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Spectrum of quantum KdV hierarchy in the semiclassical limit

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Abstract: We employ semiclassical quantization to calculate spectrum of quantum KdV charges in the limit of large central charge c. Classically, KdV charges Q_{2n-1} generate completely integrable dynamics on the co-adjoint orbit of the Virasoro algebra. They can be expressed in terms of action variables I_k, e.g. as a power series expansion. Quantum-mechanically this series becomes the expansion in $1/c$, while action variables become integer-valued quantum numbers n_i. Crucially, classical expression, which is homogeneous in I_k, acquires quantum corrections that include terms of subleading powers in n_k. At first two non-trivial orders in $1/c$ expansion these "quantum" terms can be fixed from the analytic form of Q_{2n-1} acting on the primary states. In this way we find explicit expression for the spectrum of Q_{2n-1} up to first three orders in $1/c$ expansion. We apply this result to study thermal expectation values of Q_{2n-1} and free energy of the KdV Generalized Gibbs Ensemble.

Keywords: Conformal and W Symmetry, Conformal Field Models in String Theory, Integrable Field Theories

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1 Introduction

Conformal invariance in two dimensions is a very powerful tool which gives rise to many non-pertubative relations constraining dynamics of 2d CFTs. Among them is universality of stress-energy tensor sector [1], namely any correlation function which includes only stress-energy tensor and its descendants depends only on central charge c but not on any other details of the theory. An analytic form of all such correlators can in principle be found in a recursive form [2]. The stress-energy sector can be regarded as integrable, even if the whole theory is understood to be chaotic [3]. This can be justified formally by noting there is an infinite number of mutually commuting quantum KdV charges [4–6] — local charges Q_{2n-1} of the form

$$Q_{2n-1} = \frac{1}{2\pi} \int_0^{2\pi} T_{2n}(\varphi)\, d\varphi, \tag{1.1}$$

where the densities T_{2n} are appropriately regularized polynomials in stress-energy tensor $T(\varphi)$ and its derivatives. First charge

$$Q_1 = L_0 - \frac{c}{24} = \frac{1}{2\pi} \int_0^{2\pi} T\, d\varphi \tag{1.2}$$

is the CFT Hamiltonian. (Here and below we consider 2d CFT on a cylinder. Because of standard factorization into left and right-moving sectors we restrict the discussion to one sector only.) Interest in integrable structure of 2d CFT stress-energy sector has been reignited recently in the context of Eigenstate Thermalization Hypothesis (ETH) [7]. Following original works [8–16] it has been conjectured and confirmed in [17] that 2d CFTs exhibit generalized ETH with the local equilibrium being described by qKdV Generalized Gibbs Ensemble (GGE). Schematically the role of qKdV charges is as follows. The CFT Hamiltonian (1.2) is highly degenerate with all CFT descendant states of the form

$$|E\rangle = L_{-m_1} \dots L_{-m_k}|\Delta\rangle, \qquad \sum_{i=1}^{k} m_i = m \tag{1.3}$$

sharing the same energy $E = \Delta + m - c/24$. Since all Q_{2n-1} commute, they can be simultaneously diagonalized giving rise to mathematically unique "integrable" basis of eigenstates. Unlike the energy eigenstates of the form (1.3), which fail the ETH, integrable eigenstates carry specific values of Q_{2k-1}-charges and obey generalized ETH. This novel role of qKdV symmetries motivates the question of "solving" integrable structure, i.e. evaluating spectrum of qKdV charges and finding integrable eigenstates, which would allow detailed studies of generalized ETH and qKdV GGE thermodynamics.

In certain sense the question of finding qKdV spectra can be regarded as solved: there is not one but two distinct ways to write an algebraic Bethe-ansatz reducing the problem of finding spectra to a bunch of algebraic equations [18, 19]. In practice complexity of these equations grows very rapidly with the level m (1.3), making this approach useless in the context of ETH, at least so far. The ETH holds in thermodynamic limit, it may not and does not hold beyond that regime. Thermodynamic limit assumes the length of the spatial circle L goes to infinity, with the energy density E/L kept fixed. Using rescaling, one can always bring the circle to unit radius, the notations we use throughout the paper. The energy E then must go to infinity as L^2 with $L \to \infty$ being an auxiliary parameter keeping track of corrections to various ETH-related identities. For any given primary state $|\Delta\rangle$ this essentially means the descendant level m must be taken to infinity, i.e. we arrive exactly at the limit where algebraic Bethe equations become most difficult.

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A progress was achieved by taking an additional limit of large central charge. In this case Q_{2k-1}-eigenstates, akin to (1.3), can be parametrized by a set of natural numbers, which can be conveniently combined into a Young tableau [16].[1] It is most convenient to use representation when $n_k \geq 0$ for $k = 1, 2, \ldots$ counts the number of rows of length k,

$$|n_i\rangle \equiv |n_1, \ldots\rangle, \qquad \sum_{k=1}^{\infty} k\, n_k = m. \tag{1.4}$$

We emphasize (1.4) are eigenstates of Q_{2n-1} and thus differ from (1.3). Corresponding eigenvalues at leading order were conjectured in [22][2]

$$Q_{2n-1}|n_i\rangle = \mathsf{Q}_{2n-1}|n_i\rangle, \tag{1.5}$$

$$\mathsf{Q}_{2n-1} = \tilde{\Delta}^n + \sum_{p=0}^{n-1} \xi_n^p\, \tilde{\Delta}^{n-1-p}\, \tilde{c}^p \left(\sum_{k=1}^{\infty} k^{2p+1} n_k + \frac{\zeta(-2p-1)}{2}\right) + O(\tilde{c}^{n-2}),$$

$$\xi_n^p = \frac{(2n-1)\sqrt{\pi}\,\Gamma(n+1)}{2\,\Gamma(p+3/2)\Gamma(n-p)}, \qquad \tilde{\Delta} = \Delta - \tilde{c}, \qquad \tilde{c} = \frac{c-1}{24}. \tag{1.6}$$

Here we assume the scaling when $c \to \infty$ while $\tilde{\Delta}/\tilde{c} = h$ is kept fixed. No thermodynamic limit is assumed. This is the limit of holographic correspondence, when CFT is dual to semiclassical gravity. The holographic picture provides an easy derivation for the leading $1/c$ terms in (1.6) and provides interpretation for n_k as the boson occupation numbers of the boundary gravitons, see appendix A. From the mathematical point of view simplicity of eigenstates parametrization with help of Young tableaux as well as relatively simple form of (1.6) can be readily understood from the semiclassical quantization of the co-adjoint orbit of Virasoro algebra. Indeed, as is explained in [20] in the large c limit Virasoro algebra can be understood in quasi-classical terms, as quantization of the Kirillov-Kostant-Souriau symplectic form. Because of U(1) symmetry semiclassical quantization of Q_1 is exact,

$$Q_1 = \tilde{\Delta} + \left(\sum_{k=1}^{\infty} k\, n_k - \frac{1}{24}\right) = \Delta + m - \frac{c}{24}, \tag{1.7}$$

but for all higher $Q_{2n-1}, n > 1$ it is not. It is a perturbation series in $1/\tilde{c}$, which plays the effective role of Planck constant. In this paper we develop a perturbative scheme to obtain the spectrum of Q_{2n-1} as a series in $1/\tilde{c}$ expansion and calculate first two non-trivial terms. The result is summarized in (4.15).

In the strict $c \to \infty$ limit when the problem becomes classical, CFT stress-tensor T can be substituted by an element of the co-adjoint orbit of Virasoro algebra $\frac{24}{c}u$, where u is a potential of an auxiliary periodic Schrödinger equation. Then quantum KdV charges (1.1) reduce to conventional KdV Hamiltonians of the periodic problem

$$Q_{2n-1} = \frac{1}{2\pi}\int_0^{2\pi} (u^n + \ldots)\, d\varphi, \tag{1.8}$$

[1] Appearance of n_k to parametrize the eigenstates can be understood from the Virasoro algebra, which in the large c limit reduces to a product of Heisenberg algebras, with n_k being the corresponding quantum numbers [20, 21].

[2] Since ref. [22] was working in the regime of both large central charge and thermodynamic limit $Q_1 \propto cL^2$, it only conjectured the term linear in n_k, as the n_k-independent term is $1/L^2$ suppressed.

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which we denote the same as the quantum ones, as it clear from the constant which, classical or quantum version, we had in mind. For the states with large but finite level m number of non-zero n_k will also be finite. At the classical level this corresponds to finite-zone potentials u, which form a finite-dimensional symplectic manifold equipped with the structure of a completely integrable system. Hamiltonians Q_{2n-1} can be re-expressed in terms of the action variables I_k and the orbit invariant h,

$$Q_{2n-1} = h^n + \sum_{k=1}^{\infty} \sum_{j=0}^{n-1} \xi_n^j \, h^{n-1-j} k^{2j+1} \, I_k + O(I^2) \tag{1.9}$$

which at semiclassical level become integral quantum numbers $I_k \to n_k/\tilde{c}$. It is then easy to see that (1.9) becomes (1.6), up to an overall factor $\tilde{c}^n$ and certain corrections. At each power of $1/\tilde{c}$ classical expression $Q_{2n-1}(h, I_k)$ predicts only leading power of n_k while all subleading powers are "quantum corrections" which must be fixed separately.

At leading $1/\tilde{c}$ order quantum correction is just n_k-independent constant term proportional to $\zeta(-2p-1)/2$, see (1.6). It can be fixed trivially by introducing Maslov index $I_k \to (n_k + 1/2)/\tilde{c}$, such that constant term can be formally rewritten as the vacuum energy of "quantum oscillators" with frequencies ω_k and occupation numbers n_k

$$\mathsf{Q}_{2n-1} = \tilde{\Delta}^n + \sum_{k=1}^{\infty} (n_k + 1/2)\, \omega_k + O(\tilde{c}^{n-2}), \qquad \omega_k = \sum_{j=0}^{n-1} \xi_n^j \, \tilde{\Delta}^{n-1-j} \, \tilde{c}^j \, k^{2p+1}. \tag{1.10}$$

Unfortunately this simple trick fails beyond the leading order in $1/\tilde{c}$. At $1/\tilde{c}^2$ order one has to fix both constant and linear in n_k terms, while simple $I_k \to (n_k + 1/2)/\tilde{c}$ substitution leads to incorrect results.

We propose and verify up to $1/\tilde{c}^2$ order that the subleading "quantum correction" terms can be unambiguously fixed starting from the analytic expression in terms of $1/\tilde{c}$ pertubative series of the eigenvalues Q^0_{2n-1} of Q_{2n-1} acting on the primary state $|\Delta\rangle$. For leading $1/\tilde{c}$ term this statement is trivial — taking all $n_k = 0$ yields the constant term, which is simply leading $1/\tilde{c}$ term in Q^0_{2n-1}. At the $1/\tilde{c}^2$ order this statement is more nuanced: naively Q^0_{2n-1} only fixes the constant term with all $n_k = 0$, but we show linear in n_k terms can be also fixed starting from Q^0_{2n-1}. As a result we obtain spectrum of Q_{2n-1} up to first three orders in $1/c$ expansion, including the leading Δ^n term. We then apply the obtained result to evaluate thermal expectation values of Q_{2n-1}, free energy of the KdV Generalized Gibbs Ensemble, and the asymptotic expansion of the quantum transfer matrix acting on a primary state $|\Delta\rangle$, all at first few leading orders in $1/c$.

The paper is organized as follows. In section 2 we discuss classic completely integrable system associated with the finite zone potentials and evaluate $Q_{2n-1}(h, I_k)$ as a perturbative series in I_k. In section 3 we discuss analytic form of Q_{2k-1} acting on primary states. These two pieces are combined in section 4 where we employ semiclassical quantization to obtain the spectrum of qKdV charges in the first three orders of $1/\tilde{c}$ expansion. We also perform consistency checks, confirming our result. Section 5 is devoted to applications of the obtained result. In section 5.1 we calculate thermal expectation values of Q_{2n-1} and fix two leading orders in $1/c$ of the associated differential operator $\mathcal{D}_n$

$$\mathrm{Tr}_\Delta(Q_{2k-1}\, q^{Q_1}) = \mathcal{D}_n \chi_\Delta, \qquad \chi_\Delta \equiv \mathrm{Tr}_\Delta(q^{Q_1}). \tag{1.11}$$

In section 5.2 we discussed KdV Generalized Gibbs Ensemble and calculate its free energy $-\ln Z_{\rm GGE}$,

$$Z_{\rm GGE} = {\rm Tr}\, e^{-\sum_n \mu_{2n-1} Q_{2n-1}}, \tag{1.12}$$

at leading order in $1/c$. In section 5.3 we use the asymptotic expansion to calculate the quantum transfer matrix acting on a primary state at first two orders in $1/c$ expansion. We use analytic continuation to extend the validity beyond the asymptotic regime, but notice that certain non-pertubative terms are missing. We conclude with a discussion in section 6. The paper also includes a number of appendices. Appendix A provides an easy derivation of (1.6) by quantizing boundary gravitons of semiclassical gravity in AdS$_3$. Appendix B evaluates $Q_{2n-1}(h, I_k)$ at first two orders in I_k by explicitly introducing normal coordinates at the origin of the co-adjoint orbit of the Virasoro algebra. Appendix C provides technical details concerning Novikov's one-zone potentials. Appendix D develops the technique of dealing with the multi-zone potentials in the limit of the infinitesimally small zones. Finally, appendix E provides the details of calculating the spectrum of Q_{2n-1} acting on primary states based on ODE/IM correspondence.

2 Calculation of $Q_{2n-1}(h, I_k)$

In this section our goal is to find expression for Q_{2n-1} in terms of the orbit invariant h and action variables I_k, by expanding pertubatively up to cubic order in I_k,

$$\begin{aligned} Q_{2n-1} = h^n + \sum_k f_k^{(n,1)} I_k + f_k^{(n,2)} I_k^2 + f_k^{(n,3)} I_k^3 + \sum_{k<\ell} f_{k,\ell}^{(n,2)} I_k I_\ell \\ + \sum_{k\neq\ell} f_{k,\ell}^{(n,3)} I_k^2 I_\ell + \sum_{k<\ell<p} f_{k,\ell,p}^{(n,3)} I_k I_\ell I_p + \mathcal{O}(I^4). \end{aligned} \tag{2.1}$$

Coefficients f are h-dependent. First three $f_k^{(n,1)}, f_k^{(n,2)}, f_k^{(n,3)}$ will be found using one-zone potentials in section 2.2. Using two-zone potentials we will find $f_{k,\ell}^{(n,2)}$ and $f_{k,\ell}^{(n,3)}$ in section 2.3, while coefficient $f_{k,\ell,p}^{(n,3)}$ will be fixed using three-zone potentials in section 2.4. An alternative brute-force derivation of (2.1) up to quadratic order in I_k is given in the appendix B.

2.1 Finite zone potentials: an introduction

The starting point is the "Schrödinger" equation

$$-\psi'' + \frac{u}{4}\psi = \lambda\,\psi, \tag{2.2}$$

with the periodic real-valued potential $u(\varphi + 2\pi) = u(\varphi)$. For any real λ there are two linearly-independent quasi-periodic solutions

$$\psi_\pm(\varphi + 2\pi) = e^{\pm 2\pi i\, p(\lambda)} \psi_\pm(\varphi). \tag{2.3}$$

Here quasi-momentum $p(\lambda)$ could be either real or pure imaginary. Values of $\lambda \in \mathbb{R}$ for which $p(\lambda)$ is imaginary are called "forbidden zone." At the end of forbidden zones

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$p(\lambda)$ is integer or half-integer such that $\psi_\pm$ become periodic or antiperiodic and linearly dependent. Normally, for such λ, another linearly independent singular solution appears. Yet occasionally there are two linearly independent regular periodic or antiperiodic solutions for the same λ. In this case forbidden zone degenerates and disappears, with $p(\lambda)$ being real everyone in the vicinity of that point. We provide examples below.

A general potential u would have an infinite number of forbidden zones, but there are special classes when only a finite number of forbidden zones are non-degenerate, Such u are called finite zone potentials. They were introduced in a famous work [23] and often refereed to as Novikov potentials.

Example: zero zone potential. Let us consider a constant potential $u = 4\lambda_0 = Q_1$ with some real Q_1. A solution to (2.2) can be readily found

$$\psi_\pm(\varphi) = e^{\pm ip(\lambda)\varphi}, \qquad p(\lambda) = \sqrt{\lambda - \lambda_0}. \tag{2.4}$$

For any $\lambda > Q_1/4$ quasi-momentum is real, i.e. there are no forbidden zones, except for $\lambda \in (-\infty; Q_1/4)$. The solutions (2.4) are linearly independent, including $\lambda = (Q_1 + k^2)/4$ for natural k, when $\psi_\pm$ are (anti)periodic. Values $\lambda = (Q_1 + k^2)/4$ mark the ends of degenerate forbidden zones.

Example: "opening" a zone. Let us now consider the potential $u = Q_1 + \epsilon\cos(k\varphi) + O(\epsilon^2)$ where Q_1 is a constant, k is positive integer, and ϵ is some infinitesimal parameter. Using quantum mechanics perturbation theory we find at leading order that all eigenvalues of periodic and anti-periodic problems remain the same and double-degenerate, except for λ_k which splits into

$$\lambda_k^\pm = \frac{Q_1 + k^2}{4} \pm \frac{\epsilon}{2}. \tag{2.5}$$

Hence now there are two forbidden zones, $(-\infty, Q_1/4)$ and $(\lambda_k^-, \lambda_k^+)$.

Finite-zone potentials are characterized by the ends of non-degenerate zones λ_i. For the zero-zone potential above there is only one parameter $\lambda_0 = Q_1/4$. After one zone is opened, there are three parameters: "energy" of the ground state λ_0, $\lambda_1 = \lambda_k^-$ and $\lambda_2 = \lambda_k^+$. In general an m-zone potential is characterized by

$$\lambda_0 < \lambda_1 < \cdots < \lambda_{2m}, \tag{2.6}$$

with the forbidden zones $(-\infty, \lambda_0)$ and $(\lambda_{2i-1}, \lambda_{2i})$, $i = \overline{1, m}$. For each set $\{\lambda_i\}$ we can define a hyperelliptic curve

$$y^2 = \prod_{i=0}^{2m}(\lambda - \lambda_i), \tag{2.7}$$

while the quasi-momentum p being fixed in terms of its differential

$$dp = \frac{\lambda^m + r_{m-1}\lambda^{n-1} + \ldots r_0}{2\,y} d\lambda, \qquad p(\lambda_0) = 0. \tag{2.8}$$

The latter is defined in such a way that the integrals of dp over a-cycles vanish

$$\oint_{a_i} dp = 2\int_{\lambda_{2i-1}}^{\lambda_{2i}} dp = 0. \tag{2.9}$$

This fixes m coefficients $r_0, \ldots, r_{m-1}$. Furthermore for the potential associated with $\{\lambda_i\}$ to be 2π-periodic we must additionally require integrals over b-cycles

$$w_i = \oint_{b_i} dp = 2\int_{\lambda_{2i-2}}^{\lambda_{2i-1}} dp \tag{2.10}$$

to be integer-valued

$$w_i = k_i - k_{i-1}. \tag{2.11}$$

Here natural k_i satisfying $k_{i+1} > k_i$, $k_0 \equiv 0$, label opened zones. These are additional m constrains, which reduce the total number of independent parameters λ_i to $m+1$.

A given set $\{\lambda_i\}$ which satisfies (2.9), (2.11), such that only $m+1$ parameters are independent, defines periodic potential $u(\varphi)$, but in a non-unique way. Individual potentials are labeled by points of the Jacobian of curve (2.7), with all of them sharing the same spectrum. In other words isospectral potentials form an m-dimensional torus, while full space of m-zone potentials is therefore $2m+1$ dimensional.

At this point we would like to make a connection with the Virasoro algebra. Consider Hill's equation, which is "Schrodinger" equation (2.2) with $\lambda = 0$,

$$-\psi'' + \frac{u}{4}\psi = 0. \tag{2.12}$$

One can re-parametrize the circle going from φ to $\tilde{\varphi}(\varphi)$ such that $\tilde{\varphi}(\varphi+2\pi) = \tilde{\varphi}(\varphi)+2\pi$. Then wave-function and the potential also change accordingly

$$\tilde{\psi}(\tilde{\varphi}) = \psi(\varphi)\left(\frac{d\tilde{\varphi}}{d\varphi}\right)^{-1/2}, \tag{2.13}$$

$$\tilde{u}(\tilde{\varphi}) = \left(\frac{d\tilde{\varphi}}{d\varphi}\right)^{-2}\left(u + 2(S\tilde{\varphi})(\varphi)\right), \tag{2.14}$$

where Schwarzian derivative

$$(S\theta)(\varphi) \equiv \frac{\theta'''}{\theta'} - \frac{3}{2}\left(\frac{\theta''}{\theta'}\right)^2. \tag{2.15}$$

From (2.14) it is clear that u is an element from the co-adjoint orbit of Virasoro algebra with the Schwarzian derivative term appearing because of central extension [20]. All potentials $u(\varphi)$ related by circle reparametrizations, i.e. belonging to the same co-adjoint orbit share the same invariant — quasi-momentum at zero,

$$\psi(2\pi)/\psi(0) = e^{2\pi i p(0)}, \tag{2.16}$$

which is evident from (2.13). In other words

$$-4p(0)^2 = h \tag{2.17}$$

is the invariant of u characterizing the orbit itself. By choosing an appropriate $\tilde{\varphi}$ the potential always[3] can be brought to a constant form, in which case

$$\tilde{u}(\tilde{\varphi}) = h. \tag{2.18}$$

[3]An implicit assumption here is that u belongs to the regular orbit $\mathrm{diff}(\mathbb{S}^1)/\mathbb{S}^1$, which upon quantization, becomes Verma module.

The co-adjoint orbit is a symplectic space equipped with the Kirillov-Kostant-Souriau bracket

$$\frac{c}{24}\{u(\varphi),u(\varphi')\} = -2\pi\mathcal{D}\delta(\varphi-\varphi'), \qquad \mathcal{D} = \partial u + 2u\partial - 2\partial^3. \tag{2.19}$$

Here, using linearity of symplectic form we introduce a formal parameter c, which later will be identified with the CFT central charge. Any Hamiltonian flow defined by (2.19) leaves h invariant.

There is an infinite tower of the so-called KdV Hamiltonians Q_{2k-1}, which can be defined recursively with help fo Gelfand-Dikii polynomials R_n,

$$Q_{2n-1} = \frac{1}{2\pi}\int_0^{2\pi} R_n d\varphi \qquad\qquad \partial R_{n+1} = \frac{n+1}{2n+1}\mathcal{D}R_n, \tag{2.20}$$
$$R_0 = 1, \quad R_1 = u, \quad R_2 = u^2 - \frac{4}{3}\partial^2 u, \qquad R_3 = u^3 - 4u\partial^2 u - 2(\partial u)^2 + \frac{8}{5}\partial^4 u, \dots$$

Their Hamiltonian flows generate isospectral deformations of u

$$\delta u = \frac{c}{24}\{Q_{2n-1}, u\} = (2n-1)\partial R_n, \tag{2.21}$$

while they all remain in involution $\{Q_{2n-1}, Q_{2\ell-1}\} = 0$.

We now consider a space of all m-zone potentials sharing the same h. This is a $2m$-dimensional subspace within the orbit parametrized by h, which we will denote as $\mathcal{F}_m(h)$. The pullback of the symplectic form on this space is non-degenerate, hence it is also a symplectic manifold equipped with the Poisson bracket. Isospectral flows leave this manifold invariant. Upon restricting to $\mathcal{F}_m(h)$, only first n KdV Hamiltonians remain algebraically independent. The flows they generate move u along the Jacobian of (2.7), which is the Liouvillian torus of a completely integrable dynamical system defined by Q_{2n-1}, $n \le m$. In other words the geometry of $\mathcal{F}_m(h)$ is a m-dimensional torus parametrized by angle variables fibered above a base parametrized by m variables Q_{2n-1}. Alternatively, one can introduce m action variables I_k parameterizing the base and forming canonical conjugate pairs with angle variables.

In terms of dp (2.8) values of KdV charges are given by an expansion at infinity

$$Q_{2n-1} = \frac{2\Gamma(n+1)\Gamma(1/2)}{\Gamma(n+1/2)}\frac{4^n}{2\pi i}\oint_\infty dp\, \lambda^{n-1/2}, \tag{2.22}$$

while the action variables are

$$I_k = \frac{i}{\pi}\oint_{a_k} p\frac{d\lambda}{\lambda} = \frac{1}{i\pi}\oint_{a_k} dp \ln\lambda. \tag{2.23}$$

Functional dependence of Q_{2n-1} for $n > m$ on the first m ones readily follows from (2.22) and the form of dp (2.8).

Our task is conceptually trivial: we want to learn an explicit change of variables on the base of $\mathcal{F}_m(h)$ from Q_{2n-1} to I_k. The expressions for $Q_{2n-1}(h, I_k)$ is not available in the closed form, we therefore will find first few orders by expanding it in powers of I_k. There is

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one notable exception, using Riemann bilinear relation with two one-forms dp and $pd\lambda/\lambda$ one can show in full generality

$$Q_1 = h + \sum_k k\, I_k. \tag{2.24}$$

Our main approach will be based on parameterizing both Q_{2n-1} and I_k in terms of the spectral curve $i = \overline{0,m}$, with the infinitesimal $\lambda_{2i} - \lambda_{2i-1}$, and then re-expressing Q_{2n-1} in terms of I_k. There is an alternative straightforward approach, to parametrize the potential $u(\varphi)$ in terms of its Fourier modes u_ℓ, and then express both Q_{2n-1} and I_k in terms of u_ℓ. We develop this method in the appendix B and confirm the expansion (2.1) up to second order in I_k.

2.2 One-zone potentials

Before we consider one-zone potential in detail, we revisit the zero-zone potential $u = Q_1 \equiv 4\lambda_0$ and readily find differential

$$dp = \frac{d\lambda}{2\sqrt{\lambda - \lambda_0}} \tag{2.25}$$

to be defined on a Riemann sphere. This is the simplest possible case. In this case $p = \sqrt{\lambda - \lambda_0}$, $u(\varphi) = h = 4\lambda_0$ and the whole symplectic space $\mathcal{F}_0(h)$ shrinks to a point. All KdV Hamitonians are fixed by h, $Q_{2n-1} = h^n$ with all action variables identically equal to zero.

Next, we consider the differential

$$dp = \frac{(\lambda - r)d\lambda}{2\sqrt{(\lambda-\lambda_0)(\lambda-\lambda_1)(\lambda-\lambda_2)}} \tag{2.26}$$

parameterized by λ_i, r_0. It is defined on a torus — a Riemann curve of genus one. We assume that (λ_2, λ_1) correspond to k-th zone. After satisfying (2.9) and (2.11), which requires evaluating elliptic integrals, we find one-parametric family

$$\lambda_2 = \lambda_0 + \frac{k^2}{4}\theta_3(\tau)^4, \quad \lambda_1 = \lambda_0 + \frac{k^2}{4}\theta_4(\tau)^4, \quad r = \lambda_0 + \frac{k^2}{4}\theta_4(\tau)^4\left(1 + 2\frac{\partial \ln \theta_3^2(\tau)}{\partial \ln m}\right), \tag{2.27}$$

where $m = \theta_2^4(\tau)/\theta_3^4(\tau)$ and $\tau = i\tau_2$ with positive τ_2. In what follows we use[4] $q = e^{i\pi\tau}$ such that $\theta_2 = \sum_n q^{(n+1/2)^2}$, $\theta_3 = \sum_n q^{n^2}$, $\theta_4 = \sum_n (-1)^n q^{n^2}$.

To impose the orbit constraint $-4p(0)^2 = h$ it is more convenient to use the following trick. First we evaluate

$$Q_1 = 4(\lambda_0 + \lambda_1 + \lambda_2) - 8r, \tag{2.28}$$

which expresses λ_0 in terms of Q_1 and q expansion,

$$4\lambda_0 = Q_1 - k^2\left(\theta_2^4 - 4\theta_4^4 \frac{\partial \ln \theta_3^2(\tau)}{\partial \ln m}\right) = Q_1 - 32k^2q^2\left(1 + 2q^2 + 4q^4 + 4q^6 + \ldots\right), \tag{2.29}$$

[4] Our definition of q is aligned with Wolfram Mathematica. In this section q denotes modular parameter of the genus one elliptic curve $y(\lambda)$. In section 5.1 we use q to denote modular parameter of the CFT spacetime torus.

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and then use (2.23) to evaluate action variable perturbatively in q,

$$
\begin{aligned}
I_k &= \frac{2}{\pi}\int_{\lambda_1}^{\lambda_2} \frac{d\lambda(\lambda-r)\log\lambda}{\sqrt{(\lambda-\lambda_0)(\lambda-\lambda_1)(\lambda_2-\lambda)}} \\
&= \sum_{n=1}^{\infty} \frac{2(-1)^n(\lambda_1-\lambda_0)^{n+1}}{n\sqrt{\lambda_2-\lambda_0}\lambda_0^n}\left[\frac{F\left(\frac{3}{2},\frac{1}{2},1;m\right)}{F\left(\frac{1}{2},\frac{1}{2},1;m\right)}F\left(n+\frac{1}{2},\frac{1}{2},1;m\right)-F\left(n+\frac{3}{2},\frac{1}{2},1;m\right)\right].
\end{aligned}
\tag{2.30}
$$

Here $F \equiv {}_2F_1$ is the hypergeometric function such that $F\left(\frac{3}{2},\frac{1}{2},1;m\right)=\theta_3^2$.

An infinite sum over n above has to be evaluated individually for each term in q expansion. This gives I_k as a function of λ_0 and q, $I_k = \frac{32k^3q^2}{k^2+4\lambda_0}+O(q^4)$, which with help of (2.29) can be expressed as a function of Q_1 and q,

$$
\begin{aligned}
I_k &= \frac{32k^3}{k^2+Q_1}q^2+\frac{64k^3\left(17k^4+12k^2Q_1+3Q_1^2\right)}{\left(k^2+Q_1\right)^3}q^4 \\
&+\frac{128k^3\left(5k^2+Q_1\right)\left(77k^6+69k^4Q_1+27k^2Q_1^2+3Q_1^3\right)}{\left(k^2+Q_1\right)^5}q^6+\mathcal{O}(q^8).
\end{aligned}
\tag{2.31}
$$

At this point we use (2.24), which is exact, $Q_1 = h+kI_k$. Using I_k given as a q-series expansion with Q_1-dependent coefficients (2.31), with help of (2.24) we express Q_1 as a series in q with h-dependent coefficients by iteratively substituting Q_1 written as an h-dependent series in q. Once we find $Q_1 = Q_1(h,q)$, I_k can be deduced from (2.24),

$$
\begin{aligned}
I_k &= \frac{32k^3}{h+k^2}q^2+\frac{64\left(3h^2k^3+12hk^5+k^7\right)}{\left(h+k^2\right)^3}q^4 \\
&+\frac{128k^3\left(3h^4+42h^3k^2+108h^2k^4-58hk^6+k^8\right)}{\left(h+k^2\right)^5}q^6+\mathcal{O}(q^8).
\end{aligned}
\tag{2.32}
$$

At this point it is straightforward to re-express q as a h-dependent power series in I_k, $q^2 = \frac{h+k^2}{32k^3}I_k + O(I_k^2)$.

To obtain coefficients $f^{(n,i)}$ (2.1) we act as follows. From the definition (2.22) we can find Q_{2n-1} as a polynomial in λ_i and r. Using expressions for λ_i, r (2.27) and (2.29), where Q_1 is understood as a function of h,q we write Q_{2n-1} as an h-dependent power series in q. After that it is straightforward to use $q^2=q^2(h,I_k)$ to re-express Q_{2n-1} as an h-dependent power series in I_k,

$$
Q_{2n-1} = h^n + f_k^{(n,1)}I_k + f_k^{(n,2)}I_k^2 + f_k^{(n,3)}I_k^3 + \mathcal{O}(I_k^4),
\tag{2.33}
$$

thus fixing $f^{(n,i)}$,

$$f_k^{(n,1)} = \sum_{j=0}^{n-1} \frac{\sqrt{\pi}(2n-1)\Gamma(n+1)}{2\Gamma(j+\frac{3}{2})\Gamma(n-j)} h^{n-1-j} k^{2j+1} = \sum_{j=0}^{n-1} \xi_n^j\, h^{n-1-j} k^{2j+1}, \tag{2.34}$$

$$f_k^{(n,2)} = \sum_{j=0}^{n-1} \frac{\sqrt{\pi}(2n-1)\Gamma(n+1)(j(2n+1)-2n+2)}{16\Gamma\left(j+\frac{3}{2}\right)\Gamma(n-j)} h^{n-1-j} k^{2j}, \tag{2.35}$$

$$\begin{aligned} f_k^{(n,3)} &= \frac{(2n-1)n(n-1)}{64k^3} h^n \\ &+ \sum_{j=0}^{n-1} \frac{\sqrt{\pi}(2n-1)\Gamma(n+1)\,\mathsf{p}}{1536\Gamma\left(j+\frac{5}{2}\right)\Gamma(n-j)} h^{n-1-j} k^{2j-1}, \\ \mathsf{p} &= 4j^3(2n+1)(2n+3) - 2j^2(2n+1)(10n-21) - 3j(2n+3)(10n-7) + 36(n-1)(2n-1). \end{aligned} \tag{2.36}$$

More technical details about the one-zone potential calculation can be found in appendix C.

2.3 Two-zone potentials

In case of two zones the differential

$$dp = \frac{(\lambda - r_1)(\lambda - r_2)d\lambda}{2\sqrt{\prod_{i=0}^4 (\lambda - \lambda_i)}} \tag{2.37}$$

depends on seven parameters subject to 4 constraints (2.9) and (2.11). Corresponding integrals can not be evaluated analytically. We therefore proceed by expanding perturbatively, assuming both zones, and hence corresponding action variables, are small. We introduce two infinitesimal variables ϵ_1, ϵ_2 of the same order, such that $\lambda_2 - \lambda_1$ is of order ϵ_1 and $\lambda_4 - \lambda_3$ is of order ϵ_2. Action variables are quadratic in ϵ_i, $I_k \sim \epsilon_1^2, I_\ell \sim \epsilon_2^2$, where we assumed (λ_1, λ_2) and (λ_3, λ_4) correspond to k-th and ℓ-zones respectively. Our goal is to find Q_{2n-1} up to third order in the pertubative expansion in I_k, I_ℓ. Hence in what follows we must expand all quantities in ϵ_i up to sixth order. The details of this calculation can be found in appendix D.

After satisfying (2.9) and (2.11) we find λ_i for $i \geq 1$ and r_i in terms of λ_0 and ϵ_1, ϵ_2, as a perturbative expansion in ϵ_i. Then, we evaluate I_k, h and Q_{2n-1} also as function of λ_0 and ϵ_1, ϵ_2, similarly expanding in ϵ_i up to and including sixth order. By matching both sides of (2.1) we find coefficients $f_{k,\ell}^{(m,n)}$, yielding

$$f_{k,\ell}^{(n,2)} = \sum_{j=1}^{n-1} \frac{\sqrt{\pi}(2n-1)^2\Gamma(n+1)}{4\Gamma(n-j)\Gamma\left(j+\frac{3}{2}\right)} h^{n-1-j} \sum_{s=0}^{j-1} k^{2(j-s)-1} \ell^{2s+1}, \tag{2.38}$$

$$\begin{aligned} f_{k,\ell}^{(n,3)} &= \frac{\ell}{(k^2-\ell^2)^2} \left(-\frac{(2n-1)n(n-1)}{4} h^n + \sum_{j=0}^{n-1} \frac{\sqrt{\pi}(2n-1)^2\Gamma(n+1)}{64\Gamma(n-j)\Gamma\left(j+\frac{5}{2}\right)} h^{n-1-j}\,\mathsf{q} \right), \\ \mathsf{q} &= -4(2n+1)\frac{k^{2j+4}-\ell^{2j+4}}{k^2-\ell^2} + k^{2j+2}(3+2j)(j(2n+1)-4n+5) + \ell^{2j+2}2(3j+n+5) \\ &\quad + k^{2j}\ell^2(3+2j)(j(2n+1)-2n+2) + k^2\ell^{2j}(3+2j)(4n-1). \end{aligned} \tag{2.39}$$

2.4 Three-zone potentials

Extending calculations of the previous section using the technique of appendix D to the three-zone case we can fix

$$f^{(n,3)}_{k,\ell,p}=\sum_{j=0}^{n-3}\frac{\sqrt{\pi}(2n-1)^3\Gamma(n+1)(n-2-j)}{8\Gamma(n-1-j)\Gamma\left(j+\frac{7}{2}\right)}h^{n-3-j}\sum_{s_1=0}^{j}\sum_{s_2=0}^{j-s_1}k^{2j+1-2(s_1+s_2)}\ell^{2s_1+1}p^{2s_2+1}. \tag{2.40}$$

2.5 Consistency check

In case of an m-zone potential we can parametrize the differential dp with help of λ_0 and ϵ_i, $1\le i\le m$, cf. (D.1)–(D.6),

$$\lambda_i=\lambda_0+\ldots,\qquad 1\le i\le 2m, \tag{2.41}$$

$$r_i=\lambda_0+\ldots,\qquad 1\le i\le m, \tag{2.42}$$

where dots stand for ϵ_i but not λ_0-dependent terms. Similarly action variables I_k, charges Q_{2n-1} and the orbit parameter $h=-4p(0)^2$ will be some functions of λ_0 and ϵ_i. While dependence of I_k and h on λ_0 is non-trivial, since Q_{2n-1} are the coefficients of $1/\lambda$ expansion of $p(\lambda)$ at infinity and λ_0 is simply the shift of the argument of $p(\lambda)$, we find

$$Q_{2n-1}=\sum_{k=0}^{n}\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}(4\lambda_0)^{n-k}Q^0_{2k-1}. \tag{2.43}$$

Here Q^0_{2k-1} are the charges evaluated with help of (2.22) taking $\lambda_0=0$ in (2.41), (2.42). Assuming we know $Q_{2n-1}(h,I_k)$ where $h=h(\lambda_0,\epsilon_i)$ and $I_k=I_k(\lambda_0,\epsilon_i)$, one can introduce $I^0_k=I_k(0,\epsilon_i)$ such that $Q^0_{2k-1}=Q_{2k-1}(0,I^0_k)$. Here first argument is zero simply because $h(0,\epsilon_i)=0$. Then both sides of equation (2.43) become functions of λ_0 and ϵ_i, providing a non-trivial check.

There is an alternative way to use (2.43) to check the consistency of the perturbative expansion (2.1) with the coefficients found in the text. We can invert $h=h(\lambda_0,\epsilon_i)$ and $I_k=I_k(\lambda_0,\epsilon_i)$ to express both λ_0 and I^0_k via h and I_k,

$$\begin{aligned}\lambda_0=h+&\sum_k -\frac{hI_k}{k}+\frac{h\left(h+5k^2\right)I_k^2}{8k^4}-\frac{h\left(5h^2+30hk^2+41k^4\right)I_k^3}{128k^7}+\sum_{k<\ell}\frac{hI_kI_\ell}{k\ell}\\&+\sum_{k\neq\ell}\frac{hI_k^2I_\ell\left(h^2\ell^2-h\left(k^4-4k^2\ell^2+\ell^4\right)-5k^6+11k^4l^2-5k^2\ell^4\right)}{8k^4\ell(k-\ell)^2(k+\ell)^2}-\sum_{k<\ell<p}\frac{hI_kI_\ell I_p}{k\ell p}+\mathcal{O}(I^4),\\ I^0_k=I_k+&\frac{hI_k}{k^2}-\frac{hI_k^2\left(h+5k^2\right)}{8k^5}+\frac{hI_k^3\left(5h^2+30hk^2+41k^4\right)}{128k^8}-\sum_{\ell\neq k}\frac{hI_kI_\ell\left(h+k^2\right)}{k^2\ell\left(k^2-\ell^2\right)}\\&+\sum_{\ell\neq k}\frac{hI_k^2I_\ell\left(2h^2\left(-k^4+2k^2\ell^2+\ell^4\right)+hk^2\left(7k^4-14k^2\ell^2+15\ell^4\right)+k^4\left(5k^4-10k^2\ell^2+9\ell^4\right)\right)}{8k^5l\left(k^2-\ell^2\right)^3}\\&+\sum_{\ell\neq k}\frac{hI_kI_\ell^2\left(h^2\left(k^4+5k^2\ell^2-2\ell^4\right)+h\left(k^6+10k^4\ell^2-9k^2\ell^4+6l^6\right)+k^2\ell^2\left(5k^4-7k^2\ell^2+6l^4\right)\right)}{8k^2\ell^4\left(k^2-\ell^2\right)^3}\\&+\sum_{p\neq\ell\neq k}\frac{hI_kI_\ell I_p\left(2h^2+3hk^2+k^4\right)}{k^2\ell p\left(k^2-\ell^2\right)\left(k^2-p^2\right)}+\mathcal{O}(I^4).\end{aligned}$$

Now $Q^0_{2n-1}(0, I^0_k(h, I_k))$ is a function of h, I_k and (2.43) provides a non-trivial check for the coefficients in (2.1).

This check also ensures that $Q_{2n-1}(h, I_k)$ satisfy another identity

$$\frac{1}{n+1}\frac{\partial Q_{2n+1}}{\partial u_0} = Q_{2n-1}, \tag{2.44}$$

which follows from the properties of Gelfand-Dikii polynomials (2.20). Here $Q_{2n-1}[u(\varphi)]$ are understood as functionals of $u(\varphi)$ and the derivative is with respect the zero Fourier mode of $u(\varphi)$, while all other Fourier modes are kept fixed. The shift of u_0 with all other modes intact is equivalent to a shift of the spectrum by a constant, hence

$$\left(\frac{\partial}{\partial u_0}\right)_{u_\ell} = 4\left(\frac{\partial}{\partial \lambda_0}\right)_{\epsilon_i}. \tag{2.45}$$

Then (2.44) follows immediately from the right-hand-side of (2.43).

For an m-zone potential, all higher KdV charges Q_{2n-1} are some functions of first $m+1$ charges. Thus for one-zone potentials $Q_5, Q_7, \dots$ are functions of Q_1, Q_3, see e.g. section 2.4 of [24] for details. For the three-zone potentials higher Q_{2n-1} would depend on Q_1, Q_3, Q_5, Q_7 In principle this provides additional consistency check for (2.1). In practice the dependence is so complicated, it doesn't provide a useful check even for the one-zone case.

3 "Energies" of primary states via ODE/IM correspondence

In the previous section we found classical expression for Q_{2n-1} in term of action variables I_k and the orbit invariant h. Following the standard rules of semiclassical quantization I_k should be promoted to an integer quantum number, while h will become the dimension of the highest weight (primary) state Δ, marking representation of the Virasoro algebra. It is easy to see, this naive receipt fails already for the values of Q_{2n-1} on a primary state $|\Delta\rangle$. Indeed, taking all I_k to zero, we readily find $Q_{2n-1} = h^k$, which upon the naive quantization yields $\mathsf{Q}^0_{2n-1} = \Delta^n$ where

$$Q_{2n-1}|\Delta\rangle = \mathsf{Q}^0_{2n-1}|\Delta\rangle. \tag{3.1}$$

This answer is missing c-dependent terms. Explicit values of Q^0_{2n-1} for $n \le 8$ were calculated in [25] via brute-force approach, using explicit expressions for Q_{2n-1} in terms of free field representation. The pattern is clear, while Δ^n is indeed the leading term, full expression is a polynomial in both Δ and c of order n.

There is no known receipt to obtain exact Q^0_{2n-1} from the semiclassical quantization, hence our strategy will be the following. We will combine exact expression for Q^0_{2n-1} in the large c limit, which will be obtained in this section by a different method, with the classical result of section 2, to find spectrum of excited states in the large c limit in next section.

To find Q^0_{2n-1} we use ODE/IM correspondence, initiated in [18, 26] and more recently developed in [27] (also see [28]), which relates qKdV spectrum to solutions of an auxiliary Schrödinger equation

$$\partial_x^2\Psi(x) + \left(E - x^{2\alpha} - \frac{l(l+1)}{x^2}\right)\Psi(x) = 0, \tag{3.2}$$

where

$$(l+1/2)^2 = 4(\alpha+1)\tilde{\Delta}, \quad \tilde{c} = -\frac{\alpha^2}{4(\alpha+1)}. \tag{3.3}$$

Equation (3.2) can be solved using WKB approximation by systematically expanding in a small parameter. This leads to a quadratic ODE which can be solved iteratively. We delegate all details to appendix E and only write down iterative relation which defines coefficients $c_k^{(n)}$ for $n \geq 1$, $n \geq k \geq 0$,

$$\sum_{j=0}^{n}\sum_{p=0}^{j}\sum_{q=0}^{n-j}\delta_{p+q,k}c_p^{(j)}c_q^{(n-j)} - 2\left[n-k-u-\frac{n-2}{2\alpha}\right]c_{k-1}^{(n-1)} + (2k-3n+4)\,c_k^{(n-1)} = 0, \tag{3.4}$$

and we formally assumed $c_{-1}^{(n)} = c_{n+1}^{(n)} = 0$, $u^2 = -\tilde{\Delta}/\tilde{c}$, and the starting values are

$$c_0^{(0)} = -\frac{1}{\alpha}, \qquad c_0^{(1)} = -\frac{1}{2}, \quad c_1^{(1)} = \frac{1}{2\alpha} - u. \tag{3.5}$$

Coefficients $c_k^{(n)}$ determine values of Q_{2n-1} acting on primaries [27],

$$\mathsf{Q}^0_{2n-1} = \frac{(2n-1)\Gamma(n+1)}{\sqrt{\pi}\Gamma(1-\frac{2n-1}{2\alpha})4^n(\alpha+1)^n}\sum_{k=0}^{2n}c_k^{(2n)}\Gamma\left(k+\frac{3}{2}-3n\right)\Gamma\left(2n-k-\frac{2n-1}{2\alpha}\right). \tag{3.6}$$

Although this is not obvious, Q^0_{2n-1} given by (3.6) is a polynomial in terms of $\tilde{\Delta}$ and $\tilde{c}$. After some algebra we find leading order expansion

$$\mathsf{Q}^0_{2n-1} = \tilde{\Delta}^n + \sum_{j=0}^{n-1}\tilde{R}^{(1)}_{n,j}\tilde{\Delta}^{n-j-1}\,\tilde{c}^j + \sum_{j=0}^{n-2}\tilde{R}^{(2)}_{n,j}\tilde{\Delta}^{n-j-2}\,\tilde{c}^j + \sum_{j=0}^{n-3}\tilde{R}^{(3)}_{n,j}\tilde{\Delta}^{n-j-3}\,\tilde{c}^j + \mathcal{O}(\tilde{c}^{n-3}), \tag{3.7}$$

where

$$\tilde{R}^{(1)}_{n,j} = \frac{(2n-1)\sqrt{\pi}\Gamma(n+1)}{4\Gamma(j+\frac{3}{2})\Gamma(n-j)}\zeta(-2j-1) = \xi_n^j\frac{\zeta(-2j-1)}{2}, \tag{3.8}$$

$$\begin{aligned}\tilde{R}^{(2)}_{n,j} &= \frac{(2n-1)\sqrt{\pi}\Gamma(n+1)}{24\times 4\Gamma(j+\frac{5}{2})\Gamma(n-j-1)} \\ &\quad\times\left\{-6\zeta(-2j-3)\,(2j+3-(2n-1)y_1(j+1)) + 3(2n-1)\zeta_2(j)\right\},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\tilde{R}^{(3)}_{n,j} &= \frac{(2n-1)\sqrt{\pi}\Gamma(n+1)}{24^2\times 4\Gamma(j+\frac{7}{2})\Gamma(n-j-2)}\left\{6^2\zeta(-2j-5)(2j^2+7j+5) - (2n-1)r_{n,j}\right\}, \\ r_{n,j} &= 12\zeta_3(j) + 36\zeta_2(j+1)(y_1(j+2)+j+2) + 3\left(4j^2+18j+23\right)\zeta(-2j-3) \\ &\quad + 36\zeta(-2j-5)\left(y_1^2(j+2) + 2(j+2)y_1(j+2) + y_2(j+2)\right) + (2n+1)p_j.\end{aligned} \tag{3.10}$$

Functions $\zeta_2, \zeta_3, y_1, y_3$ are defined in the appendix E, where we also give values of p_j for $0 \leq j \leq 17$.

4 Spectrum of quantum Q_{2k-1}

At this point we are ready to combine classical pertubative expression for $Q_{2n-1}(h, I_k)$ (2.1) with the "energies" of primary state (3.7) to obtain Q_{2n-1} up to first two non-trivial orders in $1/\tilde{c}$ expansion.

The naive semi-classical quantization would map the co-adjoint orbit invariant h and the actions variables I_k on the classical side to dimension of the primary state Δ and the excited state quantum numbers n_k correspondingly,

$$h \to \frac{24\Delta}{c}, \qquad I_k \to \frac{24 n_k}{c}. \tag{4.1}$$

Also classical charge Q_{2n-1} should be rescaled by $(c/24)^n$. Starting from (2.1) this correctly reproduces full quantum spectrum of Q_1 and the leading Δ^n term in Q_{2n-1}. But it falls short of reproducing sub-leading terms even for the primary state (3.7). The relation between classical and quantum quantities (4.1) is only correct at the leading c order. In [22] we observed that using $c-1$ as an expansion parameter leads to more elegant expressions. This is confirmed by (3.7), which looks most naturally if written in terms of $\tilde{\Delta}$ and $\tilde{c}$. We therefore propose the following quantization map, which agrees with the naive one at leading order,

$$h \to \frac{\tilde{\Delta}}{\tilde{c}}, \qquad I_k \to \frac{n_k}{\tilde{c}}, \qquad \tilde{\Delta} = \Delta - \tilde{c}, \qquad \tilde{c} = \frac{c-1}{24}. \tag{4.2}$$

This does not solve the problem of reproducing subleadig terms in Q^0_{2n-1}, but this can be fixed, at least at first subleading order, by introducing the Maslov index, $n_k \to \tilde{n}_k = n_k + 1/2$. We thus arrive at the following map,

$$Q_{2n-1}(h, I_k) \to \mathsf{Q}_{2n-1} = \tilde{c}^n\, Q_{2n-1}(\tilde{\Delta}/\tilde{c}, (n_k + 1/2)/\tilde{c}). \tag{4.3}$$

Infinite sums due to Maslov index contributing to "vacuum energy" should be regularized using zeta-function regularization. It is now straightforward to see that we immediately reproduce the leading $1/\tilde{c}$ term (3.8),

$$\begin{aligned} Q_{2n-1} &= h^n + \sum_k f_k^{(n,1)}(h)\, I_k + O(I^2) \to \mathsf{Q}_{2n-1} = \tilde{\Delta}^n + \tilde{c}^{n-1} \sum_k f_k^{(n,1)}(\tilde{\Delta}/\tilde{c})\, \tilde{n}_k + O(\tilde{c}^{n-2}) \\ &= \tilde{\Delta}^n + \sum_k \sum_{j=0}^{n-1} \xi_n^j\, \tilde{\Delta}^{n-1-j} \tilde{c}^j k^{2j+1} (n_k + 1/2) + O(\tilde{c}^{n-2}) \\ &= \tilde{\Delta}^n + \sum_{j=0}^{n-1} \xi_n^j\, \tilde{\Delta}^{n-1-j} \tilde{c}^j \left(\sum_k k^{2j+1} n_k + \frac{\zeta(-2j-1)}{2} \right) + O(\tilde{c}^{n-2}) \end{aligned} \tag{4.4}$$

In other words, at first sub-leading order $\tilde{c}^{n-1}$ the quantization prescription (4.3) leads to (1.6) which passes all available tests: matches the spectrum of Q_1, Q_3, Q_5, Q_7 (see section 4.1 below) and thermal expectation values for $Q_9, \ldots, Q_{13}$ (see section 5.1 below) at the order $\tilde{c}^{n-1}$.

There is another way to write (4.4). We can express Q_{2n-1} as Q^0_{2n-1} plus the terms from the classical Q_{2n-1} (2.1) which non-trivially depend on I_k using the substitution (4.2), i.e. without the Maslov index,

$$\mathsf{Q}_{2n-1} = \mathsf{Q}^0_{2n-1} + \tilde{c}^{n-1} \sum_k f_k^{(n,1)}(\tilde{\Delta}/\tilde{c})\, n_k + O(\tilde{c}^{n-2}). \tag{4.5}$$

At $\tilde{c}^{n-1}$ order it is the same as (4.4).

To obtain the quantum spectrum at next order $\tilde{c}^{n-2}$, we could try the prescription (4.3), apply the zeta-function regularization and notice that many but not all terms from (3.9) are reproduced. Thus, we see that the quantization (4.3) is exact only at leading $1/\tilde{c}$ order, at higher orders the expression obtained from the classical Q_{2n-1} has to be modified as well. Indeed, starting from the classical (2.1) and using substitution (4.2) we would find that terms contributing at the order $\tilde{c}^{n-p}$ are homogeneous polynomials in n_k of order p. This is very restrictive and obviously incorrect. We already saw that even at the first sub-leading order $\tilde{c}^{n-1}$ the homogeneous (linear) in n_k terms have to be amended by a constant, i.e. $(n_k)^0$ term. This suggest the following "quantization rules": to obtain the quantum spectrum Q_{2n-1} in $1/\tilde{c}$ expansion, one starts with the classical perturbation expression (2.1) and make the substitution (4.2), together with the overall rescaling by $\tilde{c}^n$. As the order $\tilde{c}^{n-p}$ this fixes leading, homogeneous in n_k terms of order p. These terms should be amended by the sub-leading terms of order $p-1$, $p-2$, $\ldots$, 0 in n_k. These terms should be regarded as quantum corrections and should be determined separately, they do not follow from the classical answer in any simple way. More explicitly,

$$\begin{aligned}\mathsf{Q}_{2n-1} = \tilde{\Delta}^n &+ \tilde{c}^{n-1}\left(\sum_k g_k^{(1)} n_k + g^{(1)}\right) + \tilde{c}^{n-2}\left(\sum_{k_1,k_2} g^{(2)}_{k_1,k_2} n_{k_1} n_{k_2} + \sum_k g_k^{(2)} n_k + g^{(2)}\right)\\ &+ \tilde{c}^{n-3}\left(\sum_{k_1,k_2,k_3} g^{(3)}_{k_1,k_2,k_3} n_{k_1} n_{k_2} n_{k_3} + \sum_{k_1,k_2} g^{(3)}_{k_1,k_2} n_{k_1} n_{k_2} + \sum_k g_k^{(3)} n_k + g^{(3)}\right) + \ldots\end{aligned} \tag{4.6}$$

Here $g^{(p)}$ with different number of indexes denote different quantities. The leading terms $g^{(p)}_{k_1,\ldots,k_p}$ are given by classical expressions (2.1) upon the substitution (4.2)

$$g_k^{(1)} = f_k^{(n,1)}(\tilde{\Delta}/\tilde{c}), \tag{4.7}$$

$$g^{(2)}_{k\ell} = \frac{1}{2} f^{(n,2)}_{k\ell}, \qquad g^{(2)}_{kk} = f^{(n,2)}_k, \tag{4.8}$$

$$g^{(3)}_{k\ell m} = \frac{1}{6} f^{(n,3)}_{k\ell m}, \qquad g^{(3)}_{kk\ell} = \frac{1}{3} f^{(n,3)}_{k\ell}, \quad g^{(3)}_{kkk} = f^{(n,3)}_k, \tag{4.9}$$

for $k \neq \ell \neq m$ and $g^{(p)}$ are given by (3.8), (3.9), (3.10). This is essentially the generalization of (4.5) to higher orders in $1/\tilde{c}$. Coefficients $g_k^{(2)}$, $g_{k\ell}^{(3)}, g_k^{(3)}$, etc. are quantum corrections and a priory not known.

To fix $g_k^{(2)}$ we employ the following strategy, we will try to "salvage" the Maslov index quantization (4.3) by adding minimal possible terms subleading in powers of n_k,

$$\mathsf{Q}_{2n-1} = \tilde{\Delta}^n + \tilde{c}^{n-1} \sum_k g_k^{(1)} \tilde{n}_k + \tilde{c}^{n-2}\left(\sum_{k_1,k_2} g^{(2)}_{k_1,k_2} \tilde{n}_{k_1} \tilde{n}_{k_2} + \sum_k \tilde{g}_k^{(2)} \tilde{n}_k + \tilde{g}^{(2)}\right) + \ldots \tag{4.10}$$

This expression is understood in terms of the zeta-function regularization and $\tilde{g}_k^{(2)}, \tilde{g}^{(2)}$ are different from $g_k^{(2)}, g^{(2)}$. Our goal is to reproduce "vacuum energy" Q_{2n-1}^0. There is infinitely many ways to do that, for example by taking $g_k^{(2)} = 0$, $\tilde{g}^{(2)} = g^{(2)}$, but we will additionally require that the zeta-functions from (3.9) will become the sums of the form $\sum_k k^p$ in (4.10). This leads to

$$\tilde{g}_k^{(2)} = \sum_{j=0}^{n-1} \frac{1}{4} \xi_n^j \left((2n-1)y_1(j) - 2j - 1\right) \tilde{\Delta}^{n-1-j} \tilde{c}^j k^{2j+1}, \tag{4.11}$$

and very simple

$$\tilde{g}^{(2)} = -\frac{n(n-1)(2n-1)\tilde{\Delta}^{n-1}}{96\tilde{c}}. \tag{4.12}$$

This term is necessary to subtract n_k-independent $\tilde{\Delta}^{n-1}\tilde{c}^{-1}$ term coming from $\sum_k \tilde{g}_k^{(2)} \tilde{n}_k$ to match Q_{2n-1}^0 (3.7) which has no terms with the negative powers of c.

For convenience we give the full expression (4.10) explicitly

$$\begin{aligned}
\mathsf{Q}_{2n-1} = \tilde{\Delta}^n &+ \sum_k \sum_{j=0}^{n-1} \frac{(2n-1)\sqrt{\pi}\,\Gamma(n+1)}{2\Gamma(j+\frac{3}{2})\Gamma(n-j)} \tilde{\Delta}^{n-1-j} \tilde{c}^j k^{2j+1} \tilde{n}_k \\
&- \frac{n(n-1)(2n-1)\tilde{\Delta}^{n-1}}{96\tilde{c}} \\
&+ \sum_k \sum_{j=0}^{n-1} \frac{(2n-1)\sqrt{\pi}\,\Gamma(n+1)}{8\Gamma(j+\frac{3}{2})\Gamma(n-j)} \left((2n-1)y_1(j) - 2j - 1\right) \tilde{\Delta}^{n-1-j} \tilde{c}^{j-1} k^{2j+1} \tilde{n}_k \\
&- \sum_k \sum_{j=0}^{n-1} \frac{(2n-1)\sqrt{\pi}\,\Gamma(n+1)(2nj+2n-3j-2)}{16\,\Gamma\left(j+\frac{3}{2}\right)\Gamma(n-j)} \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1} k^{2j} \tilde{n}_k^2 \\
&+ \frac{1}{2} \sum_{k,\ell} \sum_{j=1}^{n-1} \frac{(2n-1)^2\sqrt{\pi}\,\Gamma(n+1)}{4\Gamma\left(j+\frac{3}{2}\right)\Gamma(n-j)} \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1} \sum_{s=0}^{j-1} k^{2(j-s)-1} \ell^{2s+1} \tilde{n}_k \tilde{n}_\ell + \mathcal{O}(c^{n-3}).
\end{aligned} \tag{4.13}$$

We conjecture this is the full quantum spectrum of Q_{2n-1} up to $\tilde{c}^{n-2}$ order and verify that it passes all available checks.

From here it is now straightforward to find Q_{2n-1} in the representation (4.6). Coefficient

$$\begin{aligned}
g_k^{(2)} &= \sum_{j=0}^{n-1} \frac{(2n-1)\sqrt{\pi}\,\Gamma(n+1)}{8\Gamma\left(j+\frac{3}{2}\right)\Gamma(n-j)} v(n,j,k) \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1}, \\
v(n,j,k) &= (2n-1) \sum_{s=0}^{j-1} \zeta(2(s-j)+1) k^{2s+1} + \left((2n-1)y_1(j) - 2j - 1\right) k^{2j+1} \\
&\quad - \frac{1}{2}(2nj + 2n - 3j - 2) k^{2j}
\end{aligned} \tag{4.14}$$

is significantly more bulky than (4.11), while the full expression is

$$\mathsf{Q}_{2n-1} = \mathsf{Q}^0_{2n-1} + \sum_k \sum_{j=0}^{n-1} \xi_n^j \tilde{\Delta}^{n-j-1} \tilde{c}^j k^{2j+1} n_k \tag{4.15}$$
$$+ \sum_{k,\ell} \sum_{j=1}^{n-1} \xi_n^j \frac{(2n-1)}{4} \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1} \sum_{s=0}^{j-1} k^{2(j-s)-1} \ell^{2s+1} n_k n_\ell$$
$$- \sum_k \sum_{j=0}^{n-1} \xi_n^j \frac{(2nj+2n-3j-2)}{8} \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1} k^{2j} n_k^2$$
$$+ \sum_k \sum_{j=0}^{n-1} \xi_n^j \frac{v(n,j,k)}{4} \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1} n_k + \mathcal{O}(c^{n-3}).$$

To summarize, we have found the (conjectured) spectrum of all qKdV charges at first two sub-leading orders in $1/c$ expansion (4.13), (4.15) and observed certain patterns which may help fix the spectrum at higher orders. Let us spell the step to find the next $1/\tilde{c}^3$ order, i.e. fix the terms of order $\tilde{c}^{n-3}$ in (4.6). The classical result for Q_{2n-1} in terms of action variables I_k was calculated up to cubic order in (2.36), (2.39), (2.40). "Energies" of primary states Q^0_{2n-1} were also calculated to this order, see eq. (3.10). Thus $g^{(3)}_{k_1k_2k_3}$ and $g^{(3)}$ are known, and to find the spectrum one would only need to fix $g^{(3)}_{k_1k_2}$ and $g^{(3)}_k$. To do that one would need to find $\tilde{g}^{(3)}_{k_1k_2}$ and $\tilde{g}^{(3)}_k$ from the expansion (4.10) to reproduce (3.10) via zeta-function regularization and minimal possible $\tilde{g}^{(3)}$, which presumably will only include terms with negative powers of $\tilde{c}$. "Restoring" $\tilde{g}^{(3)}_{k_1k_2}$ and $\tilde{g}^{(3)}_k$ from $\tilde{R}^{(3)}_{n,j}$ is not a mathematically well-posed problem. We expect that all zeta-functions $\zeta(-2j-1)$ in $\tilde{R}^{(3)}_{n,j}$ to lead to the sums $\sum_k k^{2j+1}\tilde{n}_k$ — the rule which successfully worked at second $1/\tilde{c}$ order. At third order this rule should be amended by others, as suggested by a non-polynomial dependence on k in (2.39). In practice, restoring $\tilde{g}^{(3)}_k$ from $\tilde{R}^{(3)}_{n,j}$ may require establishing the analytic form of coefficients p_j in (3.10) and then reverse-engineering corresponding k_1, k_2, k_3-dependent sums. Once hypothetical $\tilde{g}^{(3)}_{k_1k_2}$ and $\tilde{g}^{(3)}_k$, and accordingly $g^{(3)}_{k_1k_2}$ and $g^{(3)}_k$ are fixed, a non-trivial set of checks is provided by the spectrum of Q_3, Q_5, Q_7 generated by computer algebra, as well as the requirement that thermal expectation values $\langle Q_{2n-1}\rangle_q$ discussed in section 5.1 must have certain modular properties.

4.1 Computer algebra check

For $n=1$ the expansion (4.15) reduces to (1.7) which is a simple check. A more sophisticated check is provided by Q_3, Q_5 and Q_7 which are known explicitly in terms of the Virasoro algebra generators [4]

$$Q_3 = \left(L_0^2 - \frac{c+2}{12} L_0 + \frac{c(5c+22)}{2990}\right) + \tilde{Q}_3, \tag{4.16}$$
$$\tilde{Q}_3 = 2\sum_{k=1}^{\infty} L_{-k} L_k,$$
$$Q_5 = \left(L_0^3 - \frac{c+4}{8} L_0^2 + \frac{(c+2)(3c+20)}{576} L_0 - \frac{c(3c+14)(7c+68)}{290304}\right) + \tilde{Q}_5, \tag{4.17}$$

$$
\begin{aligned}
\tilde{Q}_5 = & \sum_{k,l=0}^{\infty} L_{-k-l}L_kL_l + 2\sum_{k=1,l=0}^{\infty} L_{-k}L_{k-l}L_l + \sum_{k,l=1}^{\infty} L_{-k}L_{-l}L_{k+l} + \\
& + \sum_{n=1}^{\infty}\left(\frac{c+2}{6}n^2 - \frac{c}{4} - 1\right) L_{-n}L_n - L_0^3,
\end{aligned} \tag{4.18}
$$

and [29]

$$
\begin{aligned}
Q_7 = & \sum_{k,l,m=1}^{\infty} L_{-k}L_{-l}L_{-m}L_{k+l+m} + \sum_{k,l,m=0}^{\infty} L_{-k-l-m}L_kL_lL_m \\
& + 3\sum_{\substack{k,l=1\\ m=0}}^{\infty} L_{-k}L_{-l}L_{k+l-m}L_m + 3\sum_{\substack{k=1\\ l,m=0}}^{\infty} L_{-k}L_{k-l-m}L_lL_m \\
& + \frac{8+c}{3}\left[\sum_{k,l=1}^{\infty}(k+l)lL_{-k}L_{-l}L_{k+l} + \sum_{\substack{k=1\\ l=0}}^{\infty}(k-l)kL_{-k}L_{k-l}L_l\right] \\
& + \frac{8+c}{3}\left[\sum_{k,l=0}^{\infty}(k+l)kL_{-k-l}L_kL_l + \sum_{\substack{k=0\\ l=1}}^{\infty}(k-l)kL_{-l}L_{l-k}L_k\right] \\
& + \sum_{n=1}^{\infty}\left(\frac{c^2-c-141}{90}n^4 - \frac{7c+59}{18}n^2\right)L_{-n}L_n - \left(\frac{1}{48}c^2 + \frac{53}{360}c + \frac{19}{90}\right)\tilde{Q}_3 \\
& - \left(\frac{1}{6}c+1\right)\tilde{Q}_5 - \frac{c+6}{6}L_0^3 + \frac{15c^2+194c+568}{1440}L_0^2 \\
& - \frac{(c+2)(c+10)(3c+28)}{10368}L_0 + \frac{c(3c+46)(25c^2+426c+1400)}{24883200}.
\end{aligned} \tag{4.19}
$$

Using computer algebra spectrum of Q_3, Q_5, Q_7 for all descendants at a small levels m can be evaluated explicitly, as an expansion in powers of $1/c$. The resulting expressions can be compared with the spectrum following from (4.15), which we will write in terms of quantum numbers n_k packaged as follows

$$
m_{p,r} \equiv \sum_k k^p n_k^r, \qquad m_p \equiv m_{p,1}, \qquad m \equiv m_1, \qquad h = \tilde{\Delta}/\tilde{c}, \tag{4.20}
$$

$$
\begin{aligned}
\mathsf{Q}_3 = & \tilde{\Delta}^2 + \tilde{\Delta}\left(6\,m_1 - \frac{1}{4}\right) + \tilde{c}\left(4\,m_3 + \frac{1}{60}\right) \\
& + \left(m_3 - \frac{3}{2}m_2 - \frac{1}{4}m_1\right) - \frac{3}{2}m_{2,2} + 3m_1^2 + \frac{3}{2}h(2m_1 - m_0 - m_{0,2}) + \frac{3}{320} + \mathcal{O}(1/\tilde{c}),
\end{aligned} \tag{4.21}
$$

and similarly

$$
\begin{aligned}
\mathsf{Q}_5 = & \tilde{\Delta}^3 + \left(15m_1 - \frac{5}{8}\right)\tilde{\Delta}^2 + \tilde{\Delta}\,\tilde{c}\left(20m_3 + \frac{1}{12}\right) + \tilde{c}^2\left(8m_5 - \frac{1}{63}\right) \\
& + \tilde{\Delta}\left(\frac{5}{12}(-5m_1 - 42m_2 + 44m_3) - \frac{35}{2}m_{2,2} + 25m_1^2 + \frac{15}{2}h(2m_1 - m_0 - m_{0,2}) + \frac{23}{192}\right) \\
& + \tilde{c}\left(\frac{1}{12}(m_1 - 10m_3 - 120m_4 + 64m_5) - 10m_{4,2} + 20m_1m_3 - \frac{85}{6048}\right) + \mathcal{O}(\tilde{c}^0),
\end{aligned} \tag{4.22}
$$

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and

$$
\begin{aligned}
\mathsf{Q}_7 =&\, \tilde\Delta^4+\tilde\Delta^3\left(28m_1-\frac{7}{6}\right)+\tilde\Delta^2\,\tilde c\left(56m_3+\frac{7}{30}\right)+\tilde\Delta\,\tilde c^2\left(\frac{224}{5}m_5-\frac{4}{45}\right)+\tilde c^3\left(\frac{64}{5}m_7+\frac{2}{75}\right)\\
&+\tilde\Delta^2\left(\frac{7}{6}(-7m_1-66m_2+76m_3)-77m_{2,2}+98m_1^2+21h(2m_1-m_0-m_{0,2})+\frac{259}{480}\right)\\
&+\tilde\Delta\,\tilde c\left(\frac{7}{75}(7m_1-70m_3-960m_4+688m_5)-\frac{448}{5}m_{4,2}+\frac{784}{5}m_1m_3-\frac{167}{1080}\right)\\
&+\tilde c^2\left(\frac{2}{225}(-10m_1+21m_3-210m_5-3780m_6+1704m_7)\right.\\
&\qquad\left.-\frac{168}{5}m_{6,2}+\frac{112}{5}(2m_1m_5+m_3^2)+\frac{77}{2160}\right)\\
&+\mathcal{O}(\tilde c^1).
\end{aligned}
\tag{4.23}
$$

We checked, these expressions are in agreement with the computer algebra generated spectrum for $m \le 12$, which serves as a non-trivial consistency check of (4.15).

5 Miscellaneous results

Explicit expression for the spectrum of quantum Q_{2n-1} in large c limit opens the opportunity to make progress in a number of adjacent directions. In this section we discuss several applications of our results.

5.1 Thermal expectation values of Q_{2n-1}

Our first application is toward thermal exaction value of Q_{2n-1}, i.e. averaged over the CFT Gibbs ensemble $\langle Q_{2n-1}\rangle_q \equiv \mathrm{Tr}(q^{L_0-c/24}Q_{2n-1})$. This question appears naturally, though in a more complicated form, to calculate the averaged value of Q_{2n-1} over the KdV Generalized Gibbs Ensemble (see section 5.2 below), if one wants to match the GGE chemical potentials to describe equilibration endpoint of some initial state. The expectation value $\langle Q_{2n-1}\rangle_q$, which is essentially the one-point function of T_{2n} (1.1) on the torus, exhibits modular properties and can be represented as a covariant differential operator acting on the CFT torus partition function [30]. In fact, one can average Q_{2n-1} over a particular Verma module, $\langle Q_{2n-1}\rangle_\Delta \equiv \mathrm{Tr}_\Delta(q^{L_0-c/24}Q_{2n-1})$, where sum goes over all Virasoro descendants of the primary state $|\Delta\rangle$. This sum too is a modular object and can be evaluated with help of the same differential operator

$$
\langle Q_{2n-1}\rangle_\Delta = \mathcal{D}_n\chi_\Delta, \qquad \chi_\Delta \equiv \mathrm{Tr}_\Delta(q^{L_0-c/24}) = q^{\tilde\Delta-\frac{1}{24}}/\eta, \tag{5.1}
$$

$$
\mathcal{D}_n = D^n + \sum_{j=1}^{n-1} P_n^j(c,q)D^{n-j-1}, \qquad D^n = D_{2(n-1)}\dots D_2D_0, \tag{5.2}
$$

and $D_r = q\partial_q - \frac{r}{12}E_2$ is Serre derivative. Each P_n^j is a degree j polynomial in c with each coefficient being a modular form of weight $2j+2$,

$$
P_n^j(c,q) = \sum_{k=1}^{j+1} P_{n,j}^{(k)}\tilde c^{j-k+1}E_{2j+2}^{(n,k)}(q). \tag{5.3}
$$

Here $P_{n,j}^{(k)}$ are numerical coefficients and $E_{2j+2}^{(n,k)}$ is some modular form, which is a linear combination of $E_4^a E_6^b$ with $4a+6b=2j+2$ for non-negative integer a,b, normalized such that $E_{2j+2}^{(n,j)}=1+O(q)$. For $j=1,2,3,4,6$ there is a unique modular form of the weight $2(j+1)$ and therefore for these j, independently of n and k, $E_{2j+2}^{(n,k)}=E_{2j+2}$ where

$$E_{2n}=1+\frac{2}{\zeta(1-2n)}\sigma_{2n-1}, \qquad \sigma_p=\sum_{k=1}^{\infty}\frac{k^p q^k}{1-q^k}. \tag{5.4}$$

For instance, in the simplest case of Q_3 the operator $\mathcal{D}_2$ is given by

$$\langle Q_3\rangle_\Delta=\mathcal{D}_2\chi_\Delta=\left[D^2+\frac{c}{1440}E_4\right]\chi_\Delta. \tag{5.5}$$

In this case $P_{2,1}^{(1)}=1/60$ and $P_{2,1}^{(2)}=1/1440$. Explicit expressions for $\mathcal{D}_n$ for $n\le 7$ were found in [30]. For higher n the modular form $E_{2i+2}^{(n,j)}$ and coefficients $P_{n,j}^{(k)}$ are not known.

Strictly speaking (5.1), (5.2) is an unproven ansatz proposed in [30]. We find it to be consistent with the large c spectrum of Q_{2n-1} (4.15) and fix two leading in c terms in P_n^j. To compare with (5.1), we need to calculate $\langle Q_{2n-1}\rangle_\Delta$ starting from (4.15). Here the following straightforward identities will be helpful

$$\langle\sum_{k=1}^{\infty}n_k k^p\rangle_\Delta=\sigma_p\chi_\Delta, \qquad \langle\sum_{k=1}^{\infty}n_k^2 k^p\rangle_\Delta=\left(2q\partial_q\sigma_{p-1}-\sigma_p\right)\chi_\Delta, \tag{5.6}$$

$$\langle\sum_{k=1}^{\infty}n_k k^p\sum_{\ell=1}^{\infty}n_\ell \ell^{p'}\rangle_\Delta=\left(q\partial_q\sigma_{p+p'-1}+\sigma_p\sigma_{p'}\right)\chi_\Delta, \tag{5.7}$$

where by n_k we mean the quantum numbers (1.4). Then (1.6) immediately yields

$$\langle Q_{2n-1}\rangle_\Delta=\tilde{\Delta}^n\chi_\Delta+\sum_{j=0}^{n-1}\tilde{\Delta}^{n-p-1}\tilde{c}^p\xi_n^p\left(\sigma_{2p+1}+\frac{\zeta(-2p-1)}{2}\right)\chi_\Delta+O(\tilde{c}^{n-2}), \tag{5.8}$$

where we assumed the usual limit, $h=\tilde{\Delta}/\tilde{c}$ is kept fixed while $\tilde{c}\to\infty$. Comparing this with (5.1), we immediately see that the leading $\tilde{\Delta}^n$ term is coming from (we drop χ_Δ for simplicity)

$$D^n\to(q\partial_q)^n\to\tilde{\Delta}^n. \tag{5.9}$$

Similarly we can trace origin of all $\tilde{c}^{n-1}$ terms,

$$D^n\to(q\partial_q)^n-\frac{n(n-1)}{12}E_2(q\partial_q)^{n-1}\to\tilde{\Delta}^{n-1}n\left(\sigma_1-\frac{1}{24}\right)-\tilde{\Delta}^{n-1}\frac{n(n-1)}{12}E_2=-\frac{n(2n-1)}{24}E_2,$$

which agrees with (5.8), and

$$P_{n,j}^{(1)}\tilde{c}^j E_{2j+2}^{(n,1)}D^{n-j-1}\to P_{n,j}^{(1)}\tilde{c}^j E_{2j+2}^{(n,1)}(q\partial_q)^{n-j-1}\to P_{n,j}^{(1)}\tilde{\Delta}^{n-j-1}\tilde{c}^j E_{2j+2}^{(n,1)}, \tag{5.10}$$

for $n-1\ge j>0$. From here immediately follows

$$P_{n,j}^{(1)}=\tilde{R}_{n,j}^{(1)}, \qquad E_{2j+2}^{(n,1)}=E_{2j+2}, \qquad n-1\ge j\ge 1. \tag{5.11}$$

To fix $P^{(2)}_{n,j}$ it is convenient to take $q \to 0$ limit and compare $\langle Q_{2n-1}\rangle_\Delta$ with (3.7), yielding

$$P^{(2)}_{n,1} = \tilde{R}^{(2)}_{n,0} - \frac{n(n-1)(12n^2-16n-1)}{3456} = \frac{n(n-1)(12n^2-38n+31)}{8640},$$
$$P^{(2)}_{n,j} = \tilde{R}^{(2)}_{n,j-1} + \frac{(n-j)(2(n-j)-1)}{24}P^{(1)}_{n,j-1}, \quad n-1 \geq j \geq 2.$$

Evaluation of $E^{(n,2)}_{2j+2}$ is a more challanging task and requires first using (5.6), (5.7) and then combining pieces into modular forms to match (5.1), (5.2). We note, there are terms in (4.15) proportional to $\tilde{\Delta}^{n-1}\tilde{c}^{-1}$, but (5.1) has no negative powers of c. Hence these terms must vanish after averaging, which follows from the identity $q\partial_q\sigma_{-1} - \sigma_1 = 0$ and serves as a consistency check. The final expression reads

$$\begin{aligned} P^{(2)}_{n,j}E^{(n,2)}_{2j+2} = &\frac{(2n-1)\sqrt{\pi}\Gamma(n+1)}{8\Gamma(j+3/2)\Gamma(n-j)} \\ &\times\Bigg(((2n-1)y_1(j)-2j-1)\frac{\zeta(-2j-1)}{2}E_{2j+2}-(n-1-j)\zeta(-2j+1)D_{2j}E_{2j} \\ &\quad + \frac{(2n-1)}{4}\sum_{s=1}^{j-2}\zeta(-2s-1)\zeta(-2(j-s)+1)E_{2s+2}E_{2(j-s)}\Bigg). \end{aligned} \tag{5.12}$$

It is valid for $n-1 \geq j \geq 2$. For $j=1$, there is a unique modular form $E^{(n,2)}_{2j+2} = E_4$. Also, as was mentioned above $E^{(n,2)}_{2j+2} = E_{2j+2}$ for $j = 2,3,4,6$, which can be checked straightforwardly. Because of the identities between modular forms there are other ways to write (5.12).

Explicit form of Q^0_{2n-1} up to $\tilde{c}^{n-3}$ order allows us, in principle, to calculate $P^{(3)}_{n,j}$, although calculation of $E^{(n,3)}_{2j+2}$ would require first extending (4.15) to the next $1/c$ order. Given involved form of $P^{(2)}_{n,j}$ and $E^{(2)}_{n,j}$ we do not expect the answer to be simple.

5.2 Generalized Gibbs Ensemble

Spectrum of Q_{2n-1} can help understand the qKdV generalized Gibbs ensemble (GGE)

$$\rho_{\rm GGE} = e^{-\sum_n \mu_{2n-1}Q_{2n-1}}, \qquad Z_{\rm GGE} = {\rm Tr}\,\rho_{\rm GGE}, \tag{5.13}$$

and corresponding (generalized) partition function and free energy. Earlier attempts to evaluate KdV generalized free energy include [15, 16, 22, 31]. The GGE describes local equilibrium in a state carrying specific values of qKdV charges. It is expected on general grounds that most initial states, upon equilibration, can be locally described by the GEE with the appropriate values of chemical potentials μ_{2n-1} [32]. From the mathematical point of view, it is of great interest to investigate modular properties of $Z_{\rm GGE}$, generalizing modular invariance of the conventional torus partition function $\mu_{2n-1} = 0$, for $n > 1$.

The explicit spectrum of Q_{2n-1} in the large c limit allows in principle to calculate the generalized sum over a particular Verma module

$${\rm Tr}_\Delta\, e^{-\sum_n \mu_{2n-1}Q_{2n-1}} \tag{5.14}$$

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in the "holographic limit": $h = \tilde{\Delta}/\tilde{c}$, $t_n := \mu_{2n-1}/\tilde{c}^{n-1}$ fixed, $\tilde{c} \to \infty$, by expanding the answer in powers of $1/c$. In practice sums of exponents of quadratic or higher order expressions in n_k

$$\sum_{n_k} e^{O(n^2)} \tag{5.15}$$

can not be evaluated, and we restrict our analysis to first non-trivial $1/c$ order,

$$\mathrm{Tr}_\Delta\, e^{-\sum_n \mu_{2n-1} Q_{2n-1}} = e^{-\tilde{c}\sum_n t_n h^n} \frac{e^{-\sum_n t_n \sum_{p=0}^{n-1} h^{n-1-p} \xi_n^p \zeta(-2p-1)/2}}{\prod_{k=1}^\infty \left(1 - e^{-k\sum_n (2n-1) n\, t_n\, h^{n-1}{}_2F_1(1,1-n,3/2,-k^2/h)}\right)}.$$

From here generalized partition function can be evaluated using Cardy formula (we are only writing explicitly the chiral part),

$$Z_{\mathrm{GGE}} = e^{\tilde{c} f_0 + f_1 + O(1/\tilde{c})}, \tag{5.16}$$

$$f_0 = \sum_{n=1}^\infty (2n-1)\, t_n\, h^n, \tag{5.17}$$

$$h^{1/2} = \frac{1}{2\pi}\sum_{n=1}^\infty t_n\, n\, h^n, \tag{5.18}$$

$$f_1 = -\sum_{k=1}^\infty \ln\left(1 - e^{-\gamma}\right) - \sum_{n=2}^\infty t_n\, h^{n-1}\left(\sum_{p=0}^{n-1} \xi_n^p\, h^{-p}\frac{\zeta(-2p-1)}{2} - \frac{n}{24}\right), \tag{5.19}$$

$$\gamma(k) = k\sum_{n=1}^\infty (2n-1) n\, t_n\, h^{n-1}{}_2F_1(1, 1-n, 3/2, -k^2/h). \tag{5.20}$$

Here Z_{GGE} is understood to be a function of $t_n \equiv \mu_{2n-1}\tilde{c}^{1-n}$, while h is a function of t_n satisfying (5.18). For (5.16)–(5.20) to be valid, resulting $\tilde{\Delta} = \tilde{c}\, h$ should be in the regime of validity of Cardy formula. There are at least two limits when this assumption is controllable. First, (5.16) is valid for any large c theory in the thermodynamic limit. We introduce the spatial circle radius L (we kept $L = 1$ in the paper so far) and inverse temperature β, $\mu_1 = t_1 = \beta/L$. By taking $L \to \infty$, while all other chemical potentials scale as $\mu_{2n-1} \propto t_n \sim L^{1-2n}$ to ensure that values of all $Q_{2n-1} \propto L$ are extensive, we find the saddle point value $h \sim L^2$ and $f_0, f_1 \sim$ Ł. (The scaling of f_1 follows by substituting the sum over k in (5.19) by an integral over $\kappa = k^2/h$.) In this limit second term in (5.19), the sum over n, is sub-extensive and can be neglected. We therefore arrive at the leading (extensive) contribution to f_0 and f_1 found in [22].

Second case when (5.16)–(5.20) can be trusted is in holographic theories, i.e. large c theories satisfying HKS sparseness condition [3]. From the holographic point of view f_0 is the free energy of BTZ black hole in the Euclidean classical theory of gravity with the deformed boundary conditions such that the dual CFT Hamiltonian is $H = \sum_n \mu_{2n-1} Q_{2n-1}$ [17, 24, 33]. The leading correction f_1 can be interpreted as the one-loop contribution coming from the boundary gravitons. Different solutions of (5.18) means Euclidean path integral could have numerous BTZ saddles and the condition $h(t_n) > 1/12$ necessary for the validity of Cardy formula would come automatically as the requirement of smoothness of bulk geometry.

It is possible to fine-tune chemical potentials t_n such that $\gamma(k)$ (5.20) for some k will vanish. That will render f_1 divergent, indicating higher order $1/c$ corrects are necessary to make free energy finite. Schematically, the spectrum Q_{2n-1} is an expansion in $n_k/\tilde{c}$. For the higher order corrections to contribute at the leading order, the quantum numbers n_k should be of order $\tilde{c}$. In terms of the classical problem of section 2, action variables I_k should be of order one rather than infinitesimal. In other words, leading contribution would come from a non-trivial saddle when classical $u(\varphi)$ is not a constant but some solitonic solution. Such saddles, describing black holes, which are geometrically different from the BTZ configurations, were constructed in [24] and it was shown that for certain parameters μ_{2n-1} they give leading contribution to generalized free energy. We dubbed these configurations "KdV-charged" black holes to emphasize that higher KdV charges Q_{2n-1}, even at leading order in c, are different from Q_1^n, unlike for BTZ configurations for which $u(\varphi) = u_0$ is a constant and $Q_{2n-1} \sim u_0^n$.

Theoretical control over generalized free energy in the large c limit can be used to probe modular properties of $Z_{\rm GGE}$. The currents T_{2n} (1.1) have no anomalous dimension and therefore naively $Z_{\rm GGE}$ should be invariant under modular transformation $t_1 \to t_1' = (2\pi)^2/t_1$ accompanied by

$$t_n \to (-1)^n \left(\frac{2\pi}{t_1}\right)^{2n} t_n, \quad n > 1. \tag{5.21}$$

This only holds to linear order in $t_n, n > 1$, i.e. at the level of thermal expectation values $\langle Q_{2n-1}\rangle_q$ discussed in section 5.1. At higher orders invariance is broken due to colliding T_{2n} [30]. To restore invariance of $Z_{\rm GGE}$, while working in the $c \to \infty$ limit one may require f_0 given by (5.17), (5.18) to be invariant under the hypothetical transformation $t_n \to t_n'(t_n, t_1)$, $n > 1$. More accurately, in addition to BTZ black holes described by (5.17), (5.18) we should include vacuum (thermal AdS$_3$) and KdV-charged black holes to the list of possible saddles. Given a non-trivial diagram of the Hawking-Page phase transitions, to match leading saddles, the hypothetical transformation $t_n \to t_n'(t_n, t_1)$ should be very complicated, with numerous branches of continuity. This may indicate that in the presence of higher KdV charges modular invariance of $Z_{\rm GGE}$ is not mathematically natural. Similar conclusion is recently reached in [34], which evaluated $Z_{\rm GGE}$ explicitly in the case of $c = 1/2$ free fermion model. They found that to reproduce $Z_{\rm GGE}$ in the dual channel, one needs to sum over not one but three fermion Hilbert spaces, schematically $Z_{\rm GGE}(t_1, t_3) \propto Z_1(t_1')Z_2(t_1')Z_3(t_1')$, a mathematical observation (conjecture), which so far has no physical interpretation. To summarize, failure to establish invariance of $Z_{\rm GGE}(t)$ under modular transformation supplemented by an appropriate map $t_n \to t_n'$ in both infinite c limit and for $c = 1/2$ model may suggest that it is not mathematically natural and instead covariance of $Z_{\rm GGE}(t)$ under (5.21) should be investigated.

5.3 Transfer matrix

In the classical case, as follows from (2.22), charges Q_{2n-1} encode asymptotic expansion of the quasi-momentum $p(\lambda)$. The quasi-momentum controls the eigenvalues $e^{\pm 2\pi i p(\lambda)}$ of the monodromy matrix of the differential equation (2.2). Instead of $p(\lambda)$ one can consider the trace of monodromy matrix

$$T(\lambda) = 2\cos(2\pi p(\lambda)). \tag{5.22}$$

In case of the constant potential $u(\varphi) = h$ this becomes $T(\lambda) = 2\cos(2\pi\sqrt{\lambda - h/4})$.

In quantum case $T(\lambda)$ becomes the transfer matrix, which is related to qKdV charges via an asymptotic expansion [4]

$$\ln T = \kappa\, \mu^{1/2}\, 1 - \sum_n C_n\, \mu^{1/2-n}\, Q_{2n-1}, \quad \mu \to \infty, \tag{5.23}$$

where

$$\kappa = \frac{2\sqrt{\pi}\Gamma\left(\frac{1}{2} - \frac{\xi}{2}\right)}{\Gamma\left(1 - \frac{\xi}{2}\right)} \left(\Gamma\left(1 - \beta^2\right)\right)^{1+\xi}, \tag{5.24}$$

$$C_n = \frac{\sqrt{\pi}(1+\xi)\beta^{2n}\Gamma\left((n-\frac{1}{2})(1+\xi)\right)}{\Gamma(n+1)\Gamma\left(1+(n-\frac{1}{2})\xi\right)} \left(\Gamma\left(1-\beta^2\right)\right)^{-(2n-1)(1+\xi)}, \tag{5.25}$$

$$\beta \equiv \sqrt{\frac{1-c}{24}} - \sqrt{\frac{25-c}{24}}, \qquad \xi \equiv \frac{\beta^2}{1-\beta^2}. \tag{5.26}$$

Variable μ will become spectral parameter $-\lambda$ in the classical limit. The original paper [4] introduces another variable λ, defined as $\mu \equiv \lambda^{2(1+\xi)}$. We use this definition in the reminder of this section.

We are interested in the limit $c \to \infty$, or $\beta \to 0$. Following [4] we introduce $p^2 = \beta^2\tilde{\Delta}$ which remains finite in this limit finite, $p^2 \to -\tilde{\Delta}/4\tilde{c} = -h/4$. (This is, obviously, a different quantity from the quasi-momentum $p(\lambda)$ mentioned above.) We would like to find T by summing the asymptotic expansion (5.23) while expanding it in powers of β^2 which corresponds to $1/c$ expansion. In principle we can use the spectrum (4.15) to calculate $\ln T$ acting on an excited state, but resort to a simpler calculation for $\ln T$ acting on a primary state. In this case Q_{2n-1} in (5.23) should be substituted by Q^0_{2n-1}, which we expand in powers of $\beta^2 \propto 1/c$ (3.7). The calculation is tedious and we only give the final expression

$$(\ln T)_{\rm asympt}|\Delta\rangle = 2\pi i\sqrt{p^2 - \lambda^2}\,\Phi(\lambda, p)|\Delta\rangle, \tag{5.27}$$

$$\begin{aligned}\Phi(\lambda,p) = 1 + \beta^2\Psi &- \frac{\beta^4}{48\lambda^2(p^2-\lambda^2)^3}\left[3\lambda^4p^2 + 2\pi^2\lambda^4(p^2-\lambda^2)(4p^2-3\lambda^2)\right] \\ &+ \frac{\beta^4}{2\lambda^2}\left[(2p^2+\lambda^2)\Psi^2 - 2(p^2-\lambda^2)\lambda\Psi\frac{d\Psi}{d\lambda}\right] + \mathcal{O}(\beta^6),\end{aligned} \tag{5.28}$$

$$\Psi(\lambda,p) = \frac{\lambda^2}{\lambda^2 - p^2}\left[\gamma + \frac{1}{2}\psi\left(2\sqrt{p^2-\lambda^2}\right) + \frac{1}{2}\psi\left(-2\sqrt{p^2-\lambda^2}\right)\right]. \tag{5.29}$$

Here ψ is the polygamma function.

Given analytic form of (5.27) it is tempting to extend its validity from the asymptotic regime $\lambda \to \infty$ to the vicinity of $\lambda = 0$. This is clearly wrong as even in the strict classical limit $\beta \to 0$ we do not recover correct classical expression for the trace of monodromy matrix simply from $e^{(\ln T)_{\rm asympt}}$. Yet in the limit $\beta \to 0$ the correct answer is reproduced by the following simple conjectural expression (we implicitly assume this is an eigenvalue of T acting on $|\Delta\rangle$),

$$T_{\text{guess}}(\lambda,\beta,p) = e^{(\ln T)_{\text{asympt}}} + e^{-(\ln T)_{\text{asympt}}}, \tag{5.30}$$

and we would like to check if it could be valid beyond the strict $\beta = 0$ limit. To that end we expand (5.30) in powers of β (amended by an expansion in λ), to find

$$\begin{aligned}
T_{\text{guess}} = {} & 2\cos\left(2\pi\sqrt{p^2-\lambda^2}\right) + \beta^2\left[\frac{2\pi\sin(2\pi p)}{p}\left(2\gamma+\psi(2p)+\psi(-2p)\right)\lambda^2\right.\\
&+\left(-\frac{2\pi^2\cos(2\pi p)}{p^2}\left[2\gamma+\psi(2p)+\psi(-2p)\right)\right.\\
&\left.\left.+\frac{\pi\sin(2\pi p)}{p^3}\left(2\gamma+\psi(2p)+\psi(-2p)-2p\psi^{(1)}(2p)+2p\psi^{(1)}(-2p)\right)\right)\lambda^4+O(\lambda^6)\right]\\
&+\beta^4\left[\frac{\pi\sin(2\pi p)}{12p^3}\left[3+8\pi^2p^2+12p^2\left(2\gamma+\psi(2p)+\psi(-2p)\right)^2\right]\lambda^2+O(\lambda^4)\right]+O(\beta^6).
\end{aligned}$$

Small λ expansion of the actual T is given in [4] in the explicit form in terms of the integrals of free field correlators. A comparison with T_{guess} reveals that, besides the classical β^0 term, which matches the classical expression (5.22) for a constant potential $u = h$, only $\beta^2\lambda^2$ term coincides with, while $\beta^2\lambda^4$ and $\beta^4\lambda^2$ terms do not match the correct result. We thus conclude that the conjectural expression (5.30) is missing non-perturbative terms, which are not captured by the asymptotic expansion (5.23).

6 Discussion

In this paper we obtained spectrum of quantum KdV charges Q_{2n-1} in first two non-trivial orders in $1/c$ expansion. Our result (4.13) and (4.15) is valid in the semiclassical limit of large central charge $c \to \infty$ with the ratio of Δ/c kept fixed. This limit is inspired by holographic correspondence, when CFT is dual to weakly coupled gravity. Accordingly, dynamics of stress-energy sector becomes semiclassical, with the leading (classical) contribution governed by integrable dynamics on the co-adjoint orbit of the Virasoro algebra. Under semiclassical quantization classical action variables I_k are promoted to integer quantum numbers n_k, and the spectrum of Q_{2n-1} looks most elegant in terms of variables $\tilde{\Delta}$ and $\tilde{c}$ (4.2). At each order in $1/\tilde{c}$ the quantum answer is a polynomial in n_k. Classical calculation fixes the leading term with the highest power of n_k, while all other terms should be regarded as "quantum corrections." We have seen that semiclassical quantization, combined with the values of qKdV charges Q_{2n-1} acting on primary states, is sufficient to completely fix these quantum corrections and obtain the spectrum of excited states at least in first two orders in $1/c$. We conjecture this quantization scheme can be extended to higher orders in $1/c$. We laid the groundwork for the next order $1/c^3$ by calculating classical $Q_{2n-1}(h, I_k)$ as well as "energies" on primary states Q^0_{2n-1}, albeit in the latter case not all terms are known analytically. To complete the job one would need to find analytic expressions for Q^0_{2n-1} and develop a dictionary that maps each term to an infinite sum, yielding this term back via zeta-function regularization.

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It is tempting to interpret quantization of Q_{2n-1} holographically, as a semiclassical quantization of boundary gravitons in AdS_3. We develop this picture at first $1/c$ order in the appendix A, but holographic picture does not provide any immediate insight into "quantum corrections" appearing at higher orders in $1/c$.

The obtained spectrum has several immediate applications. First, in section 5.1 we calculated two leading terms in large c expansion of the "thermal expectation values" $\langle Q_{2n-1}\rangle_\Delta \equiv \mathrm{Tr}_\Delta(q^{L_0-c/24}Q_{2n-1})$, where sum goes over a particular Verma module, and compared them with the predictions of [30]. Covariance under modular transformation of $\langle Q_{2n-1}\rangle_\Delta$ in each order in $1/c$ serves as a non-trivial check of our main result (4.15). We also fixed two leading terms in the differential operator $\mathcal{D}_n$ yielding thermal expectation values via $\langle Q_{2n-1}\rangle_\Delta = \mathcal{D}_n \mathrm{Tr}_\Delta(q^{L_0-c/24})$, see (5.11) and (5.12). Second, in section 5.2 we calculated first $1/c$ correction to generalized free energy of the qKdV Generalized Gibbs Ensemble

$$Z_{\mathrm{GGE}} = \mathrm{Tr}\, e^{-\sum_n \mu_{2n-1}Q_{2n-1}}. \tag{6.1}$$

The latter describes local equilibrium of a 2d CFT in a state carrying specific values of qKdV charges. It is of great interest to further investigate mathematical properties of Z_{GGE}, in particular covariance under modular transformation. Third, in section 5.3 using asymptotic expansion we calculated quantum transfer matrix acting on a primary state in first two non-trivial orders in $1/c$ expansion. Unfortunately the obtained expression is lacking terms non-perturbative in spectral parameter, which can not be fixed from the knowledge of spectrum of Q_{2n-1} alone.

There are several potential applications of our results, which we hope to address in the future. The obtained spectrum of Q_{2n-1} will be helpful to study generalized Eigenstate Thermalization Hypothesis of 2d CFTs [17] at the subleading order in $1/c$. We also expect the semiclassical quantization approach developed in this paper could be helpful in the context of Intermediate Long Wave hierachry, which is closely related to qKdV problem. More generally, it would be interesting to bridge the gap between the semiclassical approach of this work with the Bethe anzatz approach of [19] by taking "holographic" limit $c \to \infty$ with fixed $h = \tilde{\Delta}/\tilde{c}$ of the appropriate Bethe anzatz equations.

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A Spectrum of linear perturbations from AdS$_3$

In the classical (infinite central charge) limit gravity in AdS$_3$ can be described in terms of two functions $u(t,\varphi)$ and $\bar{u}(t,\varphi)$ living at the boundary and satisfying EOM, $\dot{u} = \partial_\varphi u$ and $\dot{\bar{u}} = -\partial_\varphi \bar{u}$. That is in the conventional case, when the dual CFT's Hamiltonian is $H = Q_1 + \bar{Q}_1 = L_0 + \bar{L}_0 - c/12$. Should the Hamiltonian be chosen to be one of the higher qKdV charges, $H = Q_{2n-1} + \bar{Q}_{2n-1}$, functions $u, \bar{u}$ will be satisfying higher KdV equations

$$\dot{u} = \frac{c}{24}\{Q_{2n-1}, \delta u\} = (2n-1)\partial R_n, \tag{A.1}$$

and similarly for $\bar{u}$ [24, 33, 35]. In this case the spectrum of Q_{2n-1} is the spectum of small fluctuations of u above the constant backgroun $u = u_0$, that corresponds to unpertubed metric in AdS$_3$. In other words, to quantize Q_{2n-1} we consider linearized EOM for small fluctuations $u = u_0 + \delta u$, where $\delta u \propto e^{i\varepsilon t + ik\varphi}$ is a flat wave. We want to find energy ε of the flat wave which satisfies the equation of motion (A.1)

$$i\varepsilon_n \delta u = (2n-1)\delta\partial R_n. \tag{A.2}$$

For example in the case $n = 1$ we have $\varepsilon_1 = k$, in case $n = 2$ we have $\varepsilon_2 = 2u_0 n + \frac{4}{3}k^3$ and so on. In general we can get from (2.20)

$$\delta\partial R_{n+1} = \frac{n+1}{2n+1}\delta(\partial u + 2u\partial - 2\partial^3)R_n = i\frac{n+1}{2n+1}(kR_n(u_0) + 2(u_0 + k^2)\delta\partial R_n). \tag{A.3}$$

Hence we find the following iterative relation for ε_n

$$\varepsilon_{n+1} = (2n-1)\frac{n+1}{2n+1}\left[2(u_0 + k^2)\varepsilon_n + ku_0^n\right], \tag{A.4}$$

where we have used that $R_n(u_0) = u_0^n$. Each ε_n is a polynomial of the form

$$\varepsilon_n = \sum_{p=0}^{n-1} \xi_n^p\, k^{2p+1} u_0^{n-1-p}, \tag{A.5}$$

where ζ_n^p satisfy

$$\xi_{n+1}^p = (2n-1)\frac{2(n+1)}{2n+1}\left(\xi_n^p + \xi_n^{p-1}\right), \tag{A.6}$$

and we defined $\xi_n^{-1} \equiv 1/2$. The solution is easy to find, cf. (1.6),

$$\xi_n^p = \frac{(2n-1)\Gamma(n+1)\Gamma(1/2)}{2\Gamma(p+3/2)\Gamma(n-p)}. \tag{A.7}$$

To match the spectrum of individual bosons $\varepsilon_n(k,u_0)$ with the spectrum of quantum Q_{2n-1} we need to restore powers of $\tilde{c}$ and make the following identification

$$Q_{2n-1} = \tilde{\Delta}^n + \tilde{c}^{n-1}\sum_k \left(n_k + \frac{1}{2}\right)\varepsilon_n(k,u_0) + \dots \tag{A.8}$$

where n_k are boson occupation numbers of boundary gravitons and $u_0 = \tilde{\Delta}/\tilde{c}$. This reproduces the spectrum of Q_{2n-1} at two first leading orders in $1/c$ and provides physical interpretation of n_k. Unfortunately the holographic picture provides no clear path to compute higher $1/c$ corrections to (A.8).

B Brute-force pertubative calculation

A straightforward but a laborious approach to evaluate Q_{2n-1} in terms of action variables I_k would be to use Fourier modes u_k of u,

$$u(\varphi) = \sum_k u_k e^{ik\varphi}, \tag{B.1}$$

to parametrize the co-adjoint orbit of Virasoro algebra, i.e. the space of potentials u sharing the same orbit invariant h (2.17). To that end u_0 should be understood as a function of u_k [20]. Then Q_{2n-1} and I_k can be expressed in terms of u_k, and consequently in terms of each other.

In terms of the Fourier modes the Poisson bracket is

$$i\frac{c}{24}\{u_k, u_\ell\} = (k-\ell)u_{k+\ell} + 2k^3\delta_{k+\ell}. \tag{B.2}$$

This coincides with the Virasoro algebra upon u_0 is shifted by a constant. At this point we introduce the orbit invariant $h(u_k)$ and express it in terms of u_k by expanding in power series

$$h = u_0 + \sum_{n=2}^{\infty} U_n, \qquad U_n = \frac{1}{n!} \sum_{\substack{p_1,\cdots,p_n \\ p_1+\cdots+p_n=0,\, p_k\neq 0}} h_{p_1,\cdots,p_n} u_{p_1}\cdots u_{p_n}. \tag{B.3}$$

After imposing $\frac{c}{24}\{h(u), u_k\} = 0$ for any k we find

$$h_{p_1,p_2} = -\frac{p_1^2+p_2^2+2h}{4(p_1^2+h)(p_2^2+h)},$$
$$h_{p_1,p_2,p_3} = \frac{p_1^2+p_2^2+p_3^2+6h}{8(p_1^2+h)(p_2^2+h)(p_3^2+h)},$$
$$h_{p_1,p_2,p_3,p_4} = -\frac{15h^4-25h^3q_2+h^2(13q_2^2-9q_4)+h(-3(q_2^3+q_3^2)+8q_2q_4)+q_2^2q_4-q_2q_3^2-4q_4^2}{den},$$

where

$$q_2 \equiv p_1p_2+p_1p_3+p_2p_3+p_1p_4+p_2p_4+p_3p_4 = -\frac{1}{2}\sum p_k^2, \tag{B.4}$$

$$q_3 \equiv p_1p_2p_3+p_1p_2p_4+p_1p_3p_4+p_2p_3p_4 = \frac{1}{3}\sum p_k^3, \tag{B.5}$$

$$q_4 \equiv p_1p_2p_3p_4 = -\frac{1}{4}\sum p_k^4+\frac{1}{2}q_2^2, \tag{B.6}$$

$$den = [(p_1+p_2)^2+(p_3+p_4)^2+2h][(p_1+p_3)^2+(p_2+p_4)^2+2h][(p_1+p_4)^2+(p_2+p_3)^2+2h] \times \prod_{k=1}^{4}((p_k^2+h). \tag{B.7}$$

Now we can get rid of $u_0 = h - \sum_{n=2}^{\infty} U_n$ and express the Poisson brackets in terms of u_k, $k \neq 0$,

$$i\frac{c}{24}\{u_k, u_\ell\} = \delta_{k+\ell}(2k)\left(k^2 + h - \sum_{n=2}^{\infty} U_n\right) + (1-\delta_{k+\ell})(k-\ell)u_{k+\ell}. \tag{B.8}$$

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Our next goal is to find symplectic form associated with the Poisson brackets

$$\omega = \frac{c}{24} \times \frac{i}{2} \sum_{k\neq 0, \ell\neq 0} \omega_{k,\ell}\, du_k \wedge du_\ell, \qquad \omega_{k,\ell} = \sum_{n=0}^{\infty} \omega^{(n)}_{k,\ell}, \tag{B.9}$$

where $\omega^{(n)}_{k,\ell}$ is an order n homogeneous polynomial in u_k. We find, order by order in u_k,

$$\omega^{(0)}_{k,\ell} = -\frac{1}{2k(k^2+h)}\delta_{k+\ell}, \tag{B.10}$$

$$\omega^{(1)}_{k,\ell} = -(1-\delta_{k+\ell})\frac{k-\ell}{4k\ell(k^2+h)(\ell^2+h)}u_{-k-\ell}, \tag{B.11}$$

$$\omega^{(2)}_{k,\ell} = -\frac{1}{8k\ell(k^2+h)(\ell^2+h)}\sum_{m\neq 0}\left[\frac{k\delta_{k+\ell}}{m^2+h}u_m u_{-m}\right. \tag{B.12}$$

$$\left. + (1-\delta_{k-m})(1-\delta_{\ell+m})\frac{(k+m)(\ell-m)}{m(m^2+h)}u_{-k+m}u_{-\ell-m}\right] \tag{B.13}$$

We now would like to introduce (rescaled) normal coordinates z_k near the origin $u_k = 0$ (which corresponds to constant $u(\varphi) = h$), such that

$$\frac{24}{c}\omega = \frac{i}{2}\sum_{k\neq 0}\frac{-1}{2k(k^2+h)}dz_k \wedge dz_{-k}, \qquad i\frac{c}{24}\{z_k, z_\ell\} = \delta_{k+\ell}(2k)(k^2+h). \tag{B.14}$$

We find

$$\begin{aligned} z_k = u_k + \frac{1}{4}\sum_{\substack{p_1+p_2=k\\ p_i\neq 0}}\frac{1}{p_1p_2}u_{p_1}u_{p_2} + \frac{1}{24}\sum_{\substack{p_1+p_2+p_3=k\\ p_i\neq 0,k}}\frac{p_1p_2+p_2p_3+p_3p_1}{p_1p_2p_3(p_1+p_2)(p_2+p_3)(p_3+p_1)}u_{p_1}u_{p_2}u_{p_3} \\ -\frac{1}{8}\sum_{\ell\neq 0,\pm k}\frac{2\ell^2+h}{\ell^2(k^2-\ell^2)(\ell^2+h)}u_k u_\ell u_{-\ell} - \frac{2k^4-k^2h+h^2}{32k^4(k^2+h)^2}u_k^2 u_{-k} + \mathcal{O}(u^4). \end{aligned} \tag{B.15}$$

This expression can be inverted

$$\begin{aligned} u_k = z_k - \frac{1}{4}\sum_{\substack{p_1+p_2=k\\ p_i\neq 0}}\frac{1}{p_1p_2}z_{p_1}z_{p_2} + \frac{1}{24}\sum_{\substack{p_1+p_2+p_3=k\\ p_i\neq 0,k}}\frac{k^2}{p_1p_2p_3(k-p_1)(k-p_2)(k-p_3)}z_{p_1}z_{p_2}z_{p_3} \\ +\frac{1}{2}\sum_{\substack{p_1+p_2=0\\ p_i\neq 0,k}}\frac{h}{p_1p_2(k-p_1)(k-p_2)[(p_1-p_2)^2+4h]}z_{p_1}z_{p_2}z_k - \frac{h(5k^2+h)}{32k^4(k^2+h)^2}z_k^2 z_{-k} + \mathcal{O}(z^4). \end{aligned} \tag{B.16}$$

We are now ready to introduce action and angles variables I_k, θ_k such that $\frac{24}{c}\omega = \sum_k dI_k \wedge d\theta_k$,

$$\frac{z_k}{\sqrt{2k(k^2+h)}} = \sqrt{I_k}e^{-i\theta_k}. \tag{B.17}$$

This leads to

$$
\begin{aligned}
I_k &= \frac{1}{2k(k^2+h)} z_k z_{-k} \qquad \text{(B.18)}\\
&= \frac{1}{2k(k^2+h)} u_k u_{-k} + \frac{1}{8k(k^2+h)} \sum_{\substack{p_1+p_2=k \\ p_i\neq 0}} \frac{1}{p_1 p_2} u_{p_1} u_{p_2} u_{-k} \\
&\quad + \frac{1}{8k(k^2+h)} \sum_{\substack{p_1+p_2=-k \\ p_i\neq 0}} \frac{1}{p_1 p_2} u_{p_1} u_{p_2} u_k + \mathcal{O}(u^4).
\end{aligned}
$$

At this point we can go back to $u_0 = h - \sum_{n=2}^{\infty} U_n$ are represent it in terms of action variables (by expressing both sides as a series in z_k),

$$
Q_1 \equiv u_0 = h + \sum_{k=1} k I_k + \mathcal{O}(z^5). \qquad \text{(B.19)}
$$

This matches the exact relation (2.24) up to the fifth order in z_k, reflecting the expansion order in (B.16).

To find Q_{2n-1} in terms of I_k we first write an iterative relation for the Fourier modes of Gelfand-Dikii polynomials, which satisfy (2.20),

$$
\begin{aligned}
R_{n,k} &\equiv \frac{1}{2\pi} \int d\varphi\, e^{-ik\varphi} R_n, \qquad \text{(B.20)}\\
R_{n+1,k} &= \frac{n+1}{2n+1} \left[2(k^2+u_0) R_{n,k} + Q_{2n-1} u_k + \frac{1}{k} \sum_{\ell\neq 0,k} (2k-\ell) u_\ell R_{n,k-\ell} \right], \quad (k\neq 0).
\end{aligned}
$$

Then, using the relation between Q_{2n-1} and R_n

$$
R_{n,k} = \frac{1}{ik(2n-1)} \frac{c}{24} \{Q_{2n-1}, u_k\} \qquad \text{(B.21)}
$$

we find

$$
\begin{aligned}
\frac{c}{24}\{Q_{2n+1}, u_k\} &= ik(n+1) Q_{2n-1} u_k + \frac{2(n+1)(k^2+u_0)}{2n-1} \frac{c}{24} \{Q_{2n-1}, u_k\} \\
&\quad + \frac{n+1}{2n-1} \sum_{\ell\neq 0,k} \frac{2k-\ell}{k-\ell} u_\ell \frac{c}{24} \{Q_{2n-1}, u_{k-\ell}\}. \qquad \text{(B.22)}
\end{aligned}
$$

We use the following ansatz for Q_{2n-1} in terms of u_k,

$$
\begin{aligned}
Q_{2n-1} &= h^n + \frac{1}{2!} \sum_{\substack{p_1+p_2=0 \\ p_i\neq 0}} q^{(n)}_{p_1,p_2} u_{p_1} u_{p_2} + \frac{1}{3!} \sum_{\substack{p_1+p_2+p_3=0 \\ p_i\neq 0}} q^{(n)}_{p_1,p_2,p_3} u_{p_1} u_{p_2} u_{p_3} \qquad \text{(B.23)}\\
&\quad + \frac{1}{4!} \sum_{\substack{p_1+p_2+p_3+p_4=0 \\ p_i\neq 0}} q^{(n)}_{p_1,p_2,p_3,p_4} u_{p_1} u_{p_2} u_{p_3} u_{p_4} + \mathcal{O}(u^5),
\end{aligned}
$$

and the iterative relation (B.22) becomes the iterative relation for $q^{(n)}_{p_1,\ldots,p_i}$ for $i=2,3,4$,

$$q^{(n+1)}_{k,-k} = \frac{2(n+1)(k^2+h)}{2n-1} q^{(n)}_{k,-k} + \frac{(n+1)h^n}{2(k^2+h)}, \tag{B.24}$$

$$q^{(n+1)}_{p_1,p_2,p_3} = \frac{2(n+1)(p_1^2+p_2^2+p_3^2+3h)}{3(2n-1)} q^{(n)}_{p_1,p_2,p_3} - \frac{(n+1)h^n(p_1^2+p_2^2+p_3^2+6h)}{8(p_1^2+h)(p_2^2+h)(p_3^2+h)} \tag{B.25}$$

$$- \frac{n+1}{3(2n-1)} \left[\left(\frac{p_1-p_2}{p_2} + \frac{p_1-p_3}{p_3} \right) q^{(n)}_{p_1,-p_1} + \text{symmetric w.r.t. } p_1,p_2,p_3. \right] \tag{B.26}$$

These can be solved as follows

$$q^{(n)}_{k,-k} = \frac{(h+k^2)^{n-2}(2n)!!}{4(2n-3)!!} \sum_{m=0}^{n-1} \frac{(2m-1)!!}{(2m)!!} \left(\frac{h}{h+k^2} \right)^m, \tag{B.27}$$

$$q^{(n)}_{p_1,p_2,p_3} = \frac{(2n)!!}{8(2n-3)!! p_1 p_2 p_3} \sum_{m=0}^{n-1} \frac{(2m-1)!!}{(2m)!!} h^m \times \left[p_1(p_1^2+h)^{n-m-2} + p_2(p_2^2+h)^{n-m-2} + p_3(p_3^2+h)^{n-m-2} \right].$$

Our goal would be to match (B.23)with the expansion

$$Q_{2n-1} = h^n + \sum_{k=1} (f^{(n,1)}_k I_k + f^{(n,2)}_k I_k^2) + \frac{1}{2} \sum_{\substack{k,\ell=1 \\ k\neq\ell}} f^{(n)}_{k,\ell} I_k I_\ell + \mathcal{O}(I^3), \tag{B.28}$$

by expressing I_k in terms of u_k using (B.18). This leads to

$$f^{(n,1)}_k = 2k(k^2+h) q^{(n)}_{k,-k} \tag{B.29}$$

and the relations for $f^{(n,2)}_k, f^{(n)}_{k\ell}$ in terms of $q^{(n)}_{p_1,\ldots,p_i}$. To fix $f^{(n,2)}_k, f^{(n)}_{k\ell}$, we would not need $q^{(n)}_{p_1,p_2,p_3,p_4}$ with arbitrary $p_1,\ldots,p_4$, but only $q^{(n)}_{k,-k,\ell,-\ell}$, including the case of $k=\ell$,

$$q^{(n)}_{k,-k,\ell,-\ell} = \frac{f^{(n)}_{k,\ell}}{4k\ell(k^2+h)(\ell^2+h)} + \frac{1}{8k^2\ell^2} \left(\frac{f^{(n,1)}_{k+\ell}}{(k+\ell)[(k+\ell)^2+h]} + \frac{f^{(n,1)}_{k-\ell}}{(k-\ell)[(k-\ell)^2+h]} \right) - \frac{1}{4k^2\ell^2(k^2-\ell^2)(k^2+h)(\ell^2+h)} \left[k(2\ell^2+h) f^{(n,1)}_k - \ell(2k^2+h) f^{(n,1)}_\ell \right], \tag{B.30}$$

and

$$q^{(n)}_{k,k,-k,-k} = \frac{f^{(n,2)}_k}{k^2(k^2+h)^2} - \frac{(2k^4-k^2h+h^2) f^{(n,1)}_k}{8k^5(k^2+h)^3} + \frac{f^{(n,1)}_{2k}}{16k^5(4k^2+h)}. \tag{B.31}$$

The iterative relation for $q^{(n)}_{k,-k,\ell,-\ell}$ is cumbersome. Instead, it is more convenient to work directly with the iterative relation in terms of $f^{(n,2)}_k$ and $f^{(n)}_{k\ell}$

$$f^{(n+1,2)}_k = \frac{2(n+1)(k^2+h)}{2n-1} f^{(n,2)}_k + \frac{(n+1)(k^2+h)}{2(2n-1)} [(4n-1)k^2 - 3h] q^{(n)}_{k,-k}, \tag{B.32}$$

and

$$f_{k,\ell}^{(n+1)} = \frac{2(n+1)(k^2+h)}{2n-1} f_{k,\ell}^{(n)} + \frac{2(n+1)(k^2+h)k\ell}{(2n-1)(k^2-\ell^2)} \left[2(k^2+h)^2 q_{k,-k}^{(n)} + (\ell^2+h)(2(k^2-\ell^2)n - k^2 - \ell^2 - 2h) q_{\ell,-\ell}^{(n)}\right]. \tag{B.33}$$

Once everything combined together we find

$$f_k^{(n,1)} = \frac{(2n)!!k(h+k^2)^{n-1}}{2(2n-3)!!} \sum_{m=0}^{n-1} \frac{(2m-1)!!}{(2m)!!} \left(\frac{h}{h+k^2}\right)^m, \tag{B.34}$$

$$f_k^{(n,2)} = -\frac{(2n)!!(h+k^2)^{n-2}}{16(2n-3)!!} \sum_{m=0}^{n-1} (3h+k^2-4k^2m) \sum_{j=0}^{m-1} \frac{(2j-1)!!}{(2j)!!} \left(\frac{h}{h+k^2}\right)^j, \tag{B.35}$$

and

$$f_{k,\ell}^{(n)} = \frac{(2n)!!k\ell}{4(2n-3)!!(k^2-\ell^2)} \sum_{j=0}^{n-1} (n-1-j) \frac{(2j-1)!!h^j}{(2j)!!} \left[(h+k^2)^{n-j-1} - (h+\ell^2)^{n-j-1}\right] + \frac{(2n)!!k\ell}{4(2n-3)!!} \sum_{m=0}^{n-1} \sum_{j=0}^{m-1} \frac{m(2j-1)!!h^j}{(2j)!!} \times \left[(h+k^2)^{m-j-1}(h+\ell^2)^{n-m-1} + (h+\ell^2)^{m-j-1}(h+k^2)^{n-m-1}\right]. \tag{B.36}$$

Although written in a different form, this result is in agreement with (2.34), (2.35), and (2.38).

C One-zone potentials: details

One-zone potentials u can be found from the condition $\{Q_3 + \alpha Q_1, u\} = 0$ for some constant α. From here we immediately find, see section 2.4 of [24],

$$\lambda_0 = -\frac{\alpha}{24} - \frac{k^2}{12}(\theta_3(\tau)^4 + \theta_4(\tau)^4), \tag{C.1}$$

$$\lambda_1 = -\frac{\alpha}{24} - \frac{k^2}{12}(\theta_2(\tau)^4 - \theta_4(\tau)^4), \tag{C.2}$$

$$\lambda_2 = -\frac{\alpha}{24} + \frac{k^2}{12}(\theta_2(\tau)^4 + \theta_3(\tau)^4). \tag{C.3}$$

Pertubatively, i.e. in the limit of small $q = e^{i\pi\tau}$, corresponding potential is

$$u = h + \frac{32k^4}{k^2+h} q^2 - 16k^2 q \cos(k\varphi) - 32k^2 q^2 \cos(2k\varphi) + \mathcal{O}(q^3). \tag{C.4}$$

There are useful relations involving Jacobi elliptic functions and hypergeometric function,

$$m := \theta_2^4(\tau)/\theta_3^4(\tau), \qquad F\left(\frac{1}{2},\frac{1}{2},1;m\right) = \theta_3(\tau)^2, \qquad \frac{F\left(\frac{1}{2},\frac{1}{2},1;1-m\right)}{F\left(\frac{1}{2},\frac{1}{2},1;m\right)} = -\frac{1}{\pi}\log q,$$

$$\frac{F\left(\frac{3}{2},\frac{1}{2},1;m\right)}{F\left(\frac{1}{2},\frac{1}{2},1;m\right)} = 1 + 2\frac{\partial \ln \theta_3^2(\tau)}{\partial \ln m}, \qquad -16\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1-q^{2n+1})^2} + 2\theta_3(\tau)^4 - 2\theta_4(\tau)^4 \frac{F\left(\frac{3}{2},\frac{1}{2},1;m\right)}{F\left(\frac{1}{2},\frac{1}{2},1;m\right)} = 0.$$

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We also list here more terms of the q-expansion of I_k,

$$I_k = \frac{32k^3q^2}{h+k^2} + \frac{64q^4\left(3h^2k^3+12hk^5+k^7\right)}{(h+k^2)^3} + \frac{128k^3q^6\left(3h^4+42h^3k^2+108h^2k^4-58hk^6+k^8\right)}{(h+k^2)^5}$$
$$+\frac{128k^3q^8\left(7h^6+156h^5k^2+1083h^4k^4+1232h^3k^6-4035h^2k^8+788hk^{10}+k^{12}\right)}{(h+k^2)^7}+\mathcal{O}(q^{10}).$$

With help of $Q_1 = h + kI_k$ immediately yields Q_1 as an h-dependent q expansion. Together with (2.29) this yields q expansion of λ_0,

$$\lambda_0 = \frac{h}{4} - \frac{8hk^2}{k^2+h}q^2 - \frac{16hk^2(h^2-9k^4)}{(k^2+h)^3}q^4 - \frac{32hk^2(h^4+2h^3k^2-32h^2k^4-98hk^6+63k^8)}{(k^2+h)^5}q^6 + \mathcal{O}(q^7).$$

The relation for I_k in terms of q can be solved for q in terms of I_k iteratively,

$$q^2 = \frac{(h+k^2)}{32k^3}I_k - \frac{(3h^2+12hk^2+k^4)}{512k^6}I_k^2 + \frac{(15h^3+87h^2k^2+105hk^4+k^6)}{8192k^9}I_k^3$$
$$-\frac{(187h^4+1402h^3k^2+3012h^2k^4+1606hk^6+k^8)}{262144k^{12}}I_k^4+\mathcal{O}(I_k^5).$$

D Perturbative calculation for finite-zone potentials

We start with the two-zone case and parametrize corresponding differential dp with help of two infinitesimal parameters ϵ_1, ϵ_2 and λ_0,

$$\lambda_1 = \lambda_0 + \frac{k^2}{4} + \epsilon_1 + a_1\epsilon_1^2 + b_1\epsilon_1\epsilon_2 + c1\epsilon_2^2 + \dots, \tag{D.1}$$
$$\lambda_2 = \lambda_0 + \frac{k^2}{4} - a\epsilon_1 + a_2\epsilon_1^2 + b_2\epsilon_1\epsilon_2 + c2\epsilon_2^2 + \dots, \tag{D.2}$$
$$\lambda_3 = \lambda_0 + \frac{\ell^2}{4} + \epsilon_2 + a_3\epsilon_1^2 + b_3\epsilon_1\epsilon_2 + c3\epsilon_2^2 + \dots, \tag{D.3}$$
$$\lambda_4 = \lambda_0 + \frac{\ell^2}{4} - b\epsilon_2 + a_4\epsilon_1^2 + b_4\epsilon_1\epsilon_2 + c4\epsilon_2^2 + \dots, \tag{D.4}$$
$$r_1 = \lambda_0 + \frac{k^2}{4} + d_1\epsilon_1^2 + e_1\epsilon_1\epsilon_2 + f_1\epsilon_2^2 + \dots, \tag{D.5}$$
$$r_2 = \lambda_0 + \frac{\ell^2}{4} + d_2\epsilon_1^2 + e_2\epsilon_1\epsilon_2 + f_2\epsilon_2^2 + \dots \tag{D.6}$$

The parametrization is redundant, with different choices related by redefinitions of ϵ_1, ϵ_2. We assume $\epsilon_1 \sim \epsilon_2$ are of the same order and in what follows we refer to expansion in ϵ_1, ϵ_2 simply as ϵ expansion. While keeping two-zone case in mind for concreteness, most of the discussion below applies to m-zone case with arbitrary m.

D.1 a-cycles

To impose a_1-cycle constraint (2.9), we need to integrate from λ_1 to λ_2. By introducing x via

$$\lambda = \frac{\lambda_2+\lambda_1}{2} + x\frac{\lambda_2-\lambda_1}{2}, \tag{D.7}$$

and then expanding in powers of ϵ we reduce the integral to standard integrals of the form

$$\int_{-1}^{1} \frac{dx\, x^{2n}}{\sqrt{1-x^2}} = \frac{\sqrt{\pi}\Gamma(n+1/2)}{\Gamma(n+1)}. \tag{D.8}$$

Provided we want to find Q_{2n-1} in terms of I_k by expanding up to p-th power, we would need to keep $2p$ terms in ϵ-expansion, up to and including ϵ^{2p}. This method works for any a-cycle integral and any number of zones.

D.2 b-cycles

We start with the b_1-cycle, which goes from λ_0 to λ_1, and introduce another variable x

$$\lambda = \lambda_1 - x(\lambda_1 - \lambda_0). \tag{D.9}$$

We can use the proximity of λ_4 to λ_3 to expand $\sqrt{(\lambda-\lambda_3)(\lambda-\lambda_4)}$ in ϵ. Now the integral of interest reduced to a sum of integrals of the form

$$\int_0^1 \frac{dx P(x)}{\sqrt{x(1-x)(16w+x)}(x-c)^r} \tag{D.10}$$

where $16w$ is a small parameter of order ϵ,

$$16w = \frac{\lambda_2 - \lambda_1}{\lambda_1 - \lambda_0}, \tag{D.11}$$

$P(x)$ is some polynomial and $c = 1 - \ell^2/k^2$ (we assumed $\ell > k$). The integral (D.10) can be related to

$$J_n(c) := \int_0^1 \frac{dx\, x^n}{\sqrt{x(1-x)(16w+x)}(x-c)} \tag{D.12}$$

by differentiating over c. To evaluate it, it is helpful to first introduce the integral

$$I_n := \int_0^1 \frac{dx\, x^n}{\sqrt{x(1-x)(16w+x)}} = \sum_{m=0}^{\infty} a_m(n) w^m + \sum_{m=n}^{\infty} b_m(n) w^m \ln w, \tag{D.13}$$

which can be expressed as formal series in w. Coefficients $a_m(n)$ for $n > m$ and $b_m(n)$ for any n, m can be found analytically

$$a_m(n) = \frac{(-16)^m \Gamma\left(m+\frac{1}{2}\right)\Gamma(n-m)}{\Gamma(m+1)\Gamma\left(-m+n+\frac{1}{2}\right)}, \quad n > m, \tag{D.14}$$

$$b_m(n) = \frac{16^m(-1)^{n+1}\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)\Gamma(m-n+1)\Gamma\left(-m+n+\frac{1}{2}\right)}. \tag{D.15}$$

To find $a_m(n)$ for $m \geq n$ we can use the iterative relation

$$I_n = (1-2n)(I_n - I_{n-1}) - \frac{1}{8}\partial_w(I_{n+1} - I_n), \tag{D.16}$$

which follows from the integration by parts, and $a_m(0)$ which can be found directly from (D.13) since the corresponding integral can be evaluated analytically. For example we find the following iterative relation for $a_n(n)$,

$$a_{m+1}(m+1) = \frac{(-1)^m 2^{2m+3}\Gamma(2m+1)}{\Gamma(m+2)^2} - \frac{8(2m+1)a_m(m)}{m+1}, \qquad a_0(0) = 0. \tag{D.17}$$

So far we are interested only in first $2p$ powers of w, we only need to worry about $a_m(n)$ with $m \leq 2p$. In our case $p = 3$ and we simply tabulate values of $a_m(n)$ for $0 \leq m \leq 6$ and $m \geq n$ for convenience

$$a_m(n) = \begin{pmatrix} 0 & & & & & & \\ 8 & 8 & & & & & \\ -84 & -104 & -112 & & & & \\ \frac{2960}{3} & 1152 & \frac{4288}{3} & \frac{4736}{3} & & & \\ -\frac{37310}{3} & -\frac{42040}{3} & -16368 & -\frac{60992}{3} & -\frac{68224}{3} & & \\ \frac{820008}{5} & 180656 & 203584 & \frac{1189248}{5} & \frac{1478144}{5} & \frac{1666048}{5} & \\ -\frac{11153912}{5} & -\frac{12097344}{5} & -\frac{7995904}{3} & -\frac{9011456}{3} & -\frac{17549824}{5} & -\frac{65468416}{15} & -\frac{74166272}{15} \end{pmatrix}.$$

Going back to (D.12), we can expand $(x-c)$ in the denominator into power series in x, thus reducing the integral to a sum of (D.13). Provided $n > 2p$ and so far we are only interested in terms of order w^r and $w^r \ln w$ with $r \leq 2p$, only relevant contributions would come from $a_m(n) w^m$ term in (D.13) with $m < n$. Corresponding coefficients are known analytically, (D.14), and can be re-summed yielding,

$$J_n(c) = -\sum_{m=0}^{2p} \frac{(-16)^m \omega^m \Gamma\left(m+\frac{1}{2}\right)\Gamma(l-m)\, {}_2F_1\left(1, l-m; l-m+\frac{1}{2}; \frac{1}{c}\right)}{c\,\Gamma(m+1)} + \mathcal{O}(w^{2p+1}).$$

Here $2F_1$ is *regularized* hypergeometric function and this expression is only valid for $n > 2p$. To extend it to smaller n we use the iterative relation, which follows from the integration by parts,

$$J_n = \frac{J_{n+1} - I_n}{c}. \tag{D.18}$$

This completes technical preliminaries as now integral over b_1 cycle can be reduced to a number of integrals J_n and their derivatives, so far we are only interested in terms of order w^r with $r \leq 2p$. Clearly, the approach above can be used to evaluate integrals over b_1 when there are more than two zones. In this case one would need to evaluate integrals

$$\int_0^1 \frac{dx\, x^n}{\sqrt{x(1-x)(16w+x)}\prod_{i=1}^{m-1}(x-c_i)}, \tag{D.19}$$

where m is the number of zones. This can be reduced to (D.12) by noting

$$\prod_{i=1}^{m-1} \frac{1}{(x-c_i)} = \sum_{i=1}^{m-1} \frac{\alpha_i}{x-c_i}, \tag{D.20}$$

with the appropriate coefficients α_i.

To evaluate the integral over b_2-cycle from λ_2 to λ_3 is more challenging because in the $\epsilon \to 0$ limit there are singularities appearing at both boundaries. There is a straightforward but complicated way. By appropriately changing variables and expanding in ϵ all terms except for $\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)}$ we reduce the calculation to the integral

$$\int_0^1 dx \frac{x^n}{\sqrt{x(1-x)(16w+x)(1+16u-x)}} \tag{D.21}$$

for positive small w, u. The indefinite integral of this kind can be evaluated analytically. Then the definite integral above can be integrated by expanding it powers of w, u (which both are of order ϵ), and keeping terms up to order $2p$. This is an involved exercise and instead one can use one of the following shortcuts.

In the particular case of two-zone potential, instead of evaluating integral over b_2, one can combine the integral over b_1 and b_2 such that the contour would enclose $\lambda_0, \ldots, \lambda_3$. Now one can deform the contour to go from λ_4 to infinity, if necessary accompanied by a circle at infinity. At this point integrand can be expanded in ϵ such that brunch-cut from λ_1 to λ_2 disappears, yielding pole singularities at $\lambda = \lambda_0 + k^2/4$. At this point corresponding integral can be rewritten as

$$\oint_{-\infty}^{-16w} dx \frac{P(x)}{\sqrt{x(1-x)(x+16w)}(x-c)^r}, \tag{D.22}$$

where $P(x)$ is some polynomial and $0 \le c \le 1$. We also emphasize that to render this integral finite, one may need to close the contour at infinity. This integral can be decomposed into a sum of integrals of the form

$$\oint_{-\infty}^{-16w} dx \frac{x^n}{\sqrt{x(1-x)(x+16w)}}, \tag{D.23}$$

and

$$\oint_{-\infty}^{-16w} dx \frac{1}{\sqrt{x(1-x)(x+16w)}(x-c)}, \tag{D.24}$$

and its derivatives. First integral can be reduced to (D.13) by deforming the contour to go from 0 to 1. Last integral can be reduced to J_0 and J_1 with help of modular transformation mapping ∞ to 1, $-16w$ to 0, and 0 to $-16w$.

$$x \to \frac{x+16w}{x-1}. \tag{D.25}$$

This shortcut works for two-zone case, but with more zones present it is not applicable. Nevetheless there is a very simple trick which make evaluation of b_2 and other b-cycles unnesessary. Indeed, to satisfy (2.9) and (2.10) for all cycles, it is sufficient to satisfy (2.9) for all cycles and (2.10) for b_1 and also impose that the expansion (D.1)–(D.6), and its generalizations for the case of more than two zones, is invariant under permutation of indexes and k_i defined in (2.11). Say, for two zones we find

$$\lambda_1 = \lambda_0 + \frac{k^2}{4} - \epsilon_1 + \frac{3\epsilon_1^2}{k^2} + \frac{4\epsilon_2^2 k^2}{\ell^2(k^2-\ell^2)} + \mathcal{O}(\epsilon^3), \tag{D.26}$$

$$\lambda_2 = \lambda_0 + \frac{k^2}{4} + \epsilon_1, \tag{D.27}$$

$$r_1 = \lambda_0 + \frac{k^2}{4} + \frac{\epsilon_1^2}{2k^2} + \frac{2\epsilon_2^2 k^2}{\ell^2(k^2-\ell^2)} + \mathcal{O}(\epsilon^3), \tag{D.28}$$

and $\lambda_{3,4}, r_2$ related to $\lambda_{1,2}, r_1$ by the exchange $\epsilon_1 \leftrightarrow \epsilon_2$ and $k \leftrightarrow \ell$. The same logic with the permutation symmetry works for any number of zones.

Above we only explicitly wrote terms up to ϵ^2, while evaluating all terms up to ϵ^6. The simple form of λ_2 above is a parametrization choice. With this choice taking $\epsilon_2 = 0$ does not close the second zone. One can check that taking

$$\epsilon_2 = -\frac{2\ell^2\epsilon_1^2}{k^2\,(k^2-\ell^2)} + \dots \tag{D.29}$$

such that $\lambda_4 = \lambda_3$ would make I_ℓ dicussed below vanish. Alternatively one could chooe ϵ_i to control the size of $\lambda_{2i} - \lambda_{2i-1}$, but with this choice both all λ_i would depend on all ϵ_i.

D.3 Evaluation of I_k, h and Q_{2n-1}

Evaluation of action variables I_k as a pertubative series in ϵ_i is straightforward. It is an integral over a-cycle and therefore can be evaluated along the lines discussed above. The only difference, in comparision with the discussion in subsection D.1, is the term $\ln\lambda$, which needs to be expanded in powers of ϵ yielding polynomials in x in the numerator of (D.8),

$$I_k = \frac{2\epsilon_1^2}{k(\lambda_0 + k^2/4)} + \mathcal{O}(\epsilon^3). \tag{D.30}$$

Again, we only keep terms up to ϵ^2 for simplicity.

Evaluation of h is also straightforward. To that end one needs to calculate $p(0)$, given by an integral from 0 to λ_0. After expanding the integrand in powers of ϵ it becomes the integral which can be evaluated in a closed form, yielding

$$h/4 = \lambda_0 + \lambda_0\left(\frac{2\epsilon_1^2}{k^2(\lambda_0+k^2/4)} + \frac{2\epsilon_2^2}{\ell^2(\lambda_0+\ell^2/4)}\right) + \mathcal{O}(\epsilon^3). \tag{D.31}$$

Finally, evaluation of Q_{2n-1} for any given n is also straightforward since λ_i are known explicitly. As a result we obtain I_k, h, Q_{2n-1} as functions of λ_0 and ϵ_i. One can then reverse-engineer coefficients in (2.1) such that it is satisfied.

E Spectrum of Q_{2n-1} acting on primaries

In this appendix we outlined calculation of Q^0_{2n} (3.6) following [27]. Starting from the Schrödinger equation (3.2), one introduces the following change of variables

$$\Psi(x) = E^{l(l-3/2)/4\alpha} w^{(l-3/2)/4} y(w), \qquad x = E^{\frac{1}{2\alpha}} w^{\frac{1}{2l}}, \tag{E.1}$$

such that (3.2) becomes

$$-\epsilon^2\partial_w^2 y+Z(w)y=0,\qquad Z(w)=w,\quad \epsilon=E^{-\frac{\alpha+1}{2\alpha}}. \tag{E.2}$$

Taking ϵ as a formal small parameter this equation can be solved via WKB expansion,

$$y(w)=e^{\frac{1}{\epsilon}S(w)},\qquad -\epsilon S''-S'^2+Z=0,\qquad S(w)=\sum_{n=0}^{\infty}\epsilon^n S_n. \tag{E.3}$$

The resulting Riccati equation can be rewritten as the iterative relation to find S'_n with $S'_0=-\sqrt{Z(w)}$. It is more convenient for what follows to make another change of variables $z=w^{\alpha/(l+1/2)}$ and introduce the polynomial ansatz

$$S'_n=i\frac{\alpha}{2l+1}z^{1-\frac{l+1/2}{\alpha}}\tilde{S}_n,\qquad \tilde{S}_n=\sum_{k=0}^{n}i^n c_k^{(n)}z^{-k+(n-1)(1-1/2\alpha)}(1-z)^{k-(3n-1)/2} \tag{E.4}$$

The Ricatti equation rewritten in terms of $c_k^{(n)}$ gives rise to (3.4), which can be used together with (3.5), to iteratively find $c_k^{(n)}$. The first few $c_k^{(n)}$ read

$$c_0^{(2)}=\frac{5}{8}\alpha,\qquad c_1^{(2)}=\frac{1}{4}(2\alpha-1),\qquad c_2^{(2)}=-\frac{1}{8\alpha}(4u^2\alpha^2-1). \tag{E.5}$$

$$c_0^{(3)}=-\frac{15\alpha^2}{8},\quad c_1^{(3)}=-\frac{9\alpha^2}{4}+\frac{9\alpha}{8},\quad c_2^{(3)}=\frac{1}{2}\alpha^2\left(u^2-1\right)+\frac{3\alpha}{4}-\frac{3}{8},\quad c_3^{(3)}=-\frac{1}{8\alpha}(4u^2\alpha^2-1). \tag{E.6}$$

To obtain Q^0_{2n-1} one needs to integrate $\tilde{S}_{2n}(w(z))$ over a Pochhammer contour γ_P,

$$\mathsf{Q}^0_{2n-1}=(-1)^n\frac{\Gamma\left(\frac{3}{2}-n-\frac{2n-1}{2\alpha}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{2n-1}{2\alpha}\right)}\frac{(2n-1)\Gamma(n+1)}{4^n(\alpha+1)^n}\check{I}_{2n-1}(\alpha,\hat{l}), \tag{E.7}$$

$$\check{I}_{2n-1}=\frac{1}{2\left(1-e^{\frac{-i\pi(2n-1)}{\alpha}}\right)}\int_{\gamma_P}dz\,\tilde{S}_{2n}(z). \tag{E.8}$$

This integral can be evaluated using,

$$(1-e^{2\pi i a})(1-e^{2\pi i b})B(a,b)=\int_{\gamma_P}dz\,z^{a-1}(1-z)^{b-1}, \tag{E.9}$$

where $B(a.b)$ is the Euler beta function. Combining everything together yields (3.6).

Evaluating Q^0_{2n-1} explicitly, using computer algebra to solve for $c_k^{(n)}$ iteratively, for small and moderate n is an easy task. To obtain $1/c$ expansion of λ^0_{2n-1} for arbitrary n requires knowing corresponding $c_k^{(n)}$ in $1/c$ expansion, i.e. in the limit of large α. This proved to be a difficult task. We obtained first three non-trivial terms of λ^0_{2n-1} in $1/\tilde{c}$ expansion (3.7), with the first two terms (3.8), (3.9) in closed analytical form. Functions y_i

and ζ_i there are defined as follows

$$y_1(j) = \sum_{\ell=0}^{j} \frac{1}{2\ell+1}, \tag{E.10}$$

$$y_2(j) = \sum_{\ell=0}^{j} \frac{1}{(2\ell+1)^2}, \tag{E.11}$$

$$\zeta_2(j) = \sum_{j_1+j_2=j} \zeta(-2j_1-1)\zeta(-2j_2-1), \tag{E.12}$$

$$\zeta_3(j) = \sum_{j_1+j_2+j_3=j} \zeta(-2j_1-1)\zeta(-2j_2-1)\zeta(-2j_3-1), \tag{E.13}$$

where sum goes only over non-negative j_1, j_2, j_3. Third term (3.10) was fixed up to one coefficient p_j, with the first several values for $0 \leq j \leq 17$ given below

$$\begin{aligned} p_j = \Big(& -\frac{31}{224}, \frac{103}{576}, -\frac{7883}{21120}, \frac{868487}{748800}, -\frac{505639}{100800}, \frac{394694297}{13708800}, -\frac{68117454019}{321753600}, \frac{4929720750223}{2540160000}, \\ & -\frac{199232137825687}{9180864000}, \frac{48745030162337923}{167650560000}, -\frac{618684597383137}{134534400}, \frac{7442737871872435019}{87783696000}, \\ & -\frac{1420749127340184137621}{788237049600}, \frac{46636700018927407368821}{1065512448000}, -\frac{19827795307778046100039}{164670105600}, \\ & \frac{21869843836862719834306038469}{587058612940800}, -\frac{314287717737094459181859168 79}{24404109649920}, \\ & \frac{4187283526052269558397574465940213}{846634880931840 00}, \dots \Big). \end{aligned}$$

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