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Mean-field BSDEs with jumps and dual representation for global risk measures

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Dedicated to Professor Alain Bensoussan on the occasion of his 80th birthday.

Abstract We study mean-field BSDEs with jumps and a generalized mean-field operator that can capture higher-order interactions. We interpret the BSDE solution as a dynamic risk measure for a representative bank whose risk attitude is influenced by the system. This influence can come in a wide class of choices, including the average system state or average intensity of system interactions. Using Fenchel−Legendre transforms, our main result is a dual representation for the expectation of the risk measure in the convex case. In particular, we exhibit its dependence on the mean-field operator.

Keywords Mean-field interactions, BSDEs, Dynamic risk measures, System influence

2020 Mathematics Subject Classification 60H10, 60H30

1. Introduction

We consider a mean-field BSDE with a mean-field operator which can accommodate several types of interactions. The dynamics is

$$-\mathrm{d}X_t = f(t,\omega,F(t,X_{t^-}),X_{t^-},Z_t,\ell_t(\cdot))\mathrm{d}t - Z_t\mathrm{d}W_t - \int_{\mathbf{E}} \ell_t(e)\tilde{N}(\mathrm{d}t,\mathrm{d}e);\ X_T=\xi, \tag{1.1}$$

where the mean-field operator F is a $\mathcal{B}([0,T])\times\mathcal{B}(L^2)-$ measurable operator from $[0,T]\times L^2$ to $\mathbf{R}$. The precise conditions are given in the next sections.

Mean-field BSDEs of type (1.1) are studied in [1], where standard existence, uniqueness, and comparison results are provided for the special case where the mean field captures the average state in the system. We also refer to the seminal paper on mean-field BSDEs and their stochastic limit approach (see [5]) and to the papers [20, 21] in which the coefficient of the

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BSDE depends, in particular, also on the law of the quadruplet of the solution of the forward-backward SDE with jumps.

In our present work, we start from the observation that in many applications (and specifically in interacting systems), it may be desirable to incorporate the intensity of system interactions as well, and not only the average state. The driver of the BSDE may contain in our case a "second-order" mean-field interaction term of the form

$$F(t, X_t) = \int_{\mathbb{R}\times\mathbb{R}} \kappa(x, x')\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}x') = \mathbb{E}\left[\kappa(X_t, X_t{}')\right], \quad ((X_t, X_t') \sim \mu_t \otimes \mu_t) \tag{1.2}$$

for a Lipschitz function $\kappa : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This can be viewed as a kernel that captures the intensity of interactions. The process $X_t{}'$ is an independent copy of the same distribution μ_t as X_t. Note that setting the kernel κ constant in its first argument, we can recover the expectation operator considered in [1] as a particular case. In section 2.2, we provide an example where the operator (1.2) represents the average intensity of interactions of nodes of the inhomogeneous random graph introduced in [4].

We refer to [2] for convergence results of finite interacting particle systems with inhomogeneous interactions to mean-field Graphon BSDEs and to [19] for convergence results of interactive particle systems based on propagation of chaos involving BSDEs. See also [18] and [17].

We explore how a solution to the mean-field BSDE (2.4) with driver f, mean-field operator F, and terminal condition $\eta \in L^2(\mathcal{F}_T)$ can be interpreted as a dynamic risk measure, defined in the usual way

$$\rho_t(\eta, T) := -X_t(\eta), \quad 0 \leqslant t \leqslant T. \tag{1.3}$$

We can think about this as a way to capture the risk of a representative bank subject to system influence: a regulator imposes the capital or liquidity to be the random variable η at time T, and the mean-field BSDE then tells us what are the acceptable levels of capital or liquidity at time t, for a given driver capturing how a representative bank's position evolves with dependence on the mean-field term, where the mean-field term is meant to represent the average intensity of interactions in the system.

Contributions and organization of the paper. Our stream of novel results starts with the *strict* comparison result in Theorem 2.11, which is then used to verify the no arbitrage condition of the dynamic risk measure. The other results in section 2 are usual properties for the mean-field BSDE of type (1.1), proven in the case of general mean-field operators. Under Lipschitz conditions, the existence, uniqueness, and the (non-strict) comparison results are standard and are provided for completeness. Armed with the comparison results, section 3.1 checks that suitable conditions for a dynamic risk measure are satisfied. We provide properties for the global dynamic risk measures such as monotonicity, consistency, and convexity under appropriate hypotheses.

Our mean-field operator is thought of as a model for the average intensity of interactions of an inhomogeneous random graph (in the limit when the graph is large). While the distribution of each banks' risk process is the same (and in this sense the system is symmetric), at each point in time, the banks have different risk positions and interact according to this inhomogeneity via the kernel function.

We introduce *global dynamic risk measures* induced by mean-field BSDEs. When the mean-field operator captures the average intensity of system interactions, the interpretation is that of a dynamic risk measure which can incorporate system influence for the representative bank.

Our main result is the dual representation for the expectation of the dynamic risk measure in the convex case, provided in section 3.2. New challenges for the proof strategy arise with respect to the classical literature as we deal with the Fenchel–Legendre transform of the mean-field functional F. We first need to establish bounds on the effective domain of this transform (Lemma 3.3). Moreover, since the mean-field operator is inside the driver, we have an additional dual variable with respect to the classical case and we need to establish new bounds for it. In particular, we provide an explicit form to the conjugacy relation of (F, F^*) and rely on a new SDE of mean-field type. The presence of the mean-field term in combination with the jumps poses additional challenges, in particular, to ensure the equivalence of the worst-case probability measure appearing in the representation and the real-world probability measure.

We obtain a representation as the expectation under a worst-case discount factor and the worst-case probability measure of the final acceptable capital level plus a penalty function. The dependence on the mean-field operator (or its Fenchel–Legendre transform) of the worst-case probability measure, discount factor, and penalty distinguishes our results from past literature on classical dynamic risk measures. Since the mean-field operator can capture the influence of the system on the representative bank (for example, via the average liquidity position in the system or the average strength of interactions), using such a dynamic risk measure can account for the system influence.

Our risk measure represents a complementary approach to systemic risk measurement, whereby we consider a "representative ban" whose risk attitude is impacted not only by its own practice but by the system as well, according to the mean-field operator. For example, the risk attitude can be different in low vs. high average system liquidity/capital or system lending intensity. To account for the impact of the system, the bank computes its risk by assuming that other banks are exchangeable copies of itself. With wide modeling choices, the driver enables risk measures that are compatible with the risk attitude of the "representative bank" under system influence, be that the average state in the system or the average intensity of interactions. This is a way to aggregate the influence of other banks in the system, and allows for a dynamic risk assessment.

Related literature. Whereas our global risk measure is not a systemic risk measure, it is a related concept. Indeed, the first generation of systemic risk models were based on mean-field interactions. For example, [6, 13, 15, 16] rely on interacting particle systems, with a particle's drift depending on the average state of the system. In [6], the authors define the state of a particle i in a system of size N, $X_t^{i,N}$, as monetary reserves. The drift of $X_t^{i,N}$ has a form

$$a\left(\frac{1}{N}\sum_{j=1}^{N} X_t^{j,N} - X_t^{i,N}\right)\mathrm{d}t. \tag{1.4}$$

When the monetary reserve reaches zero, the bank defaults. The financial interpretation of such drift is that parameter a captures the rate of interbank lending when the banks with more liquidity than average lend to banks with less liquidity than average.

Our setup is closer in spirit to an aggregation of system influence. The mean-field term operator (2.3) can be seen as a limit of $\frac{1}{N^2}\sum_{i,j\in 1,N}\kappa(X_t^{i,N}, X_t^{j,N})$, which represents the average intensity of interactions in the system, where the interactions of any two banks are state-dependent and captured by the kernel κ. We can set, for example, $\kappa(x,y)=\phi(x-y)$, where ϕ is the gaussian kernel. This leads to a tiered structure where banks with similar capital interact, while there is little interaction for banks of dissimilar size. To obtain a core-periphery structure (shown to be quite realistic [9]), we could additionally restrict the interaction of two small banks: either large banks interact with each other and small banks interact with large banks. For this case, we can choose $\kappa(x,y) = \phi(x-y)1_{x\vee y>s_0}$, which means that interaction occurs only if at least one of the banks has a size larger than s_0. Other variations of the kernel definition can capture various market structures.

We interpret the mean-field BSDE solution as a dynamic risk measure for a representative bank. The driver captures how the liquidity or financial position of an institution depends on the rest of the system and allows (in a somewhat reduced form) to account for interactions with the system. The drift form (1.4) above is one example by which the liquidity position of one bank depends on the rest of the system, and this dependence is on the average liquidity. Our setup is quite flexible in terms of modeling choices. In particular, we can use the states (e.g., liquidity positions) of the nodes to define the intensity of lending (or the existence of a link) among nodes. In the example of (1.4), the rate of interbank lending is a constant a. It would be interesting to replace this with the average intensity of interactions in the system: if the intensity of interactions is low, then there is little interbank lending. Our setup would allow for the average intensity of interactions (possibly in addition to the average state of the system) to be a critical part of the driver.

2. Mean-field BSDEs with jumps

2.1 Notation and definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let W be a one-dimensional Brownian motion. Let $\mathbf{E} := \mathbf{R}^* := \mathbf{R}\setminus\{0\}$ equipped with its Borelian σ-algebra $\mathcal{B}(\mathbf{E})$. Suppose that it is equipped with a σ-finite positive measure ν and let $N(\mathrm{d}t, \mathrm{d}e)$ be an independant Poisson random measure with compensator $\nu(\mathrm{d}e)\mathrm{d}t$. Let $\tilde{N}(\mathrm{d}t, \mathrm{d}e)$ be its compensated process. Let $\mathbb{F} = \{\mathcal{F}_t, t \geqslant 0\}$ be the completed natural filtration associated with W and N. Let $T > 0$. Let $\mathcal{P}$ be the predictable σ-algebra on $[0,T]\times\Omega$.

We use the following notation: $L^2_P(\mathcal{F}_T)$ (simply denoted by L^2) is the set of $\mathbf{R}$-valued square-integrable $\mathcal{F}_T$-measurable random variables; $\mathbb{H}^2$ is the set of real-valued predictable processes ϕ such that $\|\phi\|^2_{\mathbb{H}^2} := \mathbb{E}\left[\int_0^T \phi_t^2 \mathrm{d}t\right] < \infty$; $\mathcal{S}^2$ denotes the set of real-valued, right-continuous with left limits (RCLL), adapted processes ϕ such that $\|\phi\|^2_{\mathcal{S}^2} := \mathbb{E}(\sup_{0\leqslant t\leqslant T}|\phi_t|^2) < \infty$; We also introduce the following spaces:

- L^2_ν is the set of Borelian functions $\ell : \mathbf{E}\to\mathbf{R}$ such that $\int_{\mathbf{E}} |\ell(e)|^2\nu(\mathrm{d}e) < +\infty$. The set L^2_ν is a Hilbert space equipped with the scalar product. $\langle \ell, \ell'\rangle_\nu := \int_{\mathbf{E}} \ell(e)\ell'(e)\nu(\mathrm{d}e)$ for all $\ell, \ell' \in L^2_\nu \times L^2_\nu$, and the norm $\|\ell\|^2_\nu := \int_{\mathbf{E}} |\ell(e)|^2\nu(\mathrm{d}e)$.

- $\mathbb{H}^2_\nu$ is the set of all mappings $\ell : [0,T]\times\Omega\times\mathbf{E}\to\mathbf{R}$ that are $\mathcal{P}\otimes\mathcal{B}(\mathbf{E})/\mathcal{B}(\mathbf{R})-$ measurable and satisfy $\|\ell\|^2_{\mathbb{H}^2_\nu} := \mathbb{E}\left[\int_0^T \|\ell_t\|^2_\nu \,\mathrm{d}t\right] < \infty$, where $\ell_t(\omega, e) = \ell(t,\omega,e)$ for all $(t,\omega,e)\in[0,T]\times\Omega\times\mathbf{E}$.

Definition 2.1 *(Driver, Lipschitz driver) A function f is said to be a* driver *if*

- $f : \Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^2\times L^2_\nu \to \mathbf{R}$, $(\omega,t,y',y,z,\ell(\cdot)) \mapsto f(\omega,t,y',y,z,\ell(\cdot))$ *is* $\mathcal{P}\otimes\mathcal{B}(\mathbf{R}^2)\otimes\mathcal{B}(L^2_\nu)-$ *measurable,*

- $f(.,0,0,0,0,0) \in \mathbb{H}^2$.

A driver f is called a Lipschitz driver *if, moreover, there exists a constant $C \geqslant 0$ such that* $dt \otimes dP$*-a.s., for each* (y_1', y_1, z_1, ℓ_1), (y_2', y_2, z_2, ℓ_2),

$$\begin{aligned}&|f(\omega,t,y_1',y_1,z_1,\ell_1)-f(\omega,t,y_2',y_2,z_2,\ell_2)|\\&\leqslant C(|y_1'-y_2'|+|y_1-y_2|+|z_1-z_2|+\|\ell_1-\ell_2\|_\nu).\end{aligned}$$

We now introduce the mean-field operator F. As we see below, the canonical example is the expectation or the expected intensity of state dependent interactions.

Definition 2.2 *An operator F is said to be a mean-field operator if*

- $F : [0,T] \times L^2 \to \mathbf{R}$, $(t, X) \mapsto F(t, X)$ *is* $\mathcal{B}([0,T]) \times \mathcal{B}(L^2)-$ *measurable,*
- *For each* $t \in [0,T]$, $F(t, 0) < +\infty$.

A mean-field operator F is said to be Lipschitz *if there exists a constant $C \geqslant 0$, such that for each* $(X_1, X_2) \in L^2 \times L^2$,

$$|F(t,X_1)-F(t,X_2)| \leqslant C\|X_1-X_2\|_2, \tag{2.1}$$

where $\|\cdot\|_2$ stands for the L^2 norm.

2.2 Examples of mean-field operators

Example 2.3 *(First-order interactions) The first example for the mean-field operator that satisfies the Lipschitz condition is the expectation of a Lipschitz function of the state:* $F(t,X) := \mathbb{E}[\phi(t,X)]$ *for* $X \in L^2(\mathcal{F}, P)$, *where*

$$\phi : [0,T] \times \mathbf{R} \mapsto \mathbf{R}, \quad (t,x) \mapsto \phi(t,x)$$

is a Lipschitz function such that $\phi(t,X) \in L^2$.

Note that for a random variable $X \in L^2(\mathcal{F}, P)$, *one can consider* $F(t,X)$ *as a lifted function of the law* P_X *of* X, *but one can also consider, e.g.,* $F(t,X) = E[X\eta]$, *for some fixed* $\eta \in L^2(\mathcal{F}, P)$, *so that* $F(t,X)$ *is not necessarily only a function of* P_X.

Example 2.4 *(Second-order interactions) A more general mean-field operator is given by*

$$F(t,X) = \mathbb{E}\left[\kappa(t,X,X_1)\right], \tag{2.2}$$

where κ is a Lipschitz kernel that captures the intensity of interactions and variable X_1 is an independent copy of the same distribution as X. The expectation is understood over the product space.

We now establish links of the operator (2.2) to a dynamic version of the inhomogeneous graph model of [4].

Inhomogeneous random graph and limit of average intensity of interactions. We consider a sequence of N points $(X^{i,N})_{i=1,N}$ in $\mathbf{R}$ equipped with a Borel-probability measure μ. We assume that the empirical measure $\dfrac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ converges in probability to a measure μ as $N \to \infty$, with $\mu(\mathbf{R}) = 1$.[1] We define a kernel κ as a measurable function on $\mathbf{R} \times \mathbf{R}$. Given the sequence $(X^{i,N})_{i=1,N}$, we consider a dynamic version of the inhomogeneous random graph model (see [4, Remark 2.4]). We let $(X_t^{i,N})_{i=1,N,t\in[0,T]} \in \mathcal{S}^2$, and we assume convergence of the unidimensional empirical distributions: for all $t \in [0,T]$, $\dfrac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ converges in probability

[1] This means that for any μ-continuity set A we have that $\#\{i,\ X^{i,N} \in A\}/N \xrightarrow{p} \mu(A)$.

to a measure μ_t as $N \to \infty$. Over time, nodes i and j interact according to a Poisson process of intensity $\kappa(X_t^{i,N}, X_t^{j,N})/N$.

Following [4], we say that a kernel κ is *graphical* if the following conditions hold:

(i) κ is continuous on $\mathbf{R} \times \mathbf{R}$;
(ii) $\kappa \in \mathcal{L}^1(\mathbf{R} \times \mathbf{R})$;
(iii) $\frac{1}{N}\mathbb{E}\sum_{i,j\in 1,N} \kappa(X_t^{i,N}, X_t^{j,N})/N \longrightarrow \int_{\mathbf{R}\times\mathbf{R}} \kappa(s,s')\mu_t(\mathrm{d}s)\mu_t(\mathrm{d}s')$, as $N \to \infty$, for all $t \in \mathcal{T}$.

The first two conditions are natural technical conditions. The last condition can be interpreted as the average intensity of interactions converges to a limit as the size of the graph tends to infinity.

We can now interpret the mean-field operator in the context of an inhomogeneous random graph as the limit of average intensity of interactions in the system. In the existing literature on mean-field models, the nodes' states depend on the average state of all other nodes. Here we allow dependence on the average intensity of interactions. If nodes do not interact with each other, there is little structural reason why the nodes' states depend on others. We propose the following operator suggested by condition (iii) above:

$$F(t,X) = \int_{\mathbf{R}\times\mathbf{R}} \kappa(s,s')\mu_{X_t}(\mathrm{d}s)\mu_{X_t}(\mathrm{d}s'), \tag{2.3}$$

where μ_{X_t} is a distribution of X_t. We can check that this operator satisfies Definition 2.2 and it is Lipschitz if the kernel κ is Lipschitz, that is, if there exists a constant $C > 0$ such that for each x, y, x', y', $|\kappa(x,y) - \kappa(x',y')| \leqslant C(|x-x'| + |y-y'|)$.

2.3 Properties of mean-field BSDEs with jumps

We now introduce the BSDE with jumps, whose driver depends on the mean-field operator F.

Definition 2.5 *(Mean-field BSDEs) A solution of a mean-field BSDE with jumps with driver f, mean-field operator F, terminal time T, and terminal condition ξ in $L^2(\mathcal{F}_T)$, consists of a triplet of processes $(X, Z, l) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu$ satisfying*

$$\begin{aligned} -\mathrm{d}X_t &= f(t,\omega,F(t,X_{t-}),X_{t-},Z_t,\ell_t(\cdot))\mathrm{d}t - Z_t\mathrm{d}W_t - \int_{\mathbf{E}} \ell_t(e)\tilde{N}(\mathrm{d}t,\mathrm{d}e); \\ X_T &= \xi, \end{aligned} \tag{2.4}$$

where X is a RCLL-optional process, and Z (resp., ℓ) is an $\mathbf{R}$-valued predictable process defined on $\Omega \times [0,T]$ (resp., $\Omega \times [0,T] \times \mathbf{R}^*$) such that the stochastic integral with respect to W (resp. $\tilde{N}$) is well defined. We denote by $(X(\xi,T), Z(\xi,T), \ell(\xi,T))$ the solution of the mean-field BSDE associated with terminal conditions (T,ξ).

2.3.1 Existence and uniqueness results

We note that an existence and uniqueness result was proven in [1] in the particular case of F capturing the expectation of the state (or a function thereof).

Theorem 2.6 *(Existence and Uniqueness for mean-field BSDEs) Let f be a Lipschitz driver, F a Lipschitz mean-field operator (see Definitions 2.1 and 2.2), and ξ in $L^2(\mathcal{F}_T)$. The mean-field BSDE (2.4) admits a unique solution $(X, Z, \ell(.)) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu$.*

Proof By using a priori estimates based on the Lipschitz property of both f and F, we establish the contraction property and the convergence of the Picard iterative sequence.

Let (X_s^n, Z_s^n, ℓ_s^n) be the solution of the following iterating BSDE with jumps

$$X_t^n = \xi + \int_t^T f(s, F(s, X_{s^-}^{n-1}), X_{s^-}^n, Z_s^n, \ell_s^n)\mathrm{d}s - \int_t^T Z_s^n \mathrm{d}B_s - \int_t^T \int_{\mathbf{E}} \ell_t^n(e)\tilde{N}(\mathrm{d}t, \mathrm{d}e), \tag{2.5}$$

for $n \geqslant 1$ and $t \in [0,1]$ and where, for $n = 0$, we set $(X_s^0, Z_s^0, \ell_s^0) = (0,0,0)$. The existence and the uniqueness in each iteration is established by classical results, see [25], and we denote by Φ the resulting map $(X^n, Z^n, \ell^n) = \Phi(X^{n-1}, Z^{n-1}, \ell^{n-1})$.[2]

Let $\bar{X}_t^n = X_t^n - X_t^{n-1}$; $\bar{Z}_t^n = Z_t^n - Z_t^{n-1}$; $\bar{\ell}_t^n = \ell_t^n - \ell_t^{n-1}$. For $\beta > 0$ and ϕ in $I\!H^2$, we introduce the norm $\|\phi\|_\beta := \mathbb{E}\left[\int_0^T e^{\beta s}\phi_s^2 \mathrm{d}s\right]$ and for l in $I\!H_\nu^2$, we set $\|l\|_{\nu,\beta} := \mathbb{E}\left[\int_0^T e^{\beta s}\|l_s\|_\nu^2 \mathrm{d}s\right]$. We now show that Φ is a contraction in $I\!H^2 \times I\!H^2 \times I\!H_\nu^2$ equipped with this norm (which implies that $(X^n, Z^n, \ell^n)_{n \geqslant 0}$ is a Cauchy sequence).

By applying Itô's formula to $e^{\beta s}|X_s^n - X_s^{n-1}|^2$, $n \geqslant 1$, we have, analogously to [22, Proposition A.4],

$$\begin{aligned} & e^{\beta t}(\bar{X}_t^n)^2 + \beta \int_t^T e^{\beta s}(\bar{X}_s^n)^2 \mathrm{d}s + \int_t^T e^{\beta s}(\bar{Z}_s^n)^2 \mathrm{d}s + \int_t^T e^{\beta s}\|\bar{\ell}_s^n\|_\nu^2 \mathrm{d}s \\ &= 2\int_t^T e^{\beta s}\bar{X}_s^n[f(s, F(s, X_s^{n-1}), X_s^n, Z_s^n, \ell_s^n) - f(s, F(s, X_s^{n-2}), X_s^{n-1}, Z_s^{n-1}, \ell_s^{n-1})]\mathrm{d}s \\ &\quad - 2\int_t^T e^{\beta s}\bar{X}_s^n \bar{Z}_s^n \mathrm{d}W_s - \int_t^T e^{\beta s}\int_{\mathbb{R}^*}(2\bar{X}_{s^-}^n \bar{\ell}_s^n(u) + \bar{\ell}_s^n(u)^2)\tilde{N}(\mathrm{d}s, \mathrm{d}e). \end{aligned}$$

Taking the conditional expectation given $\mathcal{F}_t$, which we denote by $\mathbb{E}_t$ (local martingales are martingales since $X_\cdot^n, X_\cdot^{n-1} \in \mathcal{S}^2$), we get

$$\begin{aligned} & e^{\beta t}(\bar{X}_t^n)^2 + \mathbb{E}_t\left[\beta\int_t^T e^{\beta s}(\bar{X}_s^n)^2 \mathrm{d}s + \int_t^T e^{\beta s}[(\bar{Z}_s^n)^2 + \|\bar{\ell}_s^n\|_\nu^2]\mathrm{d}s\right] \\ &= 2\mathbb{E}_t\left[\int_t^T e^{\beta s}\bar{X}_s^n[f(s, F(s, X_s^{n-1}), X_s^n, Z_s^n, l_s^n) - f(s, F(s, X_s^{n-2}), X_s^{n-1}, Z_s^{n-1}, \ell_s^{n-1})]\mathrm{d}s\right]. \end{aligned}$$

Moreover,

$$\begin{aligned} & |f(s, F(s, X_s^{n-1}), X_s^n, Z_s^n, \ell_s^n) - f(s, F(s, X_s^{n-2}(\cdot)), X_s^{n-1}, Z_s^{n-1}, \ell_s^{n-1})| \\ &\leqslant C_0[|F(s, X_s^{n-1}) - F(s, X_s^{n-2})| + |\bar{X}_s^n| + |\bar{Z}_s^n| + \|\bar{\ell}_s^n\|_\nu] \\ &\leqslant C[\|\bar{X}_s^{n-1}\|_2 + |\bar{X}_s^n| + |\bar{Z}_s^n| + \|\bar{\ell}_s^n\|_\nu] = C[(\mathbb{E}|\bar{X}_s^{n-1}|^2)^{\frac{1}{2}} + |\bar{X}_s^n| + |\bar{Z}_s^n| + \|\bar{\ell}_s^n\|_\nu]. \end{aligned}$$

Now, for all real numbers x, z, l and $\varepsilon > 0$, $2x(Cx + Cz + Cl) \leqslant \dfrac{x^2}{\varepsilon^2} + \varepsilon^2(Cx + Cz + Cl)^2 \leqslant \dfrac{x^2}{\varepsilon^2} + 3\varepsilon^2(C^2x^2 + C^2z^2 + C^2\ell^2)$, and, for all η, $\mathbb{E}[2\bar{X}_s^n(\mathbb{E}|\bar{X}_s^{n-1}|^2)^{\frac{1}{2}}] \leqslant \mathbb{E}[\frac{1}{\eta^2}|\bar{X}_s^n|^2 + \eta^2\mathbb{E}|\bar{X}_s^{n-1}|^2] = \dfrac{1}{\eta^2}\mathbb{E}|\bar{X}_s^n|^2 + \eta^2\mathbb{E}|\bar{X}_s^{n-1}|^2$. Thus, we obtain that

$$\begin{aligned} & e^{\beta t}(\bar{X}_t^n)^2 + \mathbb{E}_t\left[\beta\int_t^T e^{\beta s}(\bar{X}_s^n)^2 \mathrm{d}s + \int_t^T e^{\beta s}[(\bar{Z}_s^n)^2 + \|\bar{\ell}_s^n\|_\nu^2]\mathrm{d}s\right] \\ &\leqslant \mathbb{E}_t\left[\left(\frac{C}{\eta^2} + \frac{1}{\epsilon^2}\right)\int_t^T e^{\beta s}(\bar{X}_s^n)^2 \mathrm{d}s + C\eta^2\int_t^T e^{\beta s}(\bar{X}_s^{n-1})^2 \mathrm{d}s \right. \\ &\quad \left. + 3C^2\epsilon^2\int_t^T e^{\beta s}[(\bar{X}_s^n)^2 + (\bar{Z}_s^n)^2 + \|\bar{\ell}_s^n\|_\nu^2]\mathrm{d}s\right]. \end{aligned}$$

[2] Note that in our current setting the map Φ does not depend on its last two arguments, but it would if we had a mean-field term of the form $F(s, X_s^{n-1}(\cdot), Z^{n-1}(\cdot), \ell^{n-1}(\cdot))$. Existence and uniqueness results would go similarly through in this case.

By choosing $\eta=\epsilon$, and β and ϵ such that $\beta-\frac{C+1}{\epsilon^2}-3C\epsilon^2\geqslant 2C\epsilon^2$ and $1-3C^2\epsilon^2\geqslant 2C\epsilon^2$, we obtain the contraction inequality

$$\|\bar{X}^n\|_\beta^2+\|\bar{Z}^n\|_\beta^2+\|\bar{\ell}^n\|_{\nu,\beta}^2\leqslant\frac{1}{2}(\|\bar{X}^{n-1}\|_\beta^2+\|\bar{Z}^{n-1}\|_\beta^2+\|\bar{\ell}^{n-1}\|_{\nu,\beta}^2). \tag{2.6}$$

Therefore the map ϕ is a contraction with respect to the norm $\|\cdot\|_\beta$. By the Banach fixed-point theorem, the map Φ has a unique fixed point (X,Z,ℓ). Now taking the limit in (2.5), we conclude that (X,Z,ℓ) is the unique solution of (2.4). □

2.3.2 **Comparison results**

In this section, in order to compare the first components of the solutions of two mean-field BSDEs, we need additional assumptions due to the presence of jumps and of the mean-field operator.

Assumption 2.7 *Assume that* $\mathrm{d}t\otimes\mathrm{d}F$*-a.s. for each* $(x',x,z,\ell_1,\ell_2)\in\mathbb{R}^3\times(L_\nu^2)^2$,

$$f(t,x',x,z,\ell_1)-f(t,x',x,z,\ell_2)\geqslant\langle\theta_t^{x',x,z,\ell_1,\ell_2}\,,\,\ell_1-\ell_2\rangle_\nu$$

with

$$\theta_t^{x',x,z,\ell_1,\ell_2}:[0,T]\times\Omega\times\mathbb{R}^3\times(L_\nu^2)^2\to L_\nu^2;\quad (t,\omega,x',x,z,\ell_1,\ell_2)\mapsto\theta_t^{x',x,z,\ell_1,\ell_2}(\omega,.),$$

$\mathcal{P}\otimes\mathcal{B}(\mathbf{R}^3)\otimes\mathcal{B}((L_\nu^2)^2)-$ *measurable, bounded, and satisfying* $\mathrm{d}P\otimes\mathrm{d}t\otimes\mathrm{d}\nu(u)$*-a.s., for each* $(x',x,z,\ell_1,\ell_2)\in\mathbb{R}^3\times(L_\nu^2)^2$,

$$\theta_t^{x',x,z,\ell_1,\ell_2}(u)\geqslant-1\quad\textit{and}\quad|\theta_t^{x',x,z,\ell_1,\ell_2}(u)|\leqslant\psi(u), \tag{2.7}$$

where $\psi\in L_\nu^2$.

Theorem 2.8 *(Comparison Theorem for mean-field BSDEs) Let* $f_i=f_i(\omega,t,x',x,z,l)$, $i=1,2$, *be two Lipschitz drivers, and* f_1 *satisfies Assumption 2.7. Furthermore, we assume:*

- f_2 *is non-decreasing in* x';

F *is Lipschitz on* L^2 *and satisfies the following property:*

- F *is non-decreasing in* X *in the following sense: let* $X_1,X_2\in L^2$, *if* $X_1\leqslant X_2$ *a.s., then for each* $t\in[0,T]$, $F(t,X_1)\leqslant F(t,X_2)$.

Let $\xi_1,\xi_2\in L^2$ *and denote by* (X^1,Z^1,ℓ^1) *and* (X^2,Z^2,ℓ^2) *the solution of the mean-field BSDE with jumps (2.4) associated with* (ξ_1,f_1) *and* (ξ_2,f_2). *Then, suppose that*

- $\xi_1\geqslant\xi_2$, *a.s.,*
- $f_1(\omega,t,x',x,z,\ell(\cdot))\geqslant f_2(\omega,t,x',x,z,\ell(\cdot))$, *a.s. for all* $(t,x',x,z,\ell(\cdot))\in\mathbf{R}^4\times L_\nu^2$.

Then, we have $X_t^1\geqslant X_t^2$, $\forall t\in[0,T]$ *a.s..*

Proof For $i=1,2$, let $(X_s^{i,n},Z_s^{i,n},l_s^{i,n})$ be the solution of the following iterating BSDE with jumps

$$\begin{aligned}X_t^{i,n}=\xi_i&+\int_t^T f_i(s,F(s,X_s^{i,n-1}),X_s^{i,n},Z_s^{i,n},\ell_s^{i,n})\mathrm{d}s\\&-\int_t^T Z_s^{i,n}\mathrm{d}B_s-\int_t^T\int_{\mathbf{E}}\ell_t(e)\tilde{N}(\mathrm{d}t,\mathrm{d}e),\end{aligned} \tag{2.8}$$

for $n\geqslant 1$ and $t\in[0,1]$. For $n=0$, we set $(X_s^{i,0},Z_s^{i,0},\ell_s^{i,0})=(0,0,0)$.

Now, we define

$$\tilde{f}_1^n(s,x,z,l) = f_1(s, F(s, X_s^{1,n-1}), x, z, l),$$
$$\tilde{f}_2^n(s,x,z,l) = f_2(s, F(s, X_s^{2,n-1}), x, z, l).$$

Then, we have $\tilde{f}_1^1 \geqslant \tilde{f}_2^1$ and $\tilde{f}_1^1$ satisfies the monotone assumption in [23, Theorem 4.2]. Thus by the classic comparison theorem for BSDE with jumps [23, Theorem 4.2], we have

$$X_s^{1,1} \geqslant X_s^{2,1} \text{ a.s.}, \quad s \in [0,T]. \tag{2.9}$$

Now, since f_2 is non-decreasing in x', we have

$$\begin{aligned}\tilde{f}_1^2(s,x,z,l) &= f_1(s, F(s, X_s^{1,1}), x, z, l) \geqslant f_2(s, F(s, X_s^{1,1}), x, z, l)\\ &\geqslant f_2(s, F(s, X_s^{2,1}), x, z, l) = \tilde{f}_2^2(s,x,z,l),\end{aligned}$$

where the last inequality follows from (2.9) and f_2 and F are non-decreasing. Using again the comparison results for classic BSDEs with jumps, we get

$$X_s^{1,2} \geqslant X_s^{2,2} \text{ a.s.}, \quad s \in [0,T].$$

By the same argument as above, we iteratively obtain that

$$X_s^{1,n} \geqslant X_s^{2,n} \text{ a.s.}, \quad s \in [0,T], \quad n \geqslant 1. \tag{2.10}$$

Using the proof of the existence and uniqueness result, we have that for $i = 1,2$, $(X^{i,n}, Z^{i,n}, \ell^{i,n})_{n\geqslant 0}$ converges to the respective solution with drivers $(f_i)_{i=1,2}$, call these X^i, Z^i, ℓ^i. We have that $X_t^1 \geqslant X_t^2, t \in [0,T]$ a.s. follows directly from the fact that $X_t^{1,n} \geqslant X_t^{2,n}, t \in [0,T]$ a.s.. □

Remark 2.9 *We can weaken the non-decreasing property of F, and consider it in the sense of distribution. Let $D_1(x) = \mathbb{P}(X_1 \leqslant x)$ and $D_2(x) = \mathbb{P}(X_2 \leqslant x)$. Then we say that F is non-decreasing in x, if $D_1(x) \geqslant D_2(x)$ implies $F(t, X_1) \leqslant F(t, X_2)$.*

Take Example 2.3, when $F(X) = \mathbb{E}(\phi(t,X))$, for $X \in L^2$, F is non-decreasing if ϕ is C^1 and non-decreasing. Assuming that D_1 and D_2 are sufficiently smooth, this can be verified by direct computation : $F(t,X_1) - F(t,X_2) = \mathbb{E}(\phi(t,X_1)) - \mathbb{E}(\phi(t,X_2)) = \int \phi(t,x)\mathrm{d}(D_1 - D_2)(x) = \int \frac{\partial \phi}{\partial x}(t,x)(D_2 - D_1)(x)\mathrm{d}x = \int \frac{\partial \phi}{\partial x}(t,x)[D_2(x) - D_1(x)]\mathrm{d}x$.

In the example of an operator given by an inhomogeneous random graph of section 2.2, we make the assumption that the kernel κ is ($D_1 \times D_1$ and $D_2 \times D_2-$almost everywhere) differentiable and

$$\frac{\partial^2}{\partial x \partial y}\kappa(x,y) > 0. \tag{2.11}$$

In this case, $F(t,X_1) = \int \kappa(x,y)\mathrm{d}D_1(x)\mathrm{d}D_1(y) = \int_{\mathbf{R}\times\mathbf{R}} \frac{\partial^2}{\partial x \partial y}\kappa(x,y)D_1(x)D_1(y)\mathrm{d}x\mathrm{d}y$ and the non-decreasing property of F follows from (2.11). This condition, along with the Lipschitz condition, are satisfied if one uses a truncated Gaussian kernel of coefficient σ

$$\kappa(x,y) = \begin{cases} e^{\frac{-(x-y)^2}{2\sigma^2}}, & \textit{if } |x-y| < \sigma,\\ 0, & \textit{otherwise.}\end{cases} \tag{2.12}$$

We also note that $X_1 \leqslant X_2$ a.s. implies that $D_1(x) \geqslant D_2(x)$.

Remark 2.10 *Symmetrically, if we assume one of the drivers is non-increasing in x' and F is a non-increasing operator, the arguments in Theorem 2.8 still hold.*

We now provide a strict comparison theorem, which states that under *a strict inequality* on the map θ, two solutions of the BSDEs are equal at all times if they are equal at the initial time. This theorem is used for bullet 5 in section 3.1 and is interesting in the context of no-arbitrage pricing theory.

Theorem 2.11 *(Strict comparison for mean-field BSDEs) Suppose the assumptions of Theorem 2.8 hold with strict inequality* $\theta_t^{x',x,z,\ell^1,\ell^2}(u) > -1 \ \ \mathrm{d}t \otimes \mathrm{d}P$*-a.s.. If* $X^1_{t_0} = X^2_{t_0}$ *a.s. for some* $t_0 \in [0,T]$*, then* $X^1_\cdot = X^2_\cdot$ *a.s. on* $[t_0,T]$.

Proof Let $\bar{X}_s = X_s^1 - X_s^2$; $\bar{Z}_s = Z_s^1 - Z_s^2$; $\bar{\ell}_s(u) = \ell_s^1(u) - \ell_s^2(u)$. We suppose that f_2 is non-decreasing in x' and Assumption 2.7 with strict inequality holds for f_1. Then

$$-\mathrm{d}\bar{X}_s = h_s \mathrm{d}s - \bar{Z}_s \mathrm{d}W_s - \int_{\mathbf{E}} \bar{\ell}_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e), \quad \bar{X}_T = \xi_1 - \xi_2,$$

where $h_s := f_1(s, F(s, X_s^1), X_s^1, Z_s^1, \ell_s^1) - f_2(s, F(s, X_s^2), X_s^2, Z_s^2, \ell_s^2)$.

Let $\phi(s) := f_1(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^2) - f_2(s, F(s, X_s^2), X_s^2, Z_s^2, \ell_s^2)$. Note that we have

$$h_s = \phi_s + f_1(s, F(s, X_s^1), X_s^1, Z_s^1, \ell_s^1) - f_1(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^2).$$

Below we use a classical linearization procedure. We can write

$$\begin{aligned} & f_1(s, F(s, X_s^1), X_s^1, Z_s^1, \ell_s^1) - f_1(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^2) \\ = & f_1(s, F(s, X_s^1), X_s^1, Z_s^1, \ell_s^1) - f_1(s, F(s, X_s^1), X_s^2, Z_s^1, \ell_s^1) + f_1(s, F(s, X_s^1), X_s^2, Z_s^1, \ell_s^1) \\ & - f_1(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^1) + f_1(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^1) - f_1(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^2). \end{aligned}$$

Then from the Assumption 2.7 on f_1, there exist bounded processes δ and β on $\bar{\Omega} \times [0,T]$, such that

$$f_1(s, F(s, X_s^1), X_s^1, Z_s^1, \ell_s^1) - f_1(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^2) \geqslant \delta_s \bar{X}_s + \beta_s \bar{Z}_s + \langle \theta_s, \bar{\ell}_s \rangle_\nu$$

with

$$\delta_s := \frac{f_1(s, F(s, X_s^1), X_s^1, Z_s^1, \ell_s^1) - f_1(s, F(s, X_s^1), X_s^2, Z_s^1, \ell_s^1)}{\bar{X}_s} \mathbf{1}_{\{\bar{X}_s \neq 0\}},$$

$$\beta_s := \frac{f_1(s, F(s, X_s^1), X_s^2, Z_s^1, \ell_s^1) - f_1(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^1)}{\bar{Z}_s} \mathbf{1}_{\{\bar{Z}_s \neq 0\}},$$

and θ_s is as in Assumption 2.7.

Thus we have $h_s \geqslant \phi_s + \delta_s \bar{X}_s + \beta_s \bar{Z}_s + \langle \theta_s, \bar{\ell}_s \rangle_\nu \ \ \mathrm{d}t \otimes \mathrm{d}P$-a.s.. For each $t \in [0,T]$, let $(\Gamma_{t,s})_{s\in[t,T]}$ be the unique solution of the forward SDE

$$\mathrm{d}\Gamma_{t,s} = \Gamma_{t,s^-}\big[\delta_s \mathrm{d}s + \beta_s \mathrm{d}W_s + \int_{\mathbf{E}} \theta_s(e)\tilde{N}(\mathrm{d}t,\mathrm{d}e)\big]; \quad \Gamma_{t,t} = 1.$$

By classical comparison results with respect to a linear BSDE (see, e.g., [23, Lemma 4.1]), we derive that

$$\bar{X}_{t_0} \geqslant \mathbb{E}\big[\Gamma_{t_0,t}(\xi_1 - \xi_2) + \int_{t_0}^t \Gamma_{t_0,s}\phi(s)\mathrm{d}s | \mathcal{F}_{t_0}\big], \quad t_0 \leqslant t \leqslant T.$$

By Theorem 2.8, we have $\bar{X}_t = X_t^1 - X_t^2 \geqslant 0$, and using the non-decreasing property of F, we can write

$$f_1(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^2) \geqslant f_2(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^2) \geqslant f_2(s, F(s, X_s^2), X_s^2, Z_s^2, \ell_s^2)$$

by the assumption on f_1 and f_2. We conclude the proof by pointing out the fact that

$$\phi(s) = f_1(s, F(s, X_s^1), X_s^2, Z_s^2, \ell_s^2) - f_2(s, F(s, X_s^2), X_s^2, Z_s^2, \ell_s^2) \geqslant 0$$

and that if $\theta_s(u) > -1 \mathrm{d}P \otimes \mathrm{d}s \otimes \mathrm{d}\nu(u)$-a.s., then $\Gamma_{t,s} > 0$ a.s. from [23, Corollary 3.5]. □

Remark 2.12 *We can weaken the assumption of Theorem 2.11 by assuming that the strict inequality* $\theta_t^{x',x,z,\ell^1,\ell^2}(u) > -1$ *holds only along the solutions, that is,* $\theta_t^{F(t,X_t^1),X_t^2,Z_t^2,\ell_t^1,\ell_t^2}(u) > -1 \;\; \mathrm{d}t \otimes \mathrm{d}P$*-a.s.. In the symmetric case when* f_2 *is Lipschitz and non-decreasing in* x'*, we obtain the results by assuming* $\theta_t^{F(t,X_t^2),X_t^2,Z_t^2,\ell_t^1,\ell_t^2}(u) > -1 \;\; \mathrm{d}t \otimes \mathrm{d}P$*-a.s..*

3. Global dynamic risk measures

In light of the role played by BSDEs in risk measures (see, e.g., [3, 14, 22], to name just a few, for the Brownian case, and [23, 24] for the addition of jumps), we now explore the link between mean-field BSDEs and dynamic risk measures, which we interpret in this case as global dynamic risk measures. The global dynamic risk measure can be seen as the regulatory capital or liquidity for a representative bank.

3.1 Definition and properties

Let $T > 0$ be a time horizon and f be a Lipschitz driver. Set

$$\rho_t(\eta, T) := -X_t(\eta, T), \quad 0 \leqslant t \leqslant T, \tag{3.1}$$

where $X_t(\eta, T)$ denotes the solution of the mean-field BSDE (2.4) with driver f, mean-field operator F, and terminal condition $\eta \in L^2(\mathcal{F}_T)$. We can think of η as the amount of liquidity or capital that is acceptable to a regulator at time T. The risk measure $\rho_t(\eta, T)$ is interpreted as the amount of capital needed at time t in order to be acceptable at time T. Note also that, in insurance, the functional $-\rho = X$ can represent a risk premium.

The functional $\rho : (\eta, T) \mapsto \rho_\cdot(\eta, T)$ represents a *global dynamic risk measure* induced by the mean-field BSDE with driver f and mean-field operator F. When the dependence on the time horizon T is clear from the context, we drop it from the notation and write $\rho_t(\eta)$ for $\rho_t(\eta, T)$.

We now provide properties of global dynamic risk measures, based on the comparison results of the previous section. We work under Assumption 2.7, which guarantees the monotonicity of the risk measure in the jumps. Contrary to the standard non mean-field case, the risk of a zero position may not be zero when the mean-field operator F is introduced. The first three properties, which we give for completeness, are standard and follow the pattern of the classical literature on dynamic risk measures, see, e.g., [23], in which we plug the comparison or uniqueness results specific to our case.

(1) *Consistency.* Let S be a stopping time. Then for each time t smaller than S, the risk-measure associated with position ξ and maturity T coincides with the risk-measure associated with maturity S and position $-\rho_S(\xi, T) = X_S(\xi, T)$, that is,

$$\forall t \leqslant S, \quad \rho_t(\xi, T) = \rho_t(-\rho_S(\xi, T), S) \quad \text{a.s..}$$

This corresponds to the flow property of mean-field BSDEs, which is the consequence of the uniqueness result.

(2) *Continuity.* Let $\{\theta^\alpha, \alpha \in R\}$ be a family of stopping times converging a.s. to a stopping time θ as α tends to α_0. Let $\{\xi^\alpha, \alpha \in R\}$ be a family of random variables such that $\mathbb{E}[\operatorname{ess\,sup}_\alpha (\xi^\alpha)^2] < \infty$, and for each α, ξ^α is $\mathcal{F}_{\theta^\alpha}$-measurable. Suppose also that ξ^α converges a.s. to an $\mathcal{F}_\theta$-measurable random variable ξ as α tends to α_0. Then for each stopping time S, the random variable $\rho_S(\xi^\alpha, \theta^\alpha) \to \rho_S(\xi, \theta)$ a.s. and the processes $\rho(\xi^\alpha, \theta^\alpha) \to \rho(\xi, \theta)$ in $\mathcal{S}^2$

when $\alpha \to \alpha_0$.

This property follows as in the proofs of [23, appendix], in which the a priori estimates for mean-field BSDEs can be easily extended to account for the mean-field contribution.

(3) *Monotonicity.* ρ is non-increasing with respect to ξ, i.e., for each $\xi^1, \xi^2 \in L^2$, if $\xi^1 \geqslant \xi^2$ a.s., then $\rho_t(\xi^1, T) \leqslant \rho_t(\xi^2, T)$, $0 \leqslant t \leqslant T$ a.s..

This is the direct consequence of the comparison results (Theorem 2.8).

(4) *Convexity.* Suppose f is concave with respect to (x', x, z, l). Moreover, suppose that f is non-decreasing in x' and F is a non-decreasing concave operator in X. Then the dynamic risk-measure ρ is convex, that is, for any $\lambda \in [0,1]$, $\xi^1, \xi^2 \in L^2$

$$\rho(\lambda\xi^1 + (1-\lambda)\xi^2, T) \leqslant \lambda\rho(\xi^1, T) + (1-\lambda)\rho(\xi^2, T).$$

Proof For $i = 1, 2$, let (X^i, Z^i, ℓ^i) be a solution of the mean-field BSDE (2.4) associated to terminal time T, driver f, mean-field operator F, and terminal condition ξ^i. Set $\hat{\xi} := \lambda\xi^1 + (1-\lambda)\xi^2$, $\hat{X} := \lambda X^1 + (1-\lambda)X^2$, $\hat{Z} := \lambda Z^1 + (1-\lambda)Z^2$, $\hat{\ell} := \lambda\ell^1 + (1-\lambda)\ell^2$. We have

$$\begin{aligned} -\mathrm{d}\hat{X}_t &= [\lambda f(t, F(t, X_t^1), X_t^1, Z_t^1, \ell_t^1) + (1-\lambda) f(t, F(t, X_t^2), X_t^2, Z_t^2, \ell_t^2)]\mathrm{d}t \\ &\quad - \hat{Z}_t \mathrm{d}W_t - \int_{\mathbf{E}} \hat{\ell}_t(e)\tilde{N}(\mathrm{d}t, \mathrm{d}e); \quad \hat{X}_T = \hat{\xi}. \end{aligned}$$

By the assumptions on f and F, we have

$$\begin{aligned} &\lambda f(t, F(t, X_t^1), X_t^1, Z_t^1, \ell_t^1) + (1-\lambda) f(t, F(t, X_t^2), X_t^2, Z_t^2, \ell_t^2) \\ &\leqslant f(t, \lambda F(t, X_t^1) + (1-\lambda)F(t, X_t^2), \lambda X_t^1 + (1-\lambda)X_t^2, \lambda Z_t^1 + (1-\lambda)Z_t^2, \lambda\ell_t^1 + (1-\lambda)\ell_t^2) \\ &\leqslant f(t, F(t, \lambda X_t^1 + (1-\lambda)X_t^2), \lambda X_t^1 + (1-\lambda)X_t^2, \lambda Z_t^1 + (1-\lambda)Z_t^2, \lambda\ell_t^1 + (1-\lambda)\ell_t^2) \\ &= f(t, F(t, \hat{X}_t), \hat{X}_t, \hat{Z}_t, \hat{\ell}_t). \end{aligned}$$

Thus

$$-\mathrm{d}\hat{X}_t \leqslant f(t, F(t, \hat{X}_t), \hat{X}_t, \hat{Z}_t, \hat{\ell}_t)\mathrm{d}t - \hat{Z}_t \mathrm{d}W_t - \int_{\mathbf{E}} \hat{\ell}_t(e)\tilde{N}(\mathrm{d}t, \mathrm{d}e); \ \hat{X}_T = \hat{\xi}.$$

Let $(\bar{X}, \bar{Z}, \bar{\ell})$ be a solution of the mean-field BSDE (2.4) associated to terminal time T, driver f, mean-field operator F, and terminal condition $\hat{\xi}$, i.e.,

$$-\mathrm{d}\bar{X}_t = f(t, F(t, \bar{X}_t), \bar{X}_t, \bar{Z}_t, \bar{\ell}_t)\mathrm{d}t - \bar{Z}_t \mathrm{d}W_t - \int_{\mathbf{E}} \bar{\ell}_t(e)\tilde{N}(\mathrm{d}t, \mathrm{d}e); \ \bar{X}_T = \hat{\xi}.$$

By the (extended) comparison results in Theorem 2.8, we get $\hat{X}_t \leqslant \bar{X}_t$ as desired. □

Furthermore, suppose in Assumption 2.7 that we have $\theta_t^{x', x, z, \ell^1, \ell^2}(u) > -1$. Then we have the following property.

(5) *No Arbitrage.* For each $\xi^1, \xi^2 \in L^2$, if $\xi^1 \geqslant \xi^2$ a.s. and if $\rho_t(\xi^1, T) = \rho_t(\xi^2, T)$ a.s. on $A \in \mathcal{F}_t$, then $\xi^1 = \xi^2$ a.s. on A.

This is the direct consequence of the strict comparison results (Theorem 2.11).

3.2 Dual representation for convex global risk measures

We now provide a representation for the expectation of risk measures induced by mean-field BSDEs in terms of the value of a stochastic control problem in the convex case. This dual representation is given via the supremum over a set of probability measures which are absolutely continuous with respect to F.

With the mean-field term, the dual representation relies on a careful analysis of the

Fenchel–Legendre transform F^*. We first need to establish bounds on the effective domain of this transform (Lemma 3.3). Moreover, since the mean-field operator is inside the driver, we have an additional dual variable with respect to the classical case and we need to establish new bounds for it. New elements specific to the mean-field case arise. In particular, Lemma 3.5 provides an explicit form to the conjugacy relation of (F, F^*) and relies on a new SDE of mean-field type, for which we give uniqueness and existence results in Lemma 3.4. The presence of the mean-field term in combination with the jumps poses technical challenges, in particular, to ensure the equivalence of the worst-case probability measure appearing in the representation and the real-world probability measure.

In the following, we make the following assumptions on f and F.

Assumption 3.1 *Let f be a Lipschitz driver. Suppose f is concave with respect to (x', x, z, l) and non-decreasing with respect to x' and satisfies Assumption 2.7 with strict inequality $\theta_t(u) > -1$ $\mathrm{d}t \otimes \mathrm{d}P$-a.s.. Let F be a Lipschitz mean-field operator. Suppose F is non-decreasing and concave in X, and moreover satisfies the following property: for each $s, t \in [0, T]$ and $X \in L^2$, $F(t, \mathbb{E}[X|\mathcal{F}_s]) \geqslant F(t, X)$.*

For each (ω, t), we denote by f^* the Fenchel–Legendre transform of f, defined for each $(\beta, q, \alpha_1, \alpha_2) \in \mathbf{R}^3 \times L^2_\nu$ and F^* the Fenchel–Legendre transform of F, defined for each $\delta \in L^2$, that is (see, e.g., [12]),

$$\begin{aligned} f^*(\omega, t, q, \beta, \alpha_1, \alpha_2) &= \sup_{(x', x, z, l) \in \mathbf{R}^3 \times L^2_\nu} [f(\omega, t, x', x, z, l) - qx' - \beta x - \alpha_1 z - \langle \alpha_2, l \rangle_\nu], \\ F^*(t, \delta) &= \sup_{X \in L^2} [F(t, X) - \langle X, \delta \rangle_{L^2}]. \end{aligned}$$

For each predictable processes $\alpha_t = (\alpha^1_t, \alpha^2_t(\cdot))$, let Q^α be the probability absolutely continuous with respect to P which admits Γ^α_T as density with respect to P on $\mathcal{F}_T$, where Γ^α is the solution of

$$\mathrm{d}\Gamma^\alpha_t = \Gamma^\alpha_{t-}\big(\alpha^1_t \mathrm{d}W_t + \int_{\mathbf{R}^*} \alpha^2_t(u) \mathrm{d}\tilde{N}(\mathrm{d}t, \mathrm{d}u)\big); \quad \Gamma^\alpha_0 = 1. \tag{3.2}$$

The process $W^\alpha_t := W_t - \int_0^t \alpha^1(s, \alpha_s)\mathrm{d}s$ is a Brownian motion with respect to Q^α and N is a Poisson random measure independant from W^α under Q^α with compensated process $\tilde{N}^\alpha(\mathrm{d}t, \mathrm{d}u) := \tilde{N}(\mathrm{d}t, \mathrm{d}u) - \alpha^2(t, \alpha_t, u)\nu(\mathrm{d}u)\mathrm{d}t$.

Now we define the dual set of probability measures in terms of their densities. Let $\mathcal{A}_T$ be the set of predictable processes $\alpha_s = (\alpha^1_s, \alpha^2_s)$ such that

- $\int_0^T (\alpha^1_s)^2 \mathrm{d}s + \int_0^T \|\alpha^2_s\|^2_\nu \mathrm{d}s$ is bounded.
- $\alpha^2_s(u) > -1$ $\nu(\mathrm{d}u)$-a.s..

From [23, Propositions 3.1 and 3.2], we have that for all $\alpha. \in \mathcal{A}_T$, $\Gamma^\alpha_t > 0, 0 \leqslant t \leqslant T$ a.s. and $(\Gamma^\alpha_t)_{0 \leqslant t \leqslant T} \in \mathcal{S}^2$. We therefore obtain that Q^α and P are equivalent.

Let $\bar{A}_T$ be the set of processes $(\gamma_t, \beta_t, q_t, \alpha^1_t, \alpha^2_t)$, where $(\beta_t, q_t, \alpha^1_t, \alpha^2_t)$ are predictable and γ_t is adapted, such that:

- $(f^*(\omega, t, q_t, \beta_t, \alpha^1_t, \alpha^2_t))_{t \in [0,T]}$ belongs to $I\!H^2$;
- $\alpha_t = (\alpha^1_t, \alpha^2_t(\cdot))_{t \in [0,T]}$ belongs to $\mathcal{A}_T$;
- $0 \leqslant q_t \leqslant C, \forall t \in [0, T]$, $\mathrm{d}P$ a.s.;
- The processes $(\Gamma^\alpha_t e^{-\int_0^t \gamma_s \mathrm{d}s})_{t \in [0,T]}$ belong to $I\!H^2$.

3.2.1 Technical lemmas

We begin with some technical lemmas establishing bounds on the effective domain of the Fenchel–Legendre transforms f^* and F^*.

Lemma 3.2 *For each (s,ω), the set of $(q,\beta,\alpha_1,\alpha_2)\in\mathbf{R}^3\times L^2_\nu$ such that $f^*(\omega,s,q,\beta,\alpha_1,\alpha_2)<+\infty$ is included in the set U defined by the following conditions:*

- *$q\geqslant 0$ and is bounded by C;*
- *β and α_1 are bounded by C;*
- *$\alpha_2(u)>-1$ and $|\alpha_2(u)|\leqslant C$ $\nu(\mathrm{d}u)$-a.s.,*

where C is the Lipschitz constant of f.

Proof Suppose by contradiction that $q<0$. By the definition of f^*, we have

$$f^*(t,q,\beta,\alpha_1,\alpha_2)\geqslant f(t,x',0,0,0)-x'q$$

for each x'. This holds, in particular, for $x_n:=n$, $n\in\mathbb{N}$. We thus get

$$f^*(t,q,\beta,\alpha_1,\alpha_2)\geqslant f(t,n,0,0,0)-nq\geqslant f(t,0,0,0,0)-nq,$$

where the last inequality follows by the non-decreasingness of the map f with respect to x'. By letting $n\to+\infty$ in the above inequality, we get $\lim_{n\to+\infty}f(t,0,0,0,0)-qn=+\infty$, since $q<0$. This implies that $f^*(t,q,\beta,\alpha_1,\alpha_2)=+\infty$, which provides the desired contradiction, We thus have proved that $q\geqslant 0$. The fact that q, β, and α are included in the bounded domain $[-C,C]$ is due to the uniform Lipschitz property of f. Finally, for the bounds on (α_1,α_2), the proof follows as in the classical case, see, e.g., [23, Lemma 5.4]. □

Lemma 3.3 *Assume that operator F satisfies Assumption 3.1. Then for each $t\in[0,T]$, $\{\delta\in L^2|F^*(t,\delta)<+\infty\}$ (the effective domain of F^*) satisfies the following property: $\delta\geqslant 0$ $\mathrm{d}P$ a.s. and $\|\delta\|_2\leqslant C$, where C is the Lipschitz constant of F.*

Proof Suppose $\delta\geqslant 0$ $\mathrm{d}P$-a.s. is not true. We denote $A=\{\omega\in\Omega\mid\delta(\omega)<0\}$. Then $P(A)>0$. By the definition of F^*, we have for each $X\in L^2$,

$$F^*(t,\delta)\geqslant F(t,X)-\langle X,\delta\rangle_{L^2}=F(t,X)-\mathbb{E}^P[X\delta].$$

This holds, in particular, for $X_n(\omega):=-n\delta(\omega)\mathbf{1}_A(\omega)$, where $n\in\mathbb{N}$. This gives $X_n\geqslant 0$ $\mathrm{d}P$ a.s. and thus by the non-decreasing properties of F, we obtain

$$F^*(t,\delta)\geqslant F(t,X^n)-\mathbb{E}^P[X_n\delta]\geqslant F(t,0)-\mathbb{E}^P[X_n\delta]=F(t,0)+n\int_A|\delta(\omega)|^2\mathrm{d}P(\omega).$$

By letting $n\to+\infty$ in the above inequality, we get $F^*(\delta)=+\infty$, which gives the desired contradiction, which implies $\delta\geqslant 0$ $\mathrm{d}P$ a.s.. The boundedness of δ is a direct consequence of the Lipschitz property of F. □

The following lemma establishes the existence of the solution of a particular mean-field SDE, used towards an explicit form to the conjugacy relation of (F,F^*) in Lemma 3.5 below.

Lemma 3.4 *Let $(\alpha^1_s,\alpha^2_s(\cdot))_{s\geqslant t}$ belong to $\mathcal{A}_T$, $(U_s)_{s\geqslant t}$ be adapted and uniformly bounded in L^2 and $(h_s)_{s\geqslant t}$ uniformly bounded almost surely. Then the following SDE admits a solution $(V_s)_{s\geqslant t}\in\mathcal{S}^2$.*

$$\begin{aligned} &\mathrm{d}V_s = V_{s^-}[\alpha_s^1\mathrm{d}W_s + \int_{\mathbf{R}^*}\alpha_s^2(e)\mathrm{d}\tilde{N}(\mathrm{d}s,\mathrm{d}e)] + U_s\mathbb{E}[V_s h_s]\mathrm{d}s, \quad t \leqslant s \leqslant T, \\ &V_t = 1. \end{aligned} \tag{3.3}$$

Proof We define inductively the sequence (V^n) of the processes by setting $V^0 \equiv 1$ and for $n \geqslant 1$,

$$V_u^n = 1 + \int_t^u V_s^{n-1}\mathrm{d}M_s + \int_t^u U_s\mathbb{E}[V_s^{n-1}h_s]\mathrm{d}s,$$

where $\mathrm{d}M_s = \alpha_s^1\mathrm{d}W_s + \int_{\mathbf{R}^*}\alpha_s^2(e)\mathrm{d}\tilde{N}(\mathrm{d}s,\mathrm{d}e)$. Note that V^n is adapted and right-continuous and, moreover, we have

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\leqslant s\leqslant u}|V_s^{n+1} - V_s^n|^2\right] \\ &\leqslant 2\mathbb{E}\left[\sup_{t\leqslant s\leqslant u}\left(\int_t^s (V_r^n - V_r^{n-1})\mathrm{d}M_r\right)^2 + \sup_{t\leqslant s\leqslant u}\left(\int_t^s U_r\mathbb{E}[V_r^n - V_r^{n-1}]\mathrm{d}r\right)^2\right]. \end{aligned} \tag{3.4}$$

Using the Doob and Cauchy–Schwarz inequalities, it follows that

$$\begin{aligned} \mathbb{E}\left[\sup_{t\leqslant s\leqslant u}|V_s^{n+1} - V_s^n|^2\right] &\leqslant 8\mathbb{E}\left[\left(\int_t^u (V_s^n - V_s^{n-1})\mathrm{d}M_s\right)^2\right] + 2T\mathbb{E}\left[\int_t^u |U_s\mathbb{E}[V_s^n - V_s^{n-1}]|^2\mathrm{d}s\right] \\ &\leqslant 8\mathbb{E}\left[\int_t^u (V_s^n - V_s^{n-1})^2\mathrm{d}[M,M]_s\right] + 2T\left[\int_t^u \mathbb{E}|U_s|^2|\mathbb{E}[V_s^n - V_s^{n-1}]|^2\mathrm{d}s\right]. \end{aligned} \tag{3.5}$$

We have that $\mathrm{d}[M,M]_s = (\alpha_s^1)^2\mathrm{d}s + \int_{\mathbf{R}^*}(\alpha_s^2(e))^2\mathrm{d}\tilde{N}(\mathrm{d}s,\mathrm{d}e) + \int_{\mathbf{R}^*}(\alpha_s^2(e))^2\nu(\mathrm{d}e)\mathrm{d}s$, $(\alpha_s^1, \alpha_s^2(\cdot))_{s\geqslant t}$ belongs to $\mathcal{A}_T$ and $(U_s)_{s\geqslant t}$ is uniformly bounded in L^2. We then obtain that there exists a constant K such that

$$\begin{aligned} \mathbb{E}\left[\sup_{t\leqslant s\leqslant u}|V_s^{n+1} - V_s^n|^2\right] &\leqslant 2K(4+T)\mathbb{E}\left[\int_t^u |V_s^n - V_s^{n-1}|^2\mathrm{d}s\right] \\ &\leqslant 2K(4+T)\mathbb{E}\left[\int_t^u \sup_{t\leqslant s\leqslant u}|V_s^n - V_s^{n-1}|^2\mathrm{d}s\right]. \end{aligned} \tag{3.6}$$

We set $C = 2K(4+T)$ and let $D := \mathbb{E}\left[\sup_{t\leqslant s\leqslant T}|V_s^1 - V_s^0|^2\right]$. It then follows from the above computation that for each $t \leqslant u \leqslant T$ and n,

$$\mathbb{E}\left[\sup_{t\leqslant s\leqslant u}|V_s^n - V_s^{n-1}|^2\right] \leqslant \frac{DC^nT^n}{n!}.$$

Consequently,

$$\sum_{n=1}^{\infty}\left\|\sup_{t\leqslant s\leqslant u}|V_s^n - V_s^{n-1}|\right\|_2 < \infty.$$

Thus the series $\sum_{n=1}^{\infty}\sup_{t\leqslant s\leqslant u}|V_s^n - V_s^{n-1}|$ converges a.s. and, as a result, V^n converges a.s. uniformly on every bounded interval to a right-continuous adapted process V that is a solution to (3.3). □

3.2.2 Dual representation theorem

We now give the main result of this section, the dual representation theorem for the expected risk measure. For $(\gamma, q, \beta, \alpha^1, \alpha^2) \in \bar{\mathcal{A}}_T$, we denote

$$D_{t,s}^{\beta,\gamma,q} := \exp\left(\int_t^s (\beta_u + \gamma_u \mathbf{1}_{\mathbb{E}[q_u]>0})\mathrm{d}u\right), \quad 0 \leqslant t \leqslant s \leqslant T, \tag{3.7}$$

which can be interpreted as a discount factor. We recall that the process Γ^α follows the dynamics defined in (3.2).

The following lemma gives the existence of the process $(\bar{\gamma}_s)_{s\geqslant t}$ associated to the mean-field term, which is used towards the dual representation theorem.

Lemma 3.5 *Given the predictable processes* $(X_s, \bar{q}_s, \bar{\beta}_s, \bar{\alpha}_s)_{s\geqslant t}$, *with* $\mathbb{E}[\bar{q}_s] > 0$ *for all* $s \in [0,T]$, *there exists an adapted process* $(\bar{\gamma}_s)_{s\geqslant t}$ *such that* $(\Gamma^{\bar{\alpha}_s} e^{\int_t^s \bar{\gamma}_u \mathrm{d}u})_{s\geqslant t}$ *belongs to* $\mathbb{H}^2$ *and satisfies the following equation:*

$$F(s, X_s) - \frac{\mathbb{E}^{Q^{\bar{\alpha}}}[X_s D_{t,s}^{\bar{\beta},\bar{\gamma},\bar{q}} \bar{\gamma}_s]}{\mathbb{E}^{Q^{\bar{\alpha}}}[D_{t,s}^{\bar{\beta},\bar{\gamma},\bar{q}} \bar{q}_s]} = F^*\left(s, \frac{\Gamma_s^{\bar{\alpha}} D_{t,s}^{\bar{\beta},\bar{\gamma},\bar{q}} \bar{\gamma}_s}{\mathbb{E}^{Q^{\bar{\alpha}}}[D_{t,s}^{\bar{\beta},\bar{\gamma},\bar{q}} \bar{q}_s]}\right) \text{ for each } s \in [0,T]. \tag{3.8}$$

Proof Since F is concave and Lipschitz on L^2, the conjugacy relation of (F, F^*) gives that for each s, there exists $Y_s \in L^2$ such that

$$F(s, X_s) - \mathbb{E}^P[X_s Y_s] = F^*(s, Y_s) \tag{3.9}$$

and since F is non-decreasing, by Lemma 3.3 we have $Y_s \geqslant 0$ dP a.s. and $\|Y_s\|_{L_2} \leqslant C$.

Fix now $s \in [t,T]$. For each Y in $L^2(\mathcal{F}_s)$ and feasible, by Assumption 3.1, we have for each $X \in L^2$,

$$F(s, X) - \mathbb{E}^P[XY] \leqslant F(s, \mathbb{E}[X|\mathcal{F}_s]) - \mathbb{E}^P[\mathbb{E}[X|\mathcal{F}_s]Y].$$

This gives

$$F^*(s, Y) = \sup_{X\in L^2(\mathcal{F}_T)} [F(s,X) - \langle X, Y\rangle_{L^2}] = \sup_{X\in L^2(\mathcal{F}_s)} [F(s,X) - \langle X, Y\rangle_{L^2}].$$

Therefore, we can restrict the operator F on the subspace $L^2(\mathcal{F}_s)$.[3] This implies that we can choose $Y_s \in L^2(\mathcal{F}_s)$ in (3.9), and thus it is adapted.

Now let $(V_s)_{s\geqslant t} \in \mathcal{S}^2$ be the solution of (3.3) with $U_s = e^{-\int_t^s \bar{\beta}_u \mathrm{d}u} Y_s$, $h_s = e^{\int_t^s \bar{\beta}_u \mathrm{d}u} \bar{q}_s$ and $V_t = 1$. We apply Itô's formula to $V_s(\Gamma_s^{\bar{\alpha}})^{-1}$ and we obtain

$$\begin{aligned} \mathrm{d}(V(\Gamma^{\bar{\alpha}})^{-1})_s &= V_s \mathrm{d}(\Gamma_s^{\bar{\alpha}})^{-1} + (\Gamma_s^{\bar{\alpha}})^{-1}\mathrm{d}V_s + \mathrm{d}\langle V, (\Gamma^{\bar{\alpha}})^{-1}\rangle_s \\ &= (\Gamma_s^{\bar{\alpha}})^{-1} e^{-\int_t^s \bar{\beta}_u \mathrm{d}u} Y_s \mathbb{E}[V_s e^{\int_t^s \bar{\beta}_u \mathrm{d}u} \bar{q}_s]\mathrm{d}s. \end{aligned} \tag{3.10}$$

Thus $V^1 := V(\Gamma^{\bar{\alpha}})^{-1}$ satisfies the stochastic differential equation

$$\mathrm{d}(V^1)_s = (\Gamma_s^{\bar{\alpha}})^{-1} e^{-\int_t^s \bar{\beta}_u \mathrm{d}u} Y_s \mathbb{E}[\Gamma_s^{\bar{\alpha}} V_s^1 e^{\int_t^s \bar{\beta}_u \mathrm{d}u} \bar{q}_s]\mathrm{d}s. \tag{3.11}$$

Since $\Gamma_s^{\bar{\alpha}} > 0$, $\bar{q}_s \geqslant 0$, $Y_s \geqslant 0$ dP a.s., and $V_t^1 := V_t = 1 > 0$ a.s., we have $V_s^1 > 0$ a.s.. Thus, the process $\bar{\gamma}_s$ satisfying $e^{\int_t^s \bar{\gamma}_u \mathrm{d}u} = V_s^1$ is well defined due to (3.11). We obtain that

$$\bar{\gamma}_s e^{\int_t^s \bar{\gamma}_u \mathrm{d}u}\mathrm{d}s = \mathrm{d}(e^{\int_t^s \bar{\gamma}_u \mathrm{d}u})_s = (\Gamma_s^{\bar{\alpha}})^{-1} e^{-\int_t^s \bar{\beta}_u \mathrm{d}u} Y_s \mathbb{E}[\Gamma_s^{\bar{\alpha}} e^{\int_t^s \bar{\gamma}_u \mathrm{d}u} e^{\int_t^s \bar{\beta}_u \mathrm{d}u} \bar{q}_s]\mathrm{d}s, \tag{3.12}$$

which implies that $(\bar{\gamma}_s)_{s\geqslant t}$ satisfies

$$\frac{\Gamma_s^{\bar{\alpha}} D_{t,s}^{\bar{\beta},\bar{\gamma},\bar{q}} \bar{\gamma}_s}{\mathbb{E}[\Gamma_s^{\bar{\alpha}} D_{t,s}^{\bar{\beta},\bar{\gamma},\bar{q}} \bar{q}_s]} = Y_s \ a.s.$$

[3] Note that we could have alternatively defined for each s the Fenchel–Legendre transform of the restriction of operator F on the subspace $L^2(\mathcal{F}_s)$. In this case, we conjecture that the last property of Assumption 3.1 would not be necessary.

and $\Gamma_s^{\bar{\alpha}} e^{\int_t^s \bar{\gamma}_u \mathrm{d}u} = V_s$ belongs to $I\!H^2$. Since $(Y_s)_{s\geqslant t}$ is adapted, this gives the adaptedness of $(\bar{\gamma}_s)_{s\geqslant t}$. □

We are now ready to give the main result. The expected risk measure can be interpreted as the expectation (under a worst-case discount factor and a worst-case probability measure) of the final position ξ plus a penalty function. The lemmas in the previous section ensure that the supremum is finite as the effective domain is bounded.

Theorem 3.6 *Let f and F satisfy Assumption 3.1. Then, for each $t \in [0,T]$, the expectation of the convex risk-measure ρ_t, that is, $\mathbb{E}\rho_t(.,T)$ has the following representation: for each $\xi \in L^2$,*

$$\mathbb{E}\rho_t(\xi,T) = \sup_{(\gamma,\beta,q,\alpha)\in\bar{\mathcal{A}}_T} \left[\mathbb{E}^{Q^\alpha} D_{t,T}^{\beta,\gamma,q}(-\xi) - \zeta_t(\gamma,\beta,q,\alpha,T)\right], \tag{3.13}$$

where the function ζ, called penalty function, is defined for each T and $(\gamma,q,\beta,\alpha^1,\alpha^2) \in \bar{\mathcal{A}}_T$ by

$$\zeta_t(\gamma,\beta,q,\alpha,T) := \int_t^T \Bigg(\mathbb{E}^{Q^\alpha}[D_{t,s}^{\beta,\gamma,q} f^*(s,q_s,\beta_s,\alpha_s)] + \mathbb{E}^{Q^\alpha}[D_{t,s}^{\beta,\gamma,q} q_s] F^*\left(t, \frac{\Gamma_s^\alpha D_{t,s}^{\beta,\gamma,q}\gamma_s}{\mathbb{E}^{Q^\alpha}[D_{t,s}^{\beta,\gamma} q_s]}\right)\mathbf{1}_{\mathbb{E}^{Q^\alpha}[q_s]>0}\Bigg)\mathrm{d}s$$

with Γ_s^α following the dynamics defined in (3.2). Moreover, for each $\xi \in L^2$, there exists $(\bar{\gamma}_t, \bar{q}_t, \bar{\beta}_t, \bar{\alpha}_t^1, \bar{\alpha}_t^2) \in \bar{\mathcal{A}}_T$ achieving the supremum in (3.13).

Proof For each process $(\gamma_s, q_s, \beta_s, \alpha_s^1, \alpha_s^2) \in \bar{\mathcal{A}}_T$, we apply Itô's formula to $D_{t,s}^{\beta,\gamma,q} X_s$ between t and T, where (X,Z,l) is the solution of mean-field BSDE (2.4). We obtain

$$X_t = D_{t,T}^{\beta,\gamma,q}\xi + \int_t^T D_{t,s}^{\beta,\gamma,q}[-\beta_s X_s - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s - \alpha_s^1 Z_s - \langle \alpha_s^2, l_s\rangle_\nu + f(s,F(s,X_s),X_s,Z_s,l_s)]\mathrm{d}s - \int_t^T \mathrm{d}M_s^{Q^\alpha}, \tag{3.14}$$

where $\mathrm{d}M_s^\alpha = D_{t,s}^{\beta,\gamma,q} Z_s \mathrm{d}W_s^\alpha + \int_{\mathbf{E}} D_{t,s}^{\beta,\gamma,q} l_s(e)\mathrm{d}\tilde{N}^\alpha(\mathrm{d}t,\mathrm{d}e)$.

For each $s \in [t,T]$, we have

$$\begin{aligned}&-\beta_s X_s - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s - \alpha_s^1 Z_s - \langle \alpha_s^2, l_s\rangle_\nu + f(s,F(s,X_s),X_s,Z_s,l_s)\\ =& -\beta_s X_s - q_s F(s,X_s) - \alpha_s^1 Z_s - \langle \alpha_s^2, l_s\rangle_\nu + f(s,F(s,X_s),X_s,Z_s,l_s) + (q_s F(s,X_s) - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s).\end{aligned}$$

Since $q_s \geqslant 0$ $\mathrm{d}P$ a.s., we note that

$$q_s F(s,X_s) - \gamma_s \mathbf{1}_{\mathbb{E}[q_s]>0} X_s = (q_s F(s,X_s) - \gamma_s X_s)\mathbf{1}_{\mathbb{E}[q_s]>0}.$$

By taking expectation on both sides in (3.14), we obtain that

$$\begin{aligned}\mathbb{E}[X_t] = \mathbb{E}^{Q^\alpha}\Bigg[D_{t,T}^{\beta,\gamma,q}\xi + \int_t^T D_{t,s}^{\beta,\gamma,q}[-\beta_s X_s - q_s F(X_s) - \alpha_s^1 Z_s \\ - \langle \alpha_s^2, l_s\rangle_\nu + f(s,F(s,X_s),X_s,Z_s,l_s)]\Bigg]\mathrm{d}s \\ + \int_t^T \mathbb{E}^{Q^\alpha}[D_{t,s}^{\beta,\gamma,q} q_s]\left[F(s,X_s) - \frac{\mathbb{E}^{Q^\alpha}[X_s D_{t,s}^{\beta,\gamma,q}\gamma_s]}{\mathbb{E}^{Q^\alpha}[D_{t,s}^{\beta,\gamma,q} q_s]}\right]\mathbf{1}_{\mathbb{E}[q_s]>0}\mathrm{d}s.\end{aligned}$$

Since $Q^{\bar{\alpha}}$ and P are equivalent measures and $q_s \geqslant 0$ $\mathrm{d}P$ a.s., we have $\mathbf{1}_{\mathbb{E}[q_s]>0} = \mathbf{1}_{\mathbb{E}^{Q^\alpha}[q_s]>0}$.

By the definition of Fenchel–Legendre transform, we have

$$f(s,F(s,X_s),X_s,Z_s,l_s)-q_sF(s,X_s)-\beta_sX_s-\alpha^1_sZ_s-\langle\alpha^2_s,l_s\rangle_\nu\leqslant f^*(s,q_s,\beta_s,\alpha^1_s,\alpha^2_s)$$

a.s. and

$$F(s,X_s)-\frac{\mathbb{E}^{Q^\alpha}[X_sD^{\beta,\gamma}_{t,s}\gamma_s]}{\mathbb{E}^{Q^\alpha}[D^{\beta,\gamma}_{t,s}q_s]}=F(s,X_s)-\frac{\mathbb{E}[\Gamma^\alpha_sX_sD^{\beta,\gamma}_{t,s}\gamma_s]}{\mathbb{E}[\Gamma^\alpha_sD^{\beta,\gamma,q}_{t,s}q_s]}\leqslant F^*(s,\frac{\Gamma^\alpha_sD^{\beta,\gamma}_{t,s}\gamma_s}{\mathbb{E}^{Q^\alpha}[D^{\beta,\gamma,q}_{t,s}q_s]}).$$

Since by assumption $D^{\beta,\gamma,q}_{t,s}\geqslant 0$ and $q_s\geqslant 0$ $\mathrm{d}Q^\alpha$ a.s., we obtain

$$\begin{aligned}\mathbb{E}X_t\leqslant\inf_{(\gamma,\beta,q,\alpha)\in\bar{\mathcal{A}}_T}&\mathbb{E}^{Q^\alpha}\left[D^{\beta,\gamma,q}_{t,T}\xi+\int_t^T D^{\beta,\gamma}_{t,s}f^*(s,q_s,\beta_s,\alpha^1_s,\alpha^2_s)\right]\mathrm{d}s\\&+\int_t^T\mathbb{E}^{Q^\alpha}[D^{\beta,\gamma,q}_{t,s}q_s]\left[F^*(s,\frac{\Gamma^\alpha_sD^{\beta,\gamma,q}_{t,s}\gamma_s}{\mathbb{E}^{Q^\alpha}[D^{\beta,\gamma,q}_{t,s}q_s]})\right]\mathbf{1}_{\mathbb{E}^{Q^\alpha}[q_s]>0}\mathrm{d}s.\end{aligned}\tag{3.15}$$

Recall that for any $(\omega,s)\in\Omega\times[0,T]$, f is Lipschitz, concave in (x',x,z,l), and the following conjugacy relation of (f,f^*) holds. Let U be the set introduced in Lemma 3.2, we have

$$\begin{aligned}f(\omega,s,x',x,z,l)&=\inf_{(q,\beta,\alpha^1,\alpha^2)\in\bar{U}}\{f^*(\omega,s,q,\beta,\alpha^1,\alpha^2)+qx'+\beta x+\alpha^1z+\langle\alpha^2,l\rangle_\nu\}\\&=f^*(\omega,s,\bar{q},\bar{\beta},\bar{\alpha}^1,\bar{\alpha}^2)+\bar{q}x'+\bar{\beta}x+\bar{\alpha}^1z+\langle\bar{\alpha}^2,l\rangle_\nu,\end{aligned}\tag{3.16}$$

where $\bar{U}$ is the closure of set U, that is, the set in which α_2 satisfies $\alpha_2(u)\geqslant -1$ instead of the strict inequality.

Now, since $\bar{U}$ is strongly closed and convex, we obtain that there exists $(\bar{q},\bar{\beta},\bar{\alpha}^1,\bar{\alpha}^2)\in\bar{U}$ that satisfy (3.16). Since ν is a σ-finite measure and $\mathcal{B}(\mathcal{E})$ is countably generated, by [8, Proposition 3.4.5], the space L^2_ν is separable. We can thus apply the measurable selection theorem ([11, appendix of Chapter III]) as in [23, Lemma 5.5] to assert the existence of the predictable processes $(\bar{q}_s,\bar{\beta}_s,\bar{\alpha}^1_s,\bar{\alpha}^2_s)_{s\geqslant t}$ satisfying

$$f(s,F(s,X_s),X_s,Z_s,l_s)=\bar{\beta}_sX_s+\bar{q}_sF(X_s)+\bar{\alpha}^1_sZ_s+\langle\bar{\alpha}^2_s,l_s\rangle_\nu+f^*(s,\bar{\beta}_s,\bar{q}_s,\bar{\alpha}^1_s,\bar{\alpha}^2_s)\quad a.s..\tag{3.17}$$

Similarly, since F is Lipschitz and concave, the conjugacy relation also holds for (F,F^*). Given the predictable processes $(X_s,\bar{q}_s,\bar{\beta}_s,\bar{\alpha}_s)_{s\geqslant t}\in\mathcal{S}^2\times\bar{\mathcal{A}}_T$, we now introduce

$$\tilde{q}_s=\bar{q}_s\mathbf{1}_{\mathbb{E}^{Q^{\bar{\alpha}}}[\bar{q}_s]>0}+C\mathbf{1}_{\mathbb{E}^{Q^{\bar{\alpha}}}[\bar{q}_s]=0}$$

with C the Lipschitz constant of f. By Lemma 3.5 there exists an adapted process $(\bar{\gamma}_s)_{s\geqslant t}$ such that

$$F(s,X_s)-\frac{\mathbb{E}^{Q^{\bar{\alpha}}}[X_sD^{\bar{\beta},\bar{\gamma},\bar{q}}_{t,s}\bar{\gamma}_s]}{\mathbb{E}^{Q^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma},\bar{q}}_{t,s}\tilde{q}_s]}=F^*\left(s,\frac{\Gamma^{\bar{\alpha}}_sD^{\bar{\beta},\bar{\gamma},\bar{q}}_{t,s}\bar{\gamma}_s}{\mathbb{E}^{Q^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma},\bar{q}}_{t,s}\tilde{q}_s]}\right).\tag{3.18}$$

Since $\tilde{q}_s=\bar{q}_s$ for any s such that $\mathbb{E}[\bar{q}_s]>0$, we obtain

$$\begin{aligned}\mathbb{E}X_t=\mathbb{E}^{Q^{\bar{\alpha}}}&[D^{\bar{\beta},\bar{\gamma},\bar{q}}_{t,T}\xi+\int_t^T D^{\bar{\beta},\bar{\gamma},\bar{q}}_{t,s}f^*(s,\bar{\beta}_s,q_s,\bar{\alpha}^1_s,\bar{\alpha}^2_s)\mathrm{d}s]\\&+\int_t^T\mathbb{E}^{Q^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma},\bar{q}}_{t,s}\bar{q}_s]\left[F^*(s,\frac{\Gamma^{\bar{\alpha}}_sD^{\bar{\beta},\bar{\gamma},\bar{q}}_{t,s}\bar{\gamma}_s}{\mathbb{E}^{Q^{\bar{\alpha}}}[D^{\bar{\beta},\bar{\gamma},\bar{q}}_{t,s}\bar{q}_s]})\right]\mathbf{1}_{\mathbb{E}^{Q^{\bar{\alpha}}}[\bar{q}_s]>0}\mathrm{d}s.\end{aligned}$$

Together with (3.15), we obtain (3.13).

Finally, (3.17) implies that the process $f^*(\omega,t,\bar{q}_t,\bar{\beta}_t,\bar{\alpha}^1_t,\bar{\alpha}^2_t)$ belongs to $\mathbb{H}^2_T$, since by assumption $(X,Z,l(.))\in\mathcal{S}^2\times\mathbb{H}^2\times\mathbb{H}^2_\nu$ and $(\bar{q}_t,\bar{\beta}_t,\bar{\alpha}^1_t,\bar{\alpha}^2_t)$ are bounded. □

Remark 3.7 *We note that our running examples of mean-field operators* F *satisfy the assumptions in Theorem 3.6. First, let a mean-field operator capture first-order interactions. Namely, set* $F(t,X) := \mathbb{E}[\varphi(t,X)]$ *for* $X \in L^2(\Omega, P, \mathcal{F}_T)$, *where* $\phi : [0,T] \times \mathbf{R} \mapsto \mathbf{R}, (t,x) \mapsto \varphi(t,x)$ *is a Lipschitz and concave function such that* $\varphi(t,X) \in L^2$. *Then* F *is Lipschitz and concave.*

Second, let a mean-field operator capture the average intensity of interactions in an inhomogeneous random graph as in section 2.2. For F *as in* (2.3) *and under the further assumptions that the kernel* κ *is Lipschitz and concave, we check that* F *is Lipschitz and concave. In particular, for the kernel* κ *of* (2.12) *these conditions are satisfied.*

We end this remark by making the connection of our mean-field operator (2.3) *with the economics literature. To see this, we note that*

$$\begin{aligned} F(t,X) &= \int_{\mathbf{R}\times\mathbf{R}} \kappa(x,y)\mathrm{d}D_X(x)\mathrm{d}D_X(y) \\ &= \int_{[0,1]\times[0,1]} \kappa(D_X^{-1}(u), D_X^{-1}(v))\mathrm{d}u\mathrm{d}v, \end{aligned} \tag{3.19}$$

where D_X *and* D_X^{-1} *denote the distribution function of* X *and its generalized inverse. The form in* (3.19) *of the operator appears in the literature as a utility functional quadratic in probabilities (as opposed to linear in probabilities as in the case of expected utility), see [10, Example 2.3] and [7]. In [10] this operator is an example of Shur-concave functional, and the main result of the paper is a representation theorem of* F *and its Fenchel transform* F^* *in terms of a family of non-negative affine combinations of Choquet integrals.*

4. Conclusion

We have studied mean-field BSDEs with jumps, whose driver can capture system influence with higher-order interactions. The mean-field term can capture, for example, the intensity of bilateral interactions that depend on the states of the end nodes by means of a kernel function. This opens the path towards using dynamic risk measures induced by mean-field BSDE as a complementary approach to systemic risk measurement. The expectation of the risk measure is interpreted as the necessary capital of the "representative" bank to make it acceptable. We have given a dual representation for the expectation of dynamic risk measures induced by mean-field BSDEs. The risk measure can be represented using a worst-case probability measure, discount factor, and penalty applied to the terminal financial position. The representation of the penalty function involves the Fenchel–Legendre transform of the mean-field operator.

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