cantor system with minimal sets $Y_1, \ldots, Y_k$. For any $\varepsilon > 0$ and any clopen ε-neighborhood V_i of Y_i $(i = 1, \ldots, k)$, the clopen set $V = \bigcup_{i=1}^k V_i$ satisfies the condition $\bigcup_{n=0}^{K_+} \sigma^n(V) = X$, $\bigcup_{n=0}^{K_-} \sigma^{-n}(V) = X$ for some positive integers K_+ and K_-.*

Definition 10.9. Let Z be a fixed minimal set for an aperiodic Cantor system (X, σ). We will say that two points $x, y \in X$ are *chain equivalent with respect to* Z if, for any $\varepsilon > 0$, there exist ε-chains $x \overset{\varepsilon}{\rightsquigarrow} z_0$ and $y \overset{\varepsilon}{\rightsquigarrow} z_0$ where z_0 is a point from Z.

We observe that if $x \overset{\varepsilon}{\rightsquigarrow} z_0$ and $y \overset{\varepsilon}{\rightsquigarrow} z_0$ for some point $z_0 \in Z$, then $x \overset{\varepsilon}{\rightsquigarrow} z$ and $y \overset{\varepsilon}{\rightsquigarrow} z$ for any point z because σ is minimal on Z.

LEMMA 10.10. *Let Z be a minimal set for an aperiodic Cantor system (X, σ), and let x, y be any two points from X. The following statements are equivalent.*

(i) *The points x and y are chain equivalent with respect to Z.*
(ii) *For all $\varepsilon > 0$, there exist $x \overset{\varepsilon}{\rightsquigarrow} z_1$ and $y \overset{\varepsilon}{\rightsquigarrow} z_2$ where $z_1, z_2 \in Z$.*
(iii) *For all $\varepsilon > 0$ and for all ε-neighborhoods V_ε, there exist $x \overset{\varepsilon}{\rightsquigarrow} v_1$ and $y \overset{\varepsilon}{\rightsquigarrow} v_2$ where $v_1, v_2 \in V_\varepsilon$.*

It follows from this lemma that the chain equivalence with respect to a minimal set Z is an equivalence relation on $X \times X$. We denote it by $\mathcal{E}(Z)$.

Proof. We sketch the proof of the lemma because the technique used in the proof is quite standard. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are obvious, so all that remains is to show that (iii) $\Longrightarrow$ (i). To do this, it suffices to notice that, by minimality of σ on Z, there exists a point $z \in Z$ (in fact, z can be any point from Z) such that $v_1 \overset{\varepsilon}{\rightsquigarrow} z$ and $v_2 \overset{\varepsilon}{\rightsquigarrow} z$. The result then follows Definition 10.9. □

PROPOSITION 10.11. *Suppose Z is a unique minimal set for an aperiodic homeomorphism σ of a Cantor set X. Then $\mathcal{E}(Z) = X \times X$. In other words, any two points in X are chain equivalent with respect to Z. Moreover, σ is chain transitive.*

Proof. The fact that $\mathcal{E}(Z) = X \times X$ follows directly from Lemmas 10.7 and 10.10 because the σ-orbit of any clopen neighborhood of Z covers X after finitely many iterations.

Take any two points x, y in X. Let $\varepsilon > 0$, and let V_ε be an ε-neighborhood of Z. We consider the case where $x, y \in X \setminus Z$; the other possible cases are considered similarly with obvious simplifications. By Corollary 10.8, we find positive integers K_+ and K_- such that

$$\bigcup_{n=0}^{K_+} \sigma^n(V_\varepsilon) = \bigcup_{n=0}^{K_-} \sigma^{-n}(V_\varepsilon) = X.$$

Let i be the smallest number from the set $(0, 1, \dots, K_-)$ such that $x \in \sigma^{-i}(V_\varepsilon)$, and let j be the smallest number from the set $(0, 1, \dots, K_+)$ such that $y \in \sigma^j(V_\varepsilon)$. To construct a chain C from x to y, we first find two points $w \in Z$ and $z \in Z$ such that $d(w, \sigma^{-j}(y)) < \varepsilon$ and $d(\sigma^i(x), z) < \varepsilon$. Then we take the set

$$C = (x, \sigma(x), \dots, \sigma^i(x), z, \sigma(z), \dots, \sigma^m(z), \sigma^{-j}(y), \sigma^{-j+1}(y), \dots, y),$$

where m is the smallest positive integer such that $d(\sigma^m(z), w) < \varepsilon$. Then C is a 2ε-chain for σ. This proves the chain transitivity of σ. □

Remark 10.12. (1) It follows from Theorem 10.3 that if (X, σ) is an aperiodic Cantor system with a unique minimal set, then σ is a moving homeomorphism.

(2) Another simple observation from the proved result is the fact that any aperiodic homeomorphism with a unique minimal set is the limit of a sequence of minimal homeomorphisms in the topology of uniform convergence τ_w.

Dynamical properties of an aperiodic Cantor system (X, σ) become more complicated when the system has several minimal sets $Y_1, \dots, Y_k$.

Let E_i be the subset of X defined as follows:

$$E_i = \{x \in X : \forall \varepsilon > 0\ \exists\ x \overset{\varepsilon}{\rightsquigarrow} z \text{ for some } z \in Y_i\}, \quad i = 1, \dots, k. \tag{10.3}$$

We observe that E_i contains Y_i; moreover, E_i is the $\mathcal{E}(Y_i)$-saturation of the set Y_i. The latter means that E_i is the minimal $\mathcal{E}(Y_i)$-invariant set such that every $\mathcal{E}(Y_i)$-orbit meets Z at least once.

LEMMA 10.13. *In the above notation, each E_i is a clopen subset of X, $i = 1, \dots, k$.*

Proof. First we show that E_i is open. For $x \in E_i$ and $\varepsilon > 0$, take an ε-chain $(x = x_0, x_1, \ldots, x_n = z)$ from x to $z \in /Y_i$. We prove that there exists a neighborhood V of x such that $V \subset E_i$. Choose $\delta > 0$ such that $d(\sigma(x), \sigma(y)) < \varepsilon/2$ when $y \in V := \{u : d(x, u) < \delta\}$. Then, for any $y \in V$, we see that $(y, x_1, \ldots, x_n)$ is an ε-chain from y to z.

Next, let $(u(n))$ be a sequence from E_i that converges to some point x. Suppose that for each $u(n)$ there is an ε-chain $(u(n), x_1(n), \ldots, z(n))$ from $u(n)$ to Y_i. Choose n_0 such that $d(\sigma(u(n_0)), \sigma(x)) < \varepsilon/2$. Then $(x, x_1(n), \ldots, z(n))$ is an ε-chain from x to Y_i. Hence, E_i is closed. □

Consider an aperiodic Cantor system (X, σ) with finitely many minimal sets $Y_1, \ldots, Y_k$, and let $E_1, \ldots, E_k$ be the clopen sets defined by minimal sets according to (10.3). Then, by Proposition 10.11, we have

$$X = \bigcup_{i=1}^{k} E_i.$$

LEMMA 10.14. *In the above notation,*

$$E_i \cap E_j \neq \emptyset \implies E_i = E_j, \quad i, j = 1, \ldots, k.$$

Proof. It follows from Proposition 10.11 that, for any $\varepsilon > 0$, each set E_i, and any two points $x, y \in E_i$, there exists an ε- chain from x to y. Thus, if $x_i \in E_i$, $x_j \in E_j$ are arbitrary points and $y \in E_i \cap E_j$, then there are ε-chains $x_i \overset{\varepsilon}{\rightsquigarrow} z_j$ and $x_j \overset{\varepsilon}{\rightsquigarrow} z_i$ where $z_i \in Y_i$ and $z_j \in Y_j$. □

THEOREM 10.15. *Suppose σ is an aperiodic homeomorphism, and $Y_1, \ldots, Y_k$ are minimal sets for σ on a Cantor set X. Then (X, σ) is chain transitive if and only if $E_1 = \cdots = E_k = X$.*

Proof. Let σ be a chain-transitive homeomorphism. Fix E_i and take any point $x \in E_i$. Then, for any $\varepsilon > 0$ and $y_j \in E_j$, there exists an ε-chain from x to y_j, $j = 1, \ldots, k$. Hence, $x \in E_j$ for all $j = 1, \ldots, k$. This proves the first statement.

The converse statement follows from Lemma 10.14. □

The properties of homeomorphisms with finitely many minimal sets are clearly seen for Vershik maps defined on k-minimal Bratteli diagrams. We obtain the following result from Theorem 10.15.

COROLLARY 10.16. *Let $B = (V, E, >)$ be a k-simple ordered Bratteli diagram and let (X_B, ω_B) be a Bratteli–Vershik k-minimal dynamical system defined on the path space of B. Then σ_B is a chain-transitive homeomorphism.*

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