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Optimal Brokerage Contracts in Almgren–Chriss Model with Multiple Clients*

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Abstract. This paper constructs optimal brokerage contracts for multiple (heterogeneous) clients trading a single asset whose price follows the Almgren–Chriss model. The distinctive features of this work are as follows: (i) the reservation values of the clients are determined endogenously, and (ii) the broker is allowed to not offer a contract to some of the potential clients, thus choosing her portfolio of clients strategically. We find a computationally tractable characterization of the optimal portfolios of clients (up to a digital optimization problem, which can be solved efficiently if the number of potential clients is small) and conduct numerical experiments which illustrate how these portfolios, as well as the equilibrium profits of all market participants, depend on the price impact coefficients.

Key words. brokerage, optimal contract, price impact, equilibrium

MSC codes. 91A65, 91A23, 91G99

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1. Introduction.

This paper investigates the optimal design of brokerage fees in the Almgren–Chriss model (see [1]). We consider a population of investors (a.k.a. potential clients or agents) who can either trade directly in the market (and be subject to trading costs due to their price impact) or trade via a broker (i.e., become broker's clients) who charges a contingent fee for this service. The main goal of our investigation is a tractable characterization of the optimal brokerage fees and the optimal choice of a portfolio of clients. This question is formulated as an optimal contract problem with multiple agents, where the broker plays the role of a principal who designs the fees.

The problem considered herein formally fits within the optimal contract theory, which is concerned with the design of compensation (or incentive) schemes, referred to as contracts. In the classical example of an optimal contract problem (see, among others, [11], [9]), a principal hires an agent to work on a project in exchange for a payment (contract). The payment depends on the information available to the principal, which may be affected by the agent's action. The agent chooses his action to maximize his objective, which depends on the payment promised by the principal and on the action itself (e.g., the agent may not like to work very hard). The principal aims to choose the contract so that it maximizes her objective, which also depends on the payment to the agent and on the agent's action. This leads to a pair of nested optimization problems, also known as the Stackelberg game.

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As the broker observes the orders that she receives from her client, she knows his trading strategy precisely, which implies that the first natural model for the interaction between the broker and her individual client is given by the so-called first-best optimal contact problem (see, e.g., [9]).[1] Such optimal contract problems are known to have simple solutions (especially in the case of risk-neutral preferences, as herein), and this is confirmed by Proposition 3.1, which provides the optimal brokerage fees in the present setting. However, our setting implies the following additional challenges. First, the model considered herein includes multiple agents whose objectives are coupled via their price impact. Thus, a collective response of the agents to a contract chosen by the principal is given by a collection of strategies that form a Nash equilibrium among the agents. The principal, then, chooses a contract so as to maximize her objective that depends on the associated equilibrium strategies of the agents. The optimal contract problems with multiple agents are considered, for example, in [10], [7], [12]. It is worth mentioning that we consider heterogeneous agents, as one of our goals is to study how the characteristics of the agents (i.e., their price impact coefficients) affect the optimal choice of the portfolio of clients. Another important feature that makes the present problem non-standard is the fact that the reservation value of each agent (which represents the minimum objective value that the agent must be able to attain in order to accept a proposed contract) is determined endogenously. Indeed, we naturally assume that the reservation value of each agent equals his maximum objective value in case he decides to trade directly in the market. The latter value depends on the equilibrium strategies of other agents, which in turn depend on the contract chosen by the broker. The third distinctive feature of the present work is that, unlike the classical optimal contract problems, the broker is not constrained to offer a contract to every agent and chooses her portfolio of clients strategically. In particular, one of our main questions is to determine the optimal portfolio of clients for the broker.

To the best of our knowledge, to date there exist no results on the optimal brokerage fees in the presence of price impact and multiple agents. The recent paper [3] studies a related problem in which a single agent hires a financial intermediary to trade a risky asset on his behalf and pays a fee (chosen by the agent) for this service. This is also related to earlier literature on delegated portfolio management: see, e.g., [18], [19], [16], [5], [4], [8], [13]. Despite obvious similarities, the important conceptual differences between the latter works and the present one are that, herein, (i) the fee is designed by the broker, (ii) multiple agents are present, and (iii) ex ante the trading strategy is determined by the client as opposed to the broker, who nevertheless does observe the strategy.

The rest of the paper is organized as follows. Section 2 introduces the model and the main objectives. Section 3 constructs optimal contracts (Proposition 3.1) given (arbitrary) reservation values of the agents and broker's (arbitrary) choice of clients. Section 4 describes the unique equilibrium among those agents who are not offered a contract (i.e., among independent agents). Section 5 defines the reservation values of the agents endogenously (Definition 5.6) and shows how to compute the maximum objective value of the broker given an arbitrary portfolio of clients (Theorem 5.5). The latter result allows one to find an optimal portfolio of clients for the broker by solving a digital optimization problem. Section 6 considers several

[1] Certain information asymmetry may still be natural for this problem, as discussed in Remark 2.

numerical experiments, where the aforementioned digital optimization problem is solved by an exhaustive search, and the broker's optimal portfolio of clients, as well as the profits of the broker and of the agents, are analyzed as functions of the price impact parameters.

2. The setup.

We consider N agents, each of whom can trade a single risky asset[2] that follows the Almgren–Chriss model over the time interval $[0,T]$. In addition to the agents, we assume the presence of a single broker. Each agent makes a decision (once, before the trading starts) on whether he trades the asset directly or via the broker, and these decisions are represented by the vector $\theta \in \{0,1\}^N$: $\theta_i = 1$ if and only if the ith agent trades via the broker. For convenience, we also denote by $\mathcal{N}(\theta) = \{n_1,\dots,n_r\} \subset \{1,\dots,N\}$, $0 < r < N$, the indices of the agents that trade via broker. We refer to the agents who trade via the broker as clients and to those who trade directly in the market as independent. The trading activity of an independent agent affects the price of the asset via the impact coefficients of this agent. The trading of a client is done via the broker and hence affects the price of the asset via the impact coefficients of the broker. Of course, the broker may be able to offer lower price impact to her clients, but she also charges each of them a fee for this service. Even though we introduced θ as a vector of agents' decisions, it is important to realize that these decisions are ultimately controlled by the broker, who decides whether to offer a contract to a particular agent or not (the acceptance of each offered contract is ensured by matching the reservation value of the associated agent). Therefore, in the remainder of the paper, we refer to θ as the choice of clients made by the broker.

We fix a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and consider a standard Brownian motion B on this space. Let $\mathbb{F}^B$ be the filtration generated by the Brownian motion B. We define the set of admissible controls of a single agent as

$$\mathcal{U} := \{\nu \in L^2([0,T]\times\Omega), \quad \nu \text{ is } \mathbb{F}^B\text{-adapted}\}. \tag{2.1}$$

The (controlled) inventory of agent $i = 1,\dots,N$, who uses a control ν^i, is given by the process

$$X^i_t = x^i_0 + \int_0^t \nu^i_s ds, \quad t \in [0,T].$$

Next, we recall the price process for the traded asset in the Almgren–Chriss model:

$$P_t = \mu t + \sigma B_t + \sum_{i=1}^N (1-\theta_i)\kappa_i \nu^i_t + \kappa_0 \sum_{i=1}^N \theta_i \nu^i_t + \sum_{i=1}^N (1-\theta_i)\lambda_i X^i_t + \lambda_0 \sum_{i=1}^N \theta_i X^i_t, \tag{2.2}$$

where $\{\kappa_j\}_{j=1}^N$, $\{\lambda_j\}_{j=1}^N$ are the coefficients of temporary and permanent price impacts of the agents, κ_0, λ_0 are the corresponding coefficients of the broker, $\sigma > 0$ is the volatility of the asset price, and $\mu \in \mathbb{R}$ is its drift (i.e., trading signal). For convenience, we assume that $\nu^i_T = 0$ for all i, so that the temporary impacts of the agents do not affect the terminal price.[3] We also assume that the agents and the broker have access to the same filtration $\mathbb{F}^B$ and that the constants $\{\kappa_j,\lambda_j,x^j_0\}_{j=0}^N$, μ, σ are known to all parties.

Remark 1. Equation (2.2) implies that, in the model proposed herein, the instantaneous impact is viewed as the actual price impact and, hence, affects the dynamics of the commonly

[2] A riskless asset is implicitly available but yields zero return.

[3] This is needed to simplify the notation in (2.3)–(2.4), as the agents in our setting interact through both the permanent and the temporary impacts.

observed price. There exists an alternative point view (cf. [6], [7], [12], [14], [20]), according to which the instantaneous impact represents additional execution cost, as opposed to being the actual price impact. The latter approach leads to a price model that is similar to (2.2) but has only permanent impact in the right-hand side, and where the effect of temporary impact appears in each individual optimization objective (in particular, the agents' problems are not coupled via temporary impact coefficients). We believe that the results of the present paper can be easily re-derived in such a setting. However, we choose to stick to the model given by (2.2) for two reasons. First, we prefer to not view the temporary impact as the additional execution cost, in particular, because of its superlinearity, i.e., the total sum of such execution costs of several agents trading in the same direction would be smaller than the execution cost of a strategy given by a sum of the strategies of these agents. The latter phenomenon does not have any convincing logical explanation, nor is there any clear evidence of it in practice. On the other hand, such a phenomenon would provide a strong negative incentive for the agents against trading via a broker, and hence would have a strong impact on the numerical results of this paper. Second, we view the Almgren–Chriss model as a mathematical abstraction that is meant to approximate the more realistic (but less tractable) Obizhaeva–Wang model [15], in which the price impact has exponential resilience and is captured by two parameters; hence, we view the temporary impact as the actual price impact.

Remark 2. As mentioned in the introduction, the broker clearly observes the trading strategy of her client, which eliminates the moral hazard phenomenon (appearing when the principal does not directly observe her agent's action) and, hence, makes it natural to model the interaction between the broker and her client as the first-best optimal contract problem. However, the absence of moral hazard does not exclude other forms of information asymmetry that could be natural for this interaction. For example, one could (naturally) assume that the broker does not know the constants $\{\kappa_j, \lambda_j, x_0^j\}_{j=1}^N$, μ, σ, and only knows their prior distribution, which would lead to the so-called third-best optimal contract problem (see, e.g., [9]). A slightly different version of such a model would prescribe the dynamics of (some of) these parameters (e.g., of μ which represents the trading signal) and would assume that these dynamic processes are observed by the agent (e.g., because the agent is a professional investor who can obtain a good estimate of μ) but not by the broker (who, ideally, is not supposed to be in the business of estimating trading signals). The latter extension of the model (with dynamic partially observed μ) is the subject of our ongoing investigation [2]. The other natural extensions (e.g., with partially observed $\{\kappa_j, \lambda_j, x_0^j\}_{j=1}^N$) are left for future research. It is worth mentioning that all aforementioned extensions are significantly more challenging on the mathematical level than the first- or second-best problems, due to the absence of a sufficiently general characterization of the associated optimal contracts.

Let $\xi^{n_1}, \dots, \xi^{n_r}$ be the fees that the broker charges to her clients. We assume that the fees are of the form $\xi^i = F^i(X^i, P)$, with measurable $F^i : H^1([0,T], \mathbb{R}) \times C([0,T], \mathbb{R}) \to \mathbb{R}$, where H^1 is the Sobolev space of order one, equipped with the natural norm. This definition of broker's fee implies that it is paid at the terminal time horizon T. However, as follows from Remark 4, the optimal fee constructed herein can be expressed as an integral of an adapted process (w.r.t. dt and dP_t) and, hence, can be paid as a continuous cash flow.

We assume that the time horizon T of the model is not too short, so that multiple child orders can be sent to the broker within this time horizon, allowing the agent to adjust his strategy dynamically. Even though the execution of each child order is controlled by the broker, the agent has full control of the types (buy or sell) and the sizes of these child orders. In the continuous-time approximation, we view a sequence of such child orders as the trading strategy (i.e., trading rate) ν that is fully controlled by the agent.

Every client $i \in \mathcal{N}(\theta)$ aims to choose his strategy ν^i to maximize his expected profit:

$$\begin{aligned} J^{i,\theta}(\nu^i,\nu^{-i},\xi^i) &= \mathbb{E}\left(P_T X_T^i - \frac{\lambda_i}{2}(X_T^i)^2 - \int_0^T \nu_t^i P_t dt - \xi^i\right) \\ &= \mathbb{E}\left(\int_0^T X_t^i \left[\mu + \sum_{j=1}^N (1-\theta_j)\lambda_j \nu_t^j + \lambda_0 \sum_{j=1}^N \theta_j \nu_t^j\right] dt \right. \\ &\quad \left. - \int_0^T \nu_t^i \left[\sum_{j=1}^N (1-\theta_j)\kappa_j \nu_t^j + \kappa_0 \sum_{j=1}^N \theta_j \nu_t^j\right] dt - \frac{\lambda_i}{2}(X_T^i)^2 - \xi^i\right), \end{aligned} \tag{2.3}$$

where $\nu^{-i} \in \mathcal{U}^{N-1}$ denotes the trading rate of the rest of the agents. Similarly, an independent agent $i \notin \mathcal{N}(\theta)$ maximizes his expected profit:

$$\begin{aligned} J^{i,\theta}(\nu^i,\nu^{-i},\xi^i) &= \mathbb{E}\left(P_T X_T^i - \frac{\lambda_i}{2}(X_T^i)^2 - \int_0^T \nu_t^i P_t dt\right) \\ &= \mathbb{E}\left(\int_0^T X_t^i \left[\mu + \sum_{j=1}^N (1-\theta_j)\lambda_j \nu_t^j + \lambda_0 \sum_{j=1}^N \theta_j \nu_t^j\right] dt \right. \\ &\quad \left. - \int_0^T \nu_t^i \left[\sum_{j=1}^N (1-\theta_j)\kappa_j \nu_t^j + \kappa_0 \sum_{j=1}^N \theta_j \nu_t^j\right] dt - \frac{\lambda_i}{2}(X_T^i)^2\right). \end{aligned} \tag{2.4}$$

Remark 3. In the above objectives, the penalty $-\lambda_i(X_T^i)^2/2$ corresponds to the cost of liquidation of the inventory X_T^i by agent i, over a long time period after time T. Even if the agent plans to liquidate his terminal position X_T^i with the broker, in the resulting extended game the broker will charge the agent more by increasing the deterministic component of ξ^i. Due to the optimality of the contract, this additional charge should be equal to the liquidation cost of the agent. Assuming that the agent does not know the inventories, and hence the execution plans, of the other agents, he computes his execution cost assuming that the additional drift in the asset price during his liquidation is only due to his permanent impact (the temporary impact can be ignored if the liquidation period is large). This can be interpreted as the agent being naive, in the sense that he does not take into account that other agents will be liquidating their positions at the same time, hence, he only includes his own impact in the execution cost. Under such an assumption it is well known that the expected execution cost of the agent is equal to $\lambda_i(X_T^i)^2/2$, which is added to the fees that the agent pays to the broker.

For a given combination of the strategies of other agents, we define the control problem of a client $i \in \mathcal{N}(\theta)$ and of an independent agent $i \notin \mathcal{N}(\theta)$, respectively, as

$$V^{i,\theta} := \sup_{\nu^i \in \mathcal{U}} J^{i,\theta}(\nu^i, \nu^{-i}, \xi^i), \quad i \in \mathcal{N}(\theta), \tag{2.5}$$

$$V^{i,\theta} := \sup_{\nu^i \in \mathcal{U}} J^{i,\theta}(\nu^i, \nu^{-i}), \quad i \notin \mathcal{N}(\theta). \tag{2.6}$$

Definition 2.1. *Given a choice of clients $\theta \in \{0,1\}^N$, as well as the associated indices $\mathcal{N}(\theta) = \{n_1, \dots, n_r\}$ and fees $\xi = (\xi^{n_1}, \dots, \xi^{n_r})$, we define $\mathcal{E}(\theta, \xi) \subset \mathcal{U}^N$ as the set of all agents' strategies that form Nash equilibria in the game defined by (2.5)–(2.6). Namely, $\bar{\nu} \in \mathcal{E}(\theta,\xi)$ if and only if the following two conditions hold:*

$$J^{i,\theta}(\bar{\nu}^i, \bar{\nu}^{-i}, \xi^i) \geq J^{i,\theta}(\nu, \bar{\nu}^{-i}, \xi^i) \quad \forall \nu \in \mathcal{U}, \quad \forall i \in \mathcal{N}(\theta), \tag{2.7}$$

$$J^{i,\theta}(\bar{\nu}^i, \bar{\nu}^{-i}) \geq J^{i,\theta}(\nu, \bar{\nu}^{-i}) \quad \forall \nu \in \mathcal{U}, \quad \forall i \notin \mathcal{N}(\theta). \tag{2.8}$$

The objective of the broker is given by the sum of expected fees in the best equilibrium attainable with these fees:

$$J_P^{\theta}(\xi) := \sup_{\nu \in \mathcal{E}(\theta,\xi)} \mathbb{E} \sum_{j \in \mathcal{N}(\theta)} \xi^j. \tag{2.9}$$

In the above, we make the standard assumption that, given a set of admissible contracts, the agents will choose an equilibrium that is best for the principal among all attainable equilibria. To ensure that $\mathcal{E}(\theta,\xi) \neq \emptyset$ and that the agents' reservation values are met, we introduce the set of admissible fees of the broker:

$$\Sigma(\theta) := \{(\xi^j)_{j \in \mathcal{N}(\theta)} : \mathcal{E}(\theta,\xi) \neq \emptyset \text{ and } J^j(\bar{\nu}, \xi^j) \geq R^{j,\theta} \quad \forall j \in \mathcal{N}(\theta), \forall \bar{\nu} \in \mathcal{E}(\theta,\xi)\}, \tag{2.10}$$

where $R^{j,\theta}$ is the reservation value of agent $j \in \mathcal{N}(\theta)$. Recall that each reservation value $R^{j,\theta}$ represents the alternative benefit that the agent j would receive if he does not enter into the contractual agreement with the principal (see [9] and the references therein). This value is allowed to depend on θ, as the agent's alternative profits may depend on whether the others trade via the broker or not. In general, the reservation values $\{R^{j,\theta}\}$ can be specified arbitrarily, and most of the results of the present work (up to and including Theorem 5.5) hold for any choice of the reservation values. However, in the present setting, there exists a very natural choice of such reservation values, which is presented in Definition 5.6 and is used in our numerical experiments.

Thus, we obtain the following "local" maximization problem for the broker, given a choice of clients θ:

$$V_P^{\theta} := \sup_{\xi \in \Sigma(\theta)} J_P^{\theta}(\xi) = \sup_{\xi \in \Sigma(\theta)} \sup_{\nu \in \mathcal{E}(\theta,\xi)} \mathbb{E} \sum_{j \in \mathcal{N}(\theta)} \xi^j. \tag{2.11}$$

For each θ, an optimal contract ξ^* that solves the above local maximization problem is constructed in section 3. Then, in order to find an optimal portfolio of agents, one needs to solve the "global" optimization problem of the broker, i.e., find an optimal $\theta^* \in \{0,1\}^N$ which attains the supremum in

$$V_P := \sup_{\theta \in \{0,1\}^N} V_P^\theta. \tag{2.12}$$

The above is a discrete optimization problem. We do not provide a complete solution to this problem herein, assuming instead that it can be solved by an exhaustive search in case of a reasonably small N or of a smaller subset of admissible $\{\theta\}$ (see section 6). However, even to perform such an exhaustive search, one needs to have a numerically tractable representation of the value of the local maximization problem, V_P^θ, for each θ. The latter representation is the main subject of sections 4 and 5.

3. Optimal contract for a given θ.

The main result of this section is the following proposition, which describes an optimal collection of fees offered by the broker, given a choice of clients θ and agents' reservation values $\{R^{i,\theta}\}$.

Proposition 3.1. *For any $\theta \in \{0,1\}^N$ and $\{R^{i,\theta}\}$, the fees*

$$\xi^{i,*} := -R^{i,\theta} + P_T X_T^i - \int_0^T \nu_t^i P_t dt - \frac{\lambda_i}{2}(X_T^i)^2, \quad i \in \mathcal{N}(\theta), \tag{3.1}$$

are optimal for the problem (2.11).

Remark 4. Note that $\xi^{i,*}$ can be expressed in the integral form

$$\xi^{i,*} := -R^{i,\theta} + P_0 x_0^i - \frac{\lambda_i}{2}(x_0^i)^2 + \int_0^T X_t^i dP_t - \lambda_i \int_0^T X_t^i \nu_t^i dt,$$

showing that this fee can be paid as a continuous cash flow.

Proof. It is easy to see that, with the fees (3.1), the objective of any agent $i \in \mathcal{N}$, given by (2.3), is constant and is equal to his reservation $R^{i,\theta}$. In particular, every agent $i \in \mathcal{N}$ is indifferent in which action to choose. Using this observation, let us show that, for any set of admissible fees $(\xi^j)_{j\in\mathcal{N}} \in \Sigma(\theta)$, we have the inclusion

$$\mathcal{E}(\theta,\xi) \subset \mathcal{E}(\theta,\xi^*), \tag{3.2}$$

where ξ^* is given by (3.1). Indeed, for any $(\tilde{\nu}^1,\ldots,\tilde{\nu}^N) \in \mathcal{E}(\theta,\xi)$ and $i \in \mathcal{N}$, we have

$$J^{i,\theta}(\tilde{\nu}^i,\tilde{\nu}^{-i},\xi^{i,*}) = R^{i,\theta} = \sup_{\nu^i \in \mathcal{U}} J^{i,\theta}(\nu^i,\tilde{\nu}^{-i},\xi^{i,*}),$$
$$J^{i,\theta}(\tilde{\nu}^i,\tilde{\nu}^{-i}) = \sup_{\nu^i \in \mathcal{U}} J^{i,\theta}(\nu^i,\tilde{\nu}^{-i}),$$

which implies $(\tilde{\nu}^1,\ldots,\tilde{\nu}^N) \in \mathcal{E}(\theta,\xi^*)$. Using (3.2) and the admissibility constraint in (2.10), we deduce

(3.3)

$$\begin{aligned}
V_P^\theta &= \sup_{\xi\in\Sigma(\theta)} \sup_{\nu\in\mathcal{E}(\theta,\xi)} \mathbb{E} \sum_{j\in\mathcal{N}(\theta)} \xi^j = \sup_{\xi\in\Sigma(\theta)} \sup_{\nu\in\mathcal{E}(\theta,\xi)} \Bigg[\mathbb{E} \sum_{j\in\mathcal{N}} \left(X_T^j \left(P_T - \frac{\lambda_j}{2} X_T^j \right) - \int_0^T \nu_t^j P_t^\theta dt \right) \\
&\quad - \mathbb{E} \sum_{j\in\mathcal{N}} \left(X_T^j \left(P_T - \frac{\lambda_j}{2} X_T^j \right) - \int_0^T \nu_t^j P_t^\theta dt - \xi^j \right) \Bigg] \\
&\le \sup_{\xi\in\Sigma(\theta)} \sup_{\nu\in\mathcal{E}(\theta,\xi)} \mathbb{E} \sum_{j\in\mathcal{N}} \left(X_T^j \left(P_T - \frac{\lambda_j}{2} X_T^j \right) - \int_0^T \nu_t^j P_t^\theta dt \right) - \sum_{j\in\mathcal{N}} R^{j,\theta} \\
&\le \sup_{\nu\in\mathcal{E}(\theta,\xi^*)} \mathbb{E} \left(\sum_{j\in\mathcal{N}} X_T^j \left(P_T - \frac{\lambda_j}{2} X_T^j \right) - \int_0^T \nu_t^j P_t^\theta dt \right) - \sum_{j\in\mathcal{N}} R^{j,\theta} = \sup_{\nu\in\mathcal{E}(\theta,\xi^*)} \mathbb{E} \left(\sum_{j\in\mathcal{N}} \xi^{j,*} \right) \\
&= J_P^\theta(\xi^*).
\end{aligned}$$

■

Remark 5. Note that the structure of the optimal fee ξ^* is very simple: the principal takes all the profits (and losses) from each agent and gives him back his reservation value. To show that a similar phenomenon can be observed in practice, we recall that some hedge funds consist (fully or partially) of (teams of) portfolio managers (PMs), who have close-to-complete freedom in making their investment decisions. Such a hedge fund provides investment capital to its PMs and executes their orders—the latter being reminiscent of the role of a broker in the present model. In return, the fund takes all the profits and losses of the PMs and gives them back an agreed-upon compensation. The model proposed herein clearly applies to the interaction between such a fund and its PMs, with the broker's fee being the negative of the compensation paid by the fund to a PM. This observation illustrates that the following two features, assumed or derived in the present model, are indeed observed in the real world: (i) a brokerage fee may be customized to the characteristics of a client (such as his reservation value and price impact) and (ii) the optimal fee may have the structure described in Proposition 3.1 (i.e., "take away the profits and pay back a compensation").

4. Equilibrium strategies of independent agents. Proposition 3.1 shows that, for any given θ, there exists a trivial choice of optimal contracts. This provides a solution to the broker's local problem (2.11). Nevertheless, to find the optimal choice of θ that solves the global problem (2.12), we need to compute the value function V_P^θ for each θ. This, in turn, requires the knowledge of the equilibrium strategies of the agents that correspond to the fees ξ^* constructed in Proposition 3.1, as well as the value of the broker in this Stackelberg game. The former is discussed in this section, and the latter is analyzed in section 5.

The results of this section hold for an arbitrary fixed $\theta \in \{0,1\}^N$. However, for convenience, we assume that $\mathcal{N}(\theta) = \{m+1,\dots,N\}$ with some $0 \le m \le N-1$.

Notice that, with the fees given by (3.1), the objectives of the broker's clients do not depend on their actions nor on the actions of the independent agents. Hence, any equilibrium in the subgame among the independent agents can trivially be extended to an equilibrium among all agents. This observation is made precise in Theorem 5.5, and it is only brought up here to explain why it suffices to focus on the equilibria among independent agents, which is the main subject of the remainder of this section.

We begin by noticing that the objective (2.4) of an independent agent, by design, is only affected by the actions of the broker and of her clients through the total order flow of the broker's clients, denoted

$$u := \sum_{i=m+1}^{N} \nu^i.$$

We refer to u as the broker's order flow. In particular, for the fees ξ^* constructed in Proposition 3.1, the objective (2.4) of an independent agent $i \notin \mathcal{N}$ can be rewritten as

$$\begin{aligned} J^{i,\theta}(\nu^i, \nu^{-i}) &= \tilde{J}^{i,\theta}(\nu^i, \nu^{m,-i}, u) \\ &:= \mathbb{E}\left[\int_0^T X_t^i \left(\mu + \sum_{j=1}^m \lambda_j \nu_t^j + \lambda_0 u_t\right) dt - \int_0^T \nu_t^i \left(\sum_{j=1}^m \kappa_j \nu_t^j + \kappa_0 u_t\right) dt - \frac{\lambda_i}{2}(X_T^i)^2\right] \\ &= \mathbb{E}\left[\int_0^T X_t^i \left(\mu + \sum_{j\neq i, j\leq m} \lambda_j \nu_t^j + \lambda_0 u_t\right) dt - \int_0^T \nu_t^i \left(\sum_{j=1}^m \kappa_j \nu_t^j + \kappa_0 u_t\right) dt - \frac{\lambda_i}{2}(x_0^i)^2\right], \end{aligned} \tag{4.1}$$

where $\nu^{m,-i}$denotes the vector $(\nu^1, \ldots, \nu^m)$ without the ith element and $X_t^i = x_0^i + \int_0^t \nu_s^i ds$.

The main goal of this section is to characterize all Nash equilibria among independent agents who solve

$$\sup_{\tilde{\nu}^i \in \mathcal{U}} \tilde{J}^{i,\theta}(\tilde{\nu}^i, \tilde{\nu}^{-i}, u), \quad i = 1, \ldots, m, \tag{4.2}$$

for any given order flow of the broker $u \in \mathcal{U}$. It is worth mentioning that the above equilibrium problem is similar to the ones appearing in the existing literature on multiple agents trading in a price impact model (see, e.g., [7] and the references therein). Nevertheless, there are three features that (collectively) differentiate the equilibrium problem herein: (i) the population of agents is finite and heterogeneous, (ii) the broker's strategy u is an arbitrary square-integrable (stochastic) process, and (iii) the agents interact through both permanent and temporary impact. These features bring the present problem outside the scope of the existing results, and, in particular, we have to prove existence and uniqueness of the associated equilibrium directly, which is done in the following proposition.

Proposition 4.1. *For any $u \in \mathcal{U}$, there exists a unique Nash equilibrium $(\tilde{\nu}^{1,*}, \ldots, \tilde{\nu}^{m,*})$ of (4.2), and it is given by*

$$\tilde{\nu}_t^{i,*} = \frac{1}{\kappa_i(m+1)}\left(mY_t^i - \sum_{j\neq i}^m Y_t^j - \kappa_0 u_t\right), \quad 1 \leq i \leq m, \tag{4.3}$$

where (Y, Z) is the unique solution of the BSDE:

$$\begin{aligned} dY_t^i &= -\left[\mu - \frac{\gamma - \lambda_i/\kappa_i}{m+1} Y_t^i + \sum_{j\neq i}^m \left(\frac{\lambda_j}{\kappa_j} - \frac{\gamma - \lambda_i/\kappa_i}{m+1}\right) Y_t^j + \left(\lambda_0 - \frac{\gamma - \lambda_i/\kappa_i}{m+1}\kappa_0\right) u_t\right] dt \\ &\quad + Z_t^i dB_t, \\ Y_T^i &= 0, \quad 1 \leq i \leq m, \end{aligned} \tag{4.4}$$

and

$$\gamma := \sum_{i=1}^m \frac{\lambda_i}{\kappa_i}.$$

Proof. Let us fix arbitrary $u \in \mathcal{U}$, $1 \leq i \leq m$, $\tilde{\nu}^{-i} \in \mathcal{U}^{m-1}$, and describe an optimal strategy for the agent i. Recall that the agent i maximizes the right-hand side of (4.1). We introduce his Hamiltonian:

$$H_t^i\left(\tilde{\nu}^i, \tilde{x}^i, y^i, \tilde{\nu}^{-i}\right) = \tilde{\nu}^i y^i + \tilde{x}^i \left(\mu + \lambda_0 u_t + \sum_{j\neq i} \lambda_j \tilde{\nu}^j\right) - \kappa_i \left(\tilde{\nu}^i\right)^2 - \tilde{\nu}^i \sum_{j\neq i} \kappa_j \tilde{\nu}^j - \kappa_0 u_t \tilde{\nu}^i.$$

Next, we observe that H_t^i is concave in $(\tilde{\nu}^i, \tilde{x}^i)$. Applying the stochastic maximum principle for the ith agent's problem (see, e.g., Theorem 6.4.6 in [17]), we conclude that the strategy defined by

$$\begin{aligned} \tilde{\nu}_t^i &= \frac{1}{2\kappa_i}\left(Y_t^i - \sum_{j\neq i}^m \kappa_j \tilde{\nu}_t^j - \kappa_0 u_t\right), \\ dY_t^i &= -\left(\mu + \sum_{j\neq i}^m \lambda_j \tilde{\nu}_t^j + \lambda_0 u_t\right) dt + Z_t^i dB_t, \quad Y_T^i = 0, \end{aligned} \tag{4.5}$$

is optimal.

Moreover, as the objective of the agent i is strictly concave, we conclude that (4.5) defines his unique optimal strategy, given $\tilde{\nu}^{-i} \in \mathcal{U}^{m-1}$. Applying the same argument for every agent $1 \leq i \leq m$, we deduce that any solution of the system (4.5), for $i = 1, \ldots, m$, defines a Nash equilibrium among the independent agents. By the strict concavity of the individual objectives we obtain that any Nash equilibrium is a solution to (4.5). Summing up the first equation in (4.5) over i, we obtain

$$\sum_{j=1}^m \kappa_j \tilde{\nu}_t^j = \frac{1}{m+1}\left(\sum_{j=1}^m Y_t^j - m\kappa_0 u_t\right), \tag{4.6}$$

and, in turn,

$$\tilde{\nu}_t^i = \frac{1}{\kappa_i}\left(Y_t^i - \sum_{j=1}^m \kappa_j \tilde{\nu}_t^j - \kappa_0 u_t\right) = \frac{1}{\kappa_i}\left(Y_t^i - \frac{1}{m+1}\left(\sum_{j=1}^m Y_t^j - m\kappa_0 u_t\right) - \kappa_0 u_t\right), \quad 1 \leq i \leq m.$$

Plugging the above in the second equation in (4.5), we obtain (4.4). Thus, we have shown that any Nash equilibrium among the independent agents satisfies (4.3)–(4.4). It remains to notice that (4.4) is a standard linear BSDE, and its solution is unique. The latter, in particular, yields uniqueness of the solution to (4.5) and hence the uniqueness of equilibrium. ∎

An immediate corollary of Proposition 4.1 is that, with $\mathcal{N}(\theta)=\{m+1,\dots,N\}$ and with the fees ξ^* given by (3.1), the set $\mathcal{E}(\theta,\xi^*)$ of all equilibria among the agents (see Definition 2.1) is given by

$$\mathcal{E}(\theta,\xi^*)=\left\{(\tilde\nu^{1,*}(u),\dots,\tilde\nu^{m,*}(u),\nu^{m+1},\dots,\nu^N):\, u=\sum_{i=m+1}^N \nu^i,\ \nu^{m+1},\dots,\nu^N\in\mathcal{U}\right\},$$

where $(\tilde\nu^{1,*}(u),\dots,\tilde\nu^{m,*}(u))$ are given by (4.3).

5. Optimization problem of the broker.

As in the previous section, the results of this section hold for an arbitrary fixed $\theta\in\{0,1\}^N$, but, for convenience, we assume that $\mathcal{N}(\theta)=\{m+1,\dots,N\}$ with some $0\le m\le N-1$.

Herein, we turn to the control problem of the broker. Notice that, with the fees given by (3.1) and with the strategies of the broker's clients denoted by $(\nu^{m+1},\dots,\nu^N)$, the independent agents will necessarily adapt the strategies $(\tilde\nu^{1,*}(u),\dots,\tilde\nu^{m,*}(u))$, given by (4.3) with $u=\sum_{i=m+1}^N\nu^i$, and the payoff of the broker can be written as

$$\begin{aligned}\tilde J_P^\theta(u,X_T^{m+1},\dots,X_T^N)=\mathbb{E}\Bigg[&\int_0^T\left(X_t^0+\sum_{i=m+1}^N x_0^i\right)\left(\mu+\sum_{i=1}^m\lambda_i\tilde\nu_t^{i,*}+\lambda_0u_t\right)dt\\&-\int_0^T u_t\left(\sum_{i=1}^m\kappa_i\tilde\nu_t^{i,*}+\kappa_0u_t\right)dt-\sum_{i=m+1}^N\frac{\lambda_i}{2}(X_T^i)^2-\sum_{i=m+1}^N R^{i,\theta}\Bigg],\end{aligned}\tag{5.1}$$

where

$$X_t^0:=\int_0^t u_t dt,\quad X_T^i=x_0^i+\int_0^T\nu_t^i dt.$$

This implies that the value of the broker's objective (2.9), for the fees given by (3.1), can be written as

$$J_P^\theta(\xi^*)=\sup_{u\in\mathcal{U}}\ \sup_{\substack{X_T^{m+1},\dots,X_T^N\in\mathcal{G},\\ \int_0^T u_tdt=\sum_{i=m+1}^N(X_T^i-x_0^i)}}\tilde J_P^\theta(u,X_T^{m+1},\dots,X_T^N),\tag{5.2}$$

where

$$\mathcal{G}:=\left\{X\in\mathcal{F}_T, s.t.\ X=\int_0^T u_tdt,\ \text{for some } u\in\mathcal{U}\right\}.$$

Next, we use (4.3) to deduce

$$\begin{aligned}\sum_{i=1}^m\kappa_i\tilde\nu_t^{i,*}&=\frac{m}{m+1}\sum_{i=1}^m Y_t^i-\frac{1}{m+1}\sum_{i=1}^m\left(\sum_{j=1}^m Y_t^j-Y_t^i\right)-\frac{m}{m+1}\kappa_0u_t\\&=\frac{1}{m+1}\sum_{i=1}^m Y_t^i-\frac{m}{m+1}\kappa_0u_t,\end{aligned}$$

$$\sum_{i=1}^{m}\lambda_i\tilde{\nu}_t^{i,*} = \frac{m}{m+1}\sum_{i=1}^{m}\frac{\lambda_i}{\kappa_i}Y_t^i - \frac{1}{m+1}\sum_{i=1}^{m}\frac{\lambda_i}{\kappa_i}\left(\sum_{j=1}^{m}Y_t^j - Y_t^i\right) - \frac{\gamma}{m+1}\kappa_0 u_t$$
$$= \sum_{i=1}^{m}\frac{\lambda_i}{\kappa_i}Y_t^i - \frac{\gamma}{m+1}\sum_{i=1}^{m}Y_t^i - \frac{\gamma}{m+1}\kappa_0 u_t = \sum_{i=1}^{m}\left(\frac{\lambda_i}{\kappa_i} - \frac{\gamma}{m+1}\right)Y_t^i - \frac{\gamma}{m+1}\kappa_0 u_t,$$

which leads to

$$\tilde{J}_P^\theta(u, X_T^{m+1},\ldots,X_T^N)$$
$$= \mathbb{E}\left[\int_0^T\left(X_t^0 + \sum_{i=m+1}^{N}x_0^i\right)\left(\mu + \sum_{i=1}^{m}\left[\frac{\lambda_i}{\kappa_i} - \frac{\gamma}{m+1}\right]Y_t^i + \left[\lambda_0 - \frac{\gamma\kappa_0}{m+1}\right]u_t\right)dt\right.$$
$$\left. - \frac{1}{m+1}\int_0^T u_t\left(\sum_{i=1}^{m}Y_t^i + \kappa_0 u_t\right)dt - \sum_{i=m+1}^{N}\frac{\lambda_i}{2}(X_T^i)^2 - \sum_{i=m+1}^{N}R^{i,\theta}\right],$$

where (Y, Z) is the unique solution to the linear BSDE (4.4).

Let us resolve the optimization over $X_T^{m+1},\ldots,X_T^N$ in (5.2) for each fixed u and ω. Indeed, the latter amounts to solving the quadratic minimization problem with linear constraints:

$$\inf_{X_T^{m+1},\ldots,X_T^N\in\mathbb{R}}\sum_{i=m+1}^{N}\frac{\lambda_i}{2}(X_T^i)^2$$
$$\text{s.t.}\quad \sum_{i=m+1}^{N}(X_T^i - x_0^i) = X_T^0,$$

where we recall that X_T^0 is known given u. Constructing the Lagrangian and setting its derivatives to zero, we deduce that the above infimum equals

$$\frac{1}{2\sum_{i=m+1}^{N}1/\lambda_i}\left(X_T^0 + \sum_{i=m+1}^{N}x_0^i\right)^2.$$

Thus, the broker's objective for a fixed choice of clients θ can be written as

$$J_P^\theta(\xi^*) = \sup_{u\in\mathcal{U}}\hat{J}_P^\theta(u), \tag{5.3}$$

where

$$\hat{J}_P^\theta(u) := \mathbb{E}\left[\int_0^T\left(X_t^0 + \sum_{i=m+1}^{N}x_0^i\right)\left(\mu + \sum_{i=1}^{m}\left[\frac{\lambda_i}{\kappa_i} - \frac{\gamma}{m+1}\right]Y_t^i + \left[\lambda_0 - \frac{\gamma\kappa_0}{m+1}\right]u_t\right)dt\right.$$
(5.4)
$$\left. - \frac{1}{m+1}\int_0^T u_t\left(\sum_{i=1}^{m}Y_t^i + \kappa_0 u_t\right)dt - \frac{1}{2\sum_{i=m+1}^{N}1/\lambda_i}\left(X_T^0 + \sum_{i=m+1}^{N}x_0^i\right)^2 - \sum_{i=m+1}^{N}R^{i,\theta}\right].$$

Let us denote $x_0 := \sum_{i=m+1}^{N}x_0^i$. The above expression can be viewed as a backward representation of $\hat{J}_P^\theta(u)$ as it involves Y that solves a BSDE. The following lemma establishes a convenient forward representation for $\hat{J}_P^\theta(u)$, which is used in the subsequent analysis.

Lemma 5.1. *For any $u \in \mathcal{U}$, we have*

(5.5)
$$\begin{aligned}\hat{J}^{\theta}_P(u) =&\mathbb{E}\Bigg[\int_0^T X_t^0\left(\frac{1}{m+1}\mu + \frac{2}{m+1}\tilde{\gamma}^\top C_t\right)dt + \frac{2}{m+1}\int_0^T X_t^0\tilde{\gamma}^\top e^{At}D_t\,dt - \frac{2}{m+1}E_T\,D_T\\
&+x_0\int_0^T \tilde{\gamma}^\top C_t\,dt - x_0\tilde{\gamma}^\top\int_0^T e^{At}\,dt\,D_T + x_0\tilde{\gamma}^\top\int_0^T e^{At}D_t\,dt\\
&+\frac{1}{2}\left(\frac{\lambda_0}{m+1}-\frac{2\gamma\kappa_0}{(m+1)^2}-\frac{1}{\sum_{i=m+1}^N 1/\lambda_i}\right)(X_T^0)^2+x_0\left(\lambda_0-\frac{\gamma\kappa_0}{m+1}-\frac{1}{\sum_{i=m+1}^N 1/\lambda_i}\right)X_T^0\\
&-\frac{\kappa_0}{m+1}\int_0^T (u_t)^2 dt\Bigg] + \left(\mu T x_0 - \frac{1}{2\sum_{i=m+1}^N 1/\lambda_i}x_0^2\right) - \sum_{i=m+1}^N R^{i,\theta},\end{aligned}$$

where $A\in\mathbb{R}^{m\times m}$, $\mathbf{b},\tilde{\gamma}\in\mathbb{R}^m$ are defined by

$$A_{ij} := \begin{cases} \frac{1}{m+1}(\gamma - \lambda_i/\kappa_i), & i=j,\\ -\lambda_j/\kappa_j + \frac{1}{m+1}(\gamma-\lambda_i/\kappa_i), & i\neq j,\end{cases} \tag{5.6}$$

$$\begin{aligned}\mathbf{b} &:= \left(\frac{(\gamma-\lambda_1/\kappa_1)\,\kappa_0}{m+1}-\lambda_0,\dots,\frac{(\gamma-\lambda_m/\kappa_m)\,\kappa_0}{m+1}-\lambda_0\right)^\top,\\
\tilde{\gamma} &:= \left(\frac{\lambda_1}{\kappa_1}-\frac{\gamma}{m+1},\dots,\frac{\lambda_m}{\kappa_m}-\frac{\gamma}{m+1}\right)^\top,\end{aligned} \tag{5.7}$$

and

$$C_t := \mu\, e^{At}\int_t^T e^{-As}\mathbf{1}ds, \qquad D_t := \int_0^t u_s\, e^{-As}\mathbf{b}\,ds, \quad E_t := \int_0^t X_s^0\,\tilde{\gamma}^\top e^{As}\,ds. \tag{5.8}$$

Proof. Integrating by parts and recalling (4.4), we obtain:

$$\begin{aligned}&-\frac{1}{m+1}\mathbb{E}\int_0^T u_t\sum_{i=1}^m Y_t^i\,dt = -\frac{1}{m+1}\mathbb{E}\int_0^T X_t^0\Bigg[m\mu - \sum_{i=1}^m\frac{\gamma-\lambda_i/\kappa_i}{m+1}Y_t^i\\
&+(m-1)\sum_{j=1}^m\frac{\lambda_j}{\kappa_j}Y_t^j - \frac{m\gamma-\sum_{i=1}^m\lambda_i/\kappa_i}{m+1}\sum_{j=1}^m Y_t^j + \sum_{i=1}^m\frac{\gamma-\lambda_i/\kappa_i}{m+1}Y_t^i\\
&+\left(m\lambda_0 - \sum_{i=1}^m\frac{\gamma-\lambda_i/\kappa_i}{m+1}\kappa_0\right)u_t\Bigg]dt\\
&=-\frac{1}{m+1}\mathbb{E}\int_0^T X_t^0\left[m\mu+(m-1)\sum_{j=1}^m\frac{\lambda_j}{\kappa_j}Y_t^j - \frac{m\gamma-\gamma}{m+1}\sum_{j=1}^m Y_t^j + \left(m\lambda_0-\frac{m\gamma-\gamma}{m+1}\kappa_0\right)u_t\right]dt.\end{aligned}$$

Plugging the above into the right-hand side of (5.4), we obtain

(5.9)
$$\hat{J}^{\theta}_P(u) = \mathbb{E}\left[\int_0^T X^0_t \left(\frac{1}{m+1}\mu + \frac{2}{m+1}\sum_{i=1}^m \left[\frac{\lambda_i}{\kappa_i} - \frac{\gamma}{m+1}\right] Y^i_t\right) dt\right.$$
$$+\frac{1}{2}\left(\frac{\lambda_0}{m+1} - \frac{2\gamma\kappa_0}{(m+1)^2} - \frac{1}{\sum_{i=m+1}^N 1/\lambda_i}\right)(X^0_T)^2 + x_0\left(\lambda_0 - \frac{\gamma\kappa_0}{m+1} - \frac{1}{\sum_{i=m+1}^N 1/\lambda_i}\right) X^0_T$$
$$\left.+x_0\sum_{i=1}^m \int_0^T \left(\frac{\lambda_i}{\kappa_i} - \frac{\gamma}{m+1}\right) Y^i_t dt - \frac{\kappa_0}{m+1}\int_0^T (u_t)^2 dt\right] + \left(\mu T x_0 - \frac{1}{2\sum_{i=m+1}^N 1/\lambda_i} x_0^2\right)$$
$$-\sum_{i=m+1}^N R^{i,\theta}.$$

Next, we recall that the linear BSDE (4.4) has a semiexplicit solution:

$$Y_t = \mu e^{At}\int_t^T e^{-As}\mathbf{1}ds - \mathbb{E}\left(e^{At}\int_t^T u_s e^{-As}\mathbf{b}ds \,|\, \mathcal{F}_t\right). \tag{5.10}$$

Plugging the above expression into (5.9) and recalling the definition of C_t in (5.8), we obtain

$$\hat{J}^{\theta}_P(u) = \mathbb{E}\left[\int_0^T X^0_t \left(\frac{1}{m+1}\mu + \frac{2}{m+1}\tilde{\gamma}^\top C_t - \frac{2}{m+1}\tilde{\gamma}^\top \mathbb{E}\left(e^{At}\int_t^T u_s e^{-As}\mathbf{b}\, ds|\mathcal{F}_t\right)\right) dt\right.$$
$$+x_0\tilde{\gamma}^\top \int_0^T C_t\, dt - x_0\tilde{\gamma}^\top\int_0^T \mathbb{E}\left(e^{At}\int_t^T u_s e^{-As}\mathbf{b}\, ds \,|\, \mathcal{F}_t\right) dt$$
$$+\frac{1}{2}\left(\frac{\lambda_0}{m+1} - \frac{2\gamma\kappa_0}{(m+1)^2} - \frac{1}{\sum_{i=m+1}^N 1/\lambda_i}\right)(X^0_T)^2 + x_0\left(\lambda_0 - \frac{\gamma\kappa_0}{m+1} - \frac{1}{\sum_{i=m+1}^N 1/\lambda_i}\right) X^0_T$$
$$\left.-\frac{\kappa_0}{m+1}\int_0^T (u_t)^2 dt\right] + \left(\mu T x_0 - \frac{1}{2\sum_{i=m+1}^N 1/\lambda_i} x_0^2\right) - \sum_{i=m+1}^N R^{i,\theta}.$$

Using Fubini's theorem and the tower property, we remove the conditional expectations in the right-hand side of the above. Finally, noticing that

$$\int_t^T u_s e^{-As}\mathbf{b}\, ds = D_T - D_t$$

and recalling the definition of E_T (in (5.8)), we obtain the statement of the lemma. ■

Next, we introduce our main assumption, which guarantees the concavity of the broker's objective.

Assumption 1. The parameters $\{\lambda_i, \kappa_i\}$ are such that

$$2\|\tilde{\gamma}\|\,\|\mathbf{b}\|\, e^{\|A\|_2 T} T^2 + \lambda_0 T/2 < \kappa_0, \tag{5.11}$$

where $\|\cdot\|_2$ is the 2-norm of a matrix, $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^m$, and we recall

$$\tilde{\gamma} = \left(\frac{\lambda_1}{\kappa_1} - \frac{\gamma}{m+1}, \ldots, \frac{\lambda_m}{\kappa_m} - \frac{\gamma}{m+1}\right)^\top,$$
$$\mathbf{b} = \left(\frac{(\gamma - \lambda_1/\kappa_1)\,\kappa_0}{m+1} - \lambda_0, \ldots, \frac{(\gamma - \lambda_m/\kappa_m)\,\kappa_0}{m+1} - \lambda_0\right)^\top,$$
$$\gamma = \sum_{i=1}^m \frac{\lambda_i}{\kappa_i}, \quad A_{ij} := \begin{cases} \frac{1}{m+1}(\gamma - \lambda_i/\kappa_i), & i = j, \\ -\lambda_j/\kappa_j + \frac{1}{m+1}(\gamma - \lambda_i/\kappa_i), & i \neq j, \end{cases}$$

When verifying the above assumption, it is convenient to recall that $\|A\|_2 \leq \|A\|_{\text{Frob}}$, where

$$\|A\|_{\text{Frob}} := \frac{1}{m+1}\left(\sum_{i=1}^m \left(\gamma - \frac{\lambda_i}{\kappa_i}\right)^2 + \sum_{i=1}^m \sum_{j\neq i}^m \left(\frac{\lambda_j}{\kappa_j}(m+1) - \gamma + \frac{\lambda_i}{\kappa_i}\right)^2\right)^{1/2}.$$

The next proposition shows that the broker's optimization problem (5.3) is well-posed under Assumption 1.

Proposition 5.2. *Under Assumption 1, there exists a unique maximizer u^* of $\hat{J}_P^\theta(\cdot)$ over $\mathcal{U}$. Moreover, the optimal strategy u^* is deterministic.*

Proof. Using Lemma 5.1, we deduce that $\hat{J}_P^\theta(u) = \mathbb{E} G^u$, where

$$\begin{aligned} G^u :=& \int_0^T X_t^0 \left(\frac{1}{m+1}\mu + \frac{2}{m+1}\tilde{\gamma}^\top C_t\right) dt + \frac{2}{m+1}\int_0^T X_t^0 \tilde{\gamma}^\top e^{At} D_t\, dt - \frac{2}{m+1} E_T\, D_T \\ &+ x_0 \int_0^T \tilde{\gamma}^\top C_t\, dt - x_0 \tilde{\gamma}^\top \int_0^T e^{At}\, dt\, D_T + x_0 \tilde{\gamma}^\top \int_0^T e^{At} D_t\, dt \\ &+ \frac{1}{2}\left(\frac{\lambda_0}{m+1} - \frac{2\gamma\kappa_0}{(m+1)^2} - \frac{1}{\sum_{i=m+1}^N 1/\lambda_i}\right)(X_T^0)^2 + x_0\left(\lambda_0 - \frac{\gamma\kappa_0}{m+1} - \frac{1}{\sum_{i=m+1}^N 1/\lambda_i}\right) X_T^0 \\ &- \frac{\kappa_0}{m+1}\int_0^T (u_t)^2 dt + \left(\mu T x_0 - \frac{1}{2\sum_{i=m+1}^N 1/\lambda_i} x_0^2\right) - \sum_{i=m+1}^N R^{i,\theta}. \end{aligned}$$

Next, we observe

$$\begin{aligned} \frac{2}{m+1}\left| D_T E_T - \int_0^T X_t^0 \tilde{\gamma}^\top e^{At} D_t\, dt\right| &\leq \frac{2}{m+1}\left(\int_0^T \left| X_t^0 \tilde{\gamma} e^{At} \int_t^T e^{-As}\mathbf{b}\, u_s\, ds\right| dt\right) \\ &\leq \frac{2}{m+1}\|\tilde{\gamma}\|\,\|\mathbf{b}\|\, e^{\|A\|_2 T}\left(\int_0^T |X_t^0| dt \int_0^T |u_t| dt\right) \\ &\leq \frac{2}{m+1}\|\tilde{\gamma}\|\,\|\mathbf{b}\|\, e^{\|A\|_2 T}\, T\left(\int_0^T |u_t| dt\right)^2 \\ &\leq \frac{2}{m+1}\|\tilde{\gamma}\|\,\|\mathbf{b}\|\, e^{\|A\|_2 T}\, T^2 \int_0^T u_t^2 dt, \end{aligned} \tag{5.12}$$

where we used Jensen's inequality:

$$\left(\int_0^T |u_t| dt\right)^2 \leq T \int_0^T u_t^2 dt.$$

The above inequality also yields

$$(X_T^0)^2 \le T \int_0^T (u_t)^2 dt.$$

Collecting the above, we conclude that

$$\begin{aligned}
G^u &\le \left(\frac{2}{m+1} \|\tilde{\gamma}\| \|\mathbf{b}\| e^{\|A\|_2 T} T^2 - \frac{\kappa_0}{m+1} + \frac{\lambda_0 T}{2(m+1)} \right) \int_0^T (u_t)^2 dt \\
&+ x_0 \tilde{\gamma} \int_0^T C_t dt - x_0 \tilde{\gamma} \int_0^T e^{At} dt\, D_T + x_0 \tilde{\gamma} \int_0^T e^{At} D_t dt \\
&+ \int_0^T X_t^0 \left(\frac{1}{m+1} \mu + \frac{2}{m+1} \tilde{\gamma} C_t \right) dt + x_0 \left(\lambda_0 - \frac{\gamma \kappa_0}{m+1} - \frac{1}{\sum_{i=m+1}^N 1/\lambda_i} \right) X_T^0 + \tilde{a},
\end{aligned}$$

where $\tilde{a} \in \mathbb{R}$. Notice that the second and third lines in the right-hand side of the above display are linear in u. In addition, Assumption 1 yields that the coefficient in front of $\int_0^T (u_t)^2 dt$ is strictly negative, which implies that the above expression is strictly concave as a function of $u \in \mathcal{U}_d$, where we introduced the set of deterministic strategies $\mathcal{U}_d := L^2([0,T])$. As G^u is linear-quadratic in u, we conclude that it is also strictly concave (this can be deduced easily by contradiction), which yields the statement of the proposition. ■

The above lemma shows that there is no loss of optimality in reducing the optimization problem (5.3) of the broker to the deterministic set of strategies $u \in \mathcal{U}_d = L^2([0,T])$. As the objective $\hat{J}_P^\theta$ is linear-quadratic, we can find its maximizer by setting to zero its derivative.

Lemma 5.3. *The mapping $u \mapsto \hat{J}_P^\theta(u)$ is Fréchet-differentiable w.r.t. the L^2-norm on $\mathcal{U}_d$, and its derivative is given by the following linear functional of $v \in \mathcal{U}_d$:*

$$\begin{aligned}
&D\hat{J}_P^\theta(u)(v) \\
&:= \int_0^T v_t \left(\frac{\mu(T-t)}{m+1} + p_t - \frac{2}{m+1} \tilde{\gamma} q_t + x_0 \left[\lambda_0 - \frac{\gamma\kappa_0}{m+1} - \frac{1}{\sum_{j=1}^N 1/\lambda_j} \right] - x_0 \tilde{\gamma} r_t - \frac{2\kappa_0}{m+1} u_t \right) dt,
\end{aligned}$$

where $(q,p,r) \in C([0,T], \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m)$ is the unique solution of the (noncoupled) system of ODEs:

$$\begin{aligned}
dp_t &= -\frac{2}{m+1} \tilde{\gamma}^\top Y_t dt, \quad p_T = \left(\frac{\lambda_0}{m+1} - \frac{\gamma\kappa_0}{(m+1)^2} - \frac{1}{\sum_{j=m+1}^N 1/\lambda_j} \right) X_T^0, \\
dq_t &= (-A q_t + X_t^0 \mathbf{b}) dt, \quad q_0 = \mathbf{0}, \\
dr_t &= (-A r_t + \mathbf{b}) dt, \quad r_0 = \mathbf{0},
\end{aligned} \tag{5.13}$$

where $\mathbf{b}$, $\tilde{\gamma}$, and A are defined in (5.6)–(5.7).

Proof. For any $u \in \mathcal{U}_d$ we define $X_t^{0,u} := \int_0^t u_t dt$, and we introduce Y^u defined as the unique solution of the ODE:

$$dY_t = (AY_t + \mathbf{b}u_t - \mu\mathbf{1})dt, \quad Y_T = \mathbf{0}, \tag{5.14}$$

where $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^m$. Recall that (5.14) can be written as

$$Y_t = \mu e^{At}\int_t^T e^{-As}\mathbf{1}ds - e^{At}\int_t^T u_s e^{-As}\mathbf{b}ds. \tag{5.15}$$

Let $u, v \in \mathcal{U}_d$. Then,

$$\begin{aligned}
&\hat{J}_P^\theta(u+v) - \hat{J}_P^\theta(u) = \frac{\mu}{m+1}\int_0^T X_s^{0,v}ds \\
&+ \frac{2}{m+1}\int_0^T X_s^{0,u}\left(\sum_{i=1}^m\left(\frac{\lambda_i}{\kappa_i} - \frac{1}{m+1}\gamma\right)(Y_s^{i,u+v} - Y_s^{i,u})\right)ds \\
&+ \frac{2}{m+1}\int_0^T X_s^{0,v}\left(\sum_{i=1}^m\left(\frac{\lambda_i}{\kappa_i} - \frac{1}{m+1}\gamma\right)\left(Y_s^{i,u} - e^{As}\int_s^T e^{-Ar}\mathbf{b}v_r dr\right)\right)ds - \frac{2\kappa_0}{m+1}\int_0^T u_s v_s ds \\
&\left(\frac{\lambda_0}{m+1} - \frac{2\gamma\kappa_0}{(m+1)^2} - \frac{1}{\sum_{i=m+1}^N \frac{1}{\lambda_i}}\right)X_T^{0,u}X_T^{0,v} \\
&+ \int_0^T X_s^{0,v}\left(\sum_{i=1}^m\left(\frac{\lambda_i}{\kappa_i} - \frac{1}{m+1}\gamma\right)Y_s^{i,v}\right)ds - \frac{\kappa_0}{m+1}\int_0^T (v_s)^2 ds \\
&\frac{1}{2}\left(\frac{\lambda_0}{m+1} - \frac{2\gamma\kappa_0}{(m+1)^2} - \frac{1}{\sum_{i=m+1}^N \frac{1}{\lambda_i}}\right)(X_T^{0,v})^2 + x_0\sum_{i=1}^m\int_0^T\left(\frac{\lambda_i}{\kappa_i} - \frac{\gamma}{m+1}\right)(Y_t^{i,u+v} - Y_t^{i,u})dt \\
&+ x_0\left(\lambda_0 - \frac{\gamma\kappa_0}{m+1} - \frac{1}{\sum_{i=m+1}^N \frac{1}{\lambda_i}}\right)X_T^{0,v}.
\end{aligned}$$

It is a standard exercise to check that the linear part (in u) of the right-hand side of the above gives the desired Fréchet derivative:

(5.16)

$$\begin{aligned}
D\hat{J}_P^\theta(u)(v) := &\frac{\mu}{m+1}\int_0^T X_s^{0,v}ds + \frac{2}{m+1}\int_0^T X_s^{0,u}\left(\sum_{i=1}^m\left(\frac{\lambda_i}{\kappa_i} - \frac{1}{m+1}\gamma\right)(Y_s^{i,u+v} - Y_s^{i,u})\right)ds \\
&+ \frac{2}{m+1}\int_0^T X_s^{0,v}\left(\sum_{i=1}^m\left(\frac{\lambda_i}{\kappa_i} - \frac{1}{m+1}\gamma\right)Y_s^{i,u}\right)ds \\
&+ \left(\frac{\lambda_0}{m+1} - \frac{2\gamma\kappa_0}{(m+1)^2} - \frac{1}{\sum_{i=m+1}^N \frac{1}{\lambda_i}}\right)X_T^{0,u}X_T^{0,v} \\
&- \frac{2\kappa_0}{m+1}\int_0^T u_s v_s ds + x_0\sum_{i=1}^m\int_0^T\left(\frac{\lambda_i}{\kappa_i} - \frac{\gamma}{m+1}\right)(Y_t^{i,u+v} - Y_t^{i,u})dt \\
&+ x_0\left(\lambda_0 - \frac{\gamma\kappa_0}{m+1} - \frac{1}{\sum_{i=m+1}^N \frac{1}{\lambda_i}}\right)X_T^{0,v}.
\end{aligned}$$

Next, we rewrite $DJ_P^\theta(u)$ in a more convenient way. To this end, we observe

$$\int_0^T X_t^{0,u} \sum_{i=1}^m \left(\frac{\lambda_i}{\kappa_i} - \frac{\gamma}{m+1}\right)(Y_t^{i,u+v} - Y_t^{i,u})dt = -\int_0^T X_t^{0,u}\tilde{\gamma}^T \int_t^\top e^{A(t-s)}\mathbf{b}v_s ds dt$$
$$= -\int_0^T v_t \tilde{\gamma}^T \int_0^t e^{-A(t-s)}\mathbf{b}X_s^{0,u} ds dt,$$
$$\int_0^T X_t^{0,v}\left(\sum_{i=1}^m \left(\frac{\lambda_i}{\kappa_i} - \frac{\gamma}{m+1}\right)Y_t^{i,u}\right)dt = \int_0^T \left(\sum_{i=1}^m \left(\frac{\lambda_i}{\kappa_i} - \frac{\gamma}{m+1}\right)Y_t^{i,u}\right)\int_0^t v_s ds dt$$
$$= \int_0^T v_t \int_t^T \tilde{\gamma}^\top Y_s^u ds dt,$$
$$X_T^{0,u} X_T^{0,\eta} = \int_0^T v_t (X_T^{0,u})dt,$$
$$\int_0^T (Y_t^{u+v} - Y_t^u)dt = -\int_0^T e^{At}\int_t^T v_s e^{-As}\mathbf{b} ds dt = -\int_0^T v_t \int_0^t e^{-A(t-s)}\mathbf{b} ds dt.$$

Using the above, we deduce

$$\begin{aligned} DJ_P^\theta(u)(v) := \int_0^T v_t \Bigg(& \frac{\mu(T-t)}{m+1} + \frac{2}{m+1}\tilde{\gamma}^\top \int_t^T Y_s^u ds - \frac{2}{m+1}\tilde{\gamma}^T e^{-At}\int_0^t e^{As} X_s^{0,u}\,\mathbf{b}\,ds \\ & + \left[\frac{\lambda_0}{m+1} - \frac{2\gamma\kappa_0}{(m+1)^2} - \frac{1}{\sum_{i=m+1}^N \frac{1}{\lambda_i}}\right] X_T^{0,u} + x_0\left[\lambda_0 - \frac{\gamma\kappa_0}{m+1} - \frac{1}{\sum_{j=m+1}^N \frac{1}{\lambda_j}}\right] \\ & - x_0\tilde{\gamma}^\top e^{-At}\int_0^t e^{As}\,\mathbf{b}\,ds - \frac{2\kappa_0}{m+1}u_t\Bigg)dt. \end{aligned} \tag{5.17}$$

Recalling (5.13), we obtain the statement of the lemma. ■

The above lemma allows us to characterize the optimal order flow of the broker in terms of the unique solution of a linear forward-backward system of ODEs, arising as a combination of (5.13) and

$$\begin{aligned} & dY_t = (AY_t + \mathbf{b}u_t)dt, \quad Y_T = \mathbf{0}, \\ & dX_t^0 = \Bigg[\frac{\mu(T-t)}{2\kappa_0} + \frac{(m+1)}{2\kappa_0}p_t - \frac{1}{\kappa_0}\tilde{\gamma}^\top q_t + x_0\frac{m+1}{2\kappa_0}\left(\lambda_0 - \frac{\gamma\kappa_0}{m+1} - \frac{1}{\sum_{i=m+1}^N \frac{1}{\lambda_i}}\right) \\ & \quad - \frac{m+1}{2\kappa_0}x_0\tilde{\gamma}^\top r_t\Bigg]dt, \quad X_0^0 = 0. \end{aligned} \tag{5.18}$$

Proposition 5.4. *Under Assumption 1, there exists a unique classical solution (p,q,r,Y,X^0) to (5.13), (5.18), and the optimal order flow u_t^* of the broker is equal to dX_t^0/dt.*

Proof. By Proposition 5.2, we know that there exists a unique optimal control $u^* \in \mathcal{U}_d$. Then, Lemma 5.3 implies that $X_t^{0,*} := \int_0^t u_s^* ds$ satisfies the second line of (5.18) for a.e. t, with (p^*, q^*, r^*, Y^*) defined via (5.13) and the first line of (5.18). Noticing that (p,q,r) are continuous, we conclude that X^0 is continuously differentiable and that it satisfies the second line of (5.18) for all t. Finally, for any solution (p,q,r,Y,X^0) to (5.13), (5.18), the

Frechet derivative of $\hat{J}_P^\theta$ at $u_t := dX_t^0/dt$ is zero, which implies that $u = u^*$ and in turn that $(p,q,r,Y) = (p^*,q^*,r^*,Y^*)$. ∎

The following theorem summarizes all the results we have established.

Theorem 5.5. *For any $\theta \in \{0,1\}^N$ and any $\{R^{i,\theta}\} \in \mathbb{R}^N$, the set of fees $\xi^* = \{\xi^{i,*}\}_{i\in\mathcal{N}(\theta)}$ given by (3.1) is optimal for the broker's local optimization problem (2.11). Provided the broker chooses this set of fees, the following holds.*

- *Any choice of equilibrium strategies $\{\nu^{i,*}\}_{i=1}^N \in \mathcal{E}(\theta,\xi^*)$ that is optimal for the broker has the following structure: the clients' strategies $\{\nu^{i,*}\}_{i\in\mathcal{N}(\theta)}$ can be chosen arbitrarily subject to*

$$\sum_{i\in\mathcal{N}(\theta)} \nu^{i,*} = u^*, \quad X_T^{i,*} = x_0^i + \int_0^T \nu_t^{i,*} dt = x_0^i + \frac{2}{\lambda_i \sum_{j=m+1}^N \frac{1}{\lambda_j}} \left(\int_0^T u_t^* dt \right),$$

  where u^* is defined in Proposition 5.4, and the strategies of independent agents, $\{\nu^{i,*}\}_{i\notin\mathcal{N}(\theta)}$, are determined uniquely by (4.3)–(4.4), with u^* in place of u.
- *The broker's value V_P^θ for a given θ, defined in (2.11), satisfies*

$$V_P^\theta = J_P^\theta(\xi^*) = \hat{J}_P^\theta(u^*), \tag{5.19}$$

  where $\xi^* = \{\xi^{i,*}\}_{i\in\mathcal{N}(\theta)}$ is given by (3.1), $\hat{J}_P^\theta$ is defined in (5.4), and u^* is defined in Proposition 5.4.
- *In any equilibrium $\{\nu^{j,*}\}_{j=1}^N \in \mathcal{E}(\theta,\xi^*)$, the objective value of each independent agent $i \notin \mathcal{N}(\theta)$ is given by the right-hand side of (4.1), with $\{\nu^j\}_{j=1}^N$ replaced by $\{\nu^{j,*}\}_{j=1}^N$. In addition, for any agent $i = 1,\dots,N$, his objective value is the same for any choice of $\{\nu^{j,*}\}_{j=1}^N \in \mathcal{E}(\theta,\xi^*)$ that is optimal for the broker.*

Next, we propose an endogenous definition of the clients' reservation values. Notice that the choice of reservation values $\{R^{i,\theta}\}_{i=1}^N$ does not affect the objective values $\{V^{i,\theta}\}_{i\notin\mathcal{N}(\theta)}$ of independent agents in any equilibrium that is optimal for the principal. Indeed, by the last statement of Theorem 5.5, $\{V^{i,\theta}\}_{i\notin\mathcal{N}(\theta)}$ are determined by the right-hand side of (4.1), which depends only on $\{\nu^j\}_{j\notin\mathcal{N}(\theta)}$ and on $u = \sum_{j\in\mathcal{N}(\theta)} \nu^j$. In any equilibrium that is optimal for the broker, the latter quantities are determined uniquely by (4.3)–(4.4) and by Proposition 5.4, and they do not depend on $\{R^{i,\theta}\}_i$. On the other hand, the choice of $\{R^{i,\theta}\}_{i=1}^N$ does affect the optimal fees ξ^*, given by (3.1), and in turn the objective value of the principal. Thus, in order to determine the broker's value V_P^θ (via (5.19) and (5.4)), we need to choose the reservation values $\{R^{i,\theta}\}_{i=1}^N$. Even though, in general, the reservation values can be prescribed arbitrarily, it turns out that the present setting allows for a natural constraint on these values which determines them uniquely.

The motivation for the following definition of *endogenous reservation values* is clear. Recall that the reservation value of an agent represents the alternative benefit that the agent would receive if he does not enter into the contractual agreement with the principal (see [9] and the references therein). In the present case (unlike many other optimal contract problems), there is an obvious natural way to define such a reservation value. Indeed, each potential client of the broker has an alternative opportunity to trade directly in the market (i.e., to become an independent agent), hence, the reservation value of each agent must equal the maximum objective value he can achieve by such trading.

Definition 5.6. *For every* $\theta \in \{0,1\}^N$, *we say that* $\{R^{i,\theta}\} \in \mathbb{R}^N$ *are endogenous reservation values if*

$$R^{i,\theta} = V^{i,\theta'(i)}, \quad i \in \mathcal{N}(\theta). \tag{5.20}$$

In the above, all entries of $\theta'(i)$ *are equal to those of* θ, *except for the ith entry, which is equal to zero, and* $V^{i,\theta'(i)}$ *is given by the right-hand side of* (4.1), *with* θ *replaced by* $\theta'(i)$ *and with any choice of equilibrium strategies* $\{\nu^j\}_{j=1}^N \in \mathcal{E}(\theta'(i),\xi^*)$ *that is optimal for the broker, where* $\xi^* = \{\xi^{i,*}\}_{i\in\mathcal{N}(\theta'(i))}$ *is given by* (3.1).

Note that the above definition is consistent. Indeed, for $i \notin \mathcal{N}(\theta')$, Theorem 5.5 implies that $V^{i,\theta'}$ does not depend on $\{R^{i,\theta}\}$, hence the right-hand side of (5.20) does not depend on its left-hand side. Moreover, it is clear that, for each $\theta \in \{0,1\}^N$, the endogenous reservation values are determined uniquely and can be constructed by computing $V^{i,\theta'(i)}$ for each $i \in \mathcal{N}(\theta)$. In the remainder of this paper, we assume that the reservation values are chosen to be endogenous.

Having computed the endogenous reservation values, we can use Theorem 5.5 to compute V_P^θ for each θ. Then, an optimal θ can be found by maximizing V_P^θ. The latter is accomplished by an exhaustive search, in the next section, which is realistic for small N or if the choices of θ are restricted to a small enough subset of $\{0,1\}^N$.

6. Numerical experiments.

In this section we describe five numerical simulations to study the structure of an optimal clients' portfolio for the broker, as well as the dependence of the values (i.e., the expected equilibrium profits) of the broker and of the agents on the price impact parameters $\{\kappa_i,\lambda_i\}$. In particular, we ask, Does the broker include all agents in her optimal portfolio? Does the broker prefer agents with low or high price impact coefficients? Do the agents benefit from the presence of the broker? In all experiments, we fix $\mu=1$, $T=1$, $x_0^i=0$.

6.1. Two agents, dependence of broker's value on λ.

In this subsection, we set $N=2$, fix $\kappa_0=\kappa_1=\kappa_2=10^{-1}$, $\lambda_0=10^{-3}$, and consider the dependence of the broker's value on (λ_1,λ_2). We generate $M=100$ equidistant values $(\lambda_1,\lambda_2)\in[10^{-3},5\times10^{-2}]\times[10^{-3},5\times10^{-2}]$ and show the value of the broker in Figure 1. In this experiment, the broker optimally takes both agents as her clients (i.e., $\theta^*=(1,1)$) for every value of (λ_1,λ_2).

We observe that the broker's value is larger for small (λ_1,λ_2). This can be explained as follows. Each agent benefits from the permanent impact of other agents, as they trade in the same directions (because they have the same initial inventories and observe the same signal μ). In particular, a large value of λ_1 makes the second agent more optimistic about his profits in case he decides to trade directly in the market. The latter means that the second agent has larger (endogenous) reservation value, for which he needs to be compensated by the broker, which in turn reduces the profits of the broker from taking the first agent as a client. An analogous argument applies when λ_2 is large.

6.2. Two agents, dependence of broker's value on κ.

In this subsection, we set $N=2$, fix $\lambda_0=\lambda_1=\lambda_2=10^{-2}$, $\kappa_0=10^{-2}$, and consider the dependence of the broker's value on

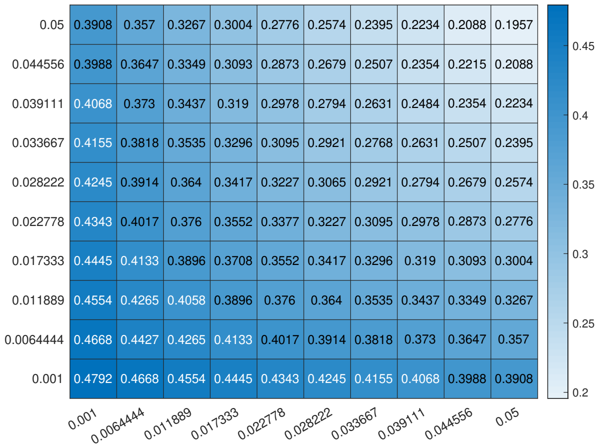

Figure 1. *Broker's value as a function of* λ_1 *(horizontal axis) and* λ_2 *(vertical axis).*

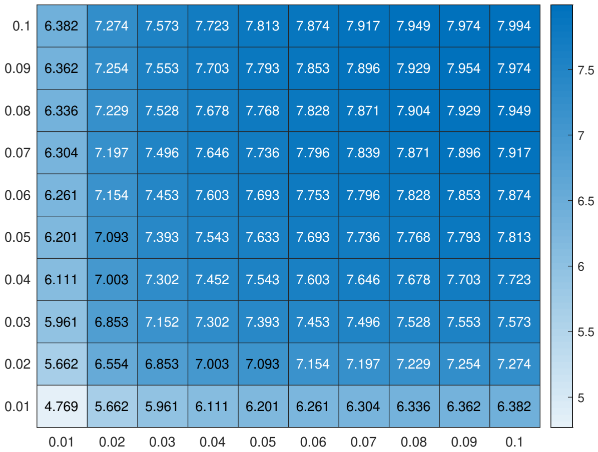

Figure 2. *Broker's value as a function of* κ_1 *(horizontal axis) and* κ_2 *(vertical axis).*

(κ_1, κ_2). We generate $M = 100$ equidistant values $(\kappa_1, \kappa_2) \in [10^{-2}, 10^{-1}] \times [10^{-2}, 10^{-1}]$ and show the value of the broker in Figure 2. In this experiment, the broker optimally takes both agents as her clients (i.e., $\theta^* = (1,1)$) for every value of (κ_1, κ_2).

We observe that the broker's value is larger for large (κ_1, κ_2). This can be explained by the fact that large κ_i reduces the value of agent i in case he decides to trade directly in the market, thus reducing his (endogenous) reservation value and in turn increasing the broker's profit from taking this agent as a client.

6.3. Many agents, choice of portfolio via λ. In this subsection, we set $N = 100$, fix $\kappa_i = 5 \times 10^{-2}$, $\lambda_0 = 10^{-4}$, $5 \times \kappa_0 = 10^{-2}$, generate $\{\lambda_i\}_{i=1}^N$ as independent realizations of a uniform random variable on $(10^{-4}, 10^{-3})$, and consider the dependence of the broker's local value function V_P^θ on her portfolio of clients $\theta \in \{0,1\}^N$. As it is too computationally expensive to compute V_P^θ for all $\theta \in \{0,1\}^N$, we restrict the analysis to portfolios θ chosen as the $100p$-percentile of agents with the highest or lowest λ_i, e.g., θ may be such that $\theta_i = 1$ if and only

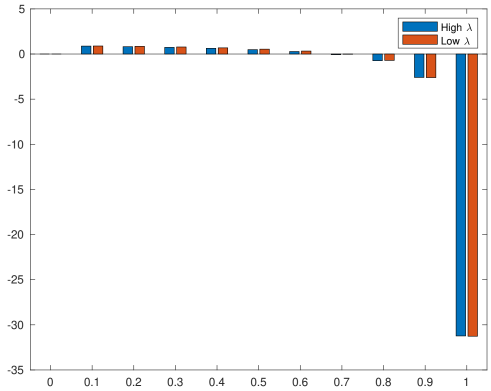

Figure 3. *Broker's local value V_P^θ when she chooses $100p\%$ of agents with the lowest (red) or highest (blue) permanent impact.*

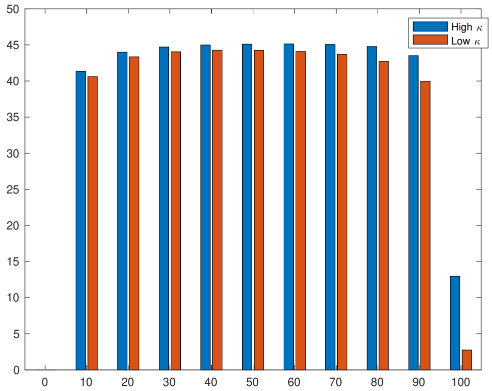

Figure 4. *Broker's local value V_P^θ when she chooses $100p\%$ of agents with the lowest (red) or highest (blue) temporary impact.*

if λ_i belongs to the bottom 10% of $\{\lambda_i\}$. The local value of the broker for $p \in [0,1]$ is shown in Figure 3.

Figure 3 shows that it is not optimal for the broker to take too many clients: in fact, it is better to take too few than too many. It is also worth mentioning that the broker is almost indifferent between choosing the agents with high or low λ, which indicates that a permanent impact coefficient is not a good metric for choosing a portfolio of clients.

6.4. Many agents, choice of portfolio via κ. In this subsection, we set $N = 100$, fix $\lambda = 10^{-4}$, $\lambda_0 = 5 \times 10^{-5}$, $\kappa_0 = 10^{-3}$, generate $\{\kappa_i\}_{i=1}^N$ as independent realizations of a uniform random variable on $(10^{-4}, 10^{-3})$, and consider the dependence of the broker's local value function V_P^θ on her portfolio of clients $\theta \in \{0,1\}^N$. As before, we restrict our analysis to portfolios θ chosen as the $100p$-percentile of agents with the highest or lowest κ_i. The local value of the broker for $p \in [0,1]$ is shown in Figure 4.

Figure 4 shows, once more, that it is not optimal for the broker to take too many clients. It also shows that choosing the agents with large κ is better than choosing those with low κ, which is natural in view of the discussion in subsection 6.2. The results of this experiment also suggest that κ is a better characteristic for choosing a portfolio of clients than λ, as the value of the broker is noticeably different in the two cases where her clients have high or low κ. This is in contrast to the results of the previous experiment.

Remark 6. It would be very interesting to derive rigorously the optimal client portfolios in conveniently chosen portfolio classes, e.g., based on the impact coefficients of the clients. We conjecture that this problem becomes tractable in the infinite-population version of the model proposed herein. Nevertheless, we leave it for future research.

6.5. Do agents benefit from the presence of a broker? In this subsection, we set $N = 8$, $\lambda_0 = 10^{-4}$, $\lambda_i = 10^{-4} + \frac{i}{8}(10^{-4} - 10^{-3})$, $\kappa_i = 10^{-2} + \frac{i}{8}(10^{-1} - 10^{-2})$, for $i = 1, \ldots, 8$, and consider the agents' values as κ_0 varies over $[0.001, 0.01]$. Figure 5 shows the agents' values less their values in the absence of a broker (the latter corresponds to $\theta = 0$)—we refer to them as relative values. Figure 6 shows the optimal portfolio of clients for each κ_0.

Figure 5 indicates that all agents benefit from the presence of a broker, as their relative values are positive. This is explained by the fact that the broker's price impact is lower than

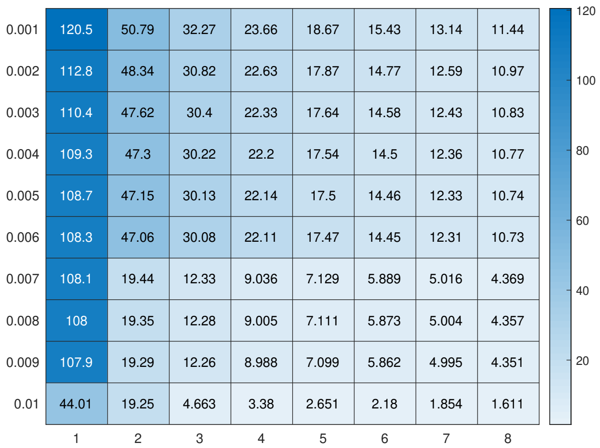

Figure 5. *Agents' relative values across different κ_0 (vertical line) and across agents (horizontal line).*

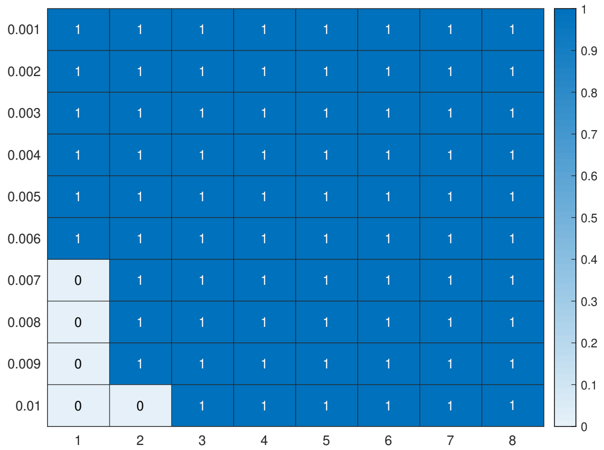

Figure 6. *θ_i^* across different κ_0 (vertical line) and across i (horizontal line).*

those of the agents. The latter allows some of the agents to reduce their trading costs by becoming broker's clients. As a result, the overall temporary impact on the price is reduced, which benefits the other (independent) agents as well.

Figures 5 and 6 show that the relative values of the agents and the optimal portfolio of clients increase as κ_0 decreases. This is natural, as small κ_0 implies larger benefit for each agent who becomes a broker's client and reduces the overall temporary impact on price (which benefits everyone).

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