GAMMA CONVERGENCE FOR THE
DE GENNES-CAHN-HILLIARD ENERGY[*]

SHIBIN DAI[†], JOSEPH RENZI[‡], AND STEVEN M. WISE[§]

Abstract. The degenerate de Gennes-Cahn-Hilliard (dGCH) equation is a model for phase separation which may more closely approximate surface diffusion than others in the limit when the thickness of the transition layer approaches zero. As a first step to understand the limiting behavior, in this paper we study the Γ–limit of the dGCH energy. We find that its Γ–limit is a constant multiple of the interface area, where the constant is determined by the de Gennes coefficient together with the double well potential. In contrast, the transition layer profile is solely determined by the double well potential.

Keywords. de Gennes-Cahn-Hilliard energy; Gamma convergence; sharp interface limit; surface diffusion.

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1. Introduction

Surface diffusion is a fourth-order, highly nonlinear geometric evolution of a surface in which the normal velocity is proportional to the surface Laplacian of the mean curvature. It plays an important role in materials science, a common example being solid state dewetting, see, for example, [35, 36]. Direct numerical approaches to surface diffusion have been developed. See, e.g., [2–4, 17, 18]. But, they inevitably face the difficulties of handling topological changes when merging or splitting occurs. In comparison, diffuse interface approaches capture the motion of interfaces implicitly and handle topological changes without any difficulty. Based on the intuition that the degenerate Cahn-Hilliard equation converges to surface diffusion when the transition layer thickness approaches zero [9, 34], diffuse-interface approximations for surface diffusion are a reasonable alternative [1, 5, 19, 24, 27, 29, 31, 33, 37, 39–41].

The conventional Cahn-Hilliard (CH) functional

$$E_{\mathrm{CH}}^{\varepsilon}(u) = \int_{\Omega} \left\{ \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right\} dx \tag{1.1}$$

is a widely used phenomenological diffuse-interface model to describe the free energy of a system that goes through phase separation [7, 8]. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$, with $n \geq 2$, $u : \Omega \to \mathbb{R}$ is the relative concentration of the two phases, $W(u)$ is a double-well potential with two equal minima at $u^- < u^+$ corresponding to the two pure phases, and $\varepsilon > 0$ is a parameter that is proportional to the thickness of the transition region between the two phases. For convenience and for systems where u is only used as an order parameter to indicate the phases, W is usually taken to be smooth, such as

$$W(u) = \gamma(u - u^+)^2 (u - u^-)^2. \tag{1.2}$$

[†]Department of Mathematics, The University of Alabama, Tuscaloosa, AL 35487-0350, USA (sdai4@ua.edu).
[‡]Department of Mathematics, The University of Alabama, Tuscaloosa, AL 35487-0350, USA (jdrenzi@crimson.ua.edu).
[§]Department of Mathematics, The University of Tennessee, Knoxville, 227 Ayres Hall. 1403 Circle Drive. Knoxville TN 37996-1320, USA (swise1@utk.edu).

Here γ is a constant which usually takes a value such that $\int_{u^-}^{u^+} \sqrt{2W(s)}\, ds = 1$. For simplicity we take $\gamma = 1$. The Cahn-Hilliard equation

$$\partial_t u = \nabla \cdot (M(u)\nabla \mu) \quad \text{for } x \in \Omega \subset \mathbb{R}^n, t \in [0, \infty), \tag{1.3}$$

$$\mu := -\varepsilon^2 \Delta u + W'(u), \tag{1.4}$$

coupled with either Neumann boundary conditions, $\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} \mu = 0$, or periodic boundary conditions, where Ω is the periodic cell, formally dissipates the free energy (1.1). The diffusion mobility $M(u)$ is nonnegative and generally depends on u. The modeling of the dependence of the diffusion mobility $M(u)$ on the phase concentration u presents additional challenges, especially when $M(u)$ is degenerate in the two pure phases $u^\pm$. For instance, $M(u)$ may take the form [9, 28, 34]

$$M(u) = |(u - u^+)(u - u^-)|^m \quad \text{for all } u \in \mathbb{R} \tag{1.5}$$

for some $m > 0$. Due to the degeneracy, it has been conjectured that there is no diffusion in the two pure phases, and the dynamics is governed by diffusion only in the transition region [9, 34]. In addition, if the initial value of u lies inside $[u^-, u^+]$, then the solution u is believed to remain in $[u^-, u^+]$, for all time. However, recent studies by one of the authors (Shibin Dai) and his collaborator Qiang Du [10–13], and by Lee, Münch, and Süli [20, 21], show that in the sharp interface limit of the degenerate Cahn-Hilliard equation, there is an undesired diffusion process in the bulk phase, in addition to the desired surface diffusion along the interface. Dai and Du studied the coarsening mechanisms by asymptotic analysis [10, 11], the coarsening rates by careful and detailed numerical simulations [12], and the wellposedness by theoretical analysis [13]. Their studies show that for a smooth potential such as (1.2) and a degenerate mobility (1.5), the sharp interface limit of the degenerate CH equations is not surface diffusion. Indeed, there is a porous medium diffusion process in both phases, in addition to surface diffusion. More precisely, for the choice $M(u) = |(u - u^+)(u - u^-)|$, the motion of the interfaces occurs in the $t_2 = O(\varepsilon^{-2})$ time scale and it can be written as

$$\nabla \cdot (|\mu_1| \nabla \mu_1) = 0 \quad \text{in } \Omega_\pm, \tag{1.6}$$

$$\mu_1 = -H \quad \text{on } \Gamma, \tag{1.7}$$

$$V_{\mathbf{n}} = \Delta_s H - \left[|\mu_1| \partial_{\mathbf{n}} \mu_1 \right]_-^+ \quad \text{on } \Gamma. \tag{1.8}$$

Here Ω_+ and Ω_- are the regions occupied by the positive and negative phases, respectively, Γ is the interface separating the two phases, H is the mean curvature, μ_1 is the chemical potential, and $\mathbf{n}$ is the normal vector, pointing toward Ω_+. If Ω_+ consists of convex components, then $H < 0$. Due to (1.6) and (1.7), we have $\mu_1 > 0$ in $\Omega_\pm$. Since $W''(u^\pm) > 0$ and $u = u^\pm + \varepsilon \mu_1 / W''(u^\pm) + O(\varepsilon^2)$, we see that $u > u^+$ in Ω_+ and $u > u^-$ in Ω_-. Consequently, the solution u is not confined in $[u^-, u^+]$ and the normal velocity of the interface is determined by surface diffusion together with a quasi-stationary porous medium diffusion process in both phases. If the degeneracy in the mobility is higher, then the porous medium diffusion process can be weakened into a process in a slower time scale, but it will not be annihilated. See also [20–22, 38] for related results and discussions.

Recently one of the authors (Wise) and his collaborators Salvalaglio and Voigt introduced a new variational diffuse interface model [32], in which they imposed a singularity in the formulation of the energy functional. Rather than formulating the free

energy as in (1.1), they define the free energy by including a de Gennes coefficient [16,25]. In the present work, we call it the de Gennes-Cahn-Hilliard (dGCH) energy

$$E_{\mathrm{dGCH}}^{\varepsilon}(u) := \int_{\Omega} \frac{1}{g_0(u)} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx, \quad \text{for all } u \in H^1(\Omega). \tag{1.9}$$

Here the double well potential $W(u)$ is defined by (1.2), and g_0 is a function of the form

$$g_0(u) = |(u - u^-)(u - u^+)|^p, \quad p > 0. \tag{1.10}$$

Of particular interest is the case when $p = 1$,

$$g_0(u) = |(u - u^-)(u - u^+)|. \tag{1.11}$$

The factor $\frac{1}{g_0}$ is called the de Gennes coefficient [16, 25], or the energy restriction function. The corresponding energy dissipation flow is the degenerate de Gennes-Cahn-Hilliard system (which was called the doubly degenerate Cahn-Hilliard system in [32]).

$$\partial_t u = \frac{1}{\varepsilon} \nabla \cdot (M_0(u) \nabla \mu), \tag{1.12}$$

$$\mu = -\varepsilon \nabla \cdot \left(\frac{1}{g_0(u)} \nabla u \right) + \frac{1}{\varepsilon g_0(u)} W'(u) - \frac{g_0'(u)}{g_0^2(u)} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right), \tag{1.13}$$

where $\mu = \delta_u E_{\mathrm{dGCH}}^{\varepsilon}$ is the chemical potential, defined by the variational derivative of $E_{\mathrm{dGCH}}^{\varepsilon}$ with respect to u. The mobility $M_0(u)$ is degenerate at $u^{\pm}$ and is defined as

$$M_0(u) = (u - u^-)^2 (u - u^+)^2. \tag{1.14}$$

The factor g_0'/g_0^2 is also singular. For $u^- < u < u^+$ we have

$$\frac{g_0'(u)}{g_0^2(u)} = -\frac{p(2u - u^+ - u^-)}{(u - u^-)^{p+1} (u^+ - u)^{p+1}}. \tag{1.15}$$

Intuitively, the singularities at $u^{\pm}$ help to keep solutions confined in $[u^-, u^+]$. But the validity of this argument remains open. The degeneracy and singularities in M_0, $1/g_0$, and g_0'/g_0^2 make numerical simulations very challenging, even for $p = 1$. One way to ease such challenges is to consider a regularization corresponding to

$$M_\alpha(u) = (u - u^-)^2 (u - u^+)^2 + \alpha \varepsilon, \tag{1.16}$$

$$g_\alpha(u) = \sqrt{(u - u^-)^2 (u - u^+)^2 + \alpha^2 \varepsilon^2}, \quad \alpha > 0. \tag{1.17}$$

The regularized system is non-degenerate and formally recovers the dGCH system (1.12) and (1.13) when $\alpha \to 0$. However, the most significant difference is that the regularization removes the mechanism which hopefully confines the values of u to be in $[u^-, u^+]$.

Numerical simulations and formal asymptotic analysis in [32] show that the degenerate dGCH Equation (1.12)–(1.13) may more closely approximate surface diffusion than others. This exciting discovery motivated us to systematically explore properties of the dGCH model. In this paper we study the sharp interface limit of the dGCH energy as $\varepsilon \to 0$ in the framework of Γ-convergence. To be more precise, we will show that the Γ-limit of the dGCH energy $E_{\mathrm{dGCH}}^{\varepsilon}$ is a scalar multiple (dependent on g and p) of the perimeter of the interface separating the two phases $\{u \approx u^+\}$ and $\{u \approx u^-\}$ when $g_0(u)$ is as stated in (1.10), (1.11), or when it is regularized as $g_\alpha(u)$ in (1.17).

When considering (1.17) we will be using $p=1$ because it is computationally the most interesting form. This is the first step for us to understand the structure of the dGCH equation, and help pave the path to eventually settle the debate about the relation between phase-field models and the surface diffusion. To start with, we extend the definition of $E_{\mathrm{dGCH}}^{\varepsilon}(u)$ to all $u \in L^1(\Omega)$ by the following generalization.

$$E_{\mathrm{dGCH}}^{\varepsilon}(u) := \begin{cases} \int_{\Omega} \frac{1}{g(u)}\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right)dx, & \text{if } u \in H^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.18}$$

Here the double well potential $W(u)$ is taken as (1.2), and $g(u)$ is taken as either $g_0(u)$ in (1.10) or (1.11), or the regularized form $g_\alpha(u)$ in (1.17).

THEOREM 1.1.   *Let $0 < p \le 1$. The Γ-limit of $E_{\mathrm{dGCH}}^{\varepsilon}$ is a scalar multiple of the perimeter of the set A on which u takes the value u^+. To be precise, the Γ-limit of $E_{\mathrm{dGCH}}^{\varepsilon}$ under the strong $L^1(\Omega)$ topology is*

$$E_{\mathrm{dGCH}}^{0}(u) := \begin{cases} \sigma(p)\mathrm{Per}(A) & \text{if } u = u^- + (u^+ - u^-)\chi_A \in BV(\Omega), \\ \infty & \text{otherwise.} \end{cases} \tag{1.19}$$

Here χ_A is the characteristic function of a set A of finite perimeter, $\mathrm{Per}(A)$ is the perimeter of A, and

$$\sigma(p) = \sqrt{2}\int_{u^-}^{u^+}|(s-u^-)(s-u^+)|^{1-p}\,ds.$$

REMARK 1.1.   In particular, the regularization of g in (1.17) does not change the corresponding Γ-convergence.

REMARK 1.2.   Due to the singularity, $E_{\mathrm{dGCH}}^{\varepsilon}(u)$ is not necessarily finite for all $u \in H^1(\Omega)$. It is an interesting question to explore the domain of $E_{\mathrm{dGCH}}^{\varepsilon}(u)$, but that is not the purpose of this paper.

REMARK 1.3.   When studying mass-preserving phase separations, it is natural to include a mass constraint, $\int_\Omega u\,dx = \text{const}$. The corresponding *Gamma* convergence results can be derived almost the same way as what we do here, except that we need to do some $O(\varepsilon)$ adjustment when constructing the recovery sequence. We omit the details.

REMARK 1.4.   We are able to show that compactness and the liminf inequality still hold for $1 < p < 3/2$.

To find the Γ-limit of a sequence of functionals, we need three results (see, e.g., [6, 23, 26]):

    (a) a compactness result that characterizes the limit functional;

    (b) a liminf inequality which tells us what the Γ-limit should be; and

    (c) a recovery sequence that satisfies a limsup inequality, which tells us under what scenario the Γ-limit can be attained.

In Section 2 we will state and prove the necessary compactness result. In Section 3 we will state and prove the liminf and limsup inequalities. Finally in Section 4 we will have a brief discussion about further topics to explore.

2. Compactness

Throughout the rest of this paper, $g(u)$ takes the form of (1.10), (1.11), or (1.17). We let ε_k be a positive sequence such that $\varepsilon_k \to 0$. Without loss of generality we assume $\varepsilon_k < 1$ for all k.

LEMMA 2.1 (Compactness).   *Let $0 < p < 3/2$. If u_k is a sequence of functions such that $\sup_k E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k) < \infty$, then there exists a subsequence, u_{k_j}, and a measurable set A of finite perimeter in Ω such that u_{k_j} converges to the following piecewise function*

$$u^- + (u^+ - u^-)\chi_A = \begin{cases} u^+ & \text{if } x \in A, \\ u^- & \text{otherwise} \end{cases}$$

strongly in $L^1(\Omega)$ and a.e. in Ω.

Proof.   Suppose $\{u_k\}$ is a sequence in $L^1(\Omega)$ such that $E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k)$ is uniformly bounded. Then there exists $M > 0$ that is independent of k such that

$$E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k) := \int_\Omega \frac{1}{g(u_k)}\left(\frac{\varepsilon_k}{2}|\nabla u_k|^2 + \frac{1}{\varepsilon_k}W(u_k)\right)dx \leq M < \infty. \tag{2.1}$$

Hence

$$\int_\Omega \frac{W(u_k)}{g(u_k)} \leq M\varepsilon_k \tag{2.2}$$

for all k, regardless of whatever form g takes. The case $p = 1$ is relatively straightforward, and has clean structures for the singular form (1.11) and the regularized form (1.17). We will first analyze this case and then study the general case $0 < p < 3/2$.

Case 1. $p = 1$ and g is in the singular form of (1.11).

For g_0 in (1.11), by the definition of W in (1.2), inequality (2.2) becomes

$$\int_\Omega |(u_k - u^-)(u_k - u^+)|\,dx = \int_\Omega \frac{W(u_k)}{g_0(u_k)}\,dx \leq M\varepsilon_k \quad \text{for all } k. \tag{2.3}$$

Since Ω is a bounded set, it is straightforward to conclude that $\int_\Omega |u_k|^2\,dx \leq C < \infty$ for some constant $C > 0$ independent of k. Hence $\{u_k\}$ is bounded in $L^2(\Omega)$ and consequently bounded in $L^1(\Omega)$: there exists $C > 0$ that depends on Ω but independent of k such that

$$\int_\Omega |u_k|\,dx \leq C < \infty \text{ for all } k. \tag{2.4}$$

Now we estimate $\{\nabla u_k\}$ in $L^1(\Omega)$. By the Arithmetic Mean-Geometric Mean (AM-GM) inequality, (2.1) implies that

$$E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k) \geq \int_\Omega \frac{2}{g_0(u_k)}\left(\frac{\varepsilon_k}{2}|\nabla u_k|^2 \cdot \frac{1}{\varepsilon_k}W(u_k)\right)^{1/2} = \sqrt{2}\int_\Omega |\nabla u_k|\,dx. \tag{2.5}$$

Thus

$$\int_\Omega |\nabla u_k|\,dx \leq M/\sqrt{2}. \tag{2.6}$$

Combining (2.4) and (2.6), we conclude that u_k is bounded in $W^{1,1}(\Omega)$. Hence there exists a subsequence u_{k_j} and a function $u \in BV(\Omega)$ such that

$$u_{k_j} \to u \quad \text{strongly in } L^1(\Omega) \text{ and a.e. in } \Omega, \tag{2.7}$$

$$\|\nabla u\|_{BV(\Omega)} \leq \liminf_{j \to \infty} \int_\Omega |\nabla u_{k_j}| \, dx. \tag{2.8}$$

Here $\|\nabla u\|_{BV(\Omega)}$ is the BV-seminorm of u. By (2.3) and Fatou's lemma, we have

$$\int_\Omega |(u - u^-)(u - u^+)| \, dx \leq \liminf_{j \to \infty} \int_\Omega |(u_{k_j} - u^-)(u_{k_j} - u^+)| = 0. \tag{2.9}$$

So u only takes two values $u^\pm$. We can write

$$u = u^- + (u^+ - u^-)\chi_A = \begin{cases} u^+ & \text{if } x \in A, \\ u^- & \text{otherwise} \end{cases} \tag{2.10}$$

where A is a set of finite perimeter because u is a shifting and rescaling of the characteristic function χ_A.

Case 2. $p = 1$ and g is in the regularized form of (1.17).

For the regularized form g_α in (1.17), define

$$\Omega_k^r := \{x \in \Omega : [(u_k - u^-)(u_k - u^+)]^2 \leq \alpha^2 \varepsilon_k^2\}. \tag{2.11}$$

Then

$$\int_{\Omega_k^r} |u - u^-| |u - u^+| \, dx \leq |\Omega_k^r| \alpha \varepsilon_k \leq \alpha |\Omega| \varepsilon_k. \tag{2.12}$$

In $\Omega \setminus \Omega_k^r$, we have

$$[(u_k - u^-)(|u_k - u^+|]^2 \geq \frac{1}{2}[(u_k - u^-)(u_k - u^+)]^2 + \frac{1}{2}\alpha^2 \varepsilon^2 = \frac{1}{2}g_\alpha^2(u_k).$$

Hence

$$\int_{\Omega \setminus \Omega_k^r} |u_k - u^-| |u_k - u^+| \, dx = \int_{\Omega \setminus \Omega_k^r} \frac{W(u_k)}{|u - u^-| |u - u^+|} \, dx$$

$$\leq \int_{\Omega \setminus \Omega_k^r} \frac{\sqrt{2} W(u_k)}{g_\alpha(u_k)} \leq \int_\Omega \frac{\sqrt{2} W(u_k)}{g_\alpha(u_k)} \leq \sqrt{2} M \varepsilon_k. \tag{2.13}$$

Combining (2.12) and (2.13) we have

$$\int_\Omega |u_k - u^-| |u_k - u^+| \, dx \leq (\alpha |\Omega| + \sqrt{2} M) \varepsilon_k \quad \text{for all } k. \tag{2.14}$$

Hence $\{u_k\}$ is bounded in $L^2(\Omega)$ and consequently bounded in $L^1(\Omega)$.

To control ∇u_k, we notice that over Ω_k^r,

$$M \geq E_{\text{dGCH}}^{\varepsilon_k}[u_k] \geq \int_{\Omega_k^r} \frac{\varepsilon_k |\nabla u_k|^2}{2\sqrt{[(u_k - u^-)(u_k - u^+)]^2 + \alpha^2 \varepsilon_k^2}} \geq \int_{\Omega_k^r} \frac{|\nabla u_k|^2}{2\sqrt{2}\alpha}.$$

Thus $\int_{\Omega_k^r} |\nabla u_k|^2 \, dx \leq 2\sqrt{2}\alpha M$ and

$$\int_{\Omega_k^r} |\nabla u_k| \, dx \leq (2\sqrt{2}\alpha M |\Omega_k^r|)^{1/2} \leq (2\sqrt{2}\alpha M |\Omega|)^{1/2}. \qquad (2.15)$$

For $\Omega \setminus \Omega_k^r$, we have

$$\begin{aligned}
M &\geq E_{\mathrm{dGCH}}^{\varepsilon_k}[u_k] \\
&\geq \int_{\Omega \setminus \Omega_k^r} \left\{ \frac{\varepsilon_k |\nabla u_k|^2}{2\sqrt{[(u_k - u^-)(u_k - u^+)]^2 + \alpha^2 \varepsilon_k^2}} \right. \\
&\qquad \left. + \frac{[|(u_k - u^-)(u_k - u^+)|]^2}{\varepsilon_k \sqrt{[(u_k - u^-)(u_k - u^+)]^2 + \alpha^2 \varepsilon_k^2}} \right\} dx \\
&\geq \int_{\Omega \setminus \Omega_k^r} \left\{ \frac{\varepsilon_k |\nabla u_k|^2}{2\sqrt{2[(u_k - u^-)(u_k - u^+)]^2}} + \frac{|(u_k - u^-)(u_k - u^+)|}{\varepsilon_k \sqrt{2}} \right\} dx \\
&\geq \int_{\Omega \setminus \Omega_k^r} |\nabla u_k| \, dx \quad \text{by the AM-GM inequality.} \qquad (2.16)
\end{aligned}$$

Combining (2.15) and (2.16), we see that $\{\nabla u_k\}$ is bounded in $L^1(\Omega)$. Combined with the $L^1(\Omega)$ bound of u, we conclude that u is bounded in $W^{1,1}(\Omega)$. The rest is similar to Case 1 except that we will use (2.14) in place of (2.3).

Case 3. $(0 < p < 3/2)$ *and g is in the singular form (1.10).*

For the general case of g defined by (1.10), inequality (2.2) translates into

$$\int_\Omega |u_k - u^+|^{2-p} |u_k - u^-|^{2-p} \, dx \leq M \varepsilon_k \quad \text{for all } k. \qquad (2.17)$$

First, we show that $\{u_k\}$ are equi-integrable. Let $\gamma > 0$ be fixed. Let k_0 be large enough so that

$$M \varepsilon_k < \lambda^{4-2p} \quad \text{for all } k \geq k_0. \qquad (2.18)$$

For any $k \geq k_0$ define

$$\Omega_k^1 := \left\{ x \in \Omega : |u_k - u^-| \geq |u^-| \text{ and } |u_k - u^+| \geq |u^+| \right\}.$$

Then in Ω_k^1 we have

$$|u_k| \leq |u_k - u^-| + |u^-| \leq 2|u_k - u^-| \text{ and } |u_k| \leq |u_k - u^+| + |u^+| \leq 2|u_k - u^+|.$$

Consequently in Ω_k^1

$$|u_k| \leq 2|u_k - u^-|^{1/2} |u_k - u^+|^{1/2}$$

and hence for all $k \geq k_0$

$$\begin{aligned}
\int_{\Omega_k^1} |u_k| \, dx &\leq 2 \int_{\Omega_k^1} |u_k - u^-|^{1/2} |u_k - u^+|^{1/2} \, dx \\
&\leq 2 \int_\Omega |u_k - u^-|^{1/2} |u_k - u^+|^{1/2} \, dx
\end{aligned}$$

$$\leq C\left(\int_\Omega |u_k - u^-|^{2-p}|u_k - u^+|^{2-p}\,dx\right)^{1/(4-2p)}$$

$$\leq C\lambda \quad \text{by (2.17) and (2.18).} \tag{2.19}$$

Here C depends on Ω and p but not on k. Now

$$\Omega \setminus \Omega_k^1 = \{x \in \Omega : |u_k - u^-| < |u^-| \text{ or } |u_k - u^+| < |u^+|\}.$$

So in $\Omega \setminus \Omega_k^1$ we have either

$$|u_k| \leq |u_k - u^-| + |u^-| \leq 2|u^-|$$

or

$$|u_k| \leq |u_k - u^+| + |u^+| \leq 2|u^+|.$$

Hence $|u_k| \leq 2(|u^-| + |u^+|)$ in $\Omega \setminus \Omega_k^1$. For any subset $G \subset \Omega \setminus \Omega_k^1$ with $|G| \leq \frac{\lambda}{2(|u^+|+|u^-|)}$,

$$\int_G |u|\,dx \leq 2(|u^-| + |u^+|)|G| \leq \lambda. \tag{2.20}$$

By (2.19) and (2.20), there exists constant C depending only on Ω and p and independent of k such that for any subset $G \subset \Omega$ with $|G| \leq \frac{\lambda}{2(|u^+|+|u^-|)}$ we have for all $k \geq k_0$

$$\int_G |u_k| \leq \int_{G \cap \Omega_k^1} |u_k|\,dx + \int_{G \setminus \Omega_k^1} |u_k|\,dx \leq C\lambda. \tag{2.21}$$

For the first $k_0 - 1$ terms of u_k, by the absolute continuity of integrals, there exists $\delta > 0$ such that for any subset $G \subset \Omega$, if $|G| \leq \delta$, then

$$\int_G |u_k|\,dx \leq \lambda, \quad k = 1, 2, \ldots, k_0 - 1. \tag{2.22}$$

This estimate together with (2.20) shows that the whole sequence u_k is equi-integrable.

Next we show that there is a subsequence of u_k that converges a.e. in Ω. By the AM-GM Inequality,

$$M \geq E_{\mathrm{dGCH}}^{\varepsilon_k}[u_k] \geq \sqrt{2}\int_\Omega |u_k - u^-|^{1-p}|u_k - u^+|^{1-p}|\nabla u_k|. \tag{2.23}$$

Let $K > 0$ be some fixed real number. Define

$$h_K(t) = \sqrt{2}\int_{u^-}^t \min\{|s - u^-|^{1-p}|s - u^+|^{1-p}, K\}\,ds. \tag{2.24}$$

Then $h_K(t)$ is a strictly increasing and Lipschitz function. Define

$$w_k(x) := h_K(u_k(x)).$$

Then $|w_k| \leq \sqrt{2}K|u_k|$ and by the chain rule,

$$|\nabla w_k| = \sqrt{2}\min\{|u_k - u^-|^{1-p}|u_k - u^+|^{1-p}, K\}|\nabla u_k|$$

$$\leq \sqrt{2}|u_k - u^-|^{1-p}|u_k - u^+|^{1-p}|\nabla u_k|. \tag{2.25}$$

By (2.23) w_k is bounded in $W^{1,1}(\Omega)$. Hence there exists a subsequence w_{k_j} that converges strongly in $L^1(\Omega)$ and a.e. in Ω to a BV function $w(x)$. Note that h_K is strictly increasing and continuous, so h_K^{-1} exists and is continuous, and the corresponding subsequence $u_{k_j} = h^{-1}(w_{k_j})$ converges a.e. in Ω to $u := h_K^{-1}(w)$. By the Egorov theorem and the equi-integrability of $\{u_{k_j}\}$, we see that u_{k_j} strongly converges in $L^1(\Omega)$ to u.

Now we show that this limit function u is in fact $BV(\Omega; \{u^-, u^+\})$. By (2.2),

$$\int_\Omega |u_{k_j} - u^-|^{2-p} |u_{k_j} - u^+|^{2-p} \, dx \leq M \varepsilon_{k_j}.$$

By Fatou's lemma,

$$\int_\Omega |u - u^-|^{2-p} |u - u^+|^{2-p} \, dx \leq \liminf_{j \to \infty} \int_\Omega |u_{k_j} - u^-|^{2-p} |u_{k_j} - u^+|^{2-p} = 0.$$

So u only takes values of u^- or u^+, and $w = h_K(u)$ takes only two values, 0, and $h_K(u^+)$. Since $w \in BV(\Omega)$, we can write $w = h_K(u^+) \chi_A$, where A is a set of finite perimeter. Consequently $u = u^- + (u^+ - u^-) \chi_A$.   □

3. Liminf and limsup inequalities

This section establishes the Γ-limit of $E_{\mathrm{dGCH}}^\varepsilon$ as stated in Theorem 1.1.

LEMMA 3.1 (Liminf Inequality).   *Let $0 < p < 3/2$. For any function $u \in L^1(\Omega)$ and any sequence $\{u_k\} \subset L^1(\Omega)$ such that $u_k \to u$ strongly in $L^1(\Omega)$, we have*

$$E_{\mathrm{dGCH}}^0(u) \leq \liminf_{k \to \infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k). \tag{3.1}$$

LEMMA 3.2 (Limsup Inequality).   *Let $0 < p \leq 1$. For any $u \in L^1(\Omega)$, there exists a sequence $\{u_k\} \subset L^1(\Omega)$ that converges to u strongly in $L^1(\Omega)$ with*

$$\limsup_{k \to \infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k) \leq E_{\mathrm{dGCH}}^0(u). \tag{3.2}$$

3.1. The proof of the liminf inequality ($0 < p < 3/2$).   If $\liminf_{k \to \infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k) = \infty$, then (3.1) is trivial. So we only need to consider the case $\liminf_{k \to \infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k) < \infty$. By extracting a subsequence if necessary we may assume without loss of generality that

$$\lim_{k \to \infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k) = \liminf_{k \to \infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k).$$

By the compactness result in Section 2, there exists a subsequence u_{k_j} and a set A of finite perimeter such that u_{k_j} converges to $u^- + (u^+ - u^-) \chi_A$ strongly in $L^1(\Omega)$ and almost everywhere in Ω. By the uniqueness of limit we conclude that actually $u = u^- + (u^+ - u^-) \chi_A$, and

$$E_{\mathrm{dGCH}}^0(u) = \sigma(p) \mathrm{Per}(A).$$

Case 1. $p = 1$ and g is in the singular form of (1.11).

By the lower semicontinuity of the BV semi-norm and (2.5), we get

$$E_{\mathrm{dGCH}}^0(u) = \sqrt{2}(u^+ - u^-) \mathrm{Per}(A) = \sqrt{2} \|u\|_{BV(\Omega)} \leq \liminf_{j \to \infty} \sqrt{2} \int_\Omega |\nabla u_{k_j}| \, dx$$

$$\leq \liminf_{j \to \infty} E_{\mathrm{dGCH}}^{\varepsilon_{k_j}}(u_{k_j}) = \liminf_{k \to \infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k).$$

Case 2: $p=1$ and g is in the regularized form of (1.17).

By the AM-GM inequality,

$$E^{\varepsilon_k}_{\mathrm{dGCH}}(u_k) \geq \int_\Omega \sqrt{\frac{2[(u_k - u^-)(u_k - u^+)]^2}{[(u_k - u^-)(u_k - u^+)]^2 + \alpha\varepsilon_k^2}} |\nabla u_k|. \tag{3.3}$$

Similar to (2.24) but without the cutoff, define

$$F_k(s) = \int_{u^-}^s \sqrt{\frac{2[(t - u^-)(t - u^+)]^2}{[(t - u^-)(t - u^+)]^2 + \alpha\varepsilon_k^2}} \, dt.$$

Then F_k is a Lipschitz function with a Lipschitz constant $\sqrt{2}$. Define

$$w_k := F_k(u_k).$$

Then $w_k \in W^{1,1}(\Omega)$ and

$$|\nabla w_k| = \sqrt{\frac{2[(u_k - u^-)(u_k - u^+)]^2}{[(u_k - u^-)(u_k - u^+)]^2 + \alpha\varepsilon_k^2}} |\nabla u_k|. \tag{3.4}$$

We will show that $w_k \to \sqrt{2}(u - u^-) = \sqrt{2}(u^+ - u^-)\chi_A$ strongly in $L^1(\Omega)$. Indeed,

$$w_k - \sqrt{2}(u - u^-) = \int_{u^-}^{u_k} \sqrt{\frac{2[(t - u^-)(t - u^+)]^2}{[(t - u^-)(t - u^+)]^2 + \alpha\varepsilon_k^2}} \, dt - \int_{u^-}^u \sqrt{2} \, dt$$

$$= \int_u^{u_k} \sqrt{\frac{2[(t - u^-)(t - u^+)]^2}{[(t - u^-)(t - u^+)]^2 + \alpha\varepsilon_k^2}} \, dt + \int_{u^-}^u \left(\sqrt{\frac{2[(t - u^-)(t - u^+)]^2}{[(t - u^-)(t - u^+)]^2 + \alpha\varepsilon_k^2}} - \sqrt{2} \right) dt.$$

Since $u = u^-$ in $\Omega \setminus A$ and $u = u^+$ in A, we have

$$\int_\Omega \left| w_k - \sqrt{2}(u - u^-) \right| dx$$

$$\leq \sqrt{2} \int_\Omega |u_k - u| \, dx + |A| \int_{u^-}^{u^+} \left| \sqrt{\frac{2[(t - u^-)(t - u^+)]^2}{[(t - u^-)(t - u^+)]^2 + \alpha\varepsilon_k^2}} - \sqrt{2} \right| dt$$

$$\to 0 \qquad \text{as } k \to \infty.$$

Here, for the convergence of the second integral above, we use the Dominated Convergence Theorem. By the lower semicontinuity of the BV norm and (3.3), (3.4), we obtain the liminf inequality

$$\sqrt{2}(u^+ - u^-)\mathrm{Per}(A) \leq \liminf_{k \to \infty} \int_\Omega |\nabla w_k| \, dx \leq \liminf_{k \to \infty} E^{\varepsilon_k}_{\mathrm{dGCH}}(u_k). \tag{3.5}$$

Case 3. $0 < p < 3/2$ and g is in the singular form (1.10).

Fix any $K > 0$. Let $h_K(t)$ be defined by (2.24) and

$$w := h_K(u), \qquad w_k := h_K(u_k) \quad \text{for all } k,.$$

Since the Lipschitz constant of h_K is $\sqrt{2}K$,

$$\int_\Omega |w_k - w| \, dx \leq \sqrt{2}K \int_\Omega |u_k - u| \, dx \to 0 \quad \text{as } k \to \infty.$$

In addition, since $w = h_K(u^+)\chi_A$, by the lower semicontinuity of the BV norm, and (2.23), (2.25), we have

$$
\begin{aligned}
h_K(u^+)\mathrm{Per}(A) &\leq \liminf_{k\to\infty} \int_\Omega |\nabla w_k|\,dx \\
&\leq \liminf_{j\to\infty} E_{\mathrm{dGCH}}^{\varepsilon_{k_j}}(u_{k_j}) = \liminf_{k\to\infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k).
\end{aligned}
$$

That is,

$$
\sqrt{2}\left(\int_{u^-}^{u^+} \min\{|(s-u^-)(s-u^+)|^{1-p}, K\}\,ds \right)\mathrm{Per}(A) \leq \liminf_{k\to\infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k).
$$

Letting $K \to \infty$, we obtain

$$
\sqrt{2}\left(\int_{u^-}^{u^+} |(s-u^-)(s-u^+)|^{1-p}\,ds \right)\mathrm{Per}(A) \leq \liminf_{k\to\infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k).
$$

This is the liminf inequality (3.1) for $0 < p < 3/2$.

3.2. The recovery sequence and the limsup inequality ($0 < p \leq 1$). Due to technical reasons we can only handle $0 < p \leq 1$ in this section. For any set $A \subset \Omega$ of finite perimeter, we have a corresponding $u := u^- + (u^+ - u^-)\chi_A \in BV(\Omega)$. We need to find a corresponding recovery sequence u_k in the sense that $u_k \to u$ strongly in $L^1(\Omega)$ and

$$
E_{\mathrm{dGCH}}^0(u) \geq \limsup_{k\to\infty} E_{\mathrm{dGCH}}^{\varepsilon_k}(u_k). \tag{3.6}
$$

Note that if we can establish such sequences for $p = 1$ and $0 < p < 1$, then the regularized dGCH will work under the same recovery sequence. We construct the sequence for $0 < p < 1$.

Suppose $u \in L^1(\Omega)$ with $E_{\mathrm{dGCH}}^0(u)$ finite. Then $u = u^- + (u^+ - u^-)\chi_A$ for some set A with finite perimeter. We wish to construct some u_k that converges to u in the $L^1(\Omega)$ sense and the limsup condition holds. There are multiple ways to construct sequences of recovery functions, see the dissertation of the second author [30]. Here we will show that the standard construction for Cahn-Hilliard energy functionals also works for the dGCH functional.

Define $\Phi_k : [u^-, u^+] \to [0, \Phi_k(u^+)]$ as follows:

$$
\Phi_k(t) = \int_{u^-}^t \frac{\varepsilon_k}{\sqrt{2W(s) + \varepsilon_k}}\,ds.
$$

Φ_k is strictly increasing and continuous and therefore invertible. Moreover, $0 < \Phi_k(u^+) < \sqrt{\varepsilon_k}(u^+ - u^-) < \infty$. Also, $\Phi_k(u^+) \to 0$ as $k \to \infty$. So, we can define $\phi_k = \Phi_k^{-1}$. We extend ϕ_k to the real line by setting $\phi_k(t) = u^-$ for $t \leq 0$ and $\phi_k(t) = u^+$ for $t > \Phi_k(u^+)$. We can find the derivative by using the inverse rule:

$$
\frac{d\phi_k}{dt} = \frac{\sqrt{2(W(\phi_k(t)) + \varepsilon_k)}}{\varepsilon_k}.
$$

Now, let $d_A(x)$ be the following function

$$
d_A(x) = \begin{cases} \mathrm{dist}(x, \partial A) & x \in A \\ -\mathrm{dist}(x, \partial A) & x \in \Omega \setminus A \end{cases}
$$

and define

$$u_k = \phi_k(d_A(x)).$$

Then $u_k \to \chi_A$ both pointwise in Ω and strongly in $L^1(\Omega)$. If ∂A is C^2, then

$$|\nabla u_k| = |\phi_k'(d_A(x) \nabla d_A(x)| = \frac{\sqrt{2(W(\phi_k(d_A(x))) + \varepsilon_k)}}{\varepsilon_k} |\nabla d_A(x)|$$
$$= \frac{\sqrt{2(W(\phi_k(d_A(x))) + \varepsilon_k)}}{\varepsilon_k}.$$

Furthermore,

$$\lim_{k \to \infty} \sup_{u^- < s < u^+} \mathcal{H}^{n-1}(\{x \in \Omega : u_k(x) = s\})$$
$$= \lim_{k \to \infty} \sup_{0 < s < \Phi_k(u^+)} \mathcal{H}^{n-1}(\{x \in \Omega : d_A(x) = \Phi_k(s)\})$$
$$= P_\Omega(A).$$

So, using the coarea formula, we get:

$$\lim_{k \to \infty} E_{\text{dGCH}}^{\varepsilon_k}(u_k)$$
$$= \int_\Omega \left(\frac{\varepsilon_k |\nabla u_k|^2}{2|u_k - u^-|^p |u_k - u^+|^p} + \frac{|u_k - u^-|^{2-p}|u_k - u^+|^{2-p}}{\varepsilon_k} \right) dx$$
$$= \lim_{k \to \infty} \int_\Omega \left(\frac{\varepsilon_k |\nabla u_k|}{2} + \frac{|u_k - u^-|^2 |u_k - u^+|^2}{\varepsilon_k |\nabla u_k|} \right) \frac{|\nabla u_k|}{|u_k - u^-|^p |u_k - u^+|^p} \, dx$$
$$= \lim_{k \to \infty} \int_\Omega \left(\frac{\sqrt{W(u_k) + \varepsilon_k}}{\sqrt{2}} + \frac{W(u_k)}{\sqrt{2(W(u_k) + \varepsilon_k)}} \right) \frac{|\nabla u_k|}{|u_k - u^-|^p |u_k - u^+|^p} \, dx$$
$$= \lim_{k \to \infty} \int_\Omega \left(\frac{2W(u_k) + \varepsilon_k}{\sqrt{2(W(u_k) + \varepsilon_k)}} \right) \frac{|\nabla u_k|}{|u_k - u^-|^p |u_k - u^+|^p} \, dx$$
$$= \lim_{k \to \infty} \int_\Omega \left(\frac{\sqrt{2W(u_k)}}{|u_k - u^-|^p |u_k - u^+|^p} \right) |\nabla u_k| \, dx$$
$$= \lim_{k \to \infty} \int_{-\infty}^\infty \left(\frac{\sqrt{2W(r)}}{|r - u^-|^p |r - u^+|^p} \right) \mathcal{H}^{n-1}(\{x \in \Omega : u_k(x) = r\}) dr$$
$$= \lim_{k \to \infty} \int_{u^-}^{u^+} \sqrt{2} |r - u^-|^{1-p} |r - u^+|^{1-p} \mathcal{H}^{n-1}(\{x \in \Omega : u_k(x) = r\}) dr$$
$$\leq \sqrt{2} P_\Omega(A) \int_{u^-}^{u^+} |r - u^-|^{1-p} |r - u^+|^{1-p} dr.$$

This is exactly what we want.

We still need to consider the case where A doesn't have C^2 boundary. Here we cite Lemma 1.15 from [23]. We can find a sequence of open A_j with ∂A_j a nonempty, compact hypersurface of class C^2 and $\mathcal{H}^{n-1}(\partial A_j \cap \partial \Omega) = 0$ such that $\chi_{A_j} \to \chi_A$ in $L^1(\Omega)$, $\text{Per}(A_j, \Omega) \to \text{Per}(A, \Omega)$, and $L^n(A_j) = L^n(A)$ for all j. We can find a sequence $\{u_{k,j}\} \subset H^1(\Omega)$ such that $u_{k,j} \to \chi_{B_j}$ with the limsup condition satisfied. Now, by a diagonalization argument, we can find a sequence of $u_{k,j}$ that still satisfies the limsup condition for u and converges to u. A similar argument works for $p = 1$. And if the argument holds for $p = 1$, the same sequence will work for the regularized version of the dGCH equation.

4. Discussion

In this paper we establish the sharp-interface limit of the dGCH energy in the framework of Γ–convergence. This is the first step to understand the limiting behavior as $\varepsilon \to 0$ of the dGCH energy. There are many open questions. One is the properties of the minimizers of the dGCH energy under various boundary conditions. For instance, under the strong anchoring conditions, it is not surprising to expect the minimizers to have similar properties as those shown in [15] for the ordinary Cahn-Hilliard energy. However, due to the de Gennes coefficient, the analysis would be much more complicated. Our ultimate goal is to analyze the limiting behavior as $\varepsilon \to 0$ of the degenerate de Gennes-Cahn-Hilliard equation. One step before that would be to study the force convergence as is done in [14] for a phase field model for molecule solvations. It will also be of interest to study a corresponding L^2 gradient flow and its degenerate version, which can be called the de Gennes-Allen-Cahn equations. We will leave these topics for future studies.

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