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Certified Control-Oriented Learning: A Coprime Factorization Approach

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***Abstract*— This paper considers the problem of learning models to be used for controller design. Using a simple example, it argues that in this scenario the objective should reflect the closed-loop, rather than open-loop distance between the learned model and the actual plant, a task that can be accomplished by using a gap metric motivated approach. This is particularly important when identifying open-loop unstable plants, since typically in this case the open-loop distance is unbounded. In this context, the paper proposes a convex optimization approach to learn its coprime factors. This approach has a dual advantage: (1) it can easily handle open-loop unstable plants, since the coprime factors are stable, and (2) it is "self certified", since a simple norm computation on the learned factors indicates whether or not a controller designed based on these factors will stabilize the actual (unknown) plant. If this test fails, it indicates that further learning is needed.**

I. Introduction

Control-oriented learning is a form of biasing the typical learning process (system identification) towards the ultimate goal of achieving certain closed-loop behavior. Jointly considering both the nominal model and a concrete uncertainty description, allows for trading off an accurate but high order model for a lower order one with larger (but well-understood) uncertainty. Doing so can significantly simplify the control design process, reduce costs, and improve reliability.

Unlike the probabilistic approach to describing the uncertainty ([1], [2]), the deterministic approach involves norm-bounding the disturbances leading to a set-membership identification approach [3]. This approach leads to interpolatory algorithms that parameterize the entire class of stable models satisfying the prior information and the available data [4]. Thus, when they fail to find a solution, they serve as a certificate of inconsistency between the data and the priors. Further, as shown in [5]–[8] these algorithms can be extended to handle mixed-domain data, and to impose minimal McMillan degree. The approach presented in this paper is based on the set-membership approach, where the control-orientation is incorporated by using a ν-gap metric motivated objective. To motivate this approach, consider the systems shown in Fig. 1. Even though the open-loop distance between G_1 and G_2 is infinite, yet, as shown there, the closed loops obtained using the controller $C = 1$ have virtually indistinguishable performance. This is due to the fact that the ν-gap between the two systems is 10^{-3} implying that a controller that stabilizes the easier plant G_1 would also stabilize G_2. Thus, if the goal is to design controllers, the easier to identify plant G_1 can be used as a proxy for G_2.

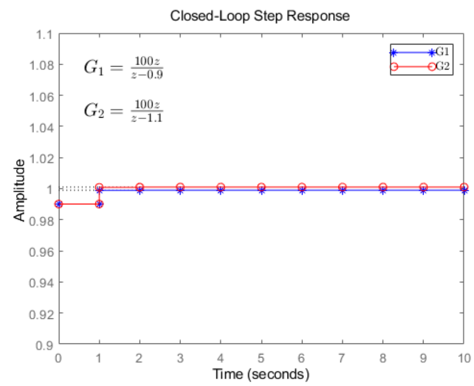

Fig. 1. The open-loop distance between the plants G_1, G_2 is ∞, yet the closed-loop responses are indistinguishable.

Identification of coprime factors has been considered since the early 1990s [9]–[12], but mostly in closed-loop settings; the knowledge of the controller is used in the factorization scheme. However, finding such a controller may not be trivial for unknown plants. The algorithm described in [13] works in open-loop. However, it leads to a bilinear formulation and requires an iterative approach that alternates between optimizing the fit and solving an $\mathcal{H}_\infty$ optimization. Furthermore, none of these approaches handles time-domain data.

Our approach directly identifies coprime factors for plants that are not necessarily strongly stabilizable. It does not need the knowledge of any controllers and yields normalized coprime factors (NCFs) directly based on the priors imposing the uncertainty bounds in the time- and frequency domains.

II. Preliminaries

A. Notation

ℓ_1 denotes the space of absolutely summable sequences. $G(z)$ is the z-transform of the sequence $g \in \ell_1$. $G(z) \doteq \sum_{i=0}^{\infty} g_i z^i$; $G(z)$ is analytic *inside* the unit disk $\mathcal{D}$. $\mathcal{H}_{\infty,\rho}$ denotes the space of functions analytic inside the disk of radius $\rho > 1$, equipped with the norm $\|G(z)\|_{\infty,\rho} \doteq \sup_{|z|<\rho} |G(z)|$. $\mathcal{H}^K_{\infty,\rho}$ denotes the K-ball in $\mathcal{H}_{\infty,\rho}$. We will denote $\mathcal{H}_{\infty,1}$ as simply $\mathcal{H}_\infty$. Given a transfer matrix $G(z)$, $G(z)^{\sim} \doteq G^T(\frac{1}{z})$. Lower-case letters (e.g., u, y, w) represent time-domain signals, upper-case letters represent their discrete Fourier transforms (U, Y, W), and bold case letters with size subscript, e.g., $\mathbf{x}_N$ represent a vector of N measurements

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of the corresponding signal "x". The symbol "$\circledast$" represents the convolution operator, $T_{\mathbf{x}_N}$ represents the Toeplitz matrix of an N-long vector $\mathbf{x}$. By a slight abuse of notation, given a transfer function G, we will denote by T_g the Toeplitz matrix associated with its impulse response sequence.

B. Normalized Coprime Factorizations and ν_{gap} metric

Consider a minimal transfer matrix $G(z)$. It is well known [14] that $G(z)$ can be factored as $G(z) = N(z)M(z)^{-1}$ where $N, M \in \mathcal{H}_\infty$ and such that $N^{\sim}N + M^{\sim}M = I$. NCFs arise in the context of robust stabilization. In particular, we will exploit the connection between NCFs and the ability of a controller to stabilize two plants G_1, G_2, based on their distance measured in terms of the ν_{gap} metric. Given $P_1 = N_1M_1^{-1}$, $P_2 = N_2M_2^{-1}$, where N_i, M_i are a NCF pair, the gap between P_1, P_2 is defined as (see Chapter 17 of [14]):

$$\nu_{\text{gap}}(P_1, P_2) = \begin{cases} \|-N_2{}^{\sim}M_1 + M_2{}^{\sim}N_1\|_\infty & \text{wno}(\Theta) = 0 \\ 1 & \text{otherwise} \end{cases}$$

where $\Theta = N_2{}^{\sim}N_1 + M_2{}^{\sim}M_1$ and wno denotes winding number. Further, if the winding number condition holds, then, for the case of SISO plants:

$$\nu_{\text{gap}}(P_1, P_2) = \max_{z=e^{j\omega}} \frac{|P_1(z) - P_2(z)|}{\sqrt{1 + |P_1(z)|^2}\sqrt{1 + |P_2(z)|^2}} \quad (1)$$

The advantage of this expression is that it can be calculated directly from the frequency response data.

Given a controller C that stabilizes P_1, define

$$b_{p_1} = \| \begin{bmatrix} C \\ I \end{bmatrix} (I + P_1C)^{-1} \begin{bmatrix} I & P_1 \end{bmatrix} \|_\infty^{-1}$$

Then, if $b_{p_1} > \nu_{\text{gap}}(P_1, P_2)$ the controller C also stabilizes P_2 (Theorem 17.8 in [14]). We will exploit this result to ascertain whether or not a controller designed using the identified model is guaranteed to stabilize the actual plant.

C. Generalized Interpolation

The generalized interpolation framework [5], [15] establishes conditions for the existence of a stable function that interpolates given time- and frequency-domain measurements. Briefly, an interpolant $G(z)$ exists if and only if a certain Hermitian matrix $\mathbf{Z}(W, \mathbf{g}, K)$ is positive semidefinite, where $\mathbf{g}$ denotes the impulse response vector for $G(z) \in \mathcal{H}_{\infty,\rho}^K$, and W its frequency response. If $\mathbf{Z}$ is rank deficient, then the interpolant is unique. When $\mathbf{Z} \succ 0$, the solution is not unique and all the interpolants can be written as LFT of a free parameter $Q(z) \in \mathcal{H}_\infty$ as follows:

$$F(z) \quad = \quad \frac{T_{11}(z)Q(z) + T_{12}(z)}{T_{21}(z)Q(z) + T_{22}(z)} \quad (2)$$

where the transfer matrices $T_{i,j}$ depend only on the problem data (an explicit expression for these matrices can be found for instance in [6]). In particular, if the free parameter $Q(z)$ is chosen as a constant, then the model order is less than or equal to the total number of measurements (counting both the time and frequency response samples). Using $Q(z) = 0$ yields the so-called *central interpolant*.

III. Problem Formulation

We consider the problem of identifying a coprime factorization NM^{-1} of a SISO transfer function G using noisy measurements of its time, and/or frequency domain response. Suppose we have collected N_t time-domain samples of input/output data $(u(t), y(t))$ and N_f frequency response samples $Y(e^{j\omega}) = G(e^{j\omega})U(e^{j\omega})$.

Time domain

Output signal: $\tilde{\mathbf{y}}_{N_t} = (\tilde{y}(1), \dots, \tilde{y}(N_t))^T$

Input signal: $\mathbf{u}_{N_t} = (u(1), \dots, u(N_t))^T$

Frequency domain

Output signal: $\tilde{\mathbf{Y}}_{N_f} = (\tilde{Y}(e^{j\omega_1}), \tilde{Y}(e^{j\omega_2}), \dots, \tilde{Y}(e^{j\omega_{N_f}}))^T$

Input signal: $\mathbf{U}_{N_f} = (U(e^{j\omega_1}), U(e^{j\omega_2}), \dots, U(e^{j\omega_{N_f}}))^T$

where $\tilde{y}(t) = y(t) + \eta_t(t)$, $\tilde{Y}(e^{j\omega}) = Y(e^{j\omega}) + \eta_f(e^{j\omega})$, and $\eta_t(t), \eta_f(e^{j\omega})$ represent the measurement noises. If the system is unstable, the data could have been collected in a closed-loop with a stabilizing controller (but measured across the plant only), or come from suitably designed open-loop experiments that keep the responses from growing too large. The identified factors must satisfy two conditions:

- *Be consistent with the measurements*. We need to ensure that the identified model $G(z) = N(z)M(z)^{-1}$ belongs to the consistency set defined by the bounds on the noises (or measurement precision).
- *Satisfy the prior knowledge regarding the nature of the system*. For the coprime factors, the prior related to the nature of $N(z)$ and $M(z)$. In particular, we impose conditions on the stability radius of N and M.

The identification problem can be stated as follows:

Problem 1: Given

(1) frequency–response measurements $(\mathbf{U}_{N_f}\tilde{\mathbf{Y}}_{N_f})$ and time–domain data points $\{\mathbf{u}_{N_t}, \tilde{\mathbf{y}}_{N_t}\}$

(2) apriori bounds $\rho > 1, \epsilon_t, \epsilon_f$

find the right coprime factors:
$N(z) \in \mathcal{H}_{\infty,\rho}^{K_N} \doteq n_1 + n_2z + \dots n_{N_t}z^{N_t} + ..$, and
$M(z) \in \mathcal{H}_{\infty,\rho}^{K_M} \doteq m_1 + m_2z + \dots m_{N_t}z^{N_t} + ..$
of a rational transfer function $G(z) \doteq N(z)M(z)^{-1} = g_1 + g_2z + \dots g_{N_t}z^{N_t} + ..$, such that

$$|\tilde{y}(t) - g(t) \circledast u(t)| \le \epsilon_t(t), t = 1, \dots, N_t$$
$$|\tilde{Y}(\omega) - G(\omega)U(\omega)| \le \epsilon_f(\omega), \omega \in (\omega_1, \dots, \omega_{N_f})$$

where $g(t)$ is the system impulse response. The $\mathcal{H}_{\infty,\rho}$ norm bounds K_N, K_M can be treated as prior knowledge. Alternatively, they can be minimized leading to the problem statement: *Given ρ and the noise bounds, what is the smallest possible value of $\| \begin{bmatrix} N(z) & M(z) \end{bmatrix} \|_{\infty,\rho}$ for a system $G(z)$ consistent with the measurements?*.

IV. Solution

The general idea behind the proposed solution is as follows:

1) Find stable $N(z)$, $M(z)$, not necessarily coprime, such that $G(z) = N(z)M(z)^{-1}$. Here, the zeros of $M(z)$ supply the unstable poles of $G(z)$ while the zeros of

$N(z)$ supply the NMP zeros of $G(z)$. As we show in the sequel, this step can be accomplished using convex optimization, by exploiting ideas from generalized interpolation theory. This can further be combined with minimum McMillan degree requirement to yield low order realizations of $N(z)$ and $M(z)$ [8].

2) A normalization step $N \to \hat{N}$, $M \to \hat{M}$ where $G_{id}(z) = \hat{N}(z)\hat{M}(z)^{-1}$ and $(\hat{N}, \hat{M})$ is a NCF, e.g., $\hat{N}^{\sim}\hat{N} + \hat{M}^{\sim}\hat{M} = 1$. This step will be accomplished via a spectral factorization.

3) A (optional) certification step where b_p, the performance of an optimal $\mathcal{H}_\infty$ controller based on the identified plant, is compared against the estimated $\nu_{\text{gap}}(G, G_{id})$ between the actual and the identified plants, G and G_{id}, respectively. If $b_p \geq \nu_{\text{gap}}(G, G_{id})$, then the optimal $\mathcal{H}_\infty$ controller synthesized using the identified plant is guaranteed to stabilize the actual one.

The top row of Figure 2 shows the block diagram corresponding to Step 1, where $w(t)$ serves as the internal signal connecting $N(z)$ and $M(z)^{-1}$. Note that our goal is to identify $M(z)$, which maps the "input" $y(t)$ to the signal $w(t)$. Coupling N and M only through w requires imposing that M is proper but not strictly proper, in order for $G = NM^{-1}$ to be proper. This can be accomplished by simply adding the interpolation constraint $M(0) = 1$.

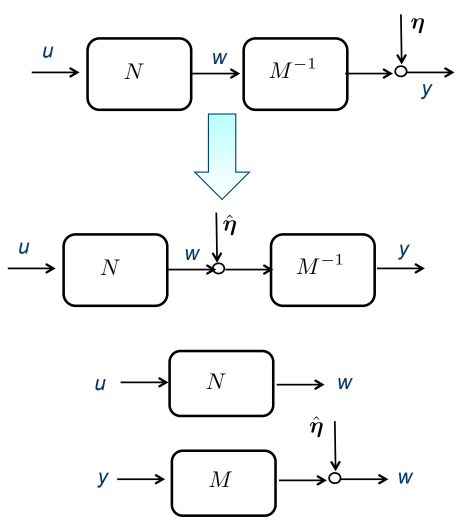

Fig. 2. Coprime factor identification setup. Internal signal $w(t)$ connects the factors N and M. The measurement noise η is handled by "walking it back" across the M^{-1} block.

A. Finding Stable N, M via Generalized Interpolation

LMI based generalized interpolation feasibility conditions are presented in [5]. We exploit these conditions for describing the feasibility of the $u \to w$, and the $y \to w$ stable dynamics. First, we generalize the conditions to the case where signal Fourier transforms $U(e^{j\omega}), Y(e^{j\omega})$ are provided for a certain $u \to y$ dynamics, rather than the measurement of the frequency response $G(e^{j\omega})$.

Lemma 1: Given N_f frequency domain points $\{z_i, U_i, Y_i\}, i = 1, \ldots, N_f$, and N_t time-domain samples y_t of the output to an input u_t, there exists $G \in \mathcal{H}^K_{\infty,\rho}$ that interpolates the frequency and time domain data (that is, $G(z_i)U_i = Y_i$ and $y = g \circledast u$, with $G(z) = g_1 + g_2 z + \ldots + g_{N_t} z^{N_t - 1} + \ldots$) if and only if

$$M_R \doteq \mathbf{\Gamma}_u^H M_0 \mathbf{\Gamma}_u - \frac{1}{K^2}\mathbf{\Gamma}_y^H M_0 \mathbf{\Gamma}_y \succeq 0 \tag{3}$$

where

$$\mathbf{\Gamma}_u = \begin{bmatrix} \mathcal{U}_f & 0 \\ 0 & I_{N_t \times N_t} \end{bmatrix} \tag{4}$$

$$\mathbf{\Gamma}_y = \begin{bmatrix} \mathcal{Y}_f & 0 \\ 0 & T^{-1}_{\mathbf{u}_{N_t}} T_{y_{N_t}} \end{bmatrix} \tag{5}$$

$$M_0 = \begin{bmatrix} Q & S_0 R^{-2} \\ R^{-2} S_0^H & R^{-2} \end{bmatrix} \tag{6}$$

$$R = \text{diag}(1, \rho, \ldots, \rho^{N_t - 1}) \tag{7}$$

$$Q = \left[\frac{\rho^2}{\rho^2 - \overline{z_i} z_j}\right]_{ij}, \; i, j = 1, 2, \ldots, N_f \tag{8}$$

$$S_0 = \left[\overline{z_i^{j-1}}\right]_{ij}, \; i = 1, \ldots, N_f, j = 1, \ldots, N_t \tag{9}$$

$$\mathcal{U}_f = \text{diag}(U(1), U(2), \ldots, U(N_f)) \tag{10}$$

$$\mathcal{Y}_f = \text{diag}(Y(1), Y(2), \ldots, Y(N_f)) \tag{11}$$

Proof: Follows from Theorems 2 and in [5] by replacing $\mathcal{W}_f$ with Y/U and factorizing to fetch the desired form. ■

Corollary 1: The consistency condition (3) is equivalent to:

$$\mathbf{Z}(U, Y, \mathbf{u}, \mathbf{y}, K) \doteq \begin{bmatrix} M_0^{-1} & \frac{1}{K}\mathbf{\Gamma}_y \\ \frac{1}{K}\mathbf{\Gamma}_y^H & \mathbf{\Gamma}_u^H M_0 \mathbf{\Gamma}_u \end{bmatrix} \succeq 0 \tag{12}$$

where $\mathbf{\Gamma}_u$, $\mathbf{\Gamma}_y$, and M_0 are defined in (4)–(6).

1) The noiseless case: In the sequel, we will assume the following apriori information: N and M have a stability radius $\rho - 1$, for some known $\rho > 1$. Equivalently, all unstable poles of G are outside the disk $|z| \geq \rho > 1$. Under this assumption, in the ideal case of noiseless data, a necessary and sufficient condition for the existence of $N, M \in \mathcal{H}_{\infty,\rho}$ such that $G = N/M$ interpolates the input/output data is given by:

Proposition 1: In the noiseless case, a necessary and sufficient condition for the existence of $N \in \mathcal{H}^{K_N}_{\infty,\rho}$ and $M \in \mathcal{H}^{K_M}_{\infty,\rho}$ such that $G = N/M$ interpolates the input output data is the existence of scalar K_N, K_M, N_t scalars $w_i, i = 0 \ldots N_{t-1}$ and N_f complex numbers $W(z_i), i = 1 \ldots N_f$ such that the following set of LMIs is feasible:

$$\mathbf{Z}(U, W_{N_f}, \mathbf{u}_{N_t}, \mathbf{w}_{N_t}, K_N) \succeq 0 \tag{13}$$

$$\mathbf{Z}(Y, W_{N_f}, \mathbf{y}_{N_t}, \mathbf{w}_{N_t}, K_M) \succeq 0 \tag{14}$$

Proof: Feasibility of (13) and (14) are sufficient for the existence of $N \in \mathcal{H}^{K_N}_{\infty,\rho}$ and $M \in \mathcal{H}^{K_M}_{\infty,\rho}$ such that $W(z_i) = N(z_i)U(z_i)$, $\mathbf{w} = T_{\mathbf{n}}\mathbf{u}$, $W(z_i) = M(z_i)Y(z_i)$, $\mathbf{w} = T_{\mathbf{m}}\mathbf{y}$. Hence $N(z_i)U(z_i) = M(z_i)Y(z_i)$ and $G \doteq N/M$ satisfies $Y(z_i) = G(z_i)U(z_i)$. Similarly, $T_{\mathbf{n}}\mathbf{u} = T_{\mathbf{m}}\mathbf{y}$. Thus, $\mathbf{y} = T_{\mathbf{m}}^{-1} T_{\mathbf{n}}\mathbf{u} = T_g\mathbf{u}$. Conversely, assume that a factorization $G = N_o/M_o$ with $N_o \in \mathcal{H}^{K_N}_{\infty,\rho}$ and $M_o \in \mathcal{H}^{K_M}_{\infty,\rho}$

exists. Then, $W = N_o(z_i)$ and $\mathbf{w} = T_{\mathbf{n}}\mathbf{u}$ satisfy the LMIs. ∎

Once W and $\mathbf{w}$ have been found, the interpolants N and M can be obtained using the formulas in [6]. Thus, in the ideal, noiseless case, finding N and M reduces to a convex semi-definite program.

2) The noisy case: Next, we consider the more realistic case where the measured output is corrupted by ℓ_∞ bounded noise. In principle, this can be addressed using Proposition 1, by searching for $K_N, K_M, W(z_i)$,$\mathbf{w}$ and signals $Y(z_i)$, y_t such that the following inequalities hold:

$$\mathbf{Z}(Y, W_{N_f}, \mathbf{y}_{N_t}, \mathbf{w}_{N_t}, K_M) \succeq 0 \tag{15}$$
$$\mathbf{Z}(U, W_{N_f}, \mathbf{u}_{N_t}, \mathbf{w}_{N_t}, K_N) \succeq 0 \tag{16}$$
$$|\tilde{Y}(z_i) - Y(z_i)| \leq \epsilon_f \text{ and } |\tilde{y}_t - y_t| \leq \epsilon_t \tag{17}$$

Note that while (16)-(17) are still LMIs, (15) is a Bilinear Matrix Inequality (BMI), due to terms of the form $\mathbf{\Gamma}_y^H M_0 \mathbf{\Gamma}_{\hat{y}}$. Given the challenges in solving BMIs, in this paper we will pursue a convex relaxation, based on using an equivalent, fictitious noise $(\hat{\eta}_f, \hat{\eta}_t)$ affecting the variable w (Fig. 2).

Proposition 2: A necessary condition for the existence of $N \in \mathcal{H}_{\infty,\rho}^{K_N}$ and $M \in \mathcal{H}_{\infty,\rho}^{K_M}$ such that $G \doteq N/M$ satisfies $|\tilde{Y}(z_i) - G(z_i)U(z_i)| \leq \epsilon_f$ and $|\tilde{y}_t - (g \circledast u)_t| \leq \epsilon_t$ is feasibility of the following set of LMIs:

$$\mathbf{Z}(\tilde{Y}, \tilde{W}_{N_f}, \tilde{\mathbf{y}}_{N_t}, \tilde{\mathbf{w}}_{N_t}, K_M) \succeq 0 \tag{18}$$
$$\mathbf{Z}(U, W_{N_f}, \mathbf{u}_{N_t}, \mathbf{w}_{N_t}, K_N) \succeq 0 \tag{19}$$
$$|\tilde{W}(z_i) - W(z_i)| \leq K_M \epsilon_f \tag{20}$$
$$|\tilde{w}_t - w_t| \leq \frac{K_M \rho}{1-\rho}\epsilon_t \tag{21}$$

Proof: If there exists $M \in \mathcal{H}_{\infty,\rho}^{K_M}$ such that $w = M(y+\eta)$ then $|\hat{\eta}(z_i)| \doteq |w(z_i) - M(z_i)\tilde{Y}(z_i)| \leq \|M\|_\infty \epsilon_f \leq K_M \epsilon_f$. Similarly, $|\eta_t| \doteq |w_t - (M \circledast \tilde{y})_t| \leq \|M\|_1 \epsilon_t \leq \frac{K_M \rho}{1-\rho}\epsilon_t$. ∎

Remark 1: While the conditions above are only necessary, consistent numerical experience shows that using these conditions to identify N, M works well in practice, leading to systems that indeed interpolate the data within (or close to) the desired noise level. Conservatism can be reduced by minimizing K_M to minimize the fictitious noise level. In case that the interpolation error is still too high after this minimization, a heuristic is to reduce the bounds ϵ_t, ϵ_f until the desired error is achieved.

3) Identification Algorithm: Proposition 2 leads to the following optimization problem:

$$\begin{aligned}
&\text{minimize } K \text{ subject to} \\
&K_M \leq K,\ K_N \leq K \\
&\mathbf{Z}(\tilde{Y}, \tilde{W}_{N_f}, \tilde{\mathbf{y}}_{N_t}, \hat{\mathbf{w}}_{N_t}, K_M) \succeq 0 \\
&\mathbf{Z}(U, W_{N_f}, \mathbf{u}_{N_t}, \mathbf{w}_{N_t}, K_N) \succeq 0 \\
&|\hat{W}(z_i) - W(z_i)| \leq K_M \epsilon_f \\
&|\hat{w}_t - w_t| \leq \frac{K_M \rho}{1-\rho}\epsilon_t
\end{aligned} \tag{22}$$

This convex minimization problem seeks to minimize $\max\{K_N, K_M\}$ subject to the interpolation constraints. Minimizing K_M leads to tighter bounds on the equivalent noise. In addition, as we discuss in Section IV-C, minimizing $\|\begin{bmatrix} N & M \end{bmatrix}\|_\infty$, an upper bound to the Hankel norm $\|\begin{bmatrix} N & M \end{bmatrix}\|_{\text{Hankel}}$, leads to larger stability margins and hence an increased chance that a controller synthesized using the identified plant will indeed stabilize the actual one.

4) Practical Considerations: Problem (22) can be solved efficiently using CVX [16]. However, the Q matrix is typically ill-conditioned unless $\rho - 1 \approx 0$ and the frequency samples span the entire Nyquist range. Conditioning is further worsened by the need to compute M_0^{-1}. We have incorporated two changes to improve conditioning:

1) Domain mapping: Map the available frequency domain sampling points $z = e^{j\omega_i}, i = 1, \ldots, N_f$ using bilinear mapping $q(z) = (z+\beta)/(\beta z+1)$ so that the points are maximally "spread out" (see [17]). The parameter β is chosen such that the distance between the endpoints in original domain is maximized through the transformation $q(z)$, that is, $\beta = \arg\max_\beta |q(e^{j\omega_1}) - q(e^{j\omega_{N_f}})|$. The data sample time gets scaled by the factor $(1-\beta)/(1+\beta)$ in the q-domain.
2) LMI replacement using the Schur complement: Replace Equation (12) with an equivalent form:

$$\hat{\mathbf{Z}}(U_f, Y_f, \mathbf{g}, K) \doteq \begin{bmatrix} I_{N_f+N_t} & \frac{1}{K}\hat{M}_0 \mathbf{\Gamma}_y \\ \frac{1}{K}\mathbf{\Gamma}_y^H \hat{M}_0^H & \mathbf{\Gamma}_u^H M_0 \mathbf{\Gamma}_u \end{bmatrix} \tag{23}$$

where $\hat{M}_0$ is the Cholesky factor of M_0.

Solving the minimization problem (22) can be be viewed as a denoising exercise wherein we remove bounded noise sequences from the measurements such that the existence of $N(z)$, and $M(z)$ becomes feasible. From the solutions for $W_{N_f}, \mathbf{m}_{N_t}, \mathbf{n}_{N_t}$, $N(z)$ and $M(z)$ can be realized as the central interpolants. Alternatively, the ideas in [8] can be used to directly seek for low order interpolants. However, this is beyond the scope of the present paper.

B. Normalizing N and M

Since N and M are SISO, common zeros can be eliminated by inspection. Let $P = N^{\sim}N + M^{\sim}M$. Note that by construction, $P(z)$ is proper (not strictly proper) since M is proper. Further, if P has a zero (pole) at z_i, then it also has a zero (pole) at $1/z_i$. Thus, $P(z)$ can be written as $P(z) = \phi(z)^{\sim}\phi(z)$ where $\phi(z)$ is stable, minimum phase. It follows that $N_r \doteq N/\phi(z)$ and $M_r \doteq M/\phi(z)$ is a NCF of G that satisfies the interpolation conditions.

C. Certifying Closed-Loop Stability

As mentioned in Section II-B, stabilization of the actual plant G when using a controller designed using the identified one G_{id} can be certified if $b_{G_{id}} > \nu_{\text{gap}}(G, G_{id})$. Note that the optimal $b_{G_{id}}$ can be calculated directly from the identified N, M using the expression ([14], Chapter 16)

$$b_{G_{id}}^{opt} = \sqrt{1 - \|\begin{bmatrix} N & M \end{bmatrix}\|_{\text{Hankel}}^2}$$

This expression gives an additional justification for the minimization of $\|\begin{bmatrix} N & M \end{bmatrix}\|_\infty$ in Algorithm (22), since the $\mathcal{H}_\infty$ norm is an upper bound of the Hankel norm.

Certifying stability requires estimating $\nu_{\text{gap}}(G, G_{id})$. This can be accomplished as follows: Let $\Psi(z) = -N_2^{\sim} M_1 + M_2^{\sim} N_1$. Since (N_2, M_2) and (N_1, M_1) are analytic inside the disk $|z| \leq \rho$, from Cauchy's integral it can be shown that $|\frac{d\Psi(z)}{dz}|_{z=e^{jw}} \leq (\rho-1)^{-1}$. Linearizing $\Psi(e^{jw})$ yields:

$$|\Psi(e^{jw})| \leq |\Psi(e^{jw_i})| + \frac{|w - w_i|}{(\rho - 1)} \tag{24}$$

The first term can be obtained directly from (1), by noting that $|G - G_{id}| \leq 2\epsilon_f$. Thus, if the frequency domain samples are sufficiently dense:

$$\nu_{\text{gap}}(G, G_{id}) \lessapprox \max_i \frac{2\epsilon_f}{\sqrt{1 + \frac{(|\tilde{Y}(z_i)| - \epsilon_f)^2}{|U(z_i)|^2}} \sqrt{1 + |\frac{N(z_i)}{M(z_i)}|^2}} + (\rho - 1)^{-1} \Delta w \tag{25}$$

If the condition $b_{G_{id}}^{opt} > \nu_{\text{gap}}(G, G_{id})$ fails, the existing experiments are insufficient to design a guaranteed stabilizing controller, and a further identification effort is required.

V. Numerical Example: SISO Unstable Plant

Consider the plant $G = 0.1(z - 1.1)/(z - 0.95)(z - 1.2)$. This plant has an NMP zero, and an unstable pole. It is not strongly stabilizable. Using NCFs is desirable here since a robustly stabilizing controller can be computed easily. The data is collected by simulating the plant in open-loop. The input profiles are chosen (after some trials) so as not to induce exponential growth in the output "too soon". The plant responses to two such inputs, one a filtered white noise and the other a filtered chirp signal, are collected. The measured responses are corrupted by additive noise. One dataset is used to perform an empirical estimation of the frequency response with Hann window (see `etfe` in MATLAB® System Identification Toolbox™ [18]). The resulting FRF contains 33 frequency samples. 50 samples of the second response are used to impose time-domain constraints. The data is shown in Figure 3. $\epsilon_t \leq 0.5$, $\epsilon_f \leq 0.15$, $\rho = 1.01$ are used as priors.

The identification approach was applied to this data in variables K_M, W, $\tilde{W}$, $\mathbf{m}$, and $\mathbf{n}$. $K_M = K_N = 20$ were determined to be feasible gain bounds. To reduce conservatism associated with the error bounds in equations (20),(21), the bounds were scaled heuristically to the values $\epsilon_f \leftarrow \epsilon_f/2, \epsilon_t \leftarrow \epsilon_t/1300$. The identified signals were used to extract parametric forms transfer functions $N(z)$, and $M(z)$ as the central interpolants, followed by $\mathcal{H}_2$ order reduction (fitting a lower order models to the frequency responses of $N(z)$ and $M(z)$). These values were finally normalized using the procedure described in section IV-B. The obtained values were:

$$N_r(z) = \frac{0.077(z - 1.093)}{(z - 0.943)(z - 0.823)}$$

$$M_r(z) = \frac{0.826(z - 0.955)(z - 1.192)}{(z - 0.943)(z - 0.823)}$$

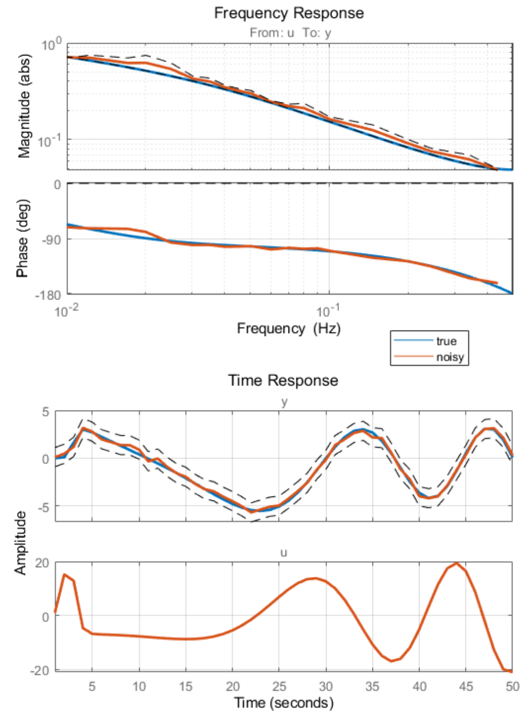

Fig. 3. Frequency and time-domain data for the example. The bounds imposed for identification are shown by black dotted lines.

The resulting impulse, and frequency responses of the NCFs $N_r(z), M_r(z)$ are shown in Figures 4 and 5.

The estimated model $G(z) = N_r(z)M_r(z)^{-1}$ is compared to the original (true) system in Fig. 6.

Computing $\nu_{\text{gap}}(G, G_{id})$ and $b_{G_{id}}$ yields $b_{G_{id}} = 0.064 > \nu_{\text{gap}}(G, G_{id}) = 0.041$. Hence an optimal $\mathcal{H}_\infty$ controller synthesized using the identified plant is guaranteed to stabilize the actual (unknown) one. Further, as shown in Fig. 7, the closed-loop responses of the plants are indeed close.

VI. Conclusions

This paper considers the problem of control-oriented learning. As illustrated with a simple example, in this scenario the objective of the identification should reflect the closed-loop, rather than open-loop distance between the learned model and the actual plant. This can be accomplished by considering a ν-gap motivated approach that seeks to identify a coprime factorization of the plant. The main result of the paper shows that these coprime factors can be identified via tractable convex optimization, by exploiting results from generalized interpolation theory. Contrary to previous approaches for identification of coprime factors, the proposed technique does not necessitate knowledge of a stabilizing controller and can handle both frequency and time domain constraints. Further, it is "self certified", since a simple norm computation on the learned factors indicates whether or not a controller designed based on these factors will stabilize the actual (unknown) system. If this test fails, it indicates that further learning is needed.

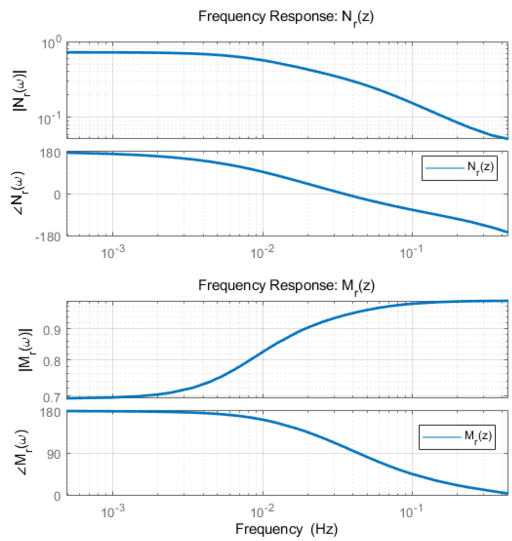

Fig. 4. Frequency responses of NCFs $N_r(z), M_r(z)$.

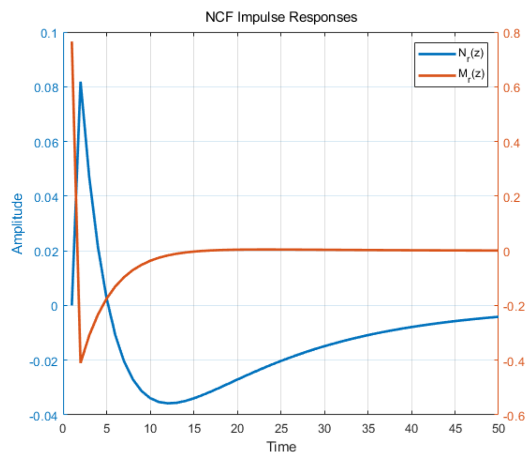

Fig. 5. Impulse responses of NCFs $N_r(z), M_r(z)$.

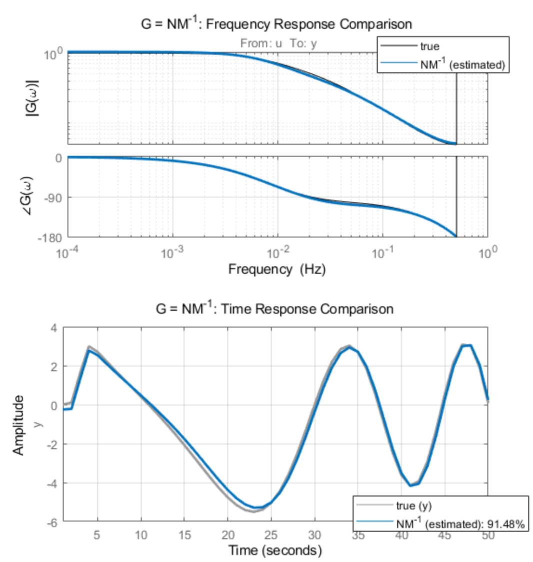

Fig. 6. Overall Identified Model $G(z) = N_r(z)M_r(z)^{-1}$.

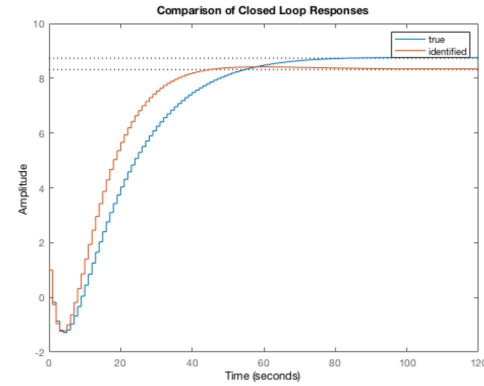

Fig. 7. Comparison of step responses obtained with a controller designed using the identified plant.

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