Global, Non-scattering solutions to the energy critical wave maps equation

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Abstract

We consider the 1-equivariant energy critical wave maps problem with two-sphere target. Using a method based on matched asymptotic expansions, we construct infinite time relaxation, blow-up, and intermediate types of solutions that have topological degree one. More precisely, for a symbol class of admissible, time-dependent length scales, we construct solutions which can be decomposed as a ground state harmonic map (soliton) re-scaled by an admissible length scale, plus radiation, and small corrections which vanish (in a suitable sense) as time approaches infinity. Our class of admissible length scales includes positive and negative powers of t, with exponents sufficiently small in absolute value. In addition, we obtain solutions with soliton length scale undergoing damped or undamped oscillations in a bounded set, or undergoing unbounded oscillations, for all sufficiently large t.

Contents

1 Introduction

We consider the wave maps equation for maps $\Phi : \mathbb{R}^{1+2} \to \mathbb{S}^2$. This wave maps equation is the Euler-Lagrange equation associated to

$$\mathcal{S}(\Phi) = \int_{\mathbb{R}^{2+1}} \langle \partial^\alpha \Phi(t,x), \partial_\alpha \Phi(t,x) \rangle_{g(\Phi(t,x))} dt dx$$

where g denotes the round metric on $\mathbb{S}^2$, and the α indices are contracted using the Minkowski metric. We consider the 1-equivariant symmetry reduction of this wave maps equation, which corresponds to writing

$$\Phi_u(t,x) = (\sin(u(t,r))\cos(\phi), \sin(u(t,r))\sin(\phi), \cos(u(t,r)))$$

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where (r, ϕ) are polar coordinates on $\mathbb{R}^2$. The resulting equation for u is the following.

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u - \frac{\sin(2u)}{2r^2} = 0, \quad r > 0 \tag{1.1}$$

Sufficiently regular solutions to (1.1) satisfy the condition that the energy $E_{WM}(u, \partial_t u)$ is independent of time, where

$$E_{WM}(u, v) = \pi \int_0^\infty \left(v^2 + (\partial_r u)^2 + \frac{\sin^2(u)}{r^2} \right) r\, dr$$

We note that the family of solitons, $Q_\lambda(r) = 2\arctan(r\lambda)$, for $\lambda > 0$, are solutions to (1.1), which minimize $E_{WM}(u, 0)$ within a class of functions u such that Φ_u has topological degree one.

The work of Shatah and Tahvildar-Zadeh, [29], studied the Cauchy problem associated to (1.1), with data (u_0, u_1) such that

$$(x_1, x_2) \mapsto (\frac{x_1 u_0(r)}{r}, \frac{x_2 u_0(r)}{r}) \in H^1_{\text{loc}}(\mathbb{R}^2), \quad (x_1, x_2) \mapsto (\frac{x_1 u_1(r)}{r}, \frac{x_2 u_1(r)}{r}) \in L^2_{\text{loc}}(\mathbb{R}^2)$$

As in the previous work of the author, [25], we will say that u is a finite energy solution to (1.1) if u is a solution to (1.1) in the sense of distributions, with $\Phi_u \in C^0_t \dot{H}^1(\mathbb{R}^2)$ and $\partial_t \Phi_u \in C^0_t L^2(\mathbb{R}^2)$. Throughout this work, we will consider the following wave equation with various right-hand sides.

$$-\partial_t^2 u + \partial_r^2 u + \frac{1}{r}\partial_r u - \frac{u}{r^2} = 0 \tag{1.2}$$

The quantity $E(u, \partial_t u)$ is formally conserved for solutions to (1.2), where

$$E(u, v) = \pi \int_0^\infty \left(v^2 + (\partial_r u)^2 + \frac{u^2}{r^2} \right) r\, dr \tag{1.3}$$

The work [5] of Cote, Kenig, Lawrie, and Schlag classified all solutions, u, to (1.1), which satisfy the condition that Φ_u has topological degree one, and $E_{WM}(Q_1, 0) < E_{WM}(u, \partial_t u) < 3E_{WM}(Q_1, 0)$. In particular, the result of [5] implies that any such solution which also exists globally in time can be decomposed as

$$Q_{\frac{1}{\lambda(t)}}(r) + \varphi_L(t, r) + \epsilon(t, r) \tag{1.4}$$

where φ_L solves (1.2), $\lambda(t) = o(t)$, $t \to \infty$, and $\epsilon \to 0$ in an appropriate sense, as $t \to \infty$. According to [5], at the time of its writing, there were no known constructions of solutions to (1.1) which can be decomposed as above, for $\lambda(t) \to 0$ or $\lambda(t) \to \infty$ as $t \to \infty$. To the knowledge of the author, the only currently known examples of such solutions with $\lambda(t) \to 0$, are those constructed in the previous work of the author, [25]. More precisely, for all $b > 0$, and all functions $\lambda_0 \in C^\infty([100, \infty))$ which satisfy the following conditions for some constants $C_l, C_m, C_{m,k} > 0$,

$$\frac{C_l}{\log^b(t)} \leq \lambda_0(t) \leq \frac{C_m}{\log^b(t)}, \quad |\lambda_0^{(k)}(t)| \leq \frac{C_{m,k}}{t^k \log^{b+1}(t)}, k \geq 1, \quad t \geq 100$$

the work [25] constructs finite energy solutions to (1.1), for t sufficiently large, of the form

$$u(t, r) = Q_{\frac{1}{\lambda(t)}}(r) + v_{rad}(t, r) + u_e(t, r)$$

where v_{rad} solves (1.2), with $E(v_{rad}, \partial_t v_{rad}) < \infty$,

$$E(u_e, \partial_t \left(Q_{\frac{1}{\lambda(t)}}(r) + u_e \right)) \leq \frac{C}{t^2 \log^{2b}(t)}$$

and

$$\lambda(t) = \lambda_0(t) + e(t), \quad |e(t)| \leq \frac{C}{\log^b(t)\sqrt{\log(\log(t))}}$$

The main result of this work can be summarized as follows. For each positive $\lambda \in C^\infty((50, \infty))$ satisfying the following for all t sufficiently large and $C_l, C_u, C_2 > 0$ sufficiently small (see (1.6) for the precise conditions)

$$\frac{-C_l}{t} \leq \frac{\lambda'(t)}{\lambda(t)} \leq \frac{C_u}{t}, \frac{|\lambda^{(k)}(t)|}{\lambda(t)} \leq \frac{C_k}{t^k}, \text{ for } k \geq 2$$

this work constructs a solution to (1.1) which can be decomposed as in (1.4) (see Theorem 1 for the precise sense in which $\epsilon \to 0$). This class of λ includes positive and negative powers of t, as well as oscillatory functions which satisfy any combination of the following (see the Remarks after Theorem 1)

$$\liminf_{t \to \infty} \lambda(t) = 0 \text{ or } \liminf_{t \to \infty} \lambda(t) = \lambda_0 > 0 \quad \text{and } \limsup_{t \to \infty} \lambda(t) = \lambda_1 > 0 \text{ or } \limsup_{t \to \infty} \lambda(t) = \infty, \quad \text{where } \lambda_0 \leq \lambda_1$$

(in addition to oscillatory λ such that $\liminf_{t\to\infty} \lambda(t) = \limsup_{t\to\infty} \lambda(t) = 0$ or ∞). The method of construction of the ansatz of this work is quite different than that used in [25], see Remark 7 after Theorem 1 (as well as Section 3) for a comparison.

To the knowledge of the author, the solutions constructed in this work are the first examples of solutions to (1.1) of the form (1.4), with $\lambda(t) \to \infty$. Our class of solutions also enlarges the symbol class of known infinite time blow-up rates for (1.1), and also includes solutions for which $\lambda(t)$ is a power of t, or oscillates as described above, see also Remarks 4, 5, 6 after Theorem 1.

Before we state our main theorem, we will have to precisely describe the set of admissible $\lambda(t)$, and this will require a short discussion of the work [16] of Krieger, Schlag, and Tataru. The work [16] constructed a continuum of finite time blow-up solutions to (1.1), with blow-up rates given by $\lambda(t) = t^{1+\nu}$, for $\nu > \frac{1}{2}$. (Here, $\lambda(t)$ is the length scale of the soliton).

When completing our approximate solution to (1.1) to an exact one, we use two important notions from [16] in this present work. First, we will use the distorted Fourier transform, $\mathcal{F}$, associated to (a conjugation of) the elliptic part of the wave equation obtained by linearizing (1.1) around Q_1. This distorted Fourier transform is defined in Section 5 of [16]. Second, we will use the "transference operator", $\mathcal{K}$, defined in Section 6 of [16] by

$$\mathcal{F}(r\partial_r u) = -2\xi\partial_\xi \mathcal{F}(u) + \mathcal{K}(\mathcal{F}(u))$$

In order to precisely describe the set of $\lambda(t)$ to which our main theorem applies, we will have to define a few absolute constants. First, we let ρ denote the density of the spectral measure of $\mathcal{F}$, as defined in Theorem 5.3 of [16]. From Proposition 5.7b of [16], there exists $C_\rho > 0$, such that, for all $y, z > 0$,

$$\frac{\rho(y)}{\rho(z)} \leqslant C_\rho \left(\frac{y}{z} + \frac{z}{y} \right)$$

Second, by Theorem 6.1, and Proposition 6.2 of [16], the operators $\mathcal{K}$ and $[\xi\partial_\xi, \mathcal{K}]$ are bounded on $L_\rho^{2,\alpha}$, for example, for $\alpha = 0, \frac{1}{2}$, where

$$||f||_{L_\rho^{2,\alpha}} = ||f(\xi)\langle\xi\rangle^\alpha \sqrt{\rho(\xi)}||_{L^2(d\xi)} \tag{1.5}$$

Now, we define Λ to be the set of positive functions $\lambda \in C^\infty((50, \infty))$ such that there exists $T_\lambda > 100$, constants $C_l, C_u, C_2 \geqslant 0$ satisfying (1.7), (1.8), and (1.9), and constants $C_k \geqslant 0$ for $k \geqslant 3$, such that the following hold for $t \geqslant T_\lambda$

$$\frac{-C_l}{t} \leqslant \frac{\lambda'(t)}{\lambda(t)} \leqslant \frac{C_u}{t}, \quad \frac{|\lambda^{(k)}(t)|}{\lambda(t)} \leqslant \frac{C_k}{t^k}, \text{ for } k \geqslant 2 \tag{1.6}$$

where, for $M := \max\{C_l, C_u\}$,

$$0 \leqslant C_u < \frac{1}{30} - \frac{C_l}{5} \tag{1.7}$$

$$\sqrt{C_\rho} \cdot \frac{179}{267} \cdot \left(M(1 + 2||\mathcal{K}||_{\mathcal{L}(L_\rho^2)}) + M^2 \left(\frac{1}{4} + 2||[\xi\partial_\xi, \mathcal{K}]||_{\mathcal{L}(L_\rho^2)} + ||\mathcal{K}||_{\mathcal{L}(L_\rho^2)}^2 \right) \right.$$
$$\left. + C_2 \left(\frac{1}{2} + ||\mathcal{K}||_{\mathcal{L}(L_\rho^2)} \right) + 4 \left(M^2 + \frac{C_2}{3}(3 + 2\pi^2) \right) \right) < \frac{1}{3} \tag{1.8}$$

$$M < \frac{1}{3\sqrt{C_\rho}} \cdot \frac{1513}{1044} \cdot \frac{1}{\left(1 + 2||\mathcal{K}||_{\mathcal{L}(L_\rho^{2,\frac{1}{2}})} \right)} \tag{1.9}$$

Remark: Given any positive function $f \in C^\infty((0, \infty))$ such that there exists $T_f > 100$ so that

$$\frac{|f^{(k)}(t)|}{f(t)} \leqslant \frac{C_{f,k}}{t^k}, \quad k \geqslant 1, \quad t \geqslant T_f \tag{1.10}$$

if, for $d > 0$, we define λ by

$$\lambda(t) = f(t^{\frac{1}{d}}), t \geqslant 50$$

then, $\lambda \in \Lambda$ for d chosen sufficiently large so that the smallness constraints (1.7) through (1.9) are satisfied. The main theorem of this paper is the following.

Theorem 1 *For all $\lambda \in \Lambda$, there exists $T_0 > 0$ such that there exists a finite energy solution, u, to (1.1) for $t \geqslant T_0$, satisfying the following properties.*

$$u(t, r) = Q_{\frac{1}{\lambda(t)}}(r) + u_e(t, r) + v_{rad}(t, r) \tag{1.11}$$

where

$$-\partial_t^2 v_{rad} + \partial_r^2 v_{rad} + \frac{1}{r}\partial_r v_{rad} - \frac{v_{rad}}{r^2} = 0, \quad E(v_{rad}, \partial_t v_{rad}) < \infty$$

and

$$E\left(u_e, \partial_t\left(Q_{\frac{1}{\lambda(t)}} + u_e\right)\right) \leqslant \frac{C\log^2(t)}{t^{2-2C_u}}$$

where C_u is as in (1.6)

Remark 1. The function u_e appearing in (1.11) satisfies

$$u_e(t,r) = u_{e,0}(t,r) + v_{6,0}(t,r)$$

where $u_{e,0}$ is fairly explicit, and $v_{6,0}$ is constructed with a fixed point argument, and satisfies

$$(t,r,\theta) \mapsto e^{i\theta} v_{6,0}(t,r) \in C_t^0\left([T_0,\infty), H^2(\mathbb{R}^2)\right)$$

$$(t,r,\theta) \mapsto e^{i\theta} \partial_t v_{6,0}(t,r) \in C_t^0\left([T_0,\infty), H^1(\mathbb{R}^2)\right)$$

where (r,θ) are polar coordinates on $\mathbb{R}^2$. This follows from the continuity of dilation on L^2, and Lemma 10.1 of [16].

Remark 2. Our class of solutions includes infinite time relaxation solutions, in other words, solutions of the form (1.11), for $\lambda(t) \to \infty$ as t approaches infinity. For example, we can apply the remark before Theorem 1 to $f(t) = t$, to get that

$$\lambda_1(t) = t^{C_u} \quad t \geqslant 50$$

is in Λ, for $C_u > 0$ sufficiently small.

Remark 3. We also have infinite time blow-up solutions, obtained by applying the remark before Theorem 1 to $f(t) = t^{-1}$, which shows that

$$\lambda_2(t) = t^{-C_l}, \quad t \geqslant 50$$

is in Λ, for $0 < C_l$ sufficiently small.

Remark 4. (Oscillatory λ, part 1). If $0 < \lambda_0 \leqslant \lambda_1$ are two real numbers, and $0 \leqslant \alpha_0, \alpha_1$ with $\alpha_1 + \alpha_0 < 1$, define λ by

$$\lambda(t) = \frac{\lambda_0 \log^{-\alpha_0}(t) + \lambda_1 \log^{\alpha_1}(t)}{2} + \frac{\left(\lambda_1 \log^{\alpha_1}(t) - \lambda_0 \log^{-\alpha_0}(t)\right)}{2} \sin(\log(\log(t)))$$

Then, $\lambda(t) \geqslant \lambda_0 \log^{-\alpha_0}(t)$. Therefore,

$$\frac{|\lambda^{(k)}(t)|}{\lambda(t)} \leqslant \frac{C_k}{t^k \log^{-\alpha_0 - \alpha_1 + 1}(t)}, \quad t \geqslant 50, \quad k \geqslant 1$$

Since $\alpha_0 + \alpha_1 < 1$, there exists $T_\lambda > 100$ so that (1.8),(1.9), and (1.7) are satisfied for $t \geqslant T_\lambda$ (for appropriate choices of C_u, C_l, C_2). So, $\lambda \in \Lambda$ and, by considering any combination of

$$\alpha_0 = 0 \text{ or } \alpha_0 > 0, \quad \text{and } \alpha_1 = 0 \text{ or } \alpha_1 > 0$$

we can satisfy any combination of the following

$$\liminf_{t\to\infty} \lambda(t) = 0 \text{ or } \liminf_{t\to\infty} \lambda(t) = \lambda_0 \quad \text{and } \limsup_{t\to\infty} \lambda(t) = \lambda_1 \text{ or } \limsup_{t\to\infty} \lambda(t) = \infty$$

Remark 5. (Oscillatory λ, part 2). We have solutions of the form (1.11), for $\lambda(t)$ with bounded (damped or undamped), or unbounded oscillations for all t sufficiently large. To ease notation, let

$$f(x,y,z) := x(1 + 2||\mathcal{K}||_{\mathcal{L}(L_\rho^2)}) + y\left(\frac{1}{4} + 2||[\xi\partial_\xi, \mathcal{K}]||_{\mathcal{L}(L_\rho^2)} + ||\mathcal{K}||_{\mathcal{L}(L_\rho^2)}^2\right) + z\left(\frac{1}{2} + ||\mathcal{K}||_{\mathcal{L}(L_\rho^2)}\right) + 4\left(y + \frac{z}{3}(3 + 2\pi^2)\right)$$

so that the constraint (1.8) is

$$\sqrt{C_\rho} \cdot \frac{179}{267} f(M, M^2, C_2) < \frac{1}{3}$$

We can, for example, let

$$0 < c_0 < \min\{\frac{2}{73}, \frac{4m}{2m+9}\}$$

and

$$0 \leqslant |a| < \frac{c_0}{2 - c_0}$$

where

$$m = \min\{\frac{1513}{3132\sqrt{C_\rho}(1 + 2||\mathcal{K}||_{\mathcal{L}(L_\rho^{2,\frac{1}{2}})})}, \frac{89}{179\sqrt{C_\rho}f(1,1,1)}\}.$$

Then, define

$$C_u = |a| + \frac{c_0}{2 - c_0} = C_l, \quad C_2 = \left(|a| + \frac{2c_0}{2 - c_0}\right)(1 + |a|)$$

and

$$\lambda_3(t) := t^a (2 + c_0 \sin(\log(t))), \quad t \geqslant 50$$

For instance $C_2 \geqslant M \geqslant M^2$. We thus have $f(M, M^2, C_2) \leqslant C_2 f(1, 1, 1)$, and this shows that (1.8),(1.9), and (1.7) are satisfied, and $\lambda_3 \in \Lambda$. When $a = 0$, this is an example of $\lambda(t)$ undergoing undamped oscillations while staying in a bounded set for all t sufficiently large, if $a < 0$, λ undergoes damped oscillations, and if $a > 0$, $\lambda(t)$ oscillates, while staying positive, but is unbounded.

Remark 6. If f_j satisfy (1.10) for $j = 1, 2$, then, since $f_j(t) > 0$, we have, for $k \geqslant 1$, and $t \geqslant T_{f_1} + T_{f_2}$,

$$\frac{|(f_1 + f_2)^{(k)}(t)|}{f_1(t) + f_2(t)} \leqslant \frac{|f_1^{(k)}(t)|}{f_1(t)} + \frac{|f_2^{(k)}(t)|}{f_2(t)} \leqslant \frac{D_k}{t^k}$$

Therefore, by the remark before Theorem 1, we get

$$\lambda_5(t) = (f_1 + f_2)(t^{\frac{1}{d}}) \in \Lambda, \quad d > 0, \text{ sufficiently large}$$

In particular, for any $c_0 > 0$, some sufficiently small $c_1, a > 0$, and a sufficiently large $d > 0$,

$$c_0 + t^{-|a|}\left(2 + c_1 \sin(\frac{\log(t)}{d})\right) \in \Lambda$$

Remark 7. We provide a quick comparison and contrast of the method used in this work and the previous work of the author [25]. In [25], the leading order part of $\lambda(t)$, say $\lambda_0(t)$ was chosen from an appropriate class of functions. Then, the radiation v_2 was inserted into the ansatz by hand, and its data was chosen so as to allow $\lambda(t) = \lambda_0(t)$ to be the leading order solution to an equation resulting from enforcing that the principal part of an error term is orthogonal to $\phi_0(\frac{\cdot}{\lambda(t)})$.

Here, the exact $\lambda(t)$ is prescribed from the beginning of the argument. An accurate approximate solution is obtained by constructing approximate solutions for small and large r, matching them in an intermediate region, and then, writing a function which interpolates between these two approximate solutions (see Section 3 for more information). This method was inspired by the discussions (in the textbooks of Nayfeh, [23], and Bender-Orszag, [3]) of matched asymptotic expansions for one-dimensional boundary value problems with a singular perturbation.

In order to be able to match our approximate solutions in the intermediate region, we need to use general solutions to various inhomogeneous wave equations, for example a particular solution which has zero Cauchy data at infinity, plus a free wave (radiation). The radiation component of our solution in this work therefore naturally arises from the fact that we need to use general solutions to inhomogeneous wave equations when doing matching, rather than being inserted into the ansatz by hand. The relation between the radiation component of our solution and $\lambda(t)$ is thus determined by a matching condition, rather than by enforcing an orthogonality condition. Despite the very different approaches to the construction of the ansatz, and different methods of determining the relation between $\lambda(t)$ and the radiation, we still have the same leading order relation in this work between $\lambda(t)$ and the radiation (for example, when we restrict attention to $\lambda(t) = \frac{1}{\log^b(t)}$ for $b > 0$ so that λ is in the admissible class of rates from [25]) as we had in [25]. (See the discussion following (3.4)).

Finally, the method of completion of our ansatz to an exact solution to (1.1) uses a simpler version of the same argument used in the previous work of the author [25]. (As we will describe in the summary of the proof, there is no orthogonality condition on the error term of our ansatz here, as opposed to [25], and this is what makes the iteration here simpler, at the expense of requiring a more accurate ansatz). We remark again that this final step of completing the ansatz to an exact solution uses the distorted Fourier transform, and the transference identity from [16].

Now, we briefly mention previous results which are related to this work. For the energy critical wave maps problem with $\mathbb{S}^m$ target, the work [32] of Tao proved global regularity at small energies. The works

[30] and [31], of Sterbenz and Tataru established a threshold theorem for the energy critical wave maps equation with a general compact Riemannian manifold target. The work [4], of Cote, Kenig, Lawrie, and Schlag, proved a threshold theorem for the 1-equivariant, energy critical wave maps problem with degree zero data, and a refined threshold, which accounts for the topological degree. The work [22], of Lawrie and Oh proved an analogous result, but without the equivariance restriction.

The works of Jendrej, and Jendrej, Lawrie [11], [13], constructed and classified, respectively, topological degree zero, threshold energy solutions to the k-equivariant wave maps equation for $k \geqslant 2$. The subsequent works of Jendrej and Lawrie, [12] and [14] give a more precise classification of the topological degree 0, threshold energy solutions to the k-equivariant critical wave map problem with $\mathbb{S}^2$ target, for $k \geqslant 4$. The work, [28], of Rodriguez, classifies threshold energy degree 0 solutions to the 1-equivariant wave maps problem, in particular obtaining a finite-time blow-up solution in this setting.

As previously mentioned, the work [16] of Krieger, Schlag, and Tataru constructed finite time blow-up solutions to (1.1) with a continuum of possible rates. The subsequent work of Gao and Krieger, [8], extended the set of solutions constructed in [16] to include ones for which $\lambda(t) = t^{1+\nu}$, for $\nu > 0$. The stability of these solutions under equivariant perturbations was studied in the work of Krieger and Miao [18], and under non-equivariant perturbations, in the recent work of Krieger, Miao, and Schlag, [19]. The works [17] and [21] of Krieger, Schlag, and Tataru are analogs of [16] for an equivariant reduction of the $4+1$-dimensional Yang-Mills equation with gauge group $SO(4)$, and the $3+1$-dimensional quintic, focusing semilinear wave equation (for which the work [7] of Donninger, Huang, Krieger, and Schlag also constructed finite time blow-up solutions with soliton length scale given by $\lambda(t) = t^{1+\nu} e^{\epsilon_0 \sin(\log(t))}$, with $\nu > 3, |\epsilon_0| \ll 1$). The work [20] extends the solutions of [21], analogously to [8] for the solutions in [16]. We also remark that the work of Perelman [24] constructs solutions of a similar form to those of [16], for the 1-equivariant Schrodinger map problem with domain $\mathbb{R}^{1+2}$ and target $\mathbb{S}^2$. In addition, the work of Bahouri, Marachli, and Perelman, [1], constructs finite time blow-up solutions to the vanishing mean curvature flow in Minkowski space, with blow-up rate given by a power of t, using a matching procedure. In addition, the work of Rodnianski and Sterbenz, [27], constructed finite time blow-up solutions to the k equivariant wave maps problem, for $k \geqslant 4$. Finite time blow-up solutions to the critical wave maps equation in all equivariance classes, as well as for the $4+1$ dimensional Yang-Mills equation with gauge group $SO(4)$ were constructed in the work [26] of Raphael and Rodnianski. The work [15] of Jendrej, Lawrie, and Rodriguez also constructed new finite time blow-up solutions to (1.1). The work of Bejenaru, Krieger, and Tataru, [2] constructed solutions to (1.1) with energy close to that of Q_1, and whose modulated soliton component has length scale bounded away from 0 and infinity, for all time.

The work [6], of Donninger and Krieger, constructed infinite time blow-up and infinite time relaxation solutions to the quintic, focusing, energy critical semilinear wave equation on $\mathbb{R}^{1+3}$, with rates $\lambda(t) = t^\mu$, where $|\mu|$ is sufficiently small, but μ can be positive or negative. The procedure used in this work is quite different than that used in [6]. We also note that the work of Gustafson, Nakanishi, and Tsai, [10], constructs solutions to the 2-equivariant harmonic map heat flow with soliton length scale having several possible asymptotic behaviors, including approaching zero, a positive constant, infinity, or having various combinations of finite or infinite $\limsup$ with positive or zero $\liminf$ as t approaches infinity.

1.1 Acknowledgments

2 Notation

2.1 Index of terms in the ansatz

Our final ansatz will involve several functions, which are listed here, along with the references to the equations they solve.

Function	Definition	Purpose
u_{ell}	(4.6)	Corrects $\partial_t^2 Q_{\frac{1}{\lambda(t)}}$ for small r
$u_{ell,2}$	(4.8)	Corrects linear error term of u_{ell}
u_w	(4.19)	Corrects $\partial_t^2 Q_{\frac{1}{\lambda(t)}}$ for large r
$u_{w,2}$	(4.49)	Corrects part of the linear error term of u_w (and contributes to matching)
$v_{2,2}$	(4.50)	Allows for third order matching
$f_{5,0}$	(4.156)	Eliminates the part of the matching error term given in (4.152)
$f_{ex,sub}$	(4.157)	Corrects part of the linear error term of u_w (but does not contribute to matching)
$f_{ell,2}$	Lemma 37	Corrects part of linear error term of $u_{ell,2}$
$f_{2,2}$	(4.166)	Corrects linear error term of $v_{2,2}$
u_{N_0}	(4.173)	Corrects principal part (given in (4.172)) of first set of nonlinear interactions
$u_{N_0,corr}$	(4.181) − (4.183)	Corrects the linear error term of u_{N_0}
$u_{N_0,corr,2}$	(4.196)	Corrects the linear error term of part of $u_{N_0,corr}$
u_{N_2}	(4.207)	Corrects the principal part of the second set of nonlinear interactions.

The first five corrections in the list above are combined into u_e and u_{wave}, which are defined in (4.139), and are further combined into u_c, which is defined in (4.141). u_c is combined with other corrections, to form u_a, which is defined in (4.167). The sum of the rest of the terms in the listing above are combined into the sum of u_n (defined in (4.203)), and u_{N_2}).

If $f \in C^k(D)$, and $a \in D$, we let $P_{k,a}(f)(x) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!}(x-a)^j$.

We use the notation

$$||f||_{\dot{H}_e^1}^2 = ||\partial_r f||_{L^2(rdr)}^2 + ||\frac{f}{r}||_{L^2(rdr)}^2$$

We let $m_{\leqslant 1}$ denote the following cutoff function.

$$m_{\leqslant 1}(x) \in C^\infty([0,\infty)), \quad 0 \leqslant m_{\leqslant 1}(x) \leqslant 1, \quad m_{\leqslant 1}(x) = \begin{cases} 1, & x \leqslant \frac{1}{2} \\ 0, & x \geqslant 1 \end{cases} \tag{2.1}$$

We will use the notation

$$I_f(s,y,\rho,\theta) = \partial_2 f(s,\sqrt{\rho^2+y^2+2\rho y \cos(\theta)})\frac{(y+\rho\cos(\theta))^2}{(\rho^2+y^2+2\rho y \cos(\theta))}$$
$$+ \frac{f(s,\sqrt{\rho^2+y^2+2\rho y \cos(\theta)})}{\sqrt{\rho^2+y^2+2\rho y \cos(\theta)}}\left(1 - \frac{(y+\rho\cos(\theta))^2}{\rho^2+y^2+2\rho y \cos(\theta)}\right), \quad s > M, y, \rho \geqslant 0, y \neq \rho, \theta \in [0,2\pi]$$
$$\tag{2.2}$$

for various choices of $M \gg 1$, and functions $f : (M,\infty) \times [0,\infty) \to \mathbb{R}$ throughout the paper.

We denote, by I_n and K_n, the modified Bessel functions of the first and second kind, respectively. We use the standard definition of $\langle \cdot \rangle$, namely

$$\langle x \rangle = \sqrt{1+x^2}$$

The wave maps equation, (1.1), linearized around $Q_1(r)$ takes the form

$$-\partial_{tt}u - L^*Lu = 0, \quad L(f) = f'(r) - \frac{\cos(Q_1(r))}{r}f(r)$$

where $(\cdot)^*$ denotes the $L^2(rdr)$ adjoint. We will also use the notation

$$L_{\frac{1}{\lambda(t)}}(f) = f'(r) - \frac{\cos(Q_1(\frac{r}{\lambda(t)}))}{r}f(r)$$

We denote by e_2 and ϕ_0 the following two linearly independent solutions to $L^*L(u) = 0$:

$$e_2(R) = \frac{R^4 + 4R^2\log(R) - 1}{2R(R^2+1)}, \quad \phi_0(R) = \frac{2R}{1+R^2} \tag{2.3}$$

We will use $\hat{}$ to denote the Hankel transform of order 1:

$$\hat{f}(\xi) = \int_0^\infty J_1(r\xi)f(r)r\,dr$$

Recalling the definition of $||f||_{L_\rho^{2,\alpha}}$ given in the introduction, (1.5), we will use the notation $||A||_{\mathcal{L}(L_\rho^{2,\alpha})}$ to denote the operator norm of a bounded operator $A : L_\rho^{2,\alpha} \to L_\rho^{2,\alpha}$. We use the standard notation for the dilogarithm function.

$$\text{Li}_2(x) = \int_0^x -\frac{\log(1-y)}{y}dy$$

3 Summary of the Proof

We start by letting $\lambda \in \Lambda$, and considering, for t sufficiently large, the modulated soliton $Q_{\frac{1}{\lambda(t)}}(r)$. The error term of the modulated soliton is $\partial_t^2 Q_{\frac{1}{\lambda(t)}}(r)$. We fix $g(t) = t^\alpha$, where α satisfies (4.2). (A priori, one might not know what constraints α must satisfy, and one could leave α general, until the very end of the argument, at which point the constraints would be clear. This is what was originally done, but, this results in many long expressions that can be greatly simplified once the constraints (4.2) are imposed. This is why we impose the constraints from the beginning of the argument. We will provide some intuition about why (4.2) has its particular form throughout this section).

Our first step is to obtain an approximate solution whose error term is small for all r, to the following linear equation

$$-\partial_{tt} u_1 + \partial_{rr} u_1 + \frac{1}{r}\partial_r u_1 - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2} u_1 = \partial_t^2 Q_1(\frac{r}{\lambda(t)})$$

Our plan is to start with a function of the form

$$u_c(t,r) = \chi_{\leqslant 1}(\frac{r}{h(t)}) u_{smallr} + (1 - \chi_{\leqslant 1}(\frac{r}{h(t)})) u_{larger}, \quad h(t) := g(t)\lambda(t) \tag{3.1}$$

where u_{smallr} and u_{larger} are good approximate solutions for $r \lesssim h(t)$ and $h(t) \lesssim r$, respectively, and whose asymptotic expansions for large t in the region $r \sim h(t)$ "match". Here, by "match", we mean that the terms at various orders in the aforementioned expansions of our corrections take the form

$$\sum_{k,l=-|j|}^{j} q_{k,l}(t) r^k \log(r)^l$$

for various choices of $j \geqslant 0$, and the difference between these sums associated to u_{smallr} and u_{larger} is equal to zero (see sections 4.3, 4.5, 4.6, 4.7).

We start with a correction to the soliton error term that we will use for $r \lesssim h(t)$. (In other words, we start with a part of u_{smallr}.) More precisely, we consider the general solution (in general, in order to do the matching process described above, one must keep sufficiently many degrees of freedom in the various corrections, which will be later fixed when we impose the matching) to the ODE

$$\partial_r^2 u_{ell} + \frac{1}{r}\partial_r u_{ell} - \frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))}{r^2} u_{ell} = \partial_t^2 Q_{\frac{1}{\lambda(t)}}(r)$$

which does not have a singularity near the origin. (This ODE was also considered in the previous works [16], [26], but we will end up choosing a different solution to this ODE than what was considered in the aforementioned works). The general solution that does not have a singularity at the origin depends on one function of t, say, $c_1(t)$, see (4.7). The linear error term associated to u_{ell} is $\partial_t^2 u_{ell}(t,r)$. (Even though u_{ell} will appear in our ansatz only after being multiplied by $\chi_{\leqslant 1}(\frac{r}{h(t)})$, it is still useful for us to consider the linear error term associated to u_{ell} alone, when trying to improve the error term in the region $r \lesssim h(t)$). Strictly speaking, this error term depends on $c_1(t)$, which is not yet chosen, but we can still give the reader an idea of the size of the error term of u_{ell} by noting that

$$|\partial_t^2 u_{ell}(t,r)| \leqslant \begin{cases} \frac{Cr\lambda(t)}{t^4} + C\frac{r}{\lambda(t)}\sum_{j=0}^{2}\frac{|c_1^{(j)}(t)|}{t^{2-j}}, & r \leqslant \lambda(t) \\ \frac{C\lambda(t)}{r}\sum_{j=0}^{2}\frac{|c_1^{(j)}(t)|}{t^{2-j}} + \frac{Cr\lambda(t)}{t^4}(1 + \log(\frac{r}{\lambda(t)})), & r \geqslant \lambda(t) \end{cases}$$

Even for $r \leqslant \lambda(t)$, the error term of u_{ell} is not quite good enough for our purposes. (The error term, F_5, of our final ansatz, satisfies the following estimates, for some $\delta > 0$ (see Lemma 54)).

$$\frac{||F_5(t,r)||_{L^2(rdr)}}{\lambda(t)^2} \leqslant \frac{C_5 \log^{30}(t)}{t^{4+2\delta}}, \quad ||L_{\frac{1}{\lambda(t)}}F_5(t,r)||_{L^2(rdr)} \leqslant \frac{C\log^2(t)}{t^{2+5\alpha}} + \frac{C\log^6(t)}{t^{4+2\alpha-3C_u}} + \frac{C\log^{30}(t)}{t^{9/2-C_l-15C_u}} \tag{3.2}$$

Therefore, we correct the error term $\partial_t^2 u_{ell}(t,r)$, with a second term in u_{smallr}, by considering a particular solution to the following ODE (see (4.10)).

$$\partial_{rr} u_{ell,2}(t,r) + \frac{1}{r}\partial_r u_{ell,2}(t,r) - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2} u_{ell,2}(t,r) = \partial_t^2 u_{ell}(t,r)$$

Defining $u_{small,r} = u_{ell} + u_{ell,2}$ will turn out to be sufficient for us. (Strictly speaking, the error terms of u_c depend not only on the quality of the matching between u_{smallr} and u_{larger} (for the terms which involve

derivatives of $\chi_{\leqslant 1}$) but also on $c_1(t)$, since $u_{ell,2}$ depends on $c_1(t)$. Also, $c_1(t)$ is chosen during a second order matching process, and is not specifically chosen to make the error of $u_{ell,2}$ small, see section 4.6).

After defining $u_{ell,2}$, we start defining the various components of u_{larger}. In particular, we consider the solution to

$$-\partial_{tt} u_w + \partial_{rr} u_w + \frac{1}{r}\partial_r u_w - \frac{u_w}{r^2} = \partial_t^2 Q_1\left(\frac{r}{\lambda(t)}\right)$$

given by

$$u_w(t,r) = v_1(t,r) + v_2(t,r)$$

where v_1 solves

$$-\partial_{tt} v_1 + \partial_{rr} v_1 + \frac{1}{r}\partial_r v_1 - \frac{v_1}{r^2} = \partial_t^2 Q_1\left(\frac{r}{\lambda(t)}\right)$$

with 0 Cauchy data at infinity, and v_2 solves the following Cauchy problem

$$\begin{cases} -\partial_{tt} v_2 + \partial_{rr} v_2 + \frac{1}{r}\partial_r v_2 - \frac{v_2}{r^2} = 0 \\ v_2(0) = 0 \\ \partial_t v_2(0,r) = v_{2,0}(r) \end{cases}$$

where $v_{2,0} \in L^2(rdr)$ will be chosen later. We remark that, in general, one needs to use general solutions, rather than particular solutions, to the inhomogeneous equations defining corrections, if one wishes to use the matching procedure described above. We also remark that, if $\lambda'(t) \neq 0$, then, $(1-\chi_{\leqslant 1}(\frac{r}{h(t)}))u_w$ has infinite kinetic energy, as does $Q_{\frac{1}{\lambda(t)}}(r)$ (because $\partial_t Q_{\frac{1}{\lambda(t)}}(r) \notin L^2(rdr)$, for $\lambda'(t) \neq 0$). However, there is sufficient cancellation in the sum $v_1(t,r) + Q_{\frac{1}{\lambda(t)}}(r)$ for large r, so that $\partial_t \left((1-\chi_{\leqslant 1}(\frac{r}{h(t)}))v_1(t,r) + Q_{\frac{1}{\lambda(t)}}(r)\right) \in L^2(rdr)$, see (4.218). (The function w_1 appearing in (4.218) is such that $\partial_t \left((1-\chi_{\leqslant 1}(\frac{r}{h(t)}))(v_1(t,r) - w_1(t,r))\right) \in L^2(rdr)$).

Next, we do the "first order matching" as follows. As long as $\widehat{v_{2,0}}(\xi)$ satisfies, for example, that $|\widehat{v_{2,0}}(\xi)| \leqslant \frac{C}{\xi^3}$, $\xi \geqslant \frac{1}{100}$ (so that the integral below in (3.3) converges), the leading contributions of $u_{ell}(t,r)$, $v_1(t,r)$, and $v_2(t,r)$ in the region $r \sim h(t)$ are, respectively:

$$u_{ell,firstorder} = \frac{r}{\lambda(t)}\left(\frac{1}{2}\lambda'(t)^2 + \lambda(t)\lambda''(t) - \lambda(t)\lambda''(t)\log\left(\frac{r}{\lambda(t)}\right)\right)$$

$$v_{1,firstorder}(t,r) = r\left(\log(2) + \frac{1}{2}\right)\lambda''(t) + r\int_t^{2t}\frac{(\lambda''(s) - \lambda''(t))}{s-t}ds + r\lambda''(t)\log\left(\frac{t}{r}\right) + r\int_{2t}^{\infty}\frac{\lambda''(s)ds}{(s-t)}$$

$$v_{2,firstorder} = \frac{-r}{4}\left(-2\int_0^{\infty}\xi\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi\right) \qquad (3.3)$$

Note that the $r\log(r)$ terms from $u_{ell,firstorder}$ and $v_{1,firstorder}$ already match. Therefore, it is possible to achieve

$$u_{ell,firstorder} = v_{1,firstorder} + v_{2,firstorder}$$

by choosing $v_{2,0}$ appropriately. In particular, we choose $v_{2,0}$ to satisfy the following equation, for all sufficiently large t:

$$\begin{aligned} F(t) &:= -2\int_0^{\infty}\xi\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi \\ &= 4\left(\left(\log(2) - \frac{1}{2}\right)\lambda''(t) + \int_t^{2t}\left(\frac{\lambda''(s) - \lambda''(t)}{s-t}\right)ds + \lambda''(t)\log\left(\frac{t}{\lambda(t)}\right) + \int_{2t}^{\infty}\frac{\lambda''(s)ds}{s-t} - \frac{\lambda'(t)^2}{2\lambda(t)}\right) \end{aligned} \qquad (3.4)$$

In particular, we have

$$\widehat{v_{2,0}}(\xi) = \frac{-1}{\pi\xi}\int_0^{\infty}H(t)\sin(t\xi)dt$$

with

$$H(t) = 4\left(\left(\log(2) - \frac{1}{2}\right)\lambda''(t) + \int_t^{2t}\left(\frac{\lambda''(s) - \lambda''(t)}{s-t}\right)ds + \lambda''(t)\log\left(\frac{t}{\lambda(t)}\right) + \int_{2t}^{\infty}\frac{\lambda''(s)ds}{s-t} - \frac{\lambda'(t)^2}{2\lambda(t)}\right)\psi(t)$$

where ψ is a relatively unimportant cutoff defined in (4.42).

The function v_2 is the leading part of our radiation in the matching region. (The other free waves $v_{2,2}$ and $v_{2,4}$ which are part of our ansatz and added later on in the argument, have much more decay in t in the entire region, for instance, $r \leqslant \frac{t}{2}$, and contribute to higher-order matching). Therefore, the relation

between the leading part of the radiation in the matching region, and $\lambda(t)$ is determined by the matching of leading terms from parts of u_{smallr} with corresponding terms from u_{larger}.

Note that the integral on the left-hand side of (3.4) is the precise integral which determined the relation between the radiation and the leading order dynamics of $\lambda(t)$ in the previous work of the author, [25] (see Section 3, pg. 6 of [25], keeping in mind that $K_1(x) = \frac{1}{x} + O\left(x|\log(x)|\right), \quad x \to 0$), though it arose from very different considerations, namely the inner product of the linear error term of v_2 with $\phi_0(\frac{\cdot}{\lambda(t)})$ rather than the near origin behavior of v_2. The fact that there is a connection between these two quantities also appears in (4.166), pg. 34 of [25].

Despite the very different approaches between the two works, if we consider the example, for $b > 0$, $\lambda(t) = \lambda_b(t) = \frac{1}{\log^b(t)}$, then, λ_b is in the admissible class of λ in the work [25] (as described in the discussion following (1.4)), and the leading order relation between $\lambda_b(t)$ and the radiation of this work is the same as that of [25] (see the third equation of pg. 6 of [25]).

We provide some intuition on why the α appearing in $g(t) = t^\alpha$ has the constraints (4.2). Note that, if $u_{ell}(t, r)$ and $u_w(t, r)$ are roughly of the same size in a region $r \sim q(t)$ for some $q(t) \gg \lambda(t)$, then, firstly, the linear error term of $u_{ell}(t, r)$ is $\partial_t^2 u_{ell}(t, r)$, and is roughly on the order of $\frac{u_{ell}(t,r)}{t^2}$. The linear error term of $u_w(t, r)$ is $\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)}))-1}{r^2} \right) u_w(t, r)$, is roughly $\frac{\lambda(t)^2}{q(t)^4} u_w(t, r)$, for $r \sim q(t)$. Since we assume that $u_{ell}(t, r)$ and $u_w(t, r)$ are roughly of the same size for $r \sim q(t)$, the linear error terms would be of comparable size if

$$ q(t) \sim \sqrt{t\lambda(t)} $$

This is not quite exactly what we have, but provides some intuition on why it is natural to expect that $\frac{1}{2} + \frac{C_l}{2}$ appears as part of (4.2). In addition, we choose to eliminate certain error terms of borderline size which become large if $g(t)$ is taken too large. In this way, certain other error terms of borderline size can be made perturbative by having $\alpha > \frac{1}{2} + \frac{C_l}{2}$. On the other hand, some other, less delicate errors become too large if $g(t)$ is taken to be too large, which is why (4.2) involves an upper bound on α as well.

We then add a second term in u_{larger}, namely, $u_{w,2} + v_{2,2}$ (see (4.49)), which corrects the linear error term associated to $w_1 + v_2$, where w_1 is part of v_1, and is defined in (4.22). We also remark that $u_{w,2}$ is a particular solution to an inhomogeneous wave equation (and it has zero Cauchy data at infinity), while $v_{2,2}$ is a free wave, which we will choose later on, as part of a higher-order matching process.

Next, we consider the terms in an expansion of u_{larger} and u_{smallr} in the matching region, which are roughly of size $r^3 \log^k(r)$, multiplied by expressions that involve roughly four derivatives of $\lambda(t)$. In particular, we note that $u_{ell,2}$, w_1, and v_2, respectively, contribute the following terms when expanded in the matching region.

$$ v_{ell,2,0,main}(t, \frac{r}{\lambda(t)}) = \frac{r^3}{8} \partial_t^2 \left(\frac{\lambda'(t)^2}{2\lambda(t)} + \lambda''(t) - \lambda''(t) \log(\frac{r}{\lambda(t)}) \right) + \frac{3}{32} r^3 \lambda''''(t) $$

$$ w_{1,cubic,main}(t, r) = \frac{3}{32} r^3 \lambda''''(t) + \frac{r^3}{8} \left(\lambda''''(t) \left(\log(2) + \frac{1}{2} \right) - \log(r) \lambda''''(t) + \log(t) \lambda''''(t) \right. $$
$$ \left. + \int_t^{2t} \frac{\lambda''''(s) - \lambda''''(t)}{s - t} ds + \int_{2t}^{\infty} \frac{\lambda''''(s) ds}{s - t} \right) $$

$$ v_{2,cubic,main}(t, r) = \frac{-r^3}{32} F''(t) $$

Note that these quantities do not involve any degrees of freedom (like, for instance, integration constants, or free waves) which can be tuned so as to guarantee matching. In fact, the coefficient of r^3 in $v_{2,cubic,main}$ is precisely one-eighth of two time derivatives of the r coefficient in $v_{2,firstorder}$ (recall (3.3)). On the other hand, the r^3 coefficient of $v_{ell,2,0,main}(t, \frac{r}{\lambda(t)})$ and that of $w_{1,cubic,main}(t, r)$ are not precisely one-eighth of two time derivatives of the r coefficients of $u_{ell,firstorder}(t, r)$ and $v_{1,firstorder}(t, r)$, respectively. However, the r^3 coefficient of the *difference* of these two functions is precisely one-eighth of two time derivatives of the r coefficient of $u_{ell,firstorder}(t, r) - v_{1,firstorder}(t, r)$. Note the important cancellation between the $\frac{3}{32} r^3 \lambda''''(t)$ terms when $v_{ell,2,0,main}(t, \frac{r}{\lambda(t)})$ and $w_{1,cubic,main}(t, r)$ are subtracted. Therefore, the matching of the r^3 terms is already accomplished with our above choice of $v_{2,0}$. In other words, by the choice of $v_{2,0}$, we have

$$ v_{ell,2,0,main}(t, \frac{r}{\lambda(t)}) - \left(w_{1,cubic,main}(t, r) + v_{2,cubic,main}(t, r) \right) = 0 $$

Next, we consider the terms arising from expansions of u_{smallr} and u_{larger} in the matching region which are roughly of size $\frac{\log^k(r)}{r}$ multiplied by roughly two derivatives of $\lambda(t)$. These terms coming from u_{ell}, u_w,

and $u_{w,2}$, respectively, are

$$v_{ell,sub,cont}(t,r)$$
$$= \frac{c_1(t)\lambda(t) - \frac{3}{2}\lambda(t)\lambda'(t)^2}{r}$$
$$- \frac{\lambda(t)^2\lambda''(t)\left(-6\log(r)(4\log(\lambda(t)) + 1) + 6\log(\lambda(t))(2\log(\lambda(t)) + 1) + 12\log^2(r) + \pi^2 + 12\right)}{6r}$$

$$v_{ex,cont}(t,r) = \frac{1}{2r}\left(\log(r)\left(-4\lambda(t)\lambda'(t)^2 - 2\lambda(t)^2\lambda''(t)\right) + 4\lambda(t)\log(\lambda(t))\lambda'(t)^2 - \lambda(t)^2\lambda''(t) + 2\lambda(t)^2\lambda''(t)\log(\lambda(t))\right)$$

and

$$u_{w,2,ell,0,cont}(t,r) = \frac{-2}{r}\left(\log(r)\left(-\lambda(t)\lambda'(t)^2 - \lambda(t)^2\lambda''(t) - 2\lambda(t)^2\log(\lambda(t))\lambda''(t)\right)\right.$$
$$+ \lambda(t)\log(\lambda(t))\lambda'(t)^2 + \lambda(t)^2\log(\lambda(t))\lambda''(t) + \lambda(t)^2\log^2(\lambda(t))\lambda''(t)$$
$$\left. + \lambda(t)^2\log^2(r)\lambda''(t) + \frac{1}{2}\lambda(t)^2\lambda''(t) - \frac{1}{12}\pi^2\lambda(t)^2\lambda''(t)\right)$$

Note that the only free parameter we can choose in this expression is c_1, which multiplies $\frac{\lambda(t)}{r}$. On the other hand, both $u_{w,2,ell,0,cont}$ and $v_{ex,cont}$ involve $\frac{\log^k(r)}{r}$ terms, for $k > 0$. It therefore appears that it is not possible to enforce

$$v_{ell,sub,cont}(t,r) = v_{ex,cont}(t,r) + u_{w,2,ell,0,cont}(t,r)$$

by choosing $c_1(t)$ appropriately. However, it turns out that the $\frac{\log(r)}{r}$ and $\frac{\log^2(r)}{r}$ terms from $v_{ell,sub,cont}$ happen to exactly match those from $v_{ex,cont} + u_{w,2,ell,0,cont}$, which therefore allows us to choose $c_1(t)$ as just discussed. (This "automatic" matching of the logarithmically higher order terms for large r is reminiscent of what we saw at the first order as well, recall the remark after (3.3)). In other words, we have

$$v_{ell,sub,cont}(t,r) - \left(u_{w,2,ell,0,cont}(t,r) + v_{ex,cont}(t,r)\right) = -\frac{\lambda(t)\left(-6c_1(t) + \left(3 + 2\pi^2\right)\lambda(t)\lambda''(t) + 9\lambda'(t)^2\right)}{6r}$$

which vanishes if

$$c_1(t) := \frac{3}{2}\lambda'(t)^2 + \frac{\lambda(t)\lambda''(t)}{2} + \frac{\pi^2}{3}\lambda(t)\lambda''(t)$$

From the point of view of constructing an approximate solution of the form (3.1), the matching conditions are needed to reduce the size of error terms involving derivatives of $\chi_{\leqslant 1}\left(\frac{r}{h(t)}\right)$. Our three matching conditions thus far reduce these error terms, but not quite enough, so that we need to do a third order matching. We remark that the previous two matching conditions are called second order matching, part 1 and 2 in the paper, since, if $\overline{g}(t) = \sqrt{\lambda(t)}t$ (again, we do not quite exactly choose $g(t) = \overline{g}(t)$, but $\overline{g}$ is the scale at which the elliptic and wave corrections are expected to have comparably sized error terms) then, the terms from the previous two matching conditions are of comparable size:

$$\frac{\overline{g}(t)^3\lambda(t)}{t^4} = \frac{\lambda(t)^{\frac{5}{2}}}{t^{5/2}}, \quad \text{and} \quad \frac{\lambda(t)^3}{t^2 \cdot \overline{g}(t)} = \frac{\lambda(t)^{5/2}}{t^{5/2}}$$

The third order matching involves comparing terms from $u_{ell,2}$, v_1, $u_{w,2}$, and $v_{2,2}$, which are on the order of $r\log^k(r)$, for $0 \leqslant k \leqslant 3$, multiplied by terms involving roughly four derivatives of λ. The computations of these terms are lengthy, and the terms themselves result in very long expressions, which we will not reproduce here. We remark that the $u_{ell,2}$ contributions are given in (4.129), and the v_1 and $u_{w,2}$ contributions are individually computed in Lemmas 12, 13, 15, (4.126) combined with Lemma 22, and (4.119), combined with Lemma 19. A very careful inspection reveals that, for $j = 1, 2, 3$ the $r\log^j(r)$ terms involved in (4.129) exactly match those terms from the v_1 and $u_{w,2}$ contributions (which we denote, in (4.130), by $u_{w,3}$) see (4.131). We therefore again see the "automatic" matching of logarithmically higher order in r (for large r) terms involved in the expansions. Precisely because of this "automatic" matching, we can choose the data for $v_{2,2}$ in a similar way for v_2 in order to do the third order matching, which is precisely stated in Proposition 2.

There are a few more corrections needed to be added to our ansatz in order to improve the linear terms, namely $f_{5,0}$, $f_{ex,sub}$, $f_{ell,2}$, and $f_{2,2}$. These corrections are not as delicate as those mentioned up to this point, so we refer the reader to Section 2.1 and the equation references therein. We also remark that the cutoff, $\chi_{\leqslant 1}(x) = 1 - \chi_{\geqslant 1}(x)$, where $\chi_{\geqslant 1}(x)$ is chosen in Lemma 27, so as to satisfy certain orthogonality conditions, for technical reasons.

At this stage, our ansatz is

$$u_a(t, r) = u_c(t, r) + f_{5,0}(t, r) + f_{ex,sub}(t, r) + f_{ell,2}(t, r) + f_{2,2}(t, r)$$

with

$$u_c(t, r) = \chi_{\leqslant 1}(\frac{r}{\lambda(t)g(t)})u_e(t, r) + \left(1 - \chi_{\leqslant 1}(\frac{r}{\lambda(t)g(t)})\right)u_{wave}(t, r)$$

and

$$u_e(t, r) = u_{ell}(t, r) + u_{ell,2}(t, r), \quad u_{wave}(t, r) = u_w(t, r) + u_{w,2}(t, r) + v_{2,2}(t, r)$$

The ansatz u_a has a linear error term, e_a, which is small enough for our purposes, see Lemma 40. We also remark that the radiation component of our ansatz thus far, is $v_2 + v_{2,2}$. On the other hand, the nonlinear error terms of u_a are not small enough. For example, the v_2 cubic self-interactions near the cone are roughly of the size

$$|\frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\sin(2v_2) - 2v_2)| \lesssim \frac{C}{t^2} \cdot \frac{1}{t^{3/2}}$$

More precisely, the set of nonlinear interactions between the terms of u_a is decomposed into $N_0 + N_1$, where N_1 is perturbative, and

$$N_0(t, r) = \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\sin(2u_{a,0}) - 2u_{a,0}) + \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\cos(2u_{a,0}) - 1)$$

where

$$u_{a,0}(t, r) = \chi_{\leqslant 1}(\frac{r}{h(t)})u_{ell}(t, r) + \left(1 - \chi_{\leqslant 1}(\frac{r}{h(t)})\right)(w_1(t, r) + v_2(t, r) + v_{2,2}(t, r))$$

First, we define u_{N_0} to be the solution to the following equation with zero Cauchy data at infinity.

$$-\partial_t^2 u_{N_0} + \partial_r^2 u_{N_0} + \frac{1}{r}\partial_r u_{N_0} - \frac{u_{N_0}}{r^2} = N_0(t, r) \tag{3.5}$$

The linear error term of u_{N_0} is $e_{N_0} = \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2}\right)u_{N_0}$, which is small for r such that $h(t) \lesssim r$ (see Lemma 43 for the precise details). Therefore, the addition of u_{N_0} reduces us to the task of eliminating an error term which is localized to the region $r \lesssim h(t)$. We carry out this task as follows. We start with solving the equation

$$\partial_{rr}u_{N_0,ell} + \frac{1}{r}\partial_r u_{N_0,ell} - \frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))}{r^2}u_{N_0,ell} = m_{\leqslant 1}(\frac{r}{2h(t)})e_{N_0}(t, r)$$

and then inserting the following function into the ansatz

$$u_{N_0,corr}(t, r) := \left(u_{N_0,ell}(t, r) - \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}\right)m_{\leqslant 1}(\frac{2r}{t}) + v_{2,4}(t, r)$$

where $v_{2,4}$ solves

$$-\partial_{tt}v_{2,4} + \partial_{rr}v_{2,4} + \frac{1}{r}\partial_r v_{2,4} - \frac{v_{2,4}}{r^2} = 0$$

and $v_{2,4}(t, r)$ matches $\frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}$ for small r. (where $m_{\leqslant 1}$ is a cutoff, and is otherwise unimportant). In particular, we choose the initial velocity, say $v_{2,5}$ of $v_{2,4}$ by requiring, for all t sufficiently large,

$$\frac{-r}{4} \cdot -2\int_0^\infty \xi \sin(t\xi)\widehat{v_{2,5}}(\xi)d\xi = \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}$$

This is analogous to how we chose the data for v_2. The point is that the error term of $v_{2,4}(t, r)$ is worst for small r, while that of $u_{N_0,ell}(t, r)$ is worst for large r. On the other hand, the error term of $v_{2,4}(t, r) - \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}$ is much smaller than the error term of $v_{2,4}(t, r)$ alone, for small r, and the error term of $u_{N_0,ell}(t, r) - \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}$ is much smaller than the error term of $u_{N_0,ell}(t, r)$ alone for large r. So, the term $\frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}$ cancels the worst behavior of both $v_{2,4}$ and $u_{N_0,ell}$ in the regions where the associated error terms are largest. This is again reminiscent of arguments related to matched asymptotic expansions for one-dimensional boundary value problems with a singular perturbation, see [23]. After adding one more term to our ansatz, namely $m_{\geqslant 1}(\frac{r}{t})u_{N_0,corr,2}(t, r)$, in

order to eliminate the linear error term associated to $v_{2,4}$ for large r, we have the improved ansatz $u_a + u_n$, where

$$u_n(t,r) := u_{N_0}(t,r) + u_{N_0,corr}(t,r) + m_{\geqslant 1}(\frac{r}{t})u_{N_0,corr,2}(t,r)$$

At this stage, the linear error terms of $u_a + u_n$ are perturbative, but, some nonlinear interactions are not. In particular, the correction u_n involves a free wave, $v_{2,4}(t,r)$, which has no more pointwise decay in t near the cone than v_2 did. The nonlinear error terms of $u_a + u_n$ are decomposed into $N_2 + N_3$, where

$$N_2(t,r) := \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(-2(u_{a0} + v_{2,4}) + \sin(2(u_{a,0} + v_{2,4})) - (\sin(2u_{a,0}) - 2u_{a,0}))$$
$$+ \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\cos(2u_{a,0} + 2v_{2,4}) - \cos(2u_{a,0}))$$

and the precise formula for N_3 is not important for the purposes of summarizing the main points of the argument. N_3 is defined in (4.205), and is perturbative.

We recall that, after adding u_a into our ansatz, we had perturbative linear error terms, and some non-perturbative nonlinear error terms, collected together in N_0. After adding the correction to N_0, namely u_n, we again have perturbative linear error terms, and some non-perturbative nonlinear error terms, and the pointwise in t decay of the nonlinear self interactions of $v_{2,4}(t,r)$ near the cone are no better than that of the nonlinear self interactions of $v_2(t,r)$ near the cone. It thus might not be clear why $u_a + u_n$ is an improvement over u_a. However, the point is that $v_{2,4}(t,r)$ has much more decay in the variable $\langle t - r \rangle$, in the region $\frac{t}{2} \leqslant r \leqslant t$, than does v_2 (compare Lemma 46, and Lemma 5). This allows us to eliminate N_2 with a correction, u_{N_2}, which solves the following equation with 0 Cauchy data at infinity.

$$-\partial_t^2 u_{N_2} + \partial_r^2 u_{N_2} + \frac{1}{r}\partial_r u_{N_2} - \frac{u_{N_2}}{r^2} = N_2(t,r)$$

It is possible to eliminate N_2 in this manner because the decay of $v_{2,4}(t,r)$ in the variable $\langle t - r \rangle$ inside the cone leads to u_{N_2} having decay in the variable $\langle t - r \rangle$ inside the light-cone, see Lemma 51. This decay of u_{N_2} is important because the linear error term of u_{N_2} is $\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)}))-1}{r^2} \right) u_{N_2}(t,r)$, which is thus most delicate in the region, for example, $r \leqslant \frac{t}{2}$, because

$$|\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) u_{N_2}(t,r)| \leqslant \frac{C\lambda(t)^2}{t^4}|u_{N_2}(t,r)|, \quad r \geqslant \frac{t}{2}$$

On the other hand, when $r \leqslant \frac{t}{2}$, then, $\frac{1}{\langle t-r \rangle} \leqslant \frac{C}{t}$, which means that the aforementioned decay of $u_{N_2}(t,r)$ in the variable $\langle t-r \rangle$, for $r < t$, makes the linear error term associated to u_{N_2} much smaller than otherwise.

Without this extra decay in the variable $\langle t-r \rangle$, we might have had to eliminate N_2 by using a matching process involving a free wave, as in $u_{N_0,corr}$, but then, the nonlinear self interactions of this free wave would yet again produce nonlinear error terms which are not perturbative from the point of view of our procedure. The key point here is that u_{N_2} has zero Cauchy data at infinity, and much more than $\frac{1}{\sqrt{t}}$ pointwise decay near the cone.

The linear error term of u_{N_2} is perturbative, see Lemma 52. Our final ansatz is

$$u_{ansatz}(t,r) = u_a(t,r) + u_n(t,r) + u_{N_2}(t,r)$$

and its nonlinear error terms are perturbative, see Lemma 53, and recall the importance of u_{N_2} eliminating N_2, while also not containing a free wave (which leads to improved decay of u_{N_2} near the cone).

At this stage, we are ready to complete the ansatz to an exact solution of (1.1). This step is done by following a simpler version of the analogous step in [25], with one extra detail. For completeness, we provide a summary of this step here. Substituting

$$u(t,r) = Q_1(\frac{r}{\lambda(t)}) + u_{ansatz}(t,r) + v_6(t,r)$$

into (1.1), we get

$$-\partial_t^2 v_6 + \partial_r^2 v_6 + \frac{1}{r}\partial_r v_6 - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}v_6 = F_5 + F_3(v_6) \tag{3.6}$$

where F_3 is defined in (5.2), and contains error terms involving v_6 both linearly and nonlinearly, and F_5 is the error term of u_{ansatz}, which we recall satisfies (3.2). We remark that (3.2) is satisfied for $\delta > 0$ partly

due to (1.7) and (4.2). We will solve (3.6) by first formally deriving the equation for y (namely (3.8)) given by

$$y(t, \xi) = \mathcal{F}(\sqrt{\cdot}\, v_6(t, \cdot\lambda(t)))(\xi\lambda(t)^2)$$

where $\mathcal{F}$ denotes the distorted Fourier transform of [16] (which is defined in section 5 of [16]). Then, we will prove that (3.8) admits a solution, say y_0 (with 0 Cauchy data at infinity) which has enough regularity to rigorously justify the statement that if v_6 given by the following expression, with $y = y_0$

$$v_6(t, r) = \sqrt{\frac{\lambda(t)}{r}} \mathcal{F}^{-1}\left(y(t, \frac{\cdot}{\lambda(t)^2})\right)\left(\frac{r}{\lambda(t)}\right), \tag{3.7}$$

then, v_6 is a solution to (3.6). We have (see also (5.4), (5.5), pg. 145 of [25])

$$\partial_{tt}y + \omega y = -\mathcal{F}(\sqrt{\cdot}\, F_5(t, \cdot\lambda(t)))(\omega\lambda(t)^2) + F_2(y)(t, \omega) - \mathcal{F}(\sqrt{\cdot}\, F_3(v_6(y))(t, \cdot\lambda(t)))(\omega\lambda(t)^2) \tag{3.8}$$

where $v_6(y)$, which appears in the argument of F_3, is the expression given in (3.7), and

$$F_2(y)(t, \omega) = -\frac{\lambda'(t)}{\lambda(t)}\partial_t y(t, \omega) + \frac{2\lambda'(t)}{\lambda(t)}\mathcal{K}\left(\partial_1 y(t, \frac{\cdot}{\lambda(t)^2})\right)(\omega\lambda(t)^2) + \left(\frac{-\lambda''(t)}{2\lambda(t)} + \frac{\lambda'(t)^2}{4\lambda(t)^2}\right) y(t, \omega)$$

$$+ \frac{\lambda''(t)}{\lambda(t)}\mathcal{K}\left(y(t, \frac{\cdot}{\lambda(t)^2})\right)(\omega\lambda(t)^2) + 2\frac{\lambda'(t)^2}{\lambda(t)^2}\left([\xi\partial_\xi, \mathcal{K}](y(t, \frac{\cdot}{\lambda(t)^2}))\right)(\omega\lambda(t)^2)$$

$$- \frac{\lambda'(t)^2}{\lambda(t)^2}\mathcal{K}\left(\mathcal{K}(y(t, \frac{\cdot}{\lambda(t)^2}))\right)(\omega\lambda(t)^2)$$

where $\mathcal{K}$ is the transference operator of [16] (which is defined in section 6 of [16]). We solve (3.8) in the space Z, defined in (5.8), by showing that the operator

$$T(y)(t, \omega) := \int_t^\infty \frac{\sin((x-t)\sqrt{\omega})}{\sqrt{\omega}}\left(-\mathcal{F}(\sqrt{\cdot}\,(F_5 + F_3(v_6(y)))(x, \cdot\lambda(x)))(\omega\lambda(x)^2) + F_2(y)(x, \omega)\right) dx \tag{3.9}$$

is a strict contraction on $\overline{B}_1(0) \subset Z$. This process essentially only uses Minkowski's inequality, as well as the following simple property of the density of the spectral measure of $\mathcal{F}$: (5.7). The reason why this iteration is simpler than that of [25] is that here, there is no orthogonality condition on the error which we need to exploit, unlike in [25]. On the other hand, there is one extra detail which is present here, that is not present in [25]. One of the estimates we have on F_2 is the following (see Lemma 56 for the complete set of estimates).

$$||F_2(t, \omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

$$\leq ||\partial_1 y(t, \omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \frac{|\lambda'(t)|}{\lambda(t)}\left(1 + 2||\mathcal{K}||_{\mathcal{L}(L^2_\rho)}\right)$$

$$+ ||y(t, \omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\left(\left(\frac{\lambda'(t)}{\lambda(t)}\right)^2\left(\frac{1}{4} + 2||[\xi\partial_\xi, \mathcal{K}]||_{\mathcal{L}(L^2_\rho)} + ||\mathcal{K}||^2_{\mathcal{L}(L^2_\rho)}\right) + \frac{|\lambda''(t)|}{\lambda(t)}\left(\frac{1}{2} + ||\mathcal{K}||_{\mathcal{L}(L^2_\rho)}\right)\right) \tag{3.10}$$

Using the symbol-type estimates on $\lambda(t)$, we see that the terms of (3.10) are of critical size, noting that (3.9) roughly loses two powers of t decay relative to its integrand. Therefore, we need the constants appearing in the symbol type estimates on λ to be sufficiently small in order to guarantee that (3.9) is a contraction. All of the terms estimated in Lemmas 57 and 56 with quantitative constants are where (1.7), (4.2), (1.8) and (1.9) are used. This detail does not appear in [25] because, there, $\frac{|\lambda^{(k)}(t)|}{\lambda(t)} \leq \frac{C_k}{t^k \log(t)}$.

4 Construction of the Ansatz

Let $\lambda \in \Lambda$. By definition of Λ, there exists $T_\lambda > 100$ such that (1.6) is true. Let $T_2 \geq e^{900}(1 + T_\lambda)$ be such that

$$t^{\frac{2}{3}}\lambda(t) < \frac{t}{100}, \quad t \geq T_2 \tag{4.1}$$

Let $T_0 \geq T_2$ be otherwise arbitrary. For the whole paper, we work in the region $t \geq T_0$, and C denotes a constant, *independent* of T_0, unless otherwise specified. We define g by

$$g(t) := t^\alpha$$

where α is any number satisfying

$$\alpha > \frac{1}{2} + \frac{C_l}{2}, \quad \alpha < \frac{2}{3} - \frac{5}{6}C_u - \frac{C_l}{6} \tag{4.2}$$

(Note that (1.7) implies that $\frac{2}{3} - \frac{5}{6}C_u - \frac{C_l}{6} > \frac{1}{2} + \frac{C_l}{2}$, so such α as above exists. We also remind the reader that some intuition behind this choice of g is given in Section 3.) We note that the definition of Λ implies that, for $x \geqslant t \geqslant T_\lambda$,

$$\left(\frac{x}{t}\right)^{-C_l} \leqslant \frac{\lambda(x)}{\lambda(t)} \leqslant \left(\frac{x}{t}\right)^{C_u}, \quad t \mapsto \frac{\lambda(t)}{t^{\frac{1}{30}}} \text{ is decreasing,} \quad \frac{\sup_{y \in [100,x]} (\lambda(y) \log(y))}{\sup_{y \in [100,t]} (\lambda(y) \log(y))} \leqslant C \left(\frac{x}{t}\right)^{C_u} \frac{\log(x)}{\log(t)} \tag{4.3}$$

All of the properties of our ansatz which will be used in the later sections are listed in the following proposition, whose proof is completed in this section.

Proposition 1 *[Approximate solution to (1.1)] There exists $T_3 > 0$ so that, for all $T_0 > T_3$, there exists $u_{ansatz} \in C^2 ([T_0, \infty) \times (0, \infty))$, satisfying Lemma 55 such that, if $u(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + u_{ansatz}(t, r)$, then,*

$$-\partial_t^2 u + \partial_r^2 u + \frac{1}{r}\partial_r u - \frac{\sin(2u)}{2r^2} = -F_5(t, r)$$

where, for

$$\delta := \frac{1}{2} min\{4\alpha - C_l - 2, 1 - 2C_u - \alpha, 4 - 5C_u - 6\alpha, \frac{1}{2} - 2C_l - 15C_u\}$$

and $C_5, C \geqslant 0$,

$$\frac{||F_5(t,r)||_{L^2(rdr)}}{\lambda(t)^2} \leqslant \frac{C_5 \log^{30}(t)}{t^{4+2\delta}}, \quad ||L_{\frac{1}{\lambda(t)}} F_5(t,r)||_{L^2(rdr)} \leqslant \frac{C \log^2(t)}{t^{2+5\alpha}} + \frac{C \log^6(t)}{t^{4+2\alpha-3C_u}} + \frac{C \log^{30}(t)}{t^{9/2-C_l-15C_u}}, \quad t \geqslant T_0$$

Moreover, there exists v_{rad} satisfying

$$-\partial_t^2 v_{rad} + \partial_r^2 v_{rad} + \frac{1}{r}\partial_r v_{rad} - \frac{v_{rad}}{r^2} = 0, \quad E(v_{rad}, \partial_t v_{rad}) < \infty$$

$$||\partial_t (Q_{\frac{1}{\lambda(t)}} + u_{ansatz} - v_{rad})||_{L^2(rdr)} \leqslant \frac{C \log(t)}{t^{1-C_u}}, \quad ||u_{ansatz} - v_{rad}||_{\dot{H}_e^1} \leqslant \frac{C \log(t)}{t^{1-C_u}} \tag{4.4}$$

Remark. The function u_{ansatz} is explicitly given in (4.209), where we refer the reader to section 2.1 for the definitions of the constituent functions. F_5 is explicitly given in (4.210) and v_{rad} is given in (4.216), in terms of $v_2, v_{2,2}$, and $v_{2,4}$, which are defined in (4.37) (with $v_{2,0}$ given in (4.43)), (4.50) (with $v_{2,3}$ given in (4.134)), and (4.189) (with $v_{2,5}$ given in (4.187)), respectively.

We begin by constructing an approximate solution to

$$-\partial_{tt}u_1 + \partial_{rr}u_1 + \frac{1}{r}\partial_r u_1 - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}u_1 = \partial_t^2 Q_1(\frac{r}{\lambda(t)}) \tag{4.5}$$

starting by matching explicit solutions to certain approximations of the operator on the left-hand side.

4.1 Small r corrections

4.1.1 First Iteration

We first consider the ODE

$$\partial_{rr}u_{ell} + \frac{1}{r}\partial_r u_{ell} - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}u_{ell} = \partial_t^2 Q_1(\frac{r}{\lambda(t)}), \quad u_{ell}(t, 0) = 0 \tag{4.6}$$

We get

$$u_{ell}(t, r) = v_{ell}(t, \frac{r}{\lambda(t)})$$

where

$$v_{ell}(t, R) = f_1(R)\lambda'(t)^2 + \lambda(t)\lambda''(t)f_2(R) + \frac{c_1(t)R}{1 + R^2} \tag{4.7}$$

and

$$f_1(R) = \frac{R(-2 + R^2)}{2(1 + R^2)}, \quad f_2(R) = \frac{R^2(-1 + 2R^2) - (-1 + R^4)\log(1 + R^2) + 2R^2 Li_2(-R^2)}{2R(1 + R^2)}$$

and c_1 will be chosen later.

4.1.2 Second iteration

We define a second order correction, $u_{ell,2}(t,r)$, which is useful in the region of small r, by the following solution to

$$\partial_{rr}u_{ell,2}(t,r) + \frac{1}{r}\partial_r u_{ell,2}(t,r) - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}u_{ell,2}(t,r) = \partial_t^2\left(v_{ell}(t,\frac{r}{\lambda(t)})\right) \tag{4.8}$$

$$u_{ell,2}(t,r) = v_{ell,2}(t,\frac{r}{\lambda(t)}) \tag{4.9}$$

where $v_{ell,2}$ is defined by

$$v_{ell,2}(t,R) = -\frac{\phi_0(R)}{2}\int_0^R \lambda(t)^2\partial_t^2\left(v_{ell}(t,\frac{r}{\lambda(t)})\right)\Big|_{r=s\lambda(t)} se_2(s)ds + e_2(R)\int_0^R \lambda(t)^2\partial_t^2\left(v_{ell}(t,\frac{r}{\lambda(t)})\right)\Big|_{r=s\lambda(t)}\frac{s\phi_0(s)}{2}ds \tag{4.10}$$

In order to compute explicit terms in an expansion of $v_{ell,2}(t,R)$ for large R, we let

$$v_{ell,lgR}(t,R) = \frac{1}{2}R\lambda'(t)^2 + R\lambda(t)(1-\log(R))\lambda''(t) \tag{4.11}$$

be the leading behavior of $v_{ell}(t,R)$ for large R (recall (4.7)), and define

$$err_0(t,r) = \lambda(t)^2\partial_t^2\left(v_{ell,lgR}(t,\frac{r}{\lambda(t)})\right) = r\lambda(t)^2\partial_t^2\left(\frac{\lambda'(t)^2}{2\lambda(t)} + \lambda''(t) - \lambda''(t)\log(\frac{r}{\lambda(t)})\right)$$

To understand $v_{ell,2}(t,R)$ for large R, we let

$$v_{ell,2,0}(t,R) = -\frac{\phi_0(R)}{2}\int_0^R err_0(t,s\lambda(t))se_2(s)ds + e_2(R)\int_0^R err_0(t,s\lambda(t))\frac{s\phi_0(s)}{2}ds$$

We get

$$v_{ell,2,0}(t,R) = f_3(R)\left(2\lambda'(t)^4 - 7\lambda(t)\lambda'(t)^2\lambda''(t) + 4\lambda(t)^2\lambda''(t)^2 + 6\lambda(t)^2\lambda'(t)\lambda'''(t)\right) + f_4(R)\lambda(t)^3\lambda''''(t) \tag{4.12}$$

where

$$f_3(R) = \frac{2\left(R^2\left(-2+6R^2+R^4\right) - 2(-1+R^4)\log(1+R^2) + 4R^2 Li_2(-R^2)\right)}{32(R+R^3)}$$

$$f_4(R) = \frac{1}{32(R+R^3)}\left(R^2\left(-12+44R^2+7R^4 - 4(-2+6R^2+R^4)\log(R)\right)\right.$$
$$+8\left(-1+R^4\right)(-1+\log(R))\log(1+R^2)$$
$$\left.+4\left(-1+4R^2+R^4-4R^2\log(R)\right)Li_2(-R^2) + 16R^2 Li_3(-R^2)\right)$$

For later convenience, we split

$$v_{ell,2,0}(t,\frac{r}{\lambda(t)}) = v_{ell,2,0,main}(t,\frac{r}{\lambda(t)}) + soln_1(t,\frac{r}{\lambda(t)})$$

and write

$$v_{ell,2}(t,R) = v_{ell,2,0,main}(t,R) + soln_1(t,R) + soln_2(t,R) \tag{4.13}$$

where

$$v_{ell,2,0,main}(t,s) = \frac{s^3\lambda(t)^3}{8}\left(\partial_t^2\left(\frac{\lambda'(t)^2}{2\lambda(t)} + \lambda''(t) + \lambda''(t)\log(\lambda(t))\right) - \lambda''''(t)\log(s\lambda(t)) + \frac{3}{4}\lambda''''(t)\right) \tag{4.14}$$

$$soln_1(t,s) = (f_3(s)-f_{3,0}(s))\left(2\lambda'(t)^4 - 7\lambda(t)\lambda'(t)^2\lambda''(t) + 4\lambda(t)^2\lambda''(t)^2 + 6\lambda(t)^2\lambda'(t)\lambda'''(t)\right)$$
$$+ (f_4(s)-f_{4,0}(s))\lambda(t)^3\lambda''''(t) \tag{4.15}$$

and

$$f_{3,0}(x) = \frac{x^3}{16}, \quad f_{4,0}(x) = \frac{1}{32}x^3(7-4\log(x))$$

Finally,

$$soln_2(t,\frac{s}{\lambda(t)}) = u_{ell,2}(t,s) - v_{ell,2,0,main}(t,\frac{s}{\lambda(t)}) - soln_1(t,\frac{s}{\lambda(t)})$$
$$= -\frac{\phi_0(\frac{s}{\lambda(t)})}{2}\int_0^{\frac{s}{\lambda(t)}} err_1(t,q\lambda(t))qe_2(q)dq + \frac{e_2(\frac{s}{\lambda(t)})}{2}\int_0^{\frac{s}{\lambda(t)}} err_1(t,q\lambda(t))q\phi_0(q)dq \tag{4.16}$$

where

$$err_1(t,r) = \lambda(t)^2 \partial_t^2 \left(v_{ell}(t, \frac{r}{\lambda(t)}) - v_{ell,lgR}(t, \frac{r}{\lambda(t)}) \right)$$

Note in particular, that

$$v_{ell,2,0}(t, \frac{r}{\lambda(t)}) = \frac{r^3}{8} \partial_t^2 \left(\frac{\lambda'(t)^2}{2\lambda(t)} + \lambda''(t) - \lambda''(t) \log(\frac{r}{\lambda(t)}) \right) + \frac{3}{32} r^3 \lambda''''(t) + O\left(r \log^2(r) \right), \quad r \to \infty \quad (4.17)$$

This observation will be important when we match the small r corrections to the large r corrections. Also, the leading behavior of $err_1(t,r)$ in the region $r \geqslant \lambda(t)$ is $err_{1,0}$ given by

$$err_{1,0}(t,r) = \lambda(t)^2 \partial_t^2 \left(\lambda(t) \left(\frac{-3}{2r} \lambda'(t)^2 + \frac{\lambda(t)\lambda''(t)}{r} \left(-(2 + \frac{\pi^2}{6}) + \log(\frac{r}{\lambda(t)}) - 2\log^2(\frac{r}{\lambda(t)}) \right) + \frac{c_1(t)}{r} \right) \right) \tag{4.18}$$

4.2 First Wave Iteration

We consider the PDE

$$-\partial_{tt} u_w + \partial_{rr} u_w + \frac{1}{r} \partial_r u_w - \frac{u_w}{r^2} = \partial_t^2 Q_1(\frac{r}{\lambda(t)}) \tag{4.19}$$

We consider the solution u_w, that can be written as

$$u_w(t,r) = v_1(t,r) + v_2(t,r) \tag{4.20}$$

where v_1 solves

$$-\partial_{tt} v_1 + \partial_{rr} v_1 + \frac{1}{r} \partial_r v_1 - \frac{v_1}{r^2} = \partial_t^2 Q_1(\frac{r}{\lambda(t)})$$

with 0 Cauchy data at infinity, and v_2 solves

$$\begin{cases} -\partial_{tt} v_2 + \partial_{rr} v_2 + \frac{1}{r} \partial_r v_2 - \frac{v_2}{r^2} = 0 \\ v_2(0) = 0 \\ \partial_t v_2(0,r) = v_{2,0}(r) \end{cases}$$

where $v_{2,0} \in L^2(rdr)$ will be chosen later. To describe $v_1(t,r)$, we decompose it as

$$v_1(t,r) = w_1(t,r) + v_{ex}(t,r) \tag{4.21}$$

where w_1 solves

$$-\partial_{tt} w_1 + \partial_{rr} w_1 + \frac{1}{r} \partial_r w_1 - \frac{w_1}{r^2} = \frac{-2\lambda''(t)}{r} \tag{4.22}$$

with 0 Cauchy data at infinity. We use the same procedure used in Section 4.2 of [25] to obtain (4.25) of [25], to derive the following expression which can be directly checked to be the solution to (4.22) with 0 Cauchy data at infinity.

$$\begin{aligned}
w_1(t,r) &= \frac{-1}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^{2\pi} \left(-2\lambda''(s) \right) \frac{(r + \rho\cos(\theta))}{r^2 + \rho^2 + 2r\rho\cos(\theta)} d\theta \\
&= \int_t^\infty ds \frac{\lambda''(s)}{r} \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \left(1 + \text{sgn}(r^2 - \rho^2) \right) \\
&= \frac{2}{r} \int_t^{t+r} ds \lambda''(s)(s-t) + \frac{2}{r} \int_{t+r}^\infty ds \lambda''(s) \left((s-t) - \sqrt{(s-t)^2 - r^2} \right) \\
&= \frac{-2}{r} \left(\lambda(t+r) - \lambda(t) - r\lambda'(t+r) \right) + \frac{2}{r} \int_{t+r}^\infty ds \lambda''(s) \left((s-t) - \sqrt{(s-t)^2 - r^2} \right)
\end{aligned} \tag{4.23}$$

(For $r \neq \rho$, we can evaluate the following integral using Cauchy's residue theorem).

$$\int_0^{2\pi} \frac{\rho\cos(\theta) + r}{\rho^2 + 2\rho r\cos(\theta) + r^2} d\theta = \frac{\pi \left(\text{sgn}\left(r^2 - \rho^2 \right) + 1 \right)}{r}$$

First, we note that $w_1(t, \cdot) \in C^\infty((0,\infty))$. This follows from

$$w_1(t,r) = \frac{-2}{r} \left(\lambda(t+r) - \lambda(t) - r\lambda'(t+r) \right) + 2r \int_1^\infty \lambda''(t+ry) \left(y - \sqrt{y^2 - 1} \right) dy \tag{4.24}$$

and the smoothness and symbol type estimates on λ. Next, we introduce some notation. Let

$$w_{1,main}(t,r) = r\left(\log(2) + \frac{1}{2}\right)\lambda''(t) + r\int_t^{2t} \frac{(\lambda''(s) - \lambda''(t))}{s - t}ds + r\lambda''(t)\log(\frac{t}{r}) + r\int_{2t}^{\infty} \frac{\lambda''(s)ds}{(s-t)} \quad (4.25)$$

$$w_{1,sub}(t,r) = w_1(t,r) - w_{1,main}(t,r) \quad (4.26)$$

$$w_{1,cubic,main}(t,r) = \frac{3}{32}r^3\lambda''''(t) + \frac{r^3}{8}\left(\lambda''''(t)\left(\log(2) + \frac{1}{2}\right) - \log(r)\lambda''''(t) + \log(t)\lambda''''(t)\right.$$
$$\left. + \int_t^{2t} \frac{\lambda''''(s) - \lambda''''(t)}{s - t}ds + \int_{2t}^{\infty} \frac{\lambda''''(s)ds}{s - t}\right)$$

The following lemma will be useful later on when we match the small r and large r corrections.

Lemma 1 For $0 \leqslant j \leqslant 8$, $0 \leqslant k \leqslant 3$ and $r \leqslant t$,

$$\left|\partial_t^j \partial_r^k \left(w_{1,sub}(t,r) - w_{1,cubic,main}(t,r) - \frac{r^5 \lambda^{(6)}(t)}{576}(5 + \log(8))\right.\right.$$
$$\left.\left. - \frac{r^5}{192}\left(\int_t^{2t} \frac{\left(\lambda^{(6)}(s) - \lambda^{(6)}(t)\right)ds}{s - t} + \lambda^{(6)}(t)\log(\frac{t}{r}) + \int_{2t}^{\infty} \frac{\lambda^{(6)}(s)ds}{s - t}\right)\right)\right|$$
$$\leqslant \frac{C\lambda(t)r^{7-k}}{t^{8+j}}(\log(t) + |\log(r)|)$$

Proof As a first step, we start with (4.24), and subtract and add the Taylor polynomial centered at 0, of degree 3, of $\lambda(t + r) - \lambda(t) - r\lambda'(t + r) + \frac{r^2}{2}\lambda''(t + r)$, regarded as a function of r for each fixed t. For the integral term of (4.24), we first note that

$$y - \sqrt{y^2 - 1} = \frac{1}{2y} + O\left(\frac{1}{y^3}\right)$$

Therefore, we have

$$2r\int_1^{\infty} \lambda''(t + ry)(y - \sqrt{y^2 - 1})dy = 2r\int_1^{\infty} \frac{\lambda''(t + ry)}{2y}dy + 2r\int_1^{\infty} \lambda''(t + ry)\left(y - \sqrt{y^2 - 1} - \frac{1}{2y}\right)dy \quad (4.27)$$

After integrating by parts twice in the second integral on the right-hand side, we get the following expression for w_1, whose second line is equal to $w_{1,main}(t,r)$, and the other terms vanish faster than $O(r\log(r))$ as r approaches zero.

$$w_1(t,r) = \frac{-2}{r}\left(\lambda(t + r) - \lambda(t) - r\lambda'(t + r) + \frac{r^2}{2}\lambda''(t + r) - \frac{r^3}{6}\lambda'''(t)\right)$$
$$+ r\left(\log(2) + \frac{1}{2}\right)\lambda''(t) + r\int_t^{2t} \frac{(\lambda''(s) - \lambda''(t))}{s - t}ds + r\lambda''(t)\log(\frac{t}{r}) + r\int_{2t}^{\infty} \frac{\lambda''(s)ds}{(s-t)}$$
$$+ r(\log(2) + \frac{1}{2})\left(\lambda''(t + r) - \lambda''(t) - r\lambda'''(t)\right)$$
$$- \left(\frac{r^2}{6} - r^2(1 + \log(\frac{1}{2}))\right)\left(\lambda'''(t + r) - \lambda'''(t)\right) - r\int_t^{t+r} \frac{\lambda''(s) - \lambda''(t) - (s - t)\lambda'''(t)}{(s - t)}ds \quad (4.28)$$
$$+ \frac{1}{r}\int_{t+r}^{\infty} \lambda'''(s)\left(\frac{(s - t)^3}{3}\left(1 - \left(1 - \frac{r^2}{(s - t)^2}\right)^{3/2}\right) - \frac{3r^2}{2(s - t)^2}\right)$$
$$+ r^2(s - t)\left(1 - \sqrt{1 - \frac{r^2}{(s - t)^2}} + \log\left(1 + \frac{\sqrt{1 - \frac{r^2}{(s - t)^2}} - 1}{2}\right)\right)\right)ds$$

Repeating this process, we make manifest the terms composing $w_{1,cubic,main}(t,r)$, and recall the notation for $P_{k,a}(f)$ given just above (2.2).

$$
\begin{aligned}
w_1(t,r) = {} & r\left(\log(2) + \frac{1}{2}\right)\lambda''(t) + r\int_t^{2t}\frac{(\lambda''(s) - \lambda''(t))}{s-t}ds + r\lambda''(t)\log(\frac{t}{r}) + r\int_{2t}^\infty\frac{\lambda''(s)ds}{s-t} \\
& - \frac{2}{r}\left(\lambda(t+r) - \lambda(t) - r\lambda'(t+r) + \frac{r^2}{2}\lambda''(t+r) - \frac{r^3}{6}\lambda'''(t) - \frac{r^4}{8}\lambda''''(t) - \frac{r^5}{20}\lambda''''' (t)\right) \\
& + \frac{r^3}{8}\lambda'''(t)\left(\log(2) + \frac{1}{2}\right) + \frac{3}{32}r^3\lambda''''(t) \\
& + r\left(\log(2) + \frac{1}{2}\right)\left(\lambda''(t+r) - P_{3,0}(r\mapsto\lambda''(t+r))\right) \\
& - r^2\left(\frac{-5}{6} + \log(2)\right)\left(\lambda'''(t+r) - P_{2,0}(r\mapsto\lambda'''(t+r))\right) \\
& - r\int_t^{t+r}\frac{(\lambda''(s) - P_{3,t}(\lambda'')(s))}{s-t}ds \\
& - \frac{r^3}{8}\log(r)\lambda''''(t) + \frac{r^3}{8}\log(t)\lambda''''(t) + \frac{r^3}{8}\int_t^{2t}\frac{\lambda''''(s) - \lambda''''(t)}{s-t}ds \\
& + \frac{r^3}{8}\int_{2t}^\infty\frac{\lambda''''(s)}{s-t}ds - \frac{r^3}{8}\int_t^{t+r}\frac{\lambda''''(s) - \lambda''''(t) - (s-t)\lambda'''''(t)}{s-t}ds \\
& - \frac{r^3}{96}(41 - 60\log(2))\left(\lambda''''(t+r) - \lambda''''(t) - r\lambda'''''(t)\right) \\
& + \left(\lambda'''''(t+r) - \lambda'''''(t)\right)r^4\frac{(299 - 420\log(2))}{1440} + \frac{1}{r}\int_{t+r}^\infty\lambda''''''(s)K(r,s-t)ds
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
K(r,w) = {} & \frac{1}{1440}\left(24w^5\left(1 - \left(\sqrt{1 - \frac{r^2}{w^2}} + \frac{r^2}{2w^2} + \frac{r^4}{8w^4}\right)\right) + 332r^2w^3\left(1 - \left(\sqrt{1 - \frac{r^2}{w^2}} + \frac{r^2}{2w^2}\right)\right)\right. \\
& - 64r^4w\left(\sqrt{1 - \frac{r^2}{w^2}} - 1\right) + 240r^2w^3\left(\log\left(\frac{1}{2}\left(1 + \sqrt{1 - \frac{r^2}{w^2}}\right)\right) + \frac{r^2}{4w^2}\right) \\
& \left. + 180r^4w\log\left(\frac{1}{2}\left(1 + \sqrt{1 - \frac{r^2}{w^2}}\right)\right)\right)
\end{aligned}
\tag{4.30}
$$

Finally, to prove the lemma, we treat (4.29) line by line. We show in detail how we obtain the leading part of terms of different forms. In the following computations, we work in the region $r \leqslant t$. By Taylor's theorem, we have

$$
\left|\frac{-2}{r}\left(\lambda(t+r) - \lambda(t) - r\lambda'(t+r) + \frac{r^2}{2}\lambda''(t+r) - \frac{r^3}{6}\lambda'''(t) - \frac{r^4}{8}\lambda''''(t) - \frac{r^5}{20}\lambda'''''(t)\right) - \left(\frac{-\lambda^{(6)}(t)r^5}{36} - \frac{\lambda^{(7)}(t)r^6}{168}\right)\right|
$$

$$
\leqslant C\frac{r^7\lambda(t)}{t^8}
$$

where we used (4.3), and the fact that $r \leqslant t$. Other terms in (4.29) of the same form are treated with the same argument. Next, we have

$$
-r\int_t^{t+r}\frac{ds}{(s-t)}\left(\frac{(s-t)^4}{24}\lambda^{(6)}(t) + \frac{(s-t)^5}{120}\lambda^{(7)}(t)\right) = -r\left(\frac{r^4}{4\cdot24}\lambda^{(6)}(t) + \frac{r^5}{600}\lambda^{(7)}(t)\right)
$$

and

$$
\left|-r\int_t^{t+r}\frac{(\lambda''(s) - P_{3,t}(\lambda'')(s))}{s-t}ds - \left(-r\int_t^{t+r}\frac{ds}{(s-t)}\left(\frac{(s-t)^4}{24}\lambda^{(6)}(t) + \frac{(s-t)^5}{120}\lambda^{(7)}(t)\right)\right)\right|
$$

$$
= \left|-r\int_t^{t+r}\frac{(\lambda''(s) - P_{5,t}(\lambda'')(s))\,ds}{s-t}\right| \leqslant C\frac{\lambda(t)r^7}{t^8}
$$

where we again used Taylor's theorem and (4.3). Again, other terms in (4.29) of the same form are treated with the same argument. Finally, we recall (4.30), and note that

$$
K(r,w) - \frac{r^6}{192w} = O\left(\frac{r^8}{w^3}\right), \quad w\to\infty
$$

Therefore, with the same procedure used in (4.27),

$$
\frac{1}{r} \int_{t+r}^{\infty} \lambda^{(6)}(s) K(r, s-t) ds
$$

$$
= \frac{r^5}{192} \left(\int_t^{2t} \frac{\left(\lambda^{(6)}(s) - \lambda^{(6)}(t) \right)}{(s-t)} ds + \lambda^{(6)}(t) \log(\frac{t}{r}) + \int_{2t}^{\infty} \frac{\lambda^{(6)}(s)}{(s-t)} ds \right)
$$

$$
- \frac{r^5}{192} \int_t^{t+r} \frac{\left(\lambda^{(6)}(s) - \lambda^{(6)}(t) - (s-t)\lambda^{(7)}(t) \right)}{(s-t)} ds - \frac{r^6}{192} \lambda^{(7)}(t) - \lambda^{(6)}(t) \frac{r^5(218 - 315 \log(2))}{2880}
$$

$$
+ \lambda^{(7)}(t) r^6 \frac{(2413 - 3465 \log(2))}{100800} + \frac{\left(\lambda^{(7)}(t+r) - \lambda^{(7)}(t) \right)}{100800} \cdot r^6 \left(2413 - 3465 \log(2) \right)
$$

$$
- \left(\lambda^{(6)}(t+r) - \lambda^{(6)}(t) - r\lambda^{(7)}(t) \right) r^5 \frac{(218 - 315 \log(2))}{2880} - r^6 \lambda^{(7)}(t) \frac{(218 - 315 \log(2))}{2880} + \mathrm{Err}
$$

where

$$
|\mathrm{Err}| \leqslant \frac{1}{r} \int_{t+r}^{\infty} |W(s)\lambda^{(8)}(s)| ds
$$

and, for $y = \frac{s-t}{r}$,

$$
W(s) = \frac{r^7}{2880} \left(-\frac{128}{35} \left(\sqrt{1 - \frac{1}{y^2}} - 1 \right) y + 15y \log \left(\frac{1}{2}\sqrt{1 - \frac{1}{y^2}} + \frac{1}{2} \right) - \frac{1779}{35} \left(\sqrt{1 - \frac{1}{y^2}} + \frac{1}{2y^2} - 1 \right) y^3 \right.
$$

$$
- \frac{1518}{35} \left(\sqrt{1 - \frac{1}{y^2}} - \left(-\frac{1}{8y^4} - \frac{1}{2y^2} + 1 \right) \right) y^5 - \frac{8}{7} \left(\sqrt{1 - \frac{1}{y^2}} + \left(\frac{1}{16y^6} + \frac{1}{8y^4} + \frac{1}{2y^2} \right) - 1 \right) y^7
$$

$$
\left. + 12 \left(2y^5 + 5y^3 \right) \left(\frac{3}{32y^4} + \frac{1}{4y^2} + \log \left(\frac{1}{2}\sqrt{1 - \frac{1}{y^2}} + \frac{1}{2} \right) \right) - \frac{45}{8y} \right)
$$

We finish the proof by noting that

$$
W(s) = O\left(\frac{r^8}{(s-t)} \right), \quad s \to \infty
$$

and estimating Err. The higher derivatives are treated similarly. $\quad\square$

We remark that $w_{1,main}(t, r)$, along with the expression of u_{ell} will end up determining the data for the free wave v_2, as $w_{1,main}(t, r)$ will turn out to be the leading contribution of w_1 in the matching region.

We also record some pointwise estimates on w_1 and some of its derivatives.

Lemma 2 *We have the following estimates. For $0 \leqslant k \leqslant 8$ and $0 \leqslant j \leqslant 3$,*

$$
|\partial_t^k \partial_r^j w_1(t, r)| \leqslant \begin{cases} \frac{Cr^{1-j}\lambda(t)(\log(t) + |\log(r)|)}{t^{2+k}}, & r \leqslant t \\ \frac{C}{r^{1+j}t^k} \sup_{x \in [t, t+r]} (\lambda(x)), & r \geqslant t \end{cases} \tag{4.31}
$$

Proof By Lemma 1, it suffices to consider (4.24) in the region $r \geqslant t$:

$$
\left| \frac{2}{r} \left(\lambda'(t+r)r - (\lambda(t+r) - \lambda(t)) \right) \right| \leqslant \frac{C}{r} \left(\lambda(t+r) + \lambda(t) + r\frac{\lambda(t+r)}{t+r} \right) \leqslant \frac{C}{r} \left(\lambda(t+r) + \lambda(t) \right)
$$

Then, we have

$$
\left| 2r \int_1^{\infty} \lambda''(t+ry) \left(y - \sqrt{y^2 - 1} \right) dy \right| \leqslant \frac{Cr\lambda(t+r)}{t+r} \int_1^{\infty} \frac{dy}{y(t+ry)} \leqslant \frac{Cr\lambda(t+r)}{r(t+r)} \frac{\log(1 + \frac{t}{r})}{\frac{t}{r}}
$$

$$
\leqslant \frac{C\lambda(t+r)}{t+r}, \quad r \geqslant t
$$

where we used (4.3). Then, using the symbol type estimates on λ, we get (4.31) for $r \geqslant t$. $\quad\square$

Next, we study v_{ex} ((4.21)) which solves the following equation with 0 Cauchy data at infinity.

$$-\partial_t^2 v_{ex} + \partial_r^2 v_{ex} + \frac{1}{r}\partial_r v_{ex} - \frac{v_{ex}}{r^2} = \partial_t^2 Q_1(\frac{r}{\lambda(t)}) + \frac{2\lambda''(t)}{r} := RHS(t,r) \tag{4.32}$$

From integral identities of [9], we have

$$\widehat{RHS}(t,\xi) = 2\xi K_0(\xi\lambda(t))\lambda(t)\lambda'(t)^2 + \frac{2\lambda''(t)}{\xi}\left(1 - \xi\lambda(t)K_1(\xi\lambda(t))\right) \tag{4.33}$$

We have the following representation formula for v_{ex}

Lemma 3 *For $r > 0$, v_{ex} is given by*

$$\begin{aligned}
v_{ex}(t,r) &= \int_t^\infty ds \int_0^\infty d\xi J_1(r\xi)\sin((t-s)\xi)\widehat{RHS}(s,\xi) \\
&= -\int_0^\infty \frac{J_1(r\xi)}{\xi}\widehat{RHS}(t,\xi)d\xi - \int_0^\infty d\xi J_1(r\xi)\int_t^\infty ds\frac{\sin((t-s)\xi)}{\xi^2}\partial_s^2\widehat{RHS}(s,\xi)
\end{aligned} \tag{4.34}$$

Proof From asymptotics of K_1, K_0 (see [9]), we get, for $k = 0,1,2$,

$$|\partial_s^k \widehat{RHS}(s,\xi)| \leqslant C\frac{\lambda(s)^2}{s^{2+k}}\frac{\xi\lambda(s)}{1+\xi^2\lambda(s)^2}\langle\log(\xi\lambda(s))\rangle \tag{4.35}$$

Using

$$|J_1(r\xi)| \leqslant \frac{C}{\sqrt{r\xi}}, \quad r > 0$$

the first integral on the right-hand side of (4.34) converges absolutely, for $r > 0$. Therefore, by the Fubini theorem, and (4.35), we get

$$\begin{aligned}
\int_t^\infty ds \int_0^\infty d\xi J_1(r\xi)\sin((t-s)\xi)\widehat{RHS}(s,\xi) &= \int_0^\infty d\xi \int_t^\infty ds J_1(r\xi)\partial_s\left(\frac{\cos((t-s)\xi)}{\xi}\right)\widehat{RHS}(s,\xi) \\
&= -\int_0^\infty \frac{J_1(r\xi)}{\xi}\widehat{RHS}(t,\xi)d\xi - \int_0^\infty d\xi J_1(r\xi)\int_t^\infty ds\frac{\sin((t-s)\xi)}{\xi^2}\partial_s^2\widehat{RHS}(s,\xi)
\end{aligned}$$

Moreover, using

$$|\sin((t-s)\xi)| \leqslant 1, \quad |J_1'(x)| \leqslant C, \quad |J_1''(x)| \leqslant \frac{C}{\sqrt{x}}, \quad x > 0, \text{ and } (4.35)$$

the dominated convergence theorem shows that, for all $r_0 > 0$, and $k = 1,2$,

$$\partial_r^k\left(\int_0^\infty d\xi J_1(r\xi)\int_t^\infty ds\frac{\sin((t-s)\xi)}{\xi^2}\partial_s^2\widehat{RHS}(s,\xi)\right)\Big|_{r=r_0} = \int_0^\infty d\xi \partial_r^k(J_1(r\xi))|_{r=r_0}\cdot\int_t^\infty ds\frac{\sin((t-s)\xi)}{\xi^2}\partial_s^2\widehat{RHS}(s,\xi)$$

A similar argument shows that we can differentiate (up to) two times under the integral sign in t. Then, by the fact that the first term on the second line of (4.34) is a solution to the following ODE (which follows, for instance, by direct computation of the integral, see (4.36))

$$\partial_r^2 f + \frac{1}{r}\partial_r f - \frac{f}{r^2} = RHS(t,r)$$

we see that v_{ex}, the solution to (4.32) with zero Cauchy data at infinity, is given by (4.34). $\quad\square$

Note that we could not have done the integration by parts in the s variable to get from the first to the second line of (4.34) if RHS was simply equal to $\partial_t^2 Q_1(\frac{r}{\lambda(t)})$, because there would be too large of a singularity of the resulting integrand at low frequencies. Also, we have

$$\begin{aligned}
v_{ex,ell}(t,r) &:= -\int_0^\infty \frac{J_1(r\xi)}{\xi}\widehat{RHS}(t,\xi)d\xi \\
&= \frac{-\left(2\lambda(t)\log(1+\frac{r^2}{\lambda(t)^2})\lambda'(t)^2 + \left(\lambda(t)^2\log(1+\frac{r^2}{\lambda(t)^2}) + r^2\log(1+\frac{\lambda(t)^2}{r^2})\right)\lambda''(t)\right)}{2r}
\end{aligned} \tag{4.36}$$

Finally, we consider v_2 solving

$$\begin{cases} -\partial_{tt}v_2 + \partial_{rr}v_2 + \frac{1}{r}\partial_r v_2 - \frac{v_2}{r^2} = 0 \\ v_2(0) = 0 \\ \partial_t v_2(0) = v_{2,0} \end{cases}$$

with $v_{2,0}$ not yet chosen. For $\widehat{v_{2,0}} \in C^0((0,\infty))$ such that $|\widehat{v_{2,0}}(\xi)| \leqslant C\left(\sqrt{\xi}(1+\xi)^{5/2}\right)^{-1}$, we have

$$
\begin{aligned}
v_2(t,r) &= \int_0^\infty J_1(r\xi)\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi = -\frac{r}{4\pi}\int_0^\pi \sin^2(\theta)\left(F(t+r\cos(\theta))+F(t-r\cos(\theta))\right)d\theta \\
&= \frac{-r}{4}F(t) - \frac{r^3}{32}F''(t) - \frac{r}{2\pi}\int_0^\pi \sin^2(\theta)\left(F(t+r\cos(\theta))-F(t)-\frac{r^2}{2}\cos^2(\theta)F''(t)\right)d\theta
\end{aligned}
\tag{4.37}
$$

where

$$
F(t) = -2\int_0^\infty \xi\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi \tag{4.38}
$$

and we used

$$
J_1(x) = \frac{x}{\pi}\int_0^\pi \cos(x\cos(\theta))\sin^2(\theta)d\theta
$$

4.3 First order matching

The error of u_{ell} in solving (4.5) is $\partial_t^2 u_{ell}(t,r)$, which is large for large values of r, because of the growth of $u_{ell}(t,r)$ for large r. On the other hand, the error of u_w in solving (4.5) is $\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)}))-1}{r^2}\right)u_w(t,r)$. The largest contributions to this error term arise from substituting the second line of (4.28) plus (4.37), expanded for $r \ll t$ into u_w in the error term. Therefore, we choose the data for v_2 in order to match the largest contributions to the error terms of u_{ell} and u_w. In particular, $v_{ell,lgR}$, defined in (4.11) and recalled here makes the largest contribution to the error term of u_{ell}:

$$
v_{ell,lgR}(t,R) = R\left(\frac{1}{2}\lambda'(t)^2 + \lambda(t)\lambda''(t) - \lambda(t)\lambda''(t)\log(R)\right) \tag{4.39}
$$

Also, as per Lemma 1, the main contribution from $w_1(t,r)$ is

$$
w_{1,main}(t,r) = r\left(\log(2)+\frac{1}{2}\right)\lambda''(t) + r\int_t^{2t}\frac{(\lambda''(s)-\lambda''(t))}{s-t}ds + r\lambda''(t)\log(\frac{t}{r}) + r\int_{2t}^\infty\frac{\lambda''(s)ds}{(s-t)}
$$

and, by (4.37), the main contribution from v_2 is

$$
v_{2,main}(t,r) = \frac{-r}{4}\left(-2\int_0^\infty \xi\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi\right) \tag{4.40}
$$

Note that the $r\log(r)$ terms from $w_{1,main}$ and (4.39) (where $R = \frac{r}{\lambda(t)}$) are the same. Therefore, it is possible to choose $v_{2,0}$ so that (for example, for all $t \geqslant 2T_\lambda$)

$$
\frac{r}{\lambda(t)}\left(\frac{1}{2}\lambda'(t)^2 + \lambda(t)\lambda''(t) - \lambda(t)\lambda''(t)\log(\frac{r}{\lambda(t)})\right) = v_{2,main}(t,r) + w_{1,main}(t,r)
$$

In other words, we choose $v_{2,0}$ to satisfy, for all $t \geqslant 2T_\lambda$,

$$
\begin{aligned}
&-2\int_0^\infty \xi\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi \\
&= 4\left(\left(\log(2)-\frac{1}{2}\right)\lambda''(t) + \int_t^{2t}\left(\frac{\lambda''(s)-\lambda''(t)}{s-t}\right)ds + \lambda''(t)\log(\frac{t}{\lambda(t)}) + \int_{2t}^\infty\frac{\lambda''(s)ds}{s-t} - \frac{\lambda'(t)^2}{2\lambda(t)}\right)
\end{aligned}
\tag{4.41}
$$

We remind the reader of the discussion following (3.4), which compares (4.41) to the relation between the radiation and $\lambda(t)$ from [25].

Since we will only need (4.41) to be true for all t sufficiently large, and since we only assume $\lambda(t)$ to be defined for $t > 50$, we use a (relatively unimportant) cutoff, $\psi \in C^\infty([0,\infty))$, such that

$$
\psi(x) = \begin{cases} 0, & x \leqslant T_\lambda \\ 1, & x \geqslant 2T_\lambda \end{cases} \quad, \quad 0 \leqslant \psi(x) \leqslant 1 \tag{4.42}
$$

where we recall that $T_\lambda > 100$ is part of the definition of Λ, see (1.6), and then get

$$
\widehat{v_{2,0}}(\xi) = \frac{-1}{\pi\xi}\int_0^\infty H(t)\sin(t\xi)dt \tag{4.43}
$$

with

$$H(t) = 4\left(\left(\log(2) - \frac{1}{2}\right)\lambda''(t) + \int_t^{2t}\left(\frac{\lambda''(s) - \lambda''(t)}{s - t}\right)ds + \lambda''(t)\log(\frac{t}{\lambda(t)}) + \int_{2t}^{\infty}\frac{\lambda''(s)ds}{s - t} - \frac{\lambda'(t)^2}{2\lambda(t)}\right)\psi(t)$$

(4.44)

We can now record some basic estimates on $\widehat{v_{2,0}}$ and v_2. We remark that the estimates of Lemma 4 below imply that the conditions on $\widehat{v_{2,0}}$ stated just above (4.37) are satisfied.

Lemma 4 *For $0 \leqslant k$, and $N \geqslant 1$, there exist $C_k, C_{k,N}$ such that*

$$\xi^k|\widehat{v_{2,0}}^{(k)}(\xi)| \leqslant \begin{cases} C_k\int_{100}^{\frac{1}{\xi}}\frac{\lambda(\sigma)\log(\sigma)d\sigma}{\sigma} + C_k\lambda(\frac{1}{\xi})\log(\frac{1}{\xi}), & \xi \leqslant \frac{1}{100} \\ \frac{C_{k,N}}{\xi^N}, & \xi > \frac{1}{100} \end{cases}$$

Proof We start with (4.44). By the mean value theorem, there exists $x \in [t, s]$ such that

$$|H(t)| \leqslant C\left(\frac{\lambda(t)}{t^2} + \int_t^{2t}|\lambda'''(x)|ds + \log(t)|\lambda''(t)| + \frac{1}{t}\int_{2t}^{\infty}\frac{\lambda(s)ds}{s^2} + \frac{\lambda(t)}{t^2}\right)|\psi(t)| \leqslant \frac{C\lambda(t)\log(t)}{t^2}\mathbb{1}_{\{t \geqslant T_\lambda\}}$$

where we used (4.3). Using the symbol-type nature of the estimates on λ, we get, for $k \geqslant 0$,

$$|H^{(k)}(t)| \leqslant C_k\mathbb{1}_{\{t \geqslant T_\lambda\}}\frac{\lambda(t)\log(t)}{t^{2+k}}$$

(4.45)

Then, for $\xi \geqslant \frac{1}{100}$, we integrate by parts in the formula

$$\widehat{v_{2,0}}(\xi) = \frac{-1}{\pi\xi}\int_0^{\infty}H(\sigma)\sin(\sigma\xi)d\sigma$$

noting that no boundary terms arise, to get that, for each $k \in \mathbb{N}$, there exists $C_k > 0$ such that

$$|\widehat{v_{2,0}}(\xi)| \leqslant \frac{C_k}{\xi^k}, \quad \xi \geqslant \frac{1}{100}$$

For $\xi \leqslant \frac{1}{100}$, we have

$$|\widehat{v_{2,0}}(\xi)| \leqslant \begin{cases} C\int_{T_\lambda}^{\frac{1}{\xi}}\frac{\lambda(\sigma)\log(\sigma)d\sigma}{\sigma} + \frac{C}{\xi}\int_{\frac{1}{\xi}}^{\infty}\frac{\lambda(\sigma)\log(\sigma)}{\sigma^2}d\sigma \leqslant C\int_{100}^{\frac{1}{\xi}}\frac{\lambda(\sigma)\log(\sigma)}{\sigma}d\sigma + C\lambda(\frac{1}{\xi})\log(\frac{1}{\xi}), & \xi \leqslant \min\{\frac{1}{100}, \frac{1}{T_\lambda}\} \\ \frac{C}{\xi}\int_{\frac{1}{\xi}}^{\infty}\frac{\lambda(\sigma)\log(\sigma)}{\sigma^2}d\sigma \leqslant C\int_{100}^{\infty}\frac{\log(\sigma)}{\sigma^{2-C_u}}d\sigma \leqslant C\lambda(\frac{1}{\xi})\log(\frac{1}{\xi}), & \frac{1}{T_\lambda} \leqslant \xi \leqslant \frac{1}{100} \end{cases}$$

where we used $T_\lambda \geqslant 100$ and (4.3) for the first case, and the fact that $\min_{x \in [100, T_\lambda]}(\lambda(x))\log(100) > 0$ for the second. In particular, the constants C depend on (the fixed function) λ, but *not* on ξ. Finally, Lemma 4 for $k > 0$ follows from (4.45) and

$$\widehat{v_{2,0}}(\xi) = \frac{-1}{\pi\xi^2}\int_0^{\infty}H(\frac{\omega}{\xi})\sin(\omega)d\omega$$

$\square$

Lemma 5 *For $0 \leqslant k \leqslant 7$, $j = 0, 1$, we have*

$$|\partial_t^k\partial_r^j v_2(t, r)| \leqslant \frac{Cr^{1-j}\lambda(t)\log(t)}{t^{2+k}}, \quad r \leqslant \frac{t}{2}$$

For all $j + k \leqslant 7$, we have the following two estimates

$$|\partial_t^k\partial_r^j v_2(t, r)| \leqslant \frac{C\log(r)}{\sqrt{r}\langle|t - r|\rangle^{\frac{1}{2}+k+j}}\sup_{x \in [100, r]}(\lambda(x)\log(x)), \quad r \geqslant \frac{t}{2}$$

and

$$|\partial_t^j\partial_r^k v_2(t, r)| \leqslant \frac{C}{\sqrt{r}}, \quad r \geqslant \frac{t}{2}$$

$$|(\partial_t + \partial_r)v_2(t, r)| \leqslant \frac{C\log(r)\sup_{x \in [100, r]}(\lambda(x)\log(x))}{r^{3/2}\sqrt{\langle t - r\rangle}}, \quad r > \frac{t}{2}$$

$$|\partial_r^2 v_2(t, r)| \leqslant \frac{Cr\lambda(t)\log(t)}{t^4}, \quad r \leqslant \frac{t}{2}$$

Proof To estimate v_2, we start with the region $r \leqslant \frac{t}{2}$. Here, we use (4.37) and (4.45), to get

$$|v_2(t,r)| \leqslant Cr \int_0^\pi |F(t + r\cos(\theta))| d\theta \leqslant Cr \int_0^\pi \frac{\lambda(t + r\cos(\theta))\log(t + r\cos(\theta))}{(t + r\cos(\theta))^2} d\theta \leqslant \frac{Cr\lambda(t)\log(t)}{t^2}$$

where we used (4.3). For the region $r \geqslant \frac{t}{2}$, we start with the first term on the right-hand side of (4.37), namely

$$v_2(t,r) = \int_0^\infty J_1(r\xi)\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi$$

One estimate which will be useful in the region $r \geqslant \frac{t}{2}$ (in particular, it is useful in the region $|t - r| \leqslant 100$) is the following. Using $|J_1(x)| \leqslant \frac{C}{\sqrt{x}}, x > 0$ (though a much better estimate is true for small $x > 0$) we get

$$|v_2(t,r)| \leqslant \frac{C}{\sqrt{r}} \int_0^{\frac{1}{100}} \frac{1}{\sqrt{\xi}} \left(\frac{\log(\frac{1}{\xi})}{\xi^{C_u}} + \int_{100}^{\frac{1}{\xi}} \frac{\log(\sigma)d\sigma}{\sigma^{1-C_u}} \right) d\xi + \frac{C}{\sqrt{r}} \int_{\frac{1}{100}}^\infty \frac{d\xi}{\xi^5} \leqslant \frac{C}{\sqrt{r}} \int_0^{\frac{1}{100}} \frac{\log(\frac{1}{\xi})d\xi}{\xi^{C_u+1/2}} + \frac{C}{\sqrt{r}} \leqslant \frac{C}{\sqrt{r}} \quad (4.46)$$

where we also used (4.3). The higher derivatives of v_2 are treated similarly. To treat the region where $|t - r| > 100$ and $r \geqslant \frac{t}{2}$, we use

$$v_2(t,r) = \int_0^{\frac{1}{r}} J_1(r\xi)\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi + \int_{\frac{1}{r}}^\infty J_1(r\xi)\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi \quad (4.47)$$

We start with

$$|\int_0^{\frac{1}{r}} J_1(r\xi)\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi| \leqslant C \int_r^\infty d\sigma \int_0^{\frac{1}{\sigma}} d\xi \frac{r\xi\lambda(\sigma)\log(\sigma)}{\sigma} + C \int_{100}^r d\sigma \int_0^{\frac{1}{r}} d\xi r\xi \frac{\lambda(\sigma)\log(\sigma)}{\sigma}$$

$$+ C \int_0^{\frac{1}{r}} d\xi r\xi\lambda(\frac{1}{\xi})\log(\frac{1}{\xi})$$

$$\leqslant \frac{Cr\lambda(r)\log(r)}{r} \int_r^\infty \frac{d\sigma}{\sigma^2} + \frac{C}{r} \int_{100}^r \frac{\lambda(\sigma)\log(\sigma)d\sigma}{\sigma} + \frac{Cr\lambda(r)\log(r)}{r^2}$$

$$\leqslant \frac{C}{r} \sup_{\sigma \in [100,r]} (\lambda(\sigma)\log(\sigma))\log(r)$$

Here, we used (4.3), and Lemma 4 in the region $\xi \leqslant \frac{1}{100}$, since $\frac{1}{r} \ll \frac{1}{100}$, along with Fubini's theorem to switch the order of the ξ and σ integrals. It remains to treat the following integral.

$$\int_{\frac{1}{r}}^\infty J_1(r\xi)\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi = -\int_{\frac{1}{r}}^\infty \frac{1}{\sqrt{\pi r\xi}} (\cos(r\xi) - \sin(r\xi))\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi$$

$$+ \int_{\frac{1}{r}}^\infty \left(J_1(r\xi) + \left(\frac{\cos(r\xi)}{\sqrt{\pi r\xi}} - \frac{\sin(r\xi)}{\sqrt{\pi r\xi}} \right) \right) \sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi$$

We start with

$$\int_{\frac{1}{r}}^\infty \frac{1}{\sqrt{\pi r\xi}} (\cos(r\xi) - \sin(r\xi))\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi$$

$$= \frac{1}{2\sqrt{\pi r}} \int_{\frac{1}{r}}^\infty \frac{1}{\sqrt{\xi}} (\sin((t+r)\xi) + \sin((t-r)\xi) + \cos((t+r)\xi) - \cos((t-r)\xi))\widehat{v_{2,0}}(\xi)d\xi$$

$$= \frac{1}{2\sqrt{\pi r}} \int_{\frac{1}{\sqrt{r}}}^\infty \left(\sin((t+r)\omega^2) + \cos((t+r)\omega^2) + \sin((t-r)\omega^2) - \cos((t-r)\omega^2) \right) \widehat{v_{2,0}}(\omega^2) \cdot 2d\omega$$

We will show in detail how to treat the term involving $\sin((t-r)\omega^2)$. The other terms can be treated with a similar argument.

$$\frac{1}{\sqrt{\pi r}} \int_{\frac{1}{\sqrt{r}}}^\infty \sin((t-r)\omega^2)\widehat{v_{2,0}}(\omega^2)d\omega = \frac{1}{\sqrt{\pi r}} \int_{\frac{1}{\sqrt{r}}}^{\frac{1}{\sqrt{|t-r|}}} \sin((t-r)\omega^2)\widehat{v_{2,0}}(\omega^2)d\omega + \frac{1}{\sqrt{\pi r}} \int_{\frac{1}{\sqrt{|t-r|}}}^\infty \sin((t-r)\omega^2)\widehat{v_{2,0}}(\omega^2)d\omega$$

For the first term, we get

$$|\frac{1}{\sqrt{\pi r}} \int_{\frac{1}{\sqrt{r}}}^{\frac{1}{\sqrt{|t-r|}}} \sin((t-r)\omega^2)\widehat{v_{2,0}}(\omega^2)d\omega| \leqslant \frac{C}{\sqrt{r}} \int_{\frac{1}{\sqrt{r}}}^{\frac{1}{\sqrt{|t-r|}}} \sup_{x \in [100,\frac{1}{\omega^2}]} (\lambda(x)\log(x))\log(\frac{1}{\omega^2})d\omega$$

$$\leqslant \frac{C \sup_{x \in [100,r]} (\lambda(x)\log(x))\log(r)}{\sqrt{r}\sqrt{|t-r|}}$$

On the other hand, we integrate by parts to get

$$|\frac{1}{\sqrt{\pi r}} \int_{\frac{1}{\sqrt{|t-r|}}}^{\infty} 2\omega \sin((t-r)\omega^2) \frac{\widehat{v_{2,0}}(\omega^2)}{2\omega} d\omega| \leqslant \frac{C}{\sqrt{r}} \left(\frac{|\widehat{v_{2,0}}(\frac{1}{|t-r|})|}{\sqrt{|t-r|}} + \int_{\frac{1}{\sqrt{|t-r|}}}^{\infty} \frac{|\widehat{v_{2,0}}'(\omega^2)| + \frac{|\widehat{v_{2,0}}(\omega^2)|}{\omega^2}}{|t-r|} d\omega \right)$$

This gives

$$|\frac{1}{\sqrt{\pi r}} \int_{\frac{1}{\sqrt{|t-r|}}}^{\infty} 2\omega \sin((t-r)\omega^2) \frac{\widehat{v_{2,0}}(\omega^2)}{2\omega} d\omega|$$

$$\leqslant \frac{C}{\sqrt{r}} \frac{|\widehat{v_{2,0}}(\frac{1}{|t-r|})|}{\sqrt{|t-r|}} + \frac{C}{\sqrt{r}|t-r|} \int_{\frac{1}{\sqrt{|t-r|}}}^{\frac{1}{10}} \frac{1}{\omega^2} \sup_{x \in [100, |t-r|]} (\lambda(x) \log(x)) \log(\frac{1}{\omega^2}) d\omega + \frac{C}{\sqrt{r}|t-r|} \int_{\frac{1}{10}}^{\infty} \frac{d\omega}{\omega^{50}}$$

$$\leqslant \frac{C \log(|t-r|)}{\sqrt{r}\sqrt{|t-r|}} \sup_{x \in [100, |t-r|]} (\lambda(x) \log(x))$$

Finally

$$|\int_{\frac{1}{r}}^{\infty} \left(J_1(r\xi) + \left(\frac{\cos(r\xi)}{\sqrt{\pi r\xi}} - \frac{\sin(r\xi)}{\sqrt{\pi r\xi}} \right) \right) \sin(t\xi) \widehat{v_{2,0}}(\xi) d\xi|$$

$$\leqslant C \int_{\frac{1}{r}}^{\infty} \frac{1}{(r\xi)^{3/2}} |\widehat{v_{2,0}}(\xi)| d\xi \leqslant C \int_{\frac{1}{r}}^{\frac{1}{100}} \frac{d\xi}{(r\xi)^{3/2}} \sup_{x \in [100, \frac{1}{\xi}]} (\lambda(x) \log(x)) \log(\frac{1}{\xi}) + C \int_{\frac{1}{100}}^{\infty} \frac{d\xi}{(r\xi)^{3/2}} \frac{1}{\xi^{101}}$$

$$\leqslant \frac{C}{r} \sup_{x \in [100, r]} (\lambda(x) \log(x)) \log(r)$$

Combining this with (4.46) finishes the estimation of $v_2(t,r)$ in the region $r \geqslant \frac{t}{2}$. The derivatives of v_2 are treated similarly. The only important difference in the procedure used to estimate $\partial_t v_2(t,r) + \partial_r v_2(t,r)$ is that we exploit the fact that $(\partial_t + \partial_r) \sin((t-r)\xi) = 0$, and similarly with $\cos((t-r)\xi)$. $\square$

Finally, we define $v_{2,sub}$ by

$$v_{2,sub}(t,r) = \frac{-r}{2\pi} \int_0^{\pi} \sin^2(\theta) \left(F(t + r\cos(\theta)) - F(t) \right) d\theta = v_2(t,r) - v_{2,main}(t,r) \tag{4.48}$$

(where we recall that F is defined in (4.38)). We also define

$$v_{2,cubic,main}(t,r) = \frac{-r^3}{32} F''(t)$$

Then, the v_2 analog of Lemma 1 is

Lemma 6 *We have the following estimates. For $0 \leqslant j \leqslant 8$, $0 \leqslant k \leqslant 2$ and $r \leqslant \frac{t}{2}$,*

$$|\partial_t^j \partial_r^k \left(v_{2,sub}(t,r) - v_{2,cubic,main}(t,r) + \frac{r^5}{768} F^{(4)}(t) \right)| \leqslant \frac{C r^{7-k} \lambda(t) \log(t)}{t^{8+j}}$$

Proof We expand (4.37) and directly estimate, as in Lemma 5 $\square$

4.4 Second Wave Iteration

The second wave correction, $u_{w,2}$ is defined as the solution to

$$-\partial_{tt} u_{w,2} + \partial_{rr} u_{w,2} + \frac{1}{r} \partial_r u_{w,2} - \frac{u_{w,2}}{r^2} = \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) (w_1 + v_2) := RHS_2(t,r) \tag{4.49}$$

with 0 Cauchy data at infinity. (We carried out the first order matching before defining $u_{w,2}$ so that we could choose $v_{2,0}$ before having to consider the equation defining $u_{w,2}$). Later on, we will add, to $u_{w,2}$, a free wave, $v_{2,2}$, solving

$$\begin{cases} -\partial_{tt} v_{2,2} + \partial_{rr} v_{2,2} + \frac{1}{r} \partial_r v_{2,2} - \frac{v_{2,2}}{r^2} = 0 \\ v_{2,2}(0,r) = 0 \\ \partial_t v_{2,2}(0,r) = v_{2,3}(r) \end{cases} \tag{4.50}$$

with $v_{2,3}$ chosen so as to satisfy a third order matching condition which we will describe later on. We start by proving estimates on $\widehat{RHS_2}$, which will allow us to justify a representation formula for $u_{w,2}$. For this, we start with the following definitions.

$$RHS_{2,1}(t,r) = \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) w_1, \quad RHS_{2,2}(t,r) = \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) v_2 \qquad (4.51)$$

Then, we have the following.

Lemma 7 *Recalling that w_1 is defined in (4.23), we have, for $2 \leqslant k \leqslant 4$ and $0 \leqslant j \leqslant 1$,*

$$\xi^j t^k |\partial_\xi^j \partial_t^k \widehat{RHS_{2,1}}(t,\xi)| + \xi^j |\partial_\xi^j \widehat{RHS_{2,1}}(t,\xi)| \leqslant \begin{cases} \frac{C\xi\lambda(t)^3 \log^2(t)}{t^2}, & \xi \leqslant \frac{1}{\lambda(t)} \\ \frac{C(\log(t) + |\log(\xi)|)}{\sqrt{\lambda(t)}\xi^{5/2}t^2}, & \frac{1}{\lambda(t)} < \xi \end{cases}$$

Proof In the regions where $\xi \leqslant \frac{1}{\lambda(t)}$, we use

$$\widehat{RHS_{2,1}}(t,\xi) = \int_0^\infty J_1(r\xi) \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) w_1(t,r) r \, dr$$

and Lemma 2. When $\xi \geqslant \frac{1}{\lambda(t)}$, we use

$$\widehat{RHS_{2,1}}(t,\xi) = \frac{1}{\xi^2} \int_0^\infty J_1(r\xi) H_1 \left(\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) w_1(t,r) \right) r \, dr$$

where

$$H_1(g) = -\partial_r^2 g - \frac{1}{r} \partial_r g + \frac{g}{r^2}$$

and then use the estimates on w_1 from Lemma 2. We use the same procedure to estimate $\partial_t^k \widehat{RHS_{2,1}}(t,\xi)$. To estimate $\partial_\xi \partial_t^k \widehat{RHS_{2,1}}(t,\xi)$, we start with

$$\widehat{RHS_{2,1}}(t,\xi) = \int_0^\infty RHS_{2,1}(t, \frac{x}{\xi}) J_1(x) \frac{x \, dx}{\xi^2}$$

differentiate under the integral, and use the symbol-type nature of the estimates in Lemma 2. $\qquad \square$

Note that some of the estimates in the following lemma can be combined into a single estimate with multiple cases of numbers of derivatives, but is presented as is for convenience.

Lemma 8 *Let*

$$f(x) = \left(\frac{\cos(2Q_1(x)) - 1}{x^2} \right) = \frac{-8}{(1+x^2)^2}$$

and

$$K(y,z) = \int_0^\infty J_1(xy) f(x) J_1(xz) x \, dx$$

Then,

$$K(y,z) = 4 \cdot \begin{cases} zI_0(z)K_1(y) - yI_1(z)K_2(y), & 0 < z < y \\ yI_0(y)K_1(z) - zI_1(y)K_2(z), & 0 < y < z \end{cases} := 4 \cdot \begin{cases} K_{y>z}(y,z), & 0 < z < y \\ K_{z>y}(y,z), & 0 < y < z \end{cases} \qquad (4.52)$$

and, we have the following estimates, for $z \neq y$, $0 \leqslant j \leqslant 1$, and $0 \leqslant k \leqslant 4$:

$$|\partial_y^j \partial_z^k K(y,z)| \leqslant \left(\max\{1, \frac{1}{y}\} \right)^j \begin{cases} Ce^{-|y-z|} \left(\frac{|z-y|}{\sqrt{zy}} + \frac{\sqrt{\max\{y,z\}}}{\min\{y,z\}^{3/2}} \right), & y, z > 1 \\ C\sqrt{y} e^{-y} z^{\frac{1}{2}((-1)^k + 1)}, & y > 1 > z \\ C\sqrt{z} e^{-z} y, & z > 1 > y \end{cases}$$

For $1 > y, z$, $y \neq z$, and $0 \leqslant j, k \leqslant 1$,

$$|\partial_y^j \partial_z^k K(y,z)| \leqslant Cy^{1-j} z^{1-k}(1 + |\log(\max\{y,z\})|), \quad |\partial_y^j \partial_z^{2+k} K(y,z)| \leqslant \frac{Cy^{-j} \min\{y,z\}}{z^k \max\{y,z\}}$$

$$|\partial_y^j \partial_z^{4+k} K(y,z)| \leqslant \begin{cases} \frac{Cz^{1-k}}{y^{1+j}}, & 1 > y > z \\ \frac{Cy^{1-j}}{z^{3+k}}, & 1 > z > y \end{cases}$$

If $1 \leqslant j \leqslant 2$, $0 \leqslant n \leqslant 1$, and $0 \leqslant k \leqslant 4 - j$, then, for $0 < \omega, \xi$ and $\omega \neq \xi$, we have

$$|\partial_\xi^n \partial_\omega^k \partial_t^j \left(K(\xi\lambda(t), \omega\lambda(t))\right)| \leqslant \frac{C\lambda(t)^k}{t^j} \left(max\{\lambda(t), \frac{1}{\xi}\}\right)^n \begin{cases} e^{-\lambda(t)|\xi - \omega|} \left(\frac{\lambda(t)^j |\xi - \omega|^{j+1}}{\sqrt{\omega\xi}} + \frac{\sqrt{max\{\xi, \omega\}}}{min\{\xi, \omega\}^{3/2}\lambda(t)}\right), & \omega, \xi > \frac{1}{\lambda(t)} \\ e^{-\lambda(t)\xi} \left(\xi\lambda(t)\right)^{1/2+j} \left(\omega\lambda(t)\right)^{\frac{1}{2}((-1)^k + 1)}, & \xi > \frac{1}{\lambda(t)} > \omega \\ e^{-\omega\lambda(t)} \xi\lambda(t) \left(\omega\lambda(t)\right)^{\frac{1}{2}+j}, & \omega > \frac{1}{\lambda(t)} > \xi \end{cases}$$

For $1 \leqslant j \leqslant 2$, $0 \leqslant n, k \leqslant 1$, and $\frac{1}{\lambda(t)} \geqslant \omega, \xi$, $\omega \neq \xi$, we have

$$|\partial_\xi^n \partial_t^j \partial_\omega^k \left(K(\xi\lambda(t), \omega\lambda(t))\right)| \leqslant \frac{C\lambda(t)^2}{t^j} \omega^{1-k} \xi^{1-n} \left(1 + |\log(max\{\xi, \omega\}\lambda(t))|\right)$$

For $0 \leqslant n, k \leqslant 1$, $1 \leqslant j \leqslant 2$, and $\frac{1}{\lambda(t)} > \omega, \xi$, $\omega \neq \xi$, we have

$$|\partial_\xi^n \partial_\omega^{2+k} \partial_t^j \left(K(\xi\lambda(t), \omega\lambda(t))\right)| \leqslant \frac{C\lambda(t)^2}{t^j} \frac{min\{\omega, \xi\}}{max\{\omega, \xi\}\omega^k \xi^n}$$

Proof We start with the table of Gradshteyn and Ryzhik, [9], entry 6.541, 1. A special case of this identity says that, for $c > 0$,

$$\int_0^\infty J_1(xy) J_1(xz) \frac{x dx}{x^2 + c^2} = \begin{cases} I_1(zc) K_1(yc), & 0 < z < y \\ I_1(yc) K_1(zc), & 0 < y < z \end{cases}$$

Upon differentiating in c (it is possible to differentiate under the integral, by the dominated convergence theorem), and setting $c = 1$, we get

$$K(y, z) = -8 \int_0^\infty \frac{J_1(xy) J_1(xz) x dx}{(1 + x^2)^2} = 4 \cdot \begin{cases} z I_0(z) K_1(y) - y I_1(z) K_2(y), & 0 < z < y \\ y I_0(y) K_1(z) - z I_1(y) K_2(z), & 0 < y < z \end{cases}$$

At this stage, the estimates on $K(y, z)$ and its derivatives follow from a straightforward application of asymptotics of Bessel functions, which can be found from numerous sources, for example, [9]. To estimate the time derivatives of $K(\xi\lambda(t), \omega\lambda(t))$, we let $V = y\partial_y + z\partial_z$, and note that

$$\partial_t \left(K(\xi\lambda(t), \omega\lambda(t))\right) = \frac{\lambda'(t)}{\lambda(t)} V(K)\Big|_{\substack{y = \xi\lambda(t) \\ z = \omega\lambda(t)}}.$$

For $1 \leqslant j \leqslant 4$, we can iterate this to obtain expressions for $\partial_t^j \left(K(\xi\lambda(t), \omega\lambda(t))\right)$ in terms of $V^n(K)$ for $1 \leqslant n \leqslant j$. Then, we note that, for all $1 \leqslant j \leqslant 4$, there exist constants $c_{k,j,w}$ such that

$$V^j(K)(y, z) = \sum_{w=0}^{j} z^w \sum_{k=0}^{j} c_{k,j,w} (y - z)^k \partial_y^k \left(\partial_y + \partial_z\right)^w K(y, z).$$

This observation (and a decomposition of the above form, except with y and z switched (note that V is symmetric in y and z)), combined with the same procedure used above, allows us to estimate the time derivatives of $K(\xi\lambda(t), \omega\lambda(t))$. $\square$

We can now estimate $\widehat{RHS_{2,2}}(t, \xi)$.

Lemma 9 *With $RHS_{2,2}$ given in (4.51), and $m_{\geqslant 1}(x) = 1 - m_{\leqslant 1}(x)$, where $m_{\leqslant 1}$ is defined in (2.1), we have*

$$\widehat{RHS_{2,2}}(t, \xi) = \int_0^\infty \widehat{v_{2,0}}(\omega) \sin(t\omega) K(\xi\lambda(t), \omega\lambda(t)) d\omega \tag{4.53}$$

and, for any $N > 10$, there exists $C_N > 0$ such that we have the following estimates for $0 \leqslant k \leqslant 1$:

$$|\partial_\xi^k \widehat{RHS_{2,2}}(t, \xi)| \leqslant \frac{C\lambda(t) \log^2(t)}{t^2} \sup_{x \in [100, \frac{t}{2}]} (\lambda(x) \log(x)) \begin{cases} \lambda(t)\xi^{1-k} \log(t), & \xi \leqslant \frac{1}{\lambda(t)} \\ \lambda(t)^k \sqrt{\xi\lambda} e^{-\xi\lambda(t)} + \frac{C_N \lambda(t)^k}{\langle \xi \rangle^N}, & \xi > \frac{1}{\lambda(t)} \end{cases}$$

$$|\partial_\xi^k \left(\partial_t^2 \widehat{RHS_{2,2}}(t, \xi) - \frac{8 \widehat{v_{2,0}}(\xi) \xi\lambda(t)^2 m_{\geqslant 1}(\xi t) \sin(\xi t)}{t^4}\right)|$$

$$\leqslant \frac{C\lambda(t) \log^2(t)}{t^4} \sup_{x \in [100, \frac{t}{2}]} (\lambda(x) \log(x)) \begin{cases} \xi^{1-k} \lambda(t) \log(t), & \xi \leqslant \frac{1}{\lambda(t)} \\ (\xi\lambda(t))^{5/2} \lambda(t)^k e^{-\xi\lambda(t)} + \frac{C_N \lambda(t)^k (1 + \lambda(t))}{\langle \xi \rangle^N}, & \xi > \frac{1}{\lambda(t)} \end{cases} \tag{4.54}$$

Proof Equation (4.53) follows from insertion of (4.37) into (4.51), and Fubini's theorem. To prove the stated estimates, we start with the decomposition

$$\partial_t^2 \widehat{RHS_{2,2}}(t,\xi) = -\int_0^\infty d\omega \widehat{v_{2,0}}(\omega)\omega^2 \sin(t\omega) K(\xi\lambda(t),\omega\lambda(t))$$
$$+ 2\int_0^\infty d\omega \widehat{v_{2,0}}(\omega)\omega \cos(\omega t)\partial_t \left(K(\xi\lambda(t),\omega\lambda(t))\right)$$
$$+ \int_0^\infty d\omega \widehat{v_{2,0}}(\omega) \sin(\omega t)\partial_t^2 \left(K(\xi\lambda(t),\omega\lambda(t))\right)$$

(Strictly speaking, we first split the integral in (4.53) over the regions $\omega \leqslant \xi$ and $\omega \geqslant \xi$, differentiate under the integral sign, and then combine the resulting integrals). Define $int_i(t,\xi)$ to be the *ith* line on the right-hand side of the expression above. Starting with int_1, we make the following decomposition (where $m_{\leqslant 1}$ is defined in (2.1))

$$int_1(t,\xi) = int_{1,a}(t,\xi) + int_{1,b}(t,\xi), \quad int_{1,a}(t,\xi) = -\int_0^\infty d\omega \widehat{v_{2,0}}(\omega)\omega^2 \sin(\omega t) m_{\leqslant 1}(\omega t) K(\xi\lambda(t),\omega\lambda(t))$$

For $int_{1,a}$, we separately treat the cases $\xi < \frac{1}{t}$, $\frac{1}{\lambda(t)} > \xi > \frac{1}{t}$, and $\xi > \frac{1}{\lambda(t)}$. In each case, we directly estimate $int_{1,a}$ using Lemmas 4 and 8. To estimate $int_{1,b}$, we decompose the integral into the regions $\omega \leqslant \xi$ and $\omega \geqslant \xi$, and integrate by parts four times in the ω variable. We recall (4.52) and note that

$$4\left(\partial_z^3 K_{y>z}(y,z) - \partial_z^3 K_{z>y}(y,z)\right)\Big|_{z=y} = \frac{8}{y}$$

which explains the form of (4.54). Then, we consider the two cases $\frac{1}{\lambda(t)} < \frac{1}{100}$, and $\frac{1}{\lambda(t)} > \frac{1}{100}$. Within each of these two cases, we then consider various regions of ξ. For example, in the case $\frac{1}{\lambda(t)} < \frac{1}{100}$, we consider the regions

$$\xi < \frac{1}{2t}, \quad \frac{1}{2t} < \xi < \frac{1}{\lambda(t)}, \quad \frac{1}{\lambda(t)} < \xi < \frac{1}{100}, \quad \frac{1}{100} < \xi$$

Then, we directly estimate the resulting integrals using Lemmas 4 and 8. After this, we combine the estimates in the separate cases $\frac{1}{\lambda(t)} < \frac{1}{100}$, and $\frac{1}{\lambda(t)} > \frac{1}{100}$. We then use the same procedure to estimate int_2, int_3 (where we integrate by parts $4 - i + 1$ times to treat $int_{i,b}$) and combine everything to get (4.54) for $k = 0$.

To estimate $\partial_\xi \partial_t^2 \widehat{RHS_{2,2}}(t,\xi)$, we start with

$$\partial_t^2 \widehat{RHS_{2,2}}(t,\xi) = \int_0^\infty \widehat{v_{2,0}}(\omega) \left(-\omega^2 \sin(t\omega) K(\xi\lambda(t),\omega\lambda(t)) + 2\omega \cos(\omega t)\partial_t \left(K(\xi\lambda(t),\omega\lambda(t))\right)\right.$$
$$\left. + \sin(\omega t)\partial_t^2 \left(K(\xi\lambda(t),\omega\lambda(t))\right)\right) m_{\leqslant 1}(\omega t) d\omega$$
$$+ \frac{8\widehat{v_{2,0}}(\xi)\xi\lambda(t)^2 m_{\geqslant 1}(\xi t)\sin(\xi t)}{t^4}$$
$$- \int_0^\infty \frac{\sin(\omega t)}{t^4}\partial_\omega^4 \left(\widehat{v_{2,0}}(\omega)\omega^2 K(\xi\lambda(t),\omega\lambda(t)) m_{\geqslant 1}(\omega t)\right) d\omega$$
$$+ 2\int_0^\infty \frac{\sin(\omega t)}{t^3}\partial_\omega^3 \left(\widehat{v_{2,0}}(\omega)\omega\partial_t \left(K(\xi\lambda(t),\omega\lambda(t))\right) m_{\geqslant 1}(\omega t)\right) d\omega$$
$$- \int_0^\infty \frac{\sin(\omega t)}{t^2}\partial_\omega^2 \left(\widehat{v_{2,0}}(\omega) m_{\geqslant 1}(\omega t)\partial_t^2 \left(K(\xi\lambda(t),\omega\lambda(t))\right)\right) d\omega$$

(4.55)

where $m_{\geqslant 1}(x) = 1 - m_{\leqslant 1}(x)$. We split each integral of (4.55) into the regions $\omega \leqslant \xi$ and $\omega \geqslant \xi$, and differentiate with respect to ξ. Finally, we use the estimates of Lemmas 8, 4 to finish the proof of (4.54). The estimation of $\partial_\xi^k \widehat{RHS_{2,2}}(t,\xi)$ for $k = 0,1$ is done similarly. $\square$

With the same procedure used to establish Lemma 3, we get that $u_{w,2}$, the solution to (4.49) with zero Cauchy data at infinity, is given by the following.

$$u_{w,2}(t,r) = \int_t^\infty ds \int_0^\infty d\xi \sin((t-s)\xi) J_1(r\xi)\widehat{RHS_2}(s,\xi)$$
$$= -\int_0^\infty J_1(r\xi)\frac{\widehat{RHS_2}(t,\xi)}{\xi}d\xi - \int_0^\infty d\xi \int_t^\infty ds \frac{\sin((t-s)\xi)}{\xi^2}\partial_s^2 \widehat{RHS_2}(s,\xi) J_1(r\xi)$$

(4.56)

Using Lemmas 2 and 5, we can use Fubini's theorem to get that the first integral in (4.56) is

$$u_{w,2,ell}(t,r) := -\int_0^\infty J_1(r\xi)\frac{\widehat{RHS_2}(t,\xi)}{\xi}d\xi = -\frac{1}{2}\left(\frac{1}{r}\int_0^r s^2 RHS_2(t,s)ds + r\int_r^\infty RHS_2(t,s)ds\right) \qquad (4.57)$$

It will be useful to compute the following function, which will turn out to contain the leading behavior of $u_{w,2,ell}$ in the matching region.

$$u_{w,2,ell,0}(t,r) := -\frac{1}{2}\left(\frac{1}{r}\int_0^r s^2 RHS_{2,0}(t,s)ds + r\int_r^\infty RHS_{2,0}(t,s)ds\right) \qquad (4.58)$$

with

$$RHS_{2,0}(t,s) := \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2}\right)s\left(f_1(t) - \lambda''(t)\log(s)\right) = \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2}\right)(w_{1,main} + v_{2,main})(t,s) \qquad (4.59)$$

and

$$f_1(t) = \lambda''(t) + \frac{\lambda'(t)^2}{2\lambda(t)} + \lambda''(t)\log(\lambda(t)) \qquad (4.60)$$

where we recall the definitions of $w_{1,main}$ and $v_{2,main}$ in (4.25) and (4.40), which are the main parts of w_1 and v_2, respectively, in the matching region. We have

$$u_{w,2,ell,0}(t,r) = \frac{2f_1(t)\lambda(t)^2\log\left(\frac{r^2}{\lambda(t)^2} + 1\right)}{r}$$
$$- \frac{\lambda(t)^2\lambda''(t)\left(\text{Li}_2\left(-\frac{r^2}{\lambda(t)^2}\right) + 2\log(r)\left(\log\left(\frac{r^2}{\lambda(t)^2} + 1\right) - \frac{r^2}{\lambda(t)^2+r^2}\right) + \log\left(\frac{r^2}{\lambda(t)^2} + 1\right)\right)}{r} \qquad (4.61)$$
$$- r\lambda''(t)\left(\frac{2\lambda(t)^2\log(r)}{\lambda(t)^2 + r^2} + \log\left(\frac{\lambda(t)^2}{r^2} + 1\right)\right)$$

The leading behavior of $u_{w,2,ell,0}$ in the matching region is $u_{w,2,ell,0,cont}$, which is defined by

$$u_{w,2,ell,0,cont}(t,r) := \frac{-2}{r}\left(\log(r)\left(-\lambda(t)\lambda'(t)^2 - \lambda(t)^2\lambda''(t) - 2\lambda(t)^2\log(\lambda(t))\lambda''(t)\right)\right.$$
$$+ \lambda(t)\log(\lambda(t))\lambda'(t)^2 + \lambda(t)^2\log(\lambda(t))\lambda''(t) + \lambda(t)^2\log^2(\lambda(t))\lambda''(t) \qquad (4.62)$$
$$\left. + \lambda(t)^2\log^2(r)\lambda''(t) + \frac{1}{2}\lambda(t)^2\lambda''(t) - \frac{1}{12}\pi^2\lambda(t)^2\lambda''(t)\right)$$

In the above computation, we have used asymptotics of Li_2, for example, from [9]. In the course of proving Proposition 2, which will occur when we do a third order matching, we will record refined estimates on the difference between $u_{w,2}$ and its leading parts in the matching region. On the other hand, we will also need estimates which are global in the spatial coordinate, of the difference between the full correction $u_{w,2}$, and the piece $u_{w,2,ell,0,cont}$. Now that we have established Lemmas 7 and 9, we can estimate $u_{w,2} - u_{w,2,ell,0,cont}$.

Lemma 10 *For $0 \leqslant k \leqslant 1$, the following two estimates are true:*

$$|\partial_r^k\left(u_{w,2}(t,r) - u_{w,2,ell}\right)(t,r)| \leqslant Cr\frac{\lambda(t)^{2-k}(1 + \lambda(t))\log^5(t+r)}{t^4}\sup_{x\in[100,\frac{t}{2}]}(\lambda(x)\log(x)), \quad r \geqslant g(t)\lambda(t) \quad (4.63)$$

and

$$|\partial_r^k u_{w,2}(t,r)| \leqslant C\lambda(t)^{\frac{1}{2}-k}(1 + \lambda(t)^{2+k})\log^2(t)\log^2(r)\frac{\sup_{x\in[100,t]}(\lambda(x)\log(x))}{\sqrt{r}t^2}, \quad r \geqslant \frac{t}{2} \qquad (4.64)$$

For $0 \leqslant j \leqslant 8$ and $0 \leqslant k \leqslant 1$, the following two estimates are true.

$$|\partial_r^k\partial_t^j\left(u_{w,2,ell} - u_{w,2,ell,0}\right)(t,r)| \leqslant \frac{Cr^{1-k}\lambda(t)^2\log(t)}{t^{4+j}}\sup_{x\in[100,t]}(\lambda(x)\log(x)), \quad r \leqslant \frac{t}{2} \qquad (4.65)$$

and

$$|\partial_r^k\left(u_{w,2,ell} - u_{w,2,ell,0}\right)(t,r)| \leqslant \frac{C\lambda(t)^2\log(r)}{r^{1+k}t^2}\sup_{x\in[100,t+r]}(\lambda(x)\log(x)), \quad r > \frac{t}{2}$$

For $0 \leqslant j, k \leqslant 5$,

$$|\partial_t^j \partial_r^k \left(u_{w,2,ell,0}(t,r) - u_{w,2,ell,0,cont}(t,r)\right)| \leqslant \begin{cases} \frac{C\lambda(t)^3 \left(\log^2(r) + \log^2(t)\right)}{r^{1+k}t^{2+j}}, & r \leqslant \lambda(t) \\ \frac{C\lambda(t)^5 \left(|\log(r)| + \log(t)\right)}{r^{3+k}t^{2+j}}, & r \geqslant \lambda(t) \end{cases} \tag{4.66}$$

For $0 \leqslant k \leqslant 1$,

$$r|\partial_r u_{w,2,ell}(t,r)| + t^k|\partial_t^k u_{w,2,ell}(t,r)| \leqslant \begin{cases} \frac{Cr\lambda(t)(\log(t) + |\log(r)|)}{t^2}, & r \leqslant \lambda(t) \\ \frac{C\lambda(t)^2 \log(t)}{rt^2} \sup_{x \in [100,t]} (\lambda(x)\log(x)), & \lambda(t) < r < \frac{t}{2} \\ \frac{C\lambda(t)^2 \log(r) \sup_{x \in [100,t+r]}(\lambda(x)\log(x))}{\sqrt{rt^{5/2}}}, & r > \frac{t}{2} \end{cases} \tag{4.67}$$

For $0 \leqslant k \leqslant 2$,

$$|\partial_t^{2+k} u_{w,2,ell}(t,r)| \leqslant \begin{cases} \frac{Cr\lambda(t)(\log(t) + |\log(r)|)}{t^{4+k}}, & r \leqslant \lambda(t) \\ \frac{C\lambda(t)^2 \log(t)}{rt^{4+k}} \sup_{x \in [100,t]} (\lambda(x)\log(x)), & \lambda(t) < r < \frac{t}{2} \\ \frac{C\lambda(t)^2 \log(r) \sup_{x \in [100,t+r]}(\lambda(x)\log(x))}{\sqrt{r}} \left(\frac{1}{r^4 \langle t-r \rangle^{\frac{1}{2}+k}} + \frac{1}{t^{9/2+k}}\right), & r > \frac{t}{2} \end{cases} \tag{4.68}$$

Finally, we have the following estimate, for all $r > 0$. (Recall the definition of E in (1.3)).

$$\sqrt{E(u_{w,2}(t), \partial_t u_{w,2})} + |u_{w,2}(t,r)| \leqslant \frac{C\log(t)}{t^{1-C_u}} \tag{4.69}$$

Proof We have

$$\begin{aligned} &u_{w,2,ell}(t,r) - u_{w,2,ell,0}(t,r) \\ &= -\frac{1}{2}\left(\frac{1}{r}\int_0^r s^2 \left(RHS_2(t,s) - RHS_{2,0}(t,s)\right) ds + r\int_r^\infty \left(RHS_2(t,s) - RHS_{2,0}(t,s)\right) ds\right) \end{aligned} \tag{4.70}$$

where we recall the definitions of $RHS_{2,0}$ and RHS_2 from (4.59), and (4.49), respectively. Inside the integrals, in the region $s \leqslant \frac{t}{2}$, we use

$$RHS_2(t,s) - RHS_{2,0}(t,s) = \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2}\right)\left(v_{2,sub}(t,s) + w_{1,sub}(t,s)\right)$$

and the estimates on $v_{2,sub}$ and $w_{1,sub}$ from Lemmas 1 and 6. (Recall the definitions of $w_{1,sub}$ and $v_{2,sub}$ in (4.26) and (4.48), respectively). On the other hand, in the region $s \geqslant \frac{t}{2}$, we use

$$(RHS_2 - RHS_{2,0})(t,s) = \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2}\right)\left(v_2(t,s) + w_1(t,s) - s\left(f_1(t) - \lambda''(t)\log(s)\right)\right)$$

where f_1 is given by (4.60), and the estimates from Lemmas 2 and 5. The only detail to note is that, when establishing (4.65) for $j \geqslant 1$, we will have terms of the following form, for $1 \leqslant k \leqslant j$.

$$-\frac{r}{2}\int_{\frac{t}{2}}^\infty g(s,t)\partial_t^k v_2(t,s) ds$$

For these, since the estimates on $\partial_t^k v_2(t,s)$ in the region $s \geqslant \frac{t}{2}$ from Lemma 5 are not a factor of t^{-k} better than the corresponding estimates on $v_2(t,s)$, we write the following (iterating as needed if $j > 2$)

$$\partial_t^k v_2(t,s) = \begin{cases} (\partial_t + \partial_s)v_2(t,s) - \partial_s v_2(t,s), & k = 1 \\ \partial_s^2 v_2 + \frac{1}{s}\partial_s v_2 - \frac{v_2}{s^2}, & k = 2 \end{cases} \tag{4.71}$$

and integrate by parts for the terms involving s derivatives, which is why (4.65) has a factor of $\frac{1}{t^j}$.

To estimate $u_{w,2} - u_{w,2,ell}$, we first note that

$$u_{w,2}(t,r) - u_{w,2,ell}(t,r) = \int_t^\infty ds \int_0^\infty d\xi \frac{\sin((s-t)\xi)}{\xi^2} \partial_s^2 \widehat{RHS_2}(s,\xi) J_1(r\xi) \tag{4.72}$$

Then, we recall (4.54), and write

$$\partial_s^2 \widehat{RHS_2}(s,\xi) = \partial_s^2 \widehat{RHS_{2,3}}(s,\xi) + \frac{8\widehat{v_{2,0}}(\xi)\xi\lambda(t)^2 m_{\geqslant 1}(\xi t)\sin(\xi t)}{t^4} \tag{4.73}$$

We split the s integral in (4.72) involving $\partial_s^2 \widehat{RHS}_{2,3}(s,\xi)$ into the regions $s - t \leqslant r$ and $s - t \geqslant r$. Then, when $s - t \leqslant r$, we split the ξ integral in (4.72) over the three regions $\xi \leqslant \frac{1}{r}$, $\frac{1}{r} \leqslant \xi \leqslant \frac{1}{s-t}$ and $\frac{1}{s-t} < \xi$. We make a similar splitting in the region $s - t \geqslant r$, with the following detail. In the region $\xi \geqslant \frac{1}{s-t}$, and $s - t \geqslant r$, we integrate by parts in the ξ variable, integrating $\sin((s - t)\xi)$, and differentiating the rest of the integrand. Then, we use

$$|J_1(x)| \leqslant \begin{cases} Cx, & x \leqslant 1 \\ \frac{C}{\sqrt{x}}, & x \geqslant 1 \end{cases}$$

along with (4.3) and Lemmas 7 and 9 to estimate each resulting integral. Recalling (4.73), we now need to consider the $\frac{8\widehat{v_{2,0}}(\xi)\xi\lambda(t)^2 m_{\geqslant 1}(\xi t)\sin(\xi t)}{t^4}$ contribution to (4.72). For this, we first use

$$\sin((s - t)\xi)\sin(\xi s) = \frac{1}{2}\left(\cos(t\xi) - \cos((2s - t)\xi)\right)$$

and then split the ξ integral in (4.72) over the regions $\xi \leqslant \frac{1}{t}$ and $\xi \geqslant \frac{1}{t}$, integrating by parts in ξ in the second integral (where we integrate $\cos(t\xi) - \cos((2s - t)\xi)$ and differentiate the rest of the integrand). This establishes (4.63) for $k = 0$. To establish (4.63) for $k = 1$, we start by differentiating (4.72) in r, and then use the same procedure as for $u_{w,2} - u_{w,2,ell}$.

To estimate $u_{w,2}(t,r)$ for $r \geqslant \frac{t}{2}$, we start with

$$u_{w,2}(t,r) = \int_t^\infty ds \int_0^\infty d\xi \sin((t - s)\xi)\widehat{RHS}_2(s,\xi)J_1(r\xi) \tag{4.74}$$

Then, we use the same procedure as above, except that, whenever $|s - t - r| \geqslant 1$, $\xi \geqslant \frac{1}{s-t}$ and $\xi \geqslant \frac{1}{r}$, we write

$$J_1(x) = \frac{\sin(x)}{\sqrt{\pi}\sqrt{x}} - \frac{\cos(x)}{\sqrt{\pi}\sqrt{x}} + J_1(x) - \left(\frac{\sin(x)}{\sqrt{\pi}\sqrt{x}} - \frac{\cos(x)}{\sqrt{\pi}\sqrt{x}}\right)$$

and use

$$\sin(y)\left(\frac{\sin(x)}{\sqrt{\pi x}} - \frac{\cos(x)}{\sqrt{\pi x}}\right) = \frac{-\sin(y - x) - \sin(x + y) + \cos(y - x) - \cos(x + y)}{2\sqrt{\pi x}}$$

with $y = (s - t)\xi$ and $x = r\xi$. Then, we integrate by parts in ξ, and estimate all of the resulting terms directly. We use the same procedure for $\partial_r u_{w,2}(t,r)$ in the region $r \geqslant \frac{t}{2}$. This yields an extra factor of $1 + \frac{1}{\lambda(t)}$ relative to the estimates for $u_{w,2}(t,r)$ in the region $r \geqslant \frac{t}{2}$. (We can differentiate under the integral sign in (4.74) by the dominated convergence theorem, and the fact that $\xi\widehat{RHS}_2(s,\xi) \in L^1_{s,\xi}((t,\infty) \times (0,\infty))$).

Finally, to obtain (4.67), we recall the definition of $u_{w,2,ell}$, in (4.57).

$$u_{w,2,ell}(t,r) = -\frac{1}{2}\left(\frac{1}{r}\int_0^r s^2 RHS_2(t,s)ds + r\int_r^\infty RHS_2(t,s)ds\right)$$

If $r \leqslant \frac{t}{2}$, we write

$$\partial_t^k u_{w,2,ell}(t,r) = \frac{-1}{2r}\int_0^r s^2 \partial_t^k RHS_2(t,s)ds - \frac{r}{2}\int_{\frac{t}{2}}^\infty \partial_t^k RHS_2(t,s)ds - \frac{r}{2}\int_r^{\frac{t}{2}} \partial_t^k RHS_2(t,s)ds \tag{4.75}$$

If $r \geqslant \frac{t}{2}$, we have

$$\partial_t^k u_{w,2,ell}(t,r) = \frac{-1}{2r}\int_0^{\frac{t}{2}} s^2 \partial_t^k RHS_2(t,s)ds - \frac{1}{2r}\int_{\frac{t}{2}}^r s^2 \partial_t^k RHS_2(t,s)ds - \frac{r}{2}\int_r^\infty \partial_t^k RHS_2(t,s)ds \tag{4.76}$$

The point of this decomposition is the following. The function $\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2}$ is a symbol in t and r. The estimates on $w_1(t,r)$ and $v_2(t,r)$ from Lemmas 2 and 5 improve by a factor of $\frac{1}{t}$ with each time derivative taken (for example, for up to 4 derivatives), in the region $r \leqslant \frac{t}{2}$. On the other hand, $v_2(t,r)$ is a free wave, and our estimates on, for example, $\partial_t v_2(t,r)$ from Lemma 5 are not a power of t better than the estimates on $v_2(t,r)$ in the region $r \geqslant \frac{t}{2}$. Therefore, when $s \geqslant \frac{t}{2}$, we rewrite $\partial_t^k v_2(t,s)$ as in (4.71) (iterating these as needed, to treat $k = 3, 4$). Then, we insert this re-writing into (4.75) and (4.76), and integrate by parts in s to remove all s derivatives from v_2 (or $(\partial_t + \partial_s)v_2$, as appropriate). A direct estimation of all of the integral

and boundary terms then gives rise to (4.67) and (4.68). The form of the estimates (4.67) and (4.68) are different in the region $r \geqslant \frac{t}{2}$ because, for sufficiently regular f,

$$-\frac{1}{2r}\int_{\frac{t}{2}}^{r}s^2 f(t,s)\partial_s v_2(t,s)ds - \frac{r}{2}\int_r^\infty f(t,s)\partial_s v_2(t,s)ds$$

$$= \frac{1}{2r}\left(\frac{t}{2}\right)^2 f(t,\frac{t}{2})v_2(t,\frac{t}{2}) + \frac{1}{2r}\int_{\frac{t}{2}}^{r} v_2(t,s)\partial_s\left(s^2 f(t,s)\right)ds + \frac{r}{2}\int_r^\infty v_2(t,s)\partial_s f(t,s)ds$$

In other words, the boundary terms at r arising from one integration by parts exactly cancel. This does not happen if $\partial_s v_2(t,s)$ in the first line is replaced by $\partial_s^k v_2(t,s)$, and we integrate by parts k times, for $k \geqslant 2$. The estimate (4.66) follows from the explicit formulae (4.61) and (4.62). Finally, we prove the energy estimate, (4.69). By the dominated convergence theorem, and fact that $\xi\widehat{RHS}_2(s,\xi) \in L^1_{s,\xi}((t,\infty)\times(0,\infty))$, we can differentiate once in either t or r under the integral in (4.74). The estimate on $||\partial_t u_{w,2}||_{L^2(rdr)}$ implied by (4.69) then follows directly from Minkowski's inequality, the L^2 isometry property of the Hankel transform of order 1, and the estimates from Lemmas 2 and 5. Next, we have

$$\partial_r u_{w,2}(t,r) + \frac{u_{w,2}(t,r)}{r} = \int_t^\infty ds \int_0^\infty d\xi \sin((t-s)\xi)\xi J_0(r\xi)\widehat{RHS}_2(s,\xi)$$

Using again Minkowski's inequality, and the L^2 isometry property of the Hankel transforms of orders 0 and 1, we get

$$||\partial_r u_{w,2}(t,r) + \frac{u_{w,2}(t,r)}{r}||_{L^2(rdr)} \leqslant C\int_t^\infty ds ||RHS_2(s,r)||_{L^2(rdr)} < \infty$$

where the last inequality is true by the estimates from Lemmas 2 and 5. Therefore, by the dominated convergence theorem,

$$\int_0^\infty \left(\partial_r u_{w,2}(t,r) + \frac{u_{w,2}(t,r)}{r}\right)^2 rdr = \lim_{M\to\infty}\int_{\frac{1}{M}}^{M}\left(\partial_r u_{w,2}(t,r) + \frac{u_{w,2}(t,r)}{r}\right)^2 rdr$$

Then, we note that

$$\left(\partial_r u_{w,2}(t,r) + \frac{u_{w,2}(t,r)}{r}\right)^2 = (\partial_r u_{w,2}(t,r))^2 + \frac{u_{w,2}(t,r)^2}{r^2} + \frac{\partial_r(u_{w,2}^2)}{r}$$

For each $M > 0$,

$$\int_{\frac{1}{M}}^{M}\frac{\partial_r(u_{w,2}^2)(t,r)}{r}rdr = u_{w,2}^2(t,M) - u_{w,2}^2(t,\frac{1}{M})$$

Again because of $\xi\widehat{RHS}_2(s,\xi) \in L^1_{s,\xi}((t,\infty)\times(0,\infty))$ and the dominated convergence theorem,

$$\lim_{r\to 0}\left(\int_t^\infty ds \int_0^\infty d\xi \sin((s-t)\xi)\widehat{RHS}_2(s,\xi)J_1(r\xi)\right) = 0$$

Then, by (4.64), we get

$$\lim_{M\to\infty}\int_{\frac{1}{M}}^{M}\frac{\partial_r(u_{w,2}^2)(t,r)}{r}rdr = 0$$

Therefore,

$$\infty > \int_0^\infty \left(\partial_r u_{w,2}(t,r) + \frac{u_{w,2}(t,r)}{r}\right)^2 rdr = \lim_{M\to\infty}\int_{\frac{1}{M}}^{M}\left((\partial_r u_{w,2}(t,r))^2 + \frac{u_{w,2}(t,r)^2}{r^2}\right)rdr$$

By the monotone convergence theorem, we thus get

$$\int_0^\infty \left((\partial_r u_{w,2}(t,r))^2 + \frac{u_{w,2}(t,r)^2}{r^2}\right)rdr \leqslant C\left(\int_t^\infty ds ||RHS_2(s,r)||_{L^2(rdr)}\right)^2$$

Finally, (4.69) follows from

$$|u_{w,2}(t,r)| \leqslant \int_t^\infty \int_0^\infty |\widehat{RHS}_2(s,\xi)|\sqrt{\xi}\frac{|J_1(r\xi)|}{\sqrt{\xi}}d\xi ds \leqslant \int_t^\infty \left(\int_0^\infty |\widehat{RHS}_2(s,\xi)|^2\xi d\xi\right)^{1/2}\left(\int_0^\infty \frac{|J_1(x)|^2}{x}dx\right)^{1/2}ds$$

$$\leqslant C\int_t^\infty ||RHS_2(s,r)||_{L^2(rdr)}ds$$

$$\tag{4.77}$$

$\square$

4.5 Second order matching, part 1

The first question is whether the choice of $v_{2,0}$ allows for matching between the next order (roughly on the order of $\frac{r^3 \lambda(t)}{t^4}$) terms in $w_1(t,r)$ and $v_{ell,2}(t, \frac{r}{\lambda(t)})$. For clarity, we recall the third, seventh, and eighth lines of (4.29), and the expression (4.17). The main contribution of $v_{ell,2,0}(t, \frac{r}{\lambda(t)})$ to the matching is

$$v_{ell,2,0,main}(t, \frac{r}{\lambda(t)}) = \frac{r^3}{8} \partial_t^2 \left(\frac{\lambda'(t)^2}{2\lambda(t)} + \lambda''(t) - \lambda''(t) \log(\frac{r}{\lambda(t)}) \right) + \frac{3}{32} r^3 \lambda''''(t)$$

The r^3 contribution of $w_1(t,r)$ is (recall (1))

$$w_{1,cubic,main}(t,r) = \frac{3}{32} r^3 \lambda''''(t) + \frac{r^3}{8} \left(\lambda''''(t) \left(\log(2) + \frac{1}{2} \right) - \log(r)\lambda''''(t) + \log(t)\lambda''''(t) \right.$$
$$\left. + \int_t^{2t} \frac{\lambda''''(s) - \lambda''''(t)}{s-t} ds + \int_{2t}^{\infty} \frac{\lambda''''(s)ds}{s-t} \right)$$

Finally, the v_2 contribution at this order is

$$v_{2,cubic,main}(t,r) = \frac{-r^3}{32} F''(t)$$

Recall that we already chose $v_{2,0}$ to allow for matching of terms of the form $rf(t)$ coming from $v_{ell}(t, \frac{r}{\lambda(t)})$ and $w_1(t,r) + v_2(t,r)$. Note that the coefficient of r^3 given in the expression above for $v_{2,cubic,main}$ is precisely one-eighth of two time derivatives of the r coefficient of v_2 (see (4.37)). Although the r^3 coefficient of $v_{ell,2,0,main}(t, \frac{r}{\lambda(t)})$ and that of $w_{1,cubic,main}(t,r)$ are not precisely one-eighth of two time derivatives of the main r coefficients in the matching region of $v_{ell}(t, \frac{r}{\lambda(t)})$ and $w_1(t,r)$, respectively, the r^3 coefficient of the *difference* of these two functions is precisely one-eighth of two time derivatives of the main r coefficient in the matching region of $v_{ell}(t, \frac{r}{\lambda(t)}) - w_1(t,r)$. Note the important cancellation between the $\frac{3}{32} r^3 \lambda''''(t)$ terms when $v_{ell,2,0,main}(t, \frac{r}{\lambda(t)})$ and $w_{1,cubic,main}(t,r)$ are subtracted. Therefore, the matching of the r^3 terms is already accomplished with our previous choice of $v_{2,0}$. In other words, by the choice of $v_{2,0}$, we have

$$v_{ell,2,0,main}(t, \frac{r}{\lambda(t)}) - \left(w_{1,cubic,main}(t,r) + v_{2,cubic,main}(t,r) \right) = 0$$

4.6 Second order matching, part 2

Next, we choose the coefficient $c_1(t)$ in (4.7). For convenience, we define $v_{ell,sub}$ by

$$v_{ell,sub}(t,R) = v_{ell}(t,R) - R \left(\frac{1}{2} \lambda'(t)^2 + \lambda(t)\lambda''(t) - \lambda(t)\lambda''(t) \log(R) \right)$$
$$= \left(2c_1(t) - 3\lambda'(t)^2 \right) f_4(R) + \lambda(t)\lambda''(t) f_3(R)$$

(4.78)

with

$$f_4(R) = \frac{R}{2(1+R^2)}$$

$$f_3(R) = \frac{R^2 \left(-3 + 2 \left(1 + R^2 \right) \log(R) \right) - \left(R^4 - 1 \right) \log(1 + R^2) + 2R^2 Li_2(-R^2)}{2R(1+R^2)}$$

The function $c_1(t)$ appears as part of the coefficient of the $\frac{1}{r}$ term in a large r expansion of $u_{ell,sub}(t,r) = v_{ell,sub}(t, \frac{r}{\lambda(t)})$. On the other hand, there are also higher order terms, of size $\frac{\log(r)}{r}$ and $\frac{\log^2(r)}{r}$ appearing in the expansion of $u_{ell,sub}(t,r)$ for large r, and $c_1(t)$ does not appear in the coefficients of these terms. However, it turns out that these terms happen to exactly match corresponding $\frac{\log(r)}{r}$ and $\frac{\log^2(r)}{r}$ terms coming from expansions of v_{ex} and $u_{w,2,ell,0}$, which means that we only need to match terms of size $\frac{1}{r}$, which can be done by choosing an appropriate $c_1(t)$. To show this, we note the following.

The main contribution of $v_{ell,sub}(t, \frac{r}{\lambda(t)})$ in the matching region is

$$v_{ell,sub,cont}(t,r)$$
$$= \frac{c_1(t)\lambda(t) - \frac{3}{2}\lambda(t)\lambda'(t)^2}{r}$$
$$- \frac{\lambda(t)^2 \lambda''(t) \left(-6\log(r)(4\log(\lambda(t)) + 1) + 6\log(\lambda(t))(2\log(\lambda(t)) + 1) + 12\log^2(r) + \pi^2 + 12 \right)}{6r}$$

(4.79)

On the other hand, the main contribution of $v_{ex}(t,r)$ in the matching region is (recall (4.36))

$$v_{ex,cont}(t,r) = \frac{1}{2r}\left(\log(r)\left(-4\lambda(t)\lambda'(t)^2 - 2\lambda(t)^2\lambda''(t)\right) + 4\lambda(t)\log(\lambda(t))\lambda'(t)^2 - \lambda(t)^2\lambda''(t) + 2\lambda(t)^2\lambda''(t)\log(\lambda(t))\right)$$
(4.80)

Finally, the main contribution of $u_{w,2,ell,0}(t,r)$ in the matching region is (recall (4.62))

$$u_{w,2,ell,0,cont}(t,r) = \frac{-2}{r}\left(\log(r)\left(-\lambda(t)\lambda'(t)^2 - \lambda(t)^2\lambda''(t) - 2\lambda(t)^2\log(\lambda(t))\lambda''(t)\right)\right.$$
$$+\lambda(t)\log(\lambda(t))\lambda'(t)^2 + \lambda(t)^2\log(\lambda(t))\lambda''(t) + \lambda(t)^2\log^2(\lambda(t))\lambda''(t)$$
$$\left. + \lambda(t)^2\log^2(r)\lambda''(t) + \frac{1}{2}\lambda(t)^2\lambda''(t) - \frac{1}{12}\pi^2\lambda(t)^2\lambda''(t)\right)$$

These expressions give

$$v_{ell,sub,cont}(t,r) - \big(u_{w,2,ell,0,cont}(t,r) + v_{ex,cont}(t,r)\big) = -\frac{\lambda(t)\left(-6c_1(t) + \left(3 + 2\pi^2\right)\lambda(t)\lambda''(t) + 9\lambda'(t)^2\right)}{6r}$$

(As previously mentioned, the $\frac{\log(r)}{r}$ and $\frac{\log^2(r)}{r}$ terms from $v_{ell,sub,cont}$, $u_{w,2,ell,0,cont}$ and $v_{ex,cont}$ cancel in the above expression). We therefore choose $c_1(t)$ so as to make

$$v_{ell,sub,cont}(t,r) - \big(u_{w,2,ell,0,cont}(t,r) + v_{ex,cont}(t,r)\big) = 0$$

by defining

$$c_1(t) = \frac{3}{2}\lambda'(t)^2 + \frac{\lambda(t)\lambda''(t)}{2} + \frac{\pi^2}{3}\lambda(t)\lambda''(t)$$
(4.81)

Note that the matching condition used to determine the initial velocity of v_2 is precisely

$$u_{ell}(t,r) - v_{ell,sub}(t,\frac{r}{\lambda(t)}) = v_2(t,r) - v_{2,sub}(t,r) + w_1(t,r) - w_{1,sub}(t,r)$$
(4.82)

where we recall that $w_{1,sub}$ and $v_{2,sub}$ were defined in (4.26) and (4.48), respectively.

4.7 Third order matching

It will be convenient for us to define

$$v_{ex,sub}(t,r) := v_{ex}(t,r) - v_{ex,ell}(t,r) = -\int_0^\infty d\xi\, J_1(r\xi)\int_t^\infty ds\,\frac{\sin((t-s)\xi)}{\xi^2}\partial_s^2\widehat{RHS}(s,\xi)$$
(4.83)

We will now choose $v_{2,2}$ (recall (4.50)) so as to match the principal terms (which are of the form $r\log^k(r)g_k(t)$) in $u_{ell,2}(t,r) - v_{ell,2,0,main}(t,\frac{r}{\lambda(t)})$ and $v_{ex,sub}(t,r) + u_{w,2}(t,r) - u_{w,2,ell,0,cont}(t,r)$. The main proposition of this section is the following.

Proposition 2 (Third order matching) *For $v_{2,2}$ defined in (4.50), where $v_{2,3}$ is defined by (4.134), and F_3 and G_3 are given in (4.131) and (4.133), respectively, we have the following estimate. For $0 \leqslant j, k \leqslant 1$ or $k = 2, j = 0$, and $g(t)\lambda(t) \leqslant r \leqslant 2g(t)\lambda(t)$,*

$$|\partial_t^j\partial_r^k\left(u_{ell,2}(t,r) - v_{ell,2,0,main}(t,\frac{r}{\lambda(t)}) - \big(u_{w,2}(t,r) - u_{w,2,ell,0,cont}(t,r) + v_{2,2}(t,r) + v_{ex} - v_{ex,cont} - q_{4,2}\big)\right)|$$

$$\leqslant \frac{C\lambda(t)^{2-k}\log(t)}{t^{2+j}g(t)^{3+k}} + \frac{Cg(t)^{2-k}\lambda(t)^{4-k}\log^4(t)\sup_{x\in[100,t]}(\lambda(x)\log(x))}{t^{5+j}}$$

where $q_{4,2}$ is defined in (4.116) and satisfies the following inequality for $k = 0,1$, and $r > 0$.

$$|\partial_t^k q_{4,2}(t,r)| + \sqrt{E(\partial_t^k q_{4,2}, \partial_t^{k+1}q_{4,2})} + ||\partial_r^2 q_{4,2}||_{L^2((\lambda(t)g(t), 2\lambda(t)g(t)), rdr)} \leqslant \frac{C\lambda(t)^5\log^3(t)}{t^5}$$

Proof We start by computing the leading parts of $u_{ell,2}(t,r) - v_{ell,2,0,main}(t,\frac{r}{\lambda(t)})$ and $v_{ex,sub}(t,r) + u_{w,2}(t,r) - u_{w,2,ell,0,cont}(t,r)$ in the matching region, $r \sim g(t)\lambda(t)$.

We recall our expressions for $v_{ex,sub}$ and $u_{w,2} - u_{w,2,ell}$:

$$v_{ex,sub}(t,r) = -\int_0^\infty d\xi J_1(r\xi) \int_t^\infty ds \frac{\sin((t-s)\xi)}{\xi^2} \partial_s^2 \widehat{RHS}(s,\xi)$$

$$u_{w,2}(t,r) - u_{w,2,ell}(t,r) = \int_t^\infty ds \int_0^\infty d\xi \frac{\sin((s-t)\xi)}{\xi^2} \partial_s^2 \widehat{RHS}_2(s,\xi) J_1(r\xi) \tag{4.84}$$

In particular, these functions solve the following equations with 0 Cauchy data at infinity

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)(u_{w,2} - u_{w,2,ell}) = \partial_t^2 u_{w,2,ell}$$

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) v_{ex,sub} = \partial_t^2 v_{ex,ell}$$

In order to compute their leading behavior in the matching region, $r \sim g(t)\lambda(t)$, we define $u_{w,2,sub,0}(t,r)$ and $v_{ex,sub,0}(t,r)$ to be the solutions to the following equations with 0 Cauchy data at infinity

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) u_{w,2,sub,0} = \partial_t^2 u_{w,2,ell,0,cont} \tag{4.85}$$

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) v_{ex,sub,0} = \partial_t^2 v_{ex,cont}$$

Each of these right-hand sides are of the following form (recall Section 4.6)

$$\frac{1}{r} \sum_{k=0}^2 f_k(t) \log^k(r)$$

In particular, the right-hand sides of the equations for $u_{w,2,sub,0}$ and $v_{ex,sub,0}$ are technically singular at $r = 0$, even though the right-hand sides of the equations for $u_{w,2} - u_{w,2,ell}$ and $v_{ex,sub}$ are not. This will not cause any problems for us, and is done so that the principal parts of $u_{w,2,sub,0}$ and $v_{ex,sub,0}$ can be exactly calculated.

We will first prove the following lemma which will lead to fairly explicit formulae for $u_{w,2,sub,0}$ and $v_{ex,sub,0}$. After the lemma we will provide some information about how the formulae arise.

Lemma 11 *Let f_k be the functions arising from expressing either $\partial_t^2 v_{ex,cont}$ or $\partial_t^2 u_{w,2,ell,0,cont}$ in the form $\frac{1}{r}\sum_{k=0}^2 f_k(t)\log^k(r)$. If u is given by*

$$u(t,r) = -\int_t^\infty f_1(s) K_0\left(\frac{r}{s-t}, r\right) ds \tag{4.86}$$

where

$$K_0(x,r) = \frac{1}{x} \begin{cases} \left(\left(1 - \sqrt{1-x^2}\right)(\log(r) - \log(x)) + \sqrt{1-x^2} - 1 + \sqrt{1-x^2}\log\left(\sqrt{1-x^2} + 1\right)\right. \\ \left. + \log\left(\sqrt{1-x^2} + 1\right) - 2\sqrt{1-x^2}\log\left(2\sqrt{1-x^2}\right)\right), \quad 0 < x < 1 \\ \left(\sqrt{x^2 - 1}\sin^{-1}\left(\frac{1}{x}\right) + \log(r) - 1\right), \quad x > 1 \end{cases}$$

then, u solves

$$-\partial_{tt} u + \partial_{rr} u + \frac{1}{r}\partial_r u - \frac{u}{r^2} = \frac{f_1(t)\log(r)}{r}, \quad r > 0$$

with 0 Cauchy data at infinity. Similarly, if u is defined by

$$u(t,r) = \int_t^\infty f_2(s) K_2\left(\frac{s-t}{r}, r\right) ds \tag{4.87}$$

with

$$K_2(a,r) = \sum_{k=0}^2 F_k(a) \log^k(r)$$

where, for $a > 1$, we have

$$F_0(a) = a\left(Li_2\left(\frac{\sqrt{a^2-1}}{a}\right) + Li_2\left(-\frac{a}{\sqrt{a^2-1}}\right) + Li_2\left(1 - 2a\left(a + \sqrt{a^2-1}\right)\right)\right.$$

$$-Li_2\left(2 - 2a\left(a + \sqrt{a^2-1}\right)\right) + 2\right) + \frac{1}{2}\left(4\sqrt{a^2-1}Li_2\left(2a\left(a - \sqrt{a^2-1}\right) - 1\right) + \frac{1}{4}a\log^2\left(1 - \frac{1}{a^2}\right)\right.$$

$$+2\left(\sqrt{a^2-1} + a\right)\log^2\left(\sqrt{a^2-1} + a\right) + 4\left(-a\log\left(2\left(a^2-1\right)\right) - \sqrt{a^2-1}\left(\log\left(a^2-1\right) - 1 + \log(4)\right)\right.$$

$$+a + a\log(a))\log\left(\sqrt{a^2-1} + a\right) + \frac{1}{6}\sqrt{a^2-1}\left(12\log\left(a^2-1\right)\left(\log\left(a^2-1\right) - 2 + \log(16)\right)\right.$$

$$-\pi^2 + 24 + 24(\log(2) - 1)\log(4)\right) - 8a\right), \quad a > 1$$

$$F_1(a) = 2\left(-\sqrt{a^2-1} - \left(\sqrt{a^2-1} + a\right)\log\left(\sqrt{a^2-1} + a\right) + \sqrt{a^2-1}\log\left(4\left(a^2-1\right)\right) + a\right), \quad a > 1$$

$$F_2(a) = \sqrt{a^2-1} - a, \quad a > 1$$

and, for $0 < a < 1$, we have

$$F_0(a) = -2\sqrt{1-a^2}Im\left(Li_2\left(\frac{1}{2} - \frac{ia}{2\sqrt{1-a^2}}\right)\right) + \pi\sqrt{1-a^2}$$

$$+\left(\frac{3\pi a}{4} - 2\sqrt{1-a^2}\right)\cos^{-1}(a) + \frac{1}{4}a\left(\tan^{-1}\left(\frac{1-2a^2}{2a\sqrt{1-a^2}}\right)\right)^2 - \sqrt{1-a^2}\log\left(4 - 4a^2\right)\sin^{-1}(a)$$

$$+\frac{1}{2}a\sin^{-1}(a)\tan^{-1}\left(\frac{1-2a^2}{2a\sqrt{1-a^2}}\right) - 2a - \frac{3\pi^2 a}{16} - a\left(arccos(a)\right)^2, \quad 0 < a < 1$$

$$F_1(a) = -2\sqrt{1-a^2}\sin^{-1}(a) + 2a, \quad 0 < a < 1$$

$$F_2(a) = -a, \quad 0 < a < 1$$

then, u solves

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u - \frac{u}{r^2} = \frac{f_2(t)\log^2(r)}{r}, \quad r > 0$$

with 0 Cauchy data at infinity.

Proof To verify that (4.86) is the claimed solution to the Cauchy problem, we start with

$$u(t,r) = -\int_0^1 K_0(\frac{1}{y},r)f_1(t+ry)rdy - \int_1^\infty K_0(\frac{1}{y},r)f_1(t+ry)rdy$$

and note that the explicit formulae for $K_0(x,r)$ in the regions $x < 1$ and $x > 1$, combined with the definitions of f_k and the Dominated convergence theorem allow us to differentiate up to two times in either t or r under the integral signs. An integration by parts then establishes the lemma. (The same procedure is used to verify the stated property of (4.87)). $\square$

Remark. To see how the formulae from the lemma can be formally derived, we could first use the same procedure used for w_1 to obtain, for example, for $u_{w,2,sub,0}$, that

$$u_{w,2,sub,0}(t,r) = \int_t^\infty u_{w,2,sub,0,s}(t,r)ds$$

where

$$u_{w,2,sub,0,s}(t,r)$$
$$= \frac{-1}{2\pi}\int_0^{s-t}\frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}}\int_0^{2\pi}d\theta\frac{(r + \rho\cos(\theta))}{r^2 + \rho^2 + 2r\rho\cos(\theta)}\sum_{k=0}^2 f_k(s)\log^k\left(\sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}\right) \quad (4.88)$$

The integral of the $k = 0$ term in the sum above has been evaluated when computing w_1, via Cauchy's residue theorem. This same procedure can not directly be applied to the integrals of the $k \neq 0$ terms. We can still explicitly compute these integrals, with a procedure involving introducing a parameter into the integrals, differentiating in this parameter, and then using Cauchy's residue theorem. This gives the following. For $r \neq \rho$, we have

$$\int_0^{2\pi}4\log(r^2 + \rho^2 + 2r\rho\cos(\theta))\frac{(r + \rho\cos(\theta))}{r^2 + \rho^2 + 2r\rho\cos(\theta)}d\theta = \begin{cases} \frac{8\pi}{r}\left(\log(r^2) + \log(1 - \frac{\rho^2}{r^2})\right), & r > \rho \\ \frac{-8\pi}{r}\log(1 - \frac{r^2}{\rho^2}), & \rho > r \end{cases}$$

$$\int_0^{2\pi} \frac{\log^2(r^2+\rho^2+2r\rho\cos(\theta))}{r^2+\rho^2+2r\rho\cos(\theta)}\,(r+\rho\cos(\theta))\,d\theta = \begin{cases} \frac{4\pi}{r}\left(\log^2(1-\frac{\rho^2}{r^2})+2\log(r)\log(r-\frac{\rho^2}{r})+\mathrm{Li}_2(\frac{\rho^2}{r^2})\right), & r > \rho \\ \frac{-4\pi}{r}\log(\rho^2-r^2)\log(1-\frac{r^2}{\rho^2}), & \rho > r \end{cases}$$

Carrying out the ρ integral in (4.88) then leads to the formulae in the Lemma statement.

Using the information from Lemma 11, we can calculate the principal parts of $u_{w,2,sub,0}(t,r)$ and $v_{ex,sub,0}(t,r)$ in the matching region, $r \sim g(t)\lambda(t)$. We describe in detail how this is done for $u_{w,2,sub,0}$, since $v_{ex,sub,0}$ can be treated in the same way. First, we introduce some notation. Let

$$K_{2,0}(\frac{s-t}{r},r) = \frac{r\left(-12\log(s-t)\log(4(s-t))+\pi^2-12\log^2(2)\right)}{24(s-t)} \tag{4.89}$$

We define $F_{k,0}$ by

$$K_{2,0}(a,r) = \sum_{k=0}^{2} F_{k,0}(a)\log^k(r) \tag{4.90}$$

We also recall the definition of $u_{w,2,ell,0,cont}$, (4.62), and write

$$\partial_t^2 u_{w,2,ell,0,cont}(t,r) = \frac{g_0''(t)}{r} + \frac{g_1''(t)\log(r)}{r} + \frac{g_2''(t)\log^2(r)}{r} \tag{4.91}$$

for

$$g_0(t) = -2\left(\frac{1}{2}\lambda(t)^2\lambda''(t) - \frac{1}{12}\pi^2\lambda(t)^2\lambda''(t) + \lambda(t)^2\lambda''(t)\log^2(\lambda(t)) + \lambda(t)^2\lambda''(t)\log(\lambda(t)) + \lambda(t)\lambda'(t)^2\log(\lambda(t))\right)$$

$$g_1(t) = -2\left(-\lambda(t)^2\lambda''(t) - 2\lambda(t)^2\lambda''(t)\log(\lambda(t)) - \lambda(t)\lambda'(t)^2\right)$$

$$g_2(t) = -2\lambda(t)^2\lambda''(t)$$

Finally, we let

$$c_0 = \int_1^{\infty} (F_0(a) - F_{0,0}(a))\,da$$

Note that the exact value of c_0 is not needed for our purposes, though the precise value of many other constants appearing in the following lemma are needed.

Lemma 12 *[Leading part of $u_{w,2,sub,0}$] Let $u_{w,2,sub,0,cont}$ be defined by*

$$u_{w,2,sub,0,cont}(t,r)$$
$$= -\frac{1}{2}g_0''(t)r - \frac{r}{2}g_0''(t)\log(\frac{t}{r}) - \frac{r}{2}\int_t^{2t}\frac{(g_0''(s)-g_0''(t))}{(s-t)}ds - \frac{r}{2}\int_{2t}^{\infty}\frac{g_0''(s)}{s-t}ds - g_0''(t)\frac{r}{4}(\log(4)-1)$$

$$- \frac{r}{16}g_1''(t)\left(-12+\pi^2+8\log(r)\right) + -g_1''(t)r\left(\frac{\log(\frac{t}{r})\log(2)}{2}+\frac{1}{4}\left(\log^2(t)-\log^2(r)\right)\right)$$

$$- \frac{r}{2}\int_{2t}^{\infty}\frac{g_1''(s)}{(s-t)}\log(2(s-t))ds - \frac{r}{2}\int_t^{2t}\left(g_1''(s)-g_1''(t)\right)\frac{\log(2(s-t))}{(s-t)}ds$$

$$+ \frac{-rg_1''(t)}{24}\left(6(\log(4)-1)\log(r)-2\pi^2+15+6\log^2(2)\right)$$

$$- \frac{rg_2''(t)}{16}\left(2\log(r)\left(4\log(r)+\pi^2-12\right)-28\zeta(3)+\pi^2+28\right)$$

$$+ g_2''(t)\frac{r}{24}\log\left(\frac{r}{t}\right)\left(4\log(t)\log(8rt)+4\log(r)\log(8r)-\pi^2+12\log^2(2)\right)$$

$$+ \int_t^{2t}\left(g_2''(s)-g_2''(t)\right)K_{2,0}(\frac{s-t}{r},r)ds + \int_{2t}^{\infty}g_2''(s)K_{2,0}(\frac{s-t}{r},r)ds$$

$$+ g_2''(t)r\left(c_0+\frac{1}{12}\left(2\pi^2-15-6\log^2(2)\right)\log(r)+\frac{1}{4}(1-\log(4))\log^2(r)\right)$$

Then, for $0 \leqslant j+k \leqslant 2$

$$|r^k t^j \partial_r^k \partial_t^j\left(u_{w,2,sub,0}(t,r)-u_{w,2,sub,0,cont}(t,r)\right)| \leqslant \frac{Cr^3\lambda(t)^3\log^5(t)}{t^6}, \quad \lambda(t)g(t) \leqslant r \leqslant 2g(t)\lambda(t)$$

Proof Using (4.91) and our calculations for w_1, along with Lemma 11, we have

$$u_{w,2,sub,0}(t,r) = \frac{-1}{r}\int_t^{t+r} g_0''(s)(s-t)ds - \frac{1}{r}\int_{t+r}^\infty g_0''(s)\left((s-t) - \sqrt{(s-t)^2 - r^2}\right)ds$$
$$- \int_t^\infty g_1''(s)K_0\left(\frac{r}{s-t}, r\right)ds + \int_t^\infty g_2''(s)K_2\left(\frac{s-t}{r}, r\right)ds \tag{4.92}$$

The first two terms on the right-hand side of the above expression are treated with the identical procedure used to study w_1, simply replacing λ in the w_1 expressions with $\frac{-g_0}{2}$. The leading part of the sum of these two terms in the matching region $r \sim g(t)\lambda(t)$ is (note that we will prove concrete estimates on the terms we claim are subleading)

$$-\frac{1}{2}g_0''(t)r - \frac{r}{2}g_0''(t)\log(\frac{t}{r}) - \frac{r}{2}\int_t^{2t}\frac{(g_0''(s) - g_0''(t))}{(s-t)}ds - \frac{r}{2}\int_{2t}^\infty \frac{g_0''(s)}{s-t}ds - g_0''(t)\frac{r}{4}(\log(4) - 1)$$

In fact, by inspecting (4.28), we have

$$\left|\frac{-1}{r}\int_t^{t+r}g_0''(s)(s-t)ds - \frac{1}{r}\int_{t+r}^\infty g_0''(s)\left((s-t) - \sqrt{(s-t)^2 - r^2}\right)ds\right.$$
$$\left. - \left(-\frac{1}{2}g_0''(t)r - \frac{r}{2}g_0''(t)\log(\frac{t}{r}) - \frac{r}{2}\int_t^{2t}\frac{(g_0''(s) - g_0''(t))}{(s-t)}ds - \frac{r}{2}\int_{2t}^\infty \frac{g_0''(s)}{s-t}ds - g_0''(t)\frac{r}{4}(\log(4) - 1)\right)\right|$$
$$\leqslant \frac{Cr^3\lambda(t)^3\log^2(t)\left(1 + \log(\frac{t}{r})\right)}{t^6}, \quad r \leqslant t$$

Estimates on the t and r derivatives are done similarly, as in the proof of Lemma 1. The third and fourth terms of (4.92) are treated with the same argument, which is essentially the same argument we used for w_1, but with a few differences in the details. For clarity, we start with the third term. We have

$$-\int_t^\infty g_1''(s)K_0\left(\frac{r}{s-t}, r\right)ds = -\int_t^{t+r}g_1''(s)K_0\left(\frac{r}{s-t}, r\right)ds - \int_{t+r}^\infty g_1''(s)K_0\left(\frac{r}{s-t}, r\right)ds \tag{4.93}$$

With $y = \frac{s-t}{r}$, we get

$$-\int_t^{t+r}g_1''(s)K_0\left(\frac{r}{s-t}, r\right)ds = -r\int_0^1 g_1''(t+ry)K_0\left(\frac{1}{y}, r\right)dy$$
$$= -r\int_0^1 g_1''(t+ry)\left(K_{0,0}(\frac{1}{y}) + K_{0,1}(\frac{1}{y})\log(r)\right)dy$$

where we note that, for $x > 1$,

$$K_0(x,r) = K_{0,0}(x) + K_{0,1}(x)\log(r), \quad K_{0,0}(x) = \frac{\sqrt{x^2-1}\sin^{-1}\left(\frac{1}{x}\right) - 1}{x}, \quad K_{0,1}(x) = \frac{1}{x}$$

Since $g(t)\lambda(t) \ll t$, the first term in the following decomposition is lower order relative to the second term, for $r \sim g(t)\lambda(t)$

$$-r\int_0^1 g_1''(t+ry)\left(K_{0,0}(\frac{1}{y}) + K_{0,1}(\frac{1}{y})\log(r)\right)dy$$
$$= -rg_1''(t)\int_0^1\left(K_{0,0}(\frac{1}{y}) + K_{0,1}(\frac{1}{y})\log(r)\right)dy - r\int_0^1\left(g_1''(t+ry) - g_1''(t)\right)\left(K_{0,0}(\frac{1}{y}) + K_{0,1}(\frac{1}{y})\log(r)\right)dy$$

We thus get that the leading contribution from $-r\int_0^1 g_1''(t+ry)\left(K_{0,0}(\frac{1}{y}) + K_{0,1}(\frac{1}{y})\log(r)\right)dy$ is

$$-\frac{r}{16}g_1''(t)\left(-12 + \pi^2 + 8\log(r)\right) \tag{4.94}$$

We treat the next term in (4.93), namely $-\int_{t+r}^\infty g_1''(s)K_0\left(\frac{r}{s-t}, r\right)ds$. For this term, we start by decomposing $K_0(x,r)$ into the leading piece for small x plus the remainder.

$$K_0(x,r) = K_{0,sm}(x,r) + O\left(x^3\left(1 + |\log(\frac{r}{x})|\right)\right), \quad x \to 0, \quad K_{0,sm}(x,r) = \frac{1}{2}x(\log(r) - \log(x) + \log(2))$$

We therefore get

$$-\int_{t+r}^{\infty} g_1''(s) K_0\left(\frac{r}{s-t}, r\right) ds = -g_1''(t) \int_{t+r}^{2t} K_{0,sm}(\frac{r}{s-t}, r) ds - \int_{t}^{2t} \left(g_1''(s) - g_1''(t)\right) K_{0,sm}(\frac{r}{s-t}, r) ds$$
$$+ \int_{t}^{t+r} \left(g_1''(s) - g_1''(t)\right) K_{0,sm}(\frac{r}{s-t}, r) ds - \int_{2t}^{\infty} g_1''(s) K_{0,sm}(\frac{r}{s-t}, r) ds \qquad (4.95)$$
$$- \int_{t+r}^{\infty} g_1''(s) \left(K_0 - K_{0,sm}\right)(\frac{r}{s-t}, r) ds$$

This decomposition also appears in the w_1 expression (4.28). The leading contribution from the first four terms of the right-hand side of (4.95) is

$$- g_1''(t) r \left(\frac{\log(\frac{t}{r}) \log(2)}{2} + \frac{1}{4}\left(\log^2(t) - \log^2(r)\right)\right)$$
$$- \frac{r}{2} \int_{2t}^{\infty} \frac{g_1''(s)}{(s-t)} \log(2(s-t)) ds - \frac{r}{2} \int_{t}^{2t} \left(g_1''(s) - g_1''(t)\right) \frac{\log(2(s-t))}{(s-t)} ds \qquad (4.96)$$

Moreover, the difference between $-\int_{t}^{\infty} g_1''(s) K_0\left(\frac{r}{s-t}, r\right) ds$ and the leading parts of the terms considered thus far is equal to

$$- \int_{t}^{t+r} g_1''(s) K_0(\frac{r}{s-t}, r) ds - \left(\frac{-r g_1''(t)}{16}\left(-12 + \pi^2 + 8\log(r)\right)\right)$$
$$- \int_{t+r}^{\infty} g_1''(s) K_0\left(\frac{r}{s-t}, r\right) ds - \left(-g_1''(t) r \left(\frac{\log(\frac{t}{r}) \log(2)}{2} + \frac{1}{4}\left(\log^2(t) - \log^2(r)\right)\right)\right.$$
$$\left. - \frac{r}{2} \int_{2t}^{\infty} \frac{g_1''(s)}{(s-t)} \log(2(s-t)) ds - \frac{r}{2} \int_{t}^{2t} \left(g_1''(s) - g_1''(t)\right) \frac{\log(2(s-t))}{(s-t)} ds\right)$$
$$= \int_{t}^{t+r} \left(g_1''(s) - g_1''(t)\right) K_{0,sm}(\frac{r}{s-t}, r) ds - \int_{t+r}^{\infty} g_1''(s) \left(K_0 - K_{0,sm}\right)(\frac{r}{s-t}, r) ds$$
$$- r \int_{0}^{1} \left(g_1''(t + ry) - g_1''(t)\right) K_0(\frac{1}{y}, r) dy$$

We have

$$- r \int_{0}^{1} \left(g_1''(t + ry) - g_1''(t)\right) K_0(\frac{1}{y}, r) dy + \int_{t}^{t+r} \left(g_1''(s) - g_1''(t)\right) K_{0,sm}(\frac{r}{s-t}, r) ds$$
$$= -r \int_{0}^{1} \left(g_1''(t + ry) - g_1''(t) - r y g_1'''(t)\right) K_0(\frac{1}{y}, r) dy + \int_{t}^{t+r} \left(g_1''(s) - g_1''(t) - (s - t) g_1'''(t)\right) K_{0,sm}(\frac{r}{s-t}, r) ds$$
$$- r^2 \int_{0}^{1} y g_1'''(t) K_0(\frac{1}{y}, r) dy + \int_{t}^{t+r} g_1'''(t)(s - t) K_{0,sm}(\frac{r}{s-t}, r) ds$$

which gives

$$\left| -r \int_{0}^{1} \left(g_1''(t + ry) - g_1''(t)\right) K_0(\frac{1}{y}, r) dy + \int_{t}^{t+r} \left(g_1''(s) - g_1''(t)\right) K_{0,sm}(\frac{r}{s-t}, r) ds\right.$$
$$\left. - \left(\frac{r^2 g_1'''(t)}{18}\left(-7 + \log(512) + 3\log(r)\right)\right)\right|$$
$$\leqslant \frac{C r^3 \left(1 + |\log(r)|\right) \lambda(t)^3 \log(t)}{t^6}$$

The higher derivatives are estimated similarly, for example, by writing

$$\int_{t}^{t+r} \left(g_1''(s) - g_1''(t) - (s - t) g_1'''(t)\right) K_{0,sm}(\frac{r}{s-t}, r) ds = \frac{r}{2} \int_{0}^{1} \left(g_1''(t + ry) - g_1''(t) - r y g_1'''(t)\right) \log(2ry) \frac{dy}{y}$$

and using the symbol-type estimates on λ to justify the differentiation under the integral, exactly as in the proof of Lemma 1. We now turn attention to the fifth term on the right-hand side of (4.95). The point here is that $(K_0 - K_{0,sm})(\frac{r}{s-t}, r)$ decays faster in $s - t$ than does $K_{0,sm}(\frac{r}{s-t}, r)$. To exploit this, we integrate by parts in s in the following integral, and the symbol type estimates on $g_1''(s)$ show that the non-boundary

terms obtained are subleading relative to the boundary terms (this will be proven once we estimate the terms that we claim are subleading)

$$-\int_{t+r}^{\infty} g_1''(s)\,(K_0 - K_{0,sm})\,(\frac{r}{s-t}, r)\,ds$$
$$= -g_1''(t+r)\int_{t+r}^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy - \int_{t+r}^{\infty}\left(\int_s^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy\right) g_1'''(s)\,ds$$

Recalling that $g(t)\lambda(t) \ll t$, the leading behavior of $-\int_{t+r}^{\infty} g_1''(s)\,(K_0 - K_{0,sm})\,(\frac{r}{s-t}, r)\,ds$ in the matching region is

$$-g_1''(t)\int_1^{\infty} (K_0 - K_{0,sm})\,(\frac{1}{q}, r)\,r\,dq = \frac{-rg_1''(t)}{24}\left(6(\log(4)-1)\log(r) - 2\pi^2 + 15 + 6\log^2(2)\right) \qquad (4.97)$$

The subleading part of $-\int_{t+r}^{\infty} g_1''(s)\,(K_0 - K_{0,sm})\,(\frac{r}{s-t}, r)\,ds$ is therefore given by

$$-\int_{t+r}^{\infty} g_1''(s)\,(K_0 - K_{0,sm})\,(\frac{r}{s-t}, r)\,ds - \left(\frac{-rg_1''(t)}{24}\left(6(\log(4)-1)\log(r) - 2\pi^2 + 15 + 6\log^2(2)\right)\right)$$
$$= -\left(g_1''(t+r) - g_1''(t)\right)\int_{t+r}^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy - \int_{t+r}^{\infty}\int_s^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy\,g_1'''(s)\,ds$$

We have

$$-\left(g_1''(t+r) - g_1''(t)\right)\int_{t+r}^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy$$
$$= -r\left(g_1''(t+r) - g_1''(t) - rg_1'''(t)\right)\int_1^{\infty} (K_0 - K_{0,sm})\,(\frac{1}{z}, r)\,dz - r^2 g_1'''(t)\int_1^{\infty} (K_0 - K_{0,sm})\,(\frac{1}{z}, r)\,dz$$

which gives

$$\left| -\left(g_1''(t+r) - g_1''(t)\right)\int_{t+r}^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy - \left(-r^2 g_1'''(t)\int_1^{\infty} (K_0 - K_{0,sm})\,(\frac{1}{z}, r)\,dz\right) \right|$$
$$\leqslant \frac{Cr^3 \log(t)\lambda(t)^3\,(1 + |\log(r)|)}{t^6}$$

and the higher derivatives are estimated similarly. On the other hand, we have

$$-\int_{t+r}^{\infty}\int_s^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy\,g_1'''(s)\,ds$$
$$= -g_1'''(t+r)\int_{t+r}^{\infty}\int_q^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy\,dq - \int_{t+r}^{\infty}\int_s^{\infty}\int_q^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,g_1''''(s)\,dy\,dq\,ds$$

Therefore, for $r \sim g(t)\lambda(t)$,

$$\left| -\int_{t+r}^{\infty}\int_s^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy\,g_1'''(s)\,ds - \left(-r^2 g_1'''(t)\int_1^{\infty} (x-1)\,(K_0 - K_{0,sm})\,(\frac{1}{x}, r)\,dx\right) \right|$$
$$\leqslant \frac{Cr^3 \log(t)\lambda(t)^3}{t^6}\,(1 + |\log(r)|)\left(\log(1 + \frac{t}{r}) + 1\right)$$

As with the previous terms, the higher derivatives of $-\int_{t+r}^{\infty}\int_s^{\infty}\int_q^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,g_1''''(s)\,dy\,dq\,ds$ can be estimated by letting $x = \frac{y-t}{r}$, then, $w = \frac{q-t}{r}$, and $z = \frac{s-t}{r}$, and differentiating under the integral. Finally, we get

$$\left| -\left(g_1''(t+r) - g_1''(t)\right)\int_{t+r}^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy - \int_{t+r}^{\infty}\int_s^{\infty} (K_0 - K_{0,sm})\,(\frac{r}{y-t}, r)\,dy\,g_1'''(s)\,ds \right.$$
$$\left. -\left(-r^2 g_1'''(t)\int_1^{\infty} x\,(K_0 - K_{0,sm})\,(\frac{1}{x}, r)\,dx\right) \right|$$
$$\leqslant \frac{Cr^3 \log(t)\lambda(t)^3\,(1 + |\log(r)|)\,(\log(1 + \frac{t}{r}) + 1)}{t^6}$$

Using

$$-r^2 g_1'''(t)\int_1^{\infty} x\,(K_0 - K_{0,sm})\,(\frac{1}{x}, r)\,dx = \frac{-r^2 g_1'''(t)}{18}\,(-7 + \log(512) + 3\log(r))$$

we finally get the following expression which identifies the leading part of $-\int_t^\infty g_1''(s)K_0(\frac{r}{s-t},r)ds$.

$$\left| -\int_t^\infty g_1''(s)K_0(\frac{r}{s-t},r)ds - \left(\frac{-rg_1''(t)}{16} \left(-12 + \pi^2 + 8\log(r) \right) \right) \right.$$
$$- \left(-g_1''(t)r \left(\frac{\log(\frac{t}{r})\log(2)}{2} + \frac{1}{4} \left(\log^2(t) - \log^2(r) \right) \right) \right.$$
$$\left. -\frac{r}{2}\int_{2t}^\infty \frac{g_1''(s)}{(s-t)}\log(2(s-t))ds - \frac{r}{2}\int_t^{2t} \left(g_1''(s) - g_1''(t) \right) \frac{\log(2(s-t))}{(s-t)}ds \right)$$
$$\left. - \left(\frac{-rg_1''(t)}{24} \left(6\left(\log(4)-1\right)\log(r) - 2\pi^2 + 15 + 6\log^2(2) \right) \right) \right|$$
$$\leqslant \frac{Cr^3\log(t)\lambda(t)^3\left(1+|\log(r)|\right)\left(\log(1+\frac{t}{r})+1\right)}{t^6}, \quad \lambda(t)g(t) \leqslant r \leqslant 2g(t)\lambda(t)$$

Notice that the r^2 terms from

$$- \left(g_1''(t+r) - g_1''(t) \right) \int_{t+r}^\infty \left(K_0 - K_{0,sm}\right)(\frac{r}{y-t},r)dy - \int_{t+r}^\infty \int_s^\infty \left(K_0 - K_{0,sm}\right)(\frac{r}{y-t},r)dy\, g_1'''(s)ds$$

and

$$-r\int_0^1 \left(g_1''(t+ry) - g_1''(t) \right) K_0(\frac{1}{y},r)dy + \int_t^{t+r} \left(g_1''(s) - g_1''(t) \right) K_{0,sm}(\frac{r}{s-t},r)ds$$

canceled to give the above expression.

The same procedure is carried out for the K_2 term in (4.92). The details are as follows. The analog of (4.94) is

$$rg_2''(t)\int_0^1 \left(F_0(a) + F_1(a)\log(r) + F_2(a)\log^2(r) \right)da$$
$$= -\frac{rg_2''(t)}{16} \left(2\log(r)\left(4\log(r) + \pi^2 - 12\right) - 28\zeta(3) + \pi^2 + 28 \right)$$

where ζ denotes the Riemann zeta function. We remark that one part of this computation involves the following.

$$\text{Im}\left(\text{Li}_2\left(\frac{1}{2} - \frac{ia}{2\sqrt{1-a^2}} \right) \right) = \int_0^a \left(\frac{x\sin^{-1}(x)}{x^2-1} - \frac{\log(4(1-x^2))}{2\sqrt{1-x^2}} \right)dx$$

and

$$\int_0^1 \frac{1}{2}\left(\cos^{-1}(x) - x\sqrt{1-x^2} \right)\left(\frac{x\sin^{-1}(x)}{x^2-1} - \frac{\log\left(4\left(1-x^2\right)\right)}{2\sqrt{1-x^2}} \right)dx = \frac{1}{32}\left(-21\zeta(3) + \pi^2 + \log(256) \right)$$

The analog of (4.96) is

$$g_2''(t)\frac{r}{24}\log\left(\frac{r}{t}\right)\left(4\log(t)\log(8rt) + 4\log(r)\log(8r) - \pi^2 + 12\log^2(2) \right)$$
$$+ \int_t^{2t} \left(g_2''(s) - g_2''(t) \right) K_{2,0}(\frac{s-t}{r},r)ds + \int_{2t}^\infty g_2''(s)K_{2,0}(\frac{s-t}{r},r)ds$$

where we recall that $K_{2,0}$ is defined in (4.89). The analog of (4.97) is

$$g_2''(t)r\int_1^\infty \left(K_2 - K_{2,0}\right)(a,r)da$$

We have

$$\int_1^\infty \left(K_2 - K_{2,0}\right)(a,r)da = c_0 + d_1\log(r) + d_2\log^2(r)$$

The exact value of c_0 is not important for our purposes, but we compute d_1 and d_2 explicitly. From direct integration,

$$d_2 = \int_1^\infty \left(\sqrt{a^2-1} - a + \frac{1}{2a} \right)da = \frac{1}{4}\left(1-\log(4)\right)$$

To compute d_1, we recall that $K_2(a,r)$ satisfies the wave equation:

$$-K_2 + r\partial_r K_2 + r^2\partial_r^2 K_2 + a\partial_a K_2 - 2ar\partial_{ar}K_2 + (a^2-1)\partial_a^2 K_2 = 0, \quad a > 1$$

and $K_2 = \sum_{k=0}^{2} F_k(a) \log^k(r)$. Integrating the equation solved by K_2 in the a variable, one relation we get is (for $M > 1$)

$$\int_1^M F_1(a)da = -\int_1^M F_2(a)da + \left(aF_1(a) + \frac{(1-a^2)}{2} F_0'(a) + \frac{aF_0(a)}{2} \right) \Big|_{a=1}^M$$

Recalling the definitions of $F_{k,0}$ in (4.90), we have

$$d_1 = \int_1^\infty \left(F_1(a) - F_{1,0}(a) \right) da = \lim_{M\to\infty} \int_1^M \left(F_1(a) - F_{1,0}(a) \right) da = \frac{1}{12} \left(2\pi^2 - 15 - 6\log^2(2) \right) \qquad (4.98)$$

Therefore,

$$g_2''(t)r \int_1^\infty \left(K_2 - K_{2,0} \right)(a,r)da = g_2''(t)r \left(c_0 + \frac{1}{12} \left(2\pi^2 - 15 - 6\log^2(2) \right) \log(r) + \frac{1}{4} \left(1 - \log(4) \right) \log^2(r) \right)$$

For estimating the subleading terms, we use the same procedure used for the K_0 integral term previously. In particular, we have

$$\begin{aligned}
| \int_t^\infty & g_2''(s) K_2(\frac{s-t}{r}, r)ds - rg_2''(t) \int_0^1 K_2(y,r)dy - rg_2''(t) \int_1^\infty (K_2 - K_{2,0})(a,r)da \\
& - g_2''(t) \int_{t+r}^{2t} K_{2,0}(\frac{s-t}{r}, r)ds - \int_t^{2t} \left(g_2''(s) - g_2''(t) \right) K_{2,0}(\frac{s-t}{r}, r)ds - \int_{2t}^\infty g_2''(s) K_{2,0}(\frac{s-t}{r}, r)ds \\
& - r^2 g_2'''(t) \int_0^1 yK_2(y,r)dy + r^2 g_2'''(t) \int_0^1 yK_{2,0}(y,r)dy - r^2 g_2'''(t) \int_1^\infty a \left(K_2 - K_{2,0} \right)(a,r)da| \\
& \leqslant \frac{Cr^3 \lambda(t)^3 \left(1 + \log^2(r) \right) \left(1 + |\log(\frac{r}{t})|^3 \right)}{t^6}, \quad r \leqslant t
\end{aligned} \qquad (4.99)$$

The higher derivatives are estimated similarly, with the same procedure used for the $g_1''(t)$ terms just studied. Finally, we will show that the last three terms on the left-hand side of the above inequality exactly cancel, just as was the case for the analogous terms arising from the K_0 integral previously. By direct computation, we have

$$\int_0^1 yK_{2,0}(y,r)dy = -\frac{\log^2(r)}{2} - \log(2)\log(r) + \log(2r) + \frac{\pi^2}{24} - 1 - \frac{1}{2}\log^2(2)$$

$$\int_0^1 yK_2(y,r)dy = -\frac{1}{3}\log^2(r) + \frac{2\log(r)}{9} + \frac{1}{18}(16\log(2) - 4) + \frac{1}{108}\left(-9\pi^2 + 88 - 96\log(2) \right)$$

Finally, using a similar procedure as in (4.98), we get

$$\begin{aligned}
\int_1^\infty \left(F_0(a) - F_{0,0}(a) \right) ada &= \lim_{M\to\infty} \left(-\frac{2}{3} \int_1^M aF_2(a)da - \frac{a^2 F_0(a)}{3} \Big|_{a=1}^M + \frac{2}{3} \int_1^M a^2 F_1'(a)da \right. \\
& \left. \quad -\frac{1}{3} \left((a^3 - a)F_0'(a) - (3a^2 - 1)F_0(a) \right) \Big|_{a=1}^M - \int_1^M F_{0,0}(a)ada \right) \\
&= \frac{\pi^2}{8} - \frac{43}{27} - \frac{1}{2}\log^2(2) + \log(2)
\end{aligned}$$

A straightforward computation then gives

$$\int_1^\infty a \left(K_2 - K_{2,0} \right)(a,r)da = -\frac{\log^2(r)}{6} + \left(\frac{7}{9} - \log(2) \right) \log(r) + \frac{\pi^2}{8} - \frac{43}{27} - \frac{1}{2}\log^2(2) + \log(2)$$

In total, we then note that

$$\begin{aligned}
& -\frac{\log^2(r)}{2} - \log(2)\log(r) + \log(2r) + \frac{\pi^2}{24} - 1 - \frac{1}{2}\log^2(2) \\
& - \left(-\frac{1}{3}\log^2(r) + \frac{2\log(r)}{9} + \frac{1}{18}(16\log(2) - 4) + \frac{1}{108}\left(-9\pi^2 + 88 - 96\log(2) \right) \right) \\
& - \left(-\frac{\log^2(r)}{6} + \left(\frac{7}{9} - \log(2) \right) \log(r) + \frac{\pi^2}{8} - \frac{43}{27} - \frac{1}{2}\log^2(2) + \log(2) \right) \\
& = 0
\end{aligned}$$

which verifies that the last three terms on the left-hand side of (4.99) exactly cancel. Combining our computations and estimates above finishes the proof of the lemma. $\quad\square$

Next, we note that

$$\partial_t^2 v_{ex,cont}(t,r) = \frac{1}{r}\left(h_0''(t) + \log(r)h_1''(t)\right)$$

where

$$h_0(t) = -2\log(\lambda(t))f_1(t) + \lambda(t)^2 f_2(t), \quad h_1(t) = 2f_1(t)$$

and

$$f_1(t) = \frac{-1}{2}\left(2\lambda(t)\lambda'(t)^2 + \lambda(t)^2\lambda''(t)\right), \quad f_2(t) = \frac{-\lambda''(t)}{2}$$

Using the same procedure as for $u_{w,2,sub,0}$ (in fact, the same computations, except with g_j replaced with h_j, for $j = 0,1$) we get the following lemma.

Lemma 13 *[Leading part of $v_{ex,sub,0}$] Let*

$$
\begin{aligned}
&v_{ex,sub,0,cont}(t,r)\\
&= \frac{-r}{2}h_0''(t) - \frac{r}{2}h_0''(t)\log(\frac{t}{r}) - \frac{r}{2}\int_t^{2t}\frac{(h_0''(s)-h_0''(t))}{(s-t)}ds\\
&\quad - \frac{r}{2}\int_{2t}^\infty\frac{h_0''(s)ds}{s-t} - rh_0''(t)\frac{(\log(4)-1)}{4} - \frac{r}{8}f_1''(t)\left(-12+\pi^2+8\log(r)\right)\\
&\quad - 2f_1''(t)\left(\frac{r\log(2)\log(\frac{t}{r})}{2} + \frac{r}{4}\left(\log^2(t)-\log^2(r)\right)\right)\\
&\quad - r\int_t^{2t}\frac{(f_1''(s)-f_1''(t))}{(s-t)}\log(2(s-t))ds\\
&\quad - r\int_{2t}^\infty f_1''(s)\left(\frac{\log(2(s-t))}{(s-t)}\right)ds - \frac{rf_1''(t)}{12}\left(6\left(\log(4)-1\right)\log(r) - 2\pi^2 + 15 + 6\log^2(2)\right)
\end{aligned}
$$

Then, for $0 \leqslant j+k \leqslant 2$,

$$\left|r^k t^j \partial_r^k \partial_t^j\left(v_{ex,sub,0} - v_{ex,sub,0,cont}\right)\right|(t,r) \leqslant \frac{Cr^3\lambda(t)^3\log^3(t)}{t^6}, \quad g(t)\lambda(t) \leqslant r \leqslant 2g(t)\lambda(t)$$

Next, we consider

$$
\begin{aligned}
&u_{w,2,ell}(t,r) - u_{w,2,ell,0}(t,r)\\
&= -\frac{1}{2}\left(\frac{1}{r}\int_0^r s^2\left(RHS_2(t,s)-RHS_{2,0}(t,s)\right)ds + r\int_r^\infty\left(RHS_2(t,s)-RHS_{2,0}(t,s)\right)ds\right)\\
&= \frac{-1}{2}\left(\frac{1}{r}\int_0^r s^2\left(RHS_2(t,s)-RHS_{2,0}(t,s)\right)ds + r\int_0^\infty\left(RHS_2(t,s)-RHS_{2,0}(t,s)\right)ds\right.\\
&\quad \left. -r\int_0^r\left(RHS_2(t,s)-RHS_{2,0}(t,s)\right)ds\right)
\end{aligned}
\tag{4.100}
$$

We proceed to study each integral term on the right-hand side. The integral over $(0,\infty)$ requires the longest argument:

Lemma 14 *We have*

$$\int_0^\infty\left(RHS_2(t,s)-RHS_{2,0}(t,s)\right)ds = \lambda(t)\int_t^\infty W_3(\frac{s-t}{\lambda(t)})\lambda''''(s)ds + \int_0^\infty d\xi\left(\frac{-8}{\xi\lambda(t)^2} + 4\xi K_2(\xi\lambda(t)) + 2\xi\right)\sin(t\xi)\widehat{v_{2,0}}(\xi)$$
$$\tag{4.101}$$

where

$$
\begin{aligned}
W_3(x) &= \frac{2}{3}\left(x\log\left(64x^6\right) + 4x^3\log(2x) + 2\left(x^2+1\right)^{3/2}\log\left(2x\left(x-\sqrt{x^2+1}\right)+1\right) + x\right)\\
&= O\left(\frac{\log(x)}{x}\right), \quad x \to \infty
\end{aligned}
$$

Proof We first note

$$r\int_0^\infty\left(RHS_2(t,s)-RHS_{2,0}(t,s)\right)ds = r\int_0^\infty\left(\left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)}))-1}{s^2}\right)(v_2+w_1) - RHS_{2,0}(t,s)\right)ds$$

Using the representation formula (4.37) for v_2, and Fubini's theorem, we have

$$\int_0^\infty \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2} \right) v_2(t,s)ds = F(t) + \int_0^\infty d\xi \left(\frac{-8}{\xi\lambda(t)^2} + 4\xi K_2(\xi\lambda(t)) + 2\xi \right) \sin(t\xi)\widehat{v_{2,0}}(\xi) \qquad (4.102)$$

where we used

$$\int_0^\infty dr \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) J_1(r\xi) = -\frac{8}{\xi\lambda(t)^2} + 4\xi K_2(\xi\lambda(t)) = -2\xi + O\left(\xi^3 \log(\xi) \right), \quad \xi \to 0 \qquad (4.103)$$

and we recall that F is defined in (4.38). Notice that the first term on the right-hand side of (4.102) is the leading part, given the smallness at low frequencies of the O term in (4.103). Next, using

$$w_1(t,r) = \frac{2}{r} \int_t^{t+r} \lambda''(s)(s-t)ds + 2r \int_1^\infty \lambda''(t+ry) \left(y - \sqrt{y^2 - 1} \right) dy$$

we get

$$\int_0^\infty \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) w_1(t,r)dr$$
$$= \int_0^\infty dw\, \lambda''(t+w)w\frac{8}{\lambda(t)^2} \left(\frac{\lambda(t)^2}{\lambda(t)^2 + w^2} - \log\left(1 + \frac{\lambda(t)^2}{w^2} \right) \right)$$
$$+ \int_0^\infty dw\, \lambda''(t+w)\frac{4}{\lambda(t)^2} \left(\frac{-2\lambda(t)^2 w}{\lambda(t)^2 + w^2} + w\log(16) + 2w\log(1 + \frac{w^2}{\lambda(t)^2}) \right. \qquad (4.104)$$
$$\left. + \frac{(\lambda(t)^2 + 2w^2)\log(1 + \frac{2w(w - \sqrt{\lambda(t)^2 + w^2})}{\lambda(t)^2})}{\sqrt{\lambda(t)^2 + w^2}} \right)$$

Using the definitions of $v_{2,sub}$, $w_{1,sub}$, and the first order matching (see, e.g. (4.39)), we get

$$RHS_{2,0}(t,r) = \frac{r}{\lambda(t)} \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) \left(\frac{1}{2}\lambda'(t)^2 + \lambda(t)\lambda''(t) - \lambda(t)\lambda''(t)\log(\frac{r}{\lambda(t)}) \right)$$

which gives

$$-\int_0^\infty RHS_{2,0}(t,s)ds = \frac{4}{\lambda(t)} \left(\frac{\lambda'(t)^2}{2} + \lambda(t)\lambda''(t) \right) \qquad (4.105)$$

Given that $RHS_{2,0}$ was the leading part of RHS_2 in the matching region, the integrals (4.102), (4.105), and (4.104) have cancellation when added together. We show this in detail now. Recalling the definition of $F(t)$, we have (for all $t \geq T_0$)

$$F(t)$$
$$= 4 \left(\left(\log(2) - \frac{1}{2} \right) \lambda''(t) + \int_t^{2t} \left(\frac{\lambda''(s) - \lambda''(t)}{s - t} \right) ds + \lambda''(t)\log(\frac{t}{\lambda(t)}) + \int_{2t}^\infty \frac{\lambda''(s)ds}{s - t} - \frac{(\lambda'(t))^2}{2\lambda(t)} \right) \qquad (4.106)$$

Therefore, the $\lambda'(t)^2$ terms from $F(t)$ and (4.105) cancel. Next, we determine the leading terms of the first term on the right-hand side of (4.104). The integral is

$$\frac{8}{\lambda(t)^2} \int_0^\infty \lambda''(t+w)w \left(\frac{\lambda(t)^2}{\lambda(t)^2 + w^2} - \log(1 + \frac{\lambda(t)^2}{w^2}) \right) dw$$

Given the decay (for large w) of the part of the integrand multiplying $\lambda''(t+w)$, we integrate by parts in w, differentiating the symbol $\lambda''(t+w)$, and integrating (backwards from infinity) the rest of the integrand. This gives

$$\frac{8}{\lambda(t)^2} \int_0^\infty \lambda''(t+w)w \left(\frac{\lambda(t)^2}{\lambda(t)^2 + w^2} - \log(1 + \frac{\lambda(t)^2}{w^2}) \right) dw$$
$$= -4\lambda''(t) - \frac{4\pi}{3}\lambda(t)\lambda'''(t) + \frac{4}{3\lambda(t)^2} \int_0^\infty \left(w\lambda(t)^2 - 2\tan^{-1}\left(\frac{\lambda(t)}{w} \right) \lambda(t)^3 - w^3 \log(1 + \frac{\lambda(t)^2}{w^2}) \right) \lambda'''(t+w)dw$$
$$\qquad (4.107)$$

Notice the cancellation between the $\lambda''(t)$ terms in the expression above and (4.105). Next, we treat the second term on the right-hand side of (4.104). We start with

$$\frac{4}{\lambda(t)^2} \int_0^t dw \lambda''(t+w) G_2(w, \lambda(t))$$

where

$$G_2(w, \lambda(t)) = \left(\frac{-2\lambda(t)^2 w}{\lambda(t)^2 + w^2} + w \log(16) + 2w \log(1 + \frac{w^2}{\lambda(t)^2}) \right.$$
$$\left. + \frac{(\lambda(t)^2 + 2w^2)}{\sqrt{\lambda(t)^2 + w^2}} \log(1 + \frac{2w(w - \sqrt{\lambda(t)^2 + w^2})}{\lambda(t)^2}) \right) = \lambda(t) G_3(\frac{w}{\lambda(t)})$$

Since there will be some cancellation between the integral under consideration and the second term in (4.106), we start by writing $\lambda''(t+w) = \lambda''(t) + \lambda''(t+w) - \lambda''(t)$, we have

$$\frac{4}{\lambda(t)^2} \int_0^t dw \lambda''(t+w) G_2(w, \lambda(t))$$
$$= -2\lambda''(t) \left(-1 + 2\log(2) - 2\log(\frac{\lambda(t)}{t}) \right) + \frac{4\lambda''(t)}{\lambda(t)^2} \lambda(t)^2 G_4(\frac{t}{\lambda(t)}) + \frac{4}{\lambda(t)^2} \int_0^t dw \left(\lambda''(t+w) - \lambda''(t) \right) G_2(w, \lambda(t))$$
$$\tag{4.108}$$

where

$$G_4(x) = \frac{1}{2} \left(2x^2 \log \left(4 \left(x^2 + 1 \right) \right) + 2x^2 \sqrt{\frac{1}{x^2} + 1} \log \left(1 - 2 \left(\sqrt{\frac{1}{x^2} + 1} - 1 \right) x^2 \right) + 2\log(2x) - 1 \right)$$
$$= O \left(\frac{\log(x)}{x^2} \right), \quad x \to \infty$$
$$\tag{4.109}$$

Note that the first term on the right-hand side of (4.108) cancels with all of the $\lambda''(t)$ terms outside the integral operators in (4.106). Next, we note that

$$G_3(x) = \frac{-1}{x} + O \left(\frac{\log(x)}{x^3} \right), \quad x \to \infty \tag{4.110}$$

Recalling that $G_2(w, \lambda(t)) = \lambda(t) G_3(\frac{w}{\lambda(t)})$, we have

$$\frac{4}{\lambda(t)^2} \int_0^t dw \left(\lambda''(t+w) - \lambda''(t) \right) G_2(w, \lambda(t))$$
$$= \frac{-4}{\lambda(t)^2} \int_0^t dw \left(\lambda''(t+w) - \lambda''(t) \right) \frac{\lambda(t)^2}{w} + \frac{4}{\lambda(t)^2} \int_0^t dw \left(\lambda''(t+w) - \lambda''(t) \right) \left(G_2(w, \lambda(t)) + \frac{\lambda(t)^2}{w} \right)$$

The first term on the right-hand side of the above expression cancels with the second term on the right-hand side of (4.106), and $G_2(w, \lambda(t)) + \frac{\lambda(t)^2}{w}$ decays much more quickly for large w than $G_2(w, \lambda(t))$ does, given (4.110). Using the same procedure as in (4.107), we get

$$\frac{4}{\lambda(t)^2} \int_0^t dw \left(\lambda''(t+w) - \lambda''(t) \right) \left(G_2(w, \lambda(t)) + \frac{\lambda(t)^2}{w} \right)$$
$$= \frac{4}{\lambda(t)^2} \left(\left(\lambda''(2t) - \lambda''(t) \right) \lambda(t)^2 G_4(\frac{t}{\lambda(t)}) \right.$$
$$\left. + \lambda(t)^3 \left(-\lambda'''(2t) W_2(\frac{t}{\lambda(t)}) + \frac{\lambda'''(t)}{3} \pi + \int_t^{2t} W_2(\frac{s-t}{\lambda(t)}) \lambda''''(s) ds \right) \right)$$
$$\tag{4.111}$$

where we recall that G_4 was defined in (4.109), and

$$W_2(x) = \frac{1}{6} \left(x^3 \log(16) + 2 \left(x^2 + 1 \right)^{3/2} \log \left(2x^2 - 2x\sqrt{x^2 + 1} + 1 \right) + 2x^3 \log \left(x^2 + 1 \right) \right.$$
$$\left. - x + x \log(64) + 6x \log(x) + 4 \tan^{-1}(\frac{1}{x}) \right)$$

Note that the second term on the right-hand side of our previous computation (4.108) cancels with the term $\frac{4}{\lambda(t)^2} \left(-\lambda''(t) \lambda(t)^2 G_4(\frac{t}{\lambda(t)}) \right)$ from (4.111).

Lastly, we consider

$$\frac{4}{\lambda(t)^2} \int_t^\infty dw \lambda''(t+w) G_2(w, \lambda(t)) = -4 \int_{2t}^\infty ds \frac{\lambda''(s)}{(s-t)} + \frac{4}{\lambda(t)^2} \int_t^\infty dw \lambda''(t+w) \left(G_2(w, \lambda(t)) + \frac{\lambda(t)^2}{w} \right)$$

The first term on the right-hand side of the above expression cancels with the fourth term on the right-hand side of (4.106). Then, we treat the following integral with the same procedure used in (4.107)

$$\frac{4}{\lambda(t)^2} \int_t^\infty dw \lambda''(t+w) \left(G_2(w, \lambda(t)) + \frac{\lambda(t)^2}{w} \right)$$
$$= \frac{4}{\lambda(t)^2} \left(-\lambda''(2t)\lambda(t)^2 G_4(\frac{t}{\lambda(t)}) + \lambda'''(2t)\lambda(t)^3 W_2(\frac{t}{\lambda(t)}) + \lambda(t)^3 \int_t^\infty dw \lambda''''(t+w) W_2(\frac{w}{\lambda(t)}) dw \right)$$

After combining all of our computations, we end up with (4.101), completing the proof of the lemma. □

Finally, to compute the principal part (in the matching region $r \sim \lambda(t)g(t)$) of the other two integrals in (4.100), we start with

$$RHS_2(t,r) - RHS_{2,0}(t,r) = \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) (v_{2,sub}(t,r) + w_{1,sub}(t,r))$$

Next, we replace $v_{2,sub}$ and $w_{1,sub}$ by their principal parts, which are $v_{2,cubic,main}$ and $w_{1,cubic,main}$, respectively. Then, we use part 1 of second order matching, which says that

$$v_{2,cubic,main}(t,r) + w_{1,cubic,main}(t,r) = v_{ell,2,0,main}(t, \frac{r}{\lambda(t)})$$

Whence, we get

$$\text{leading part of } \frac{1}{r} \int_0^r s^2 \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2} \right) v_{ell,2,0,main}(t, \frac{s}{\lambda(t)}) ds$$
$$:= (u_{w,2,ell} - u_{w,2,ell,0})_{princ,1}(t,r)$$
$$= - \left(2r \left(2j_1(t)\lambda(t)^2 + 2j_2(t)\lambda(t)^2 \log(r) - j_2(t)\lambda(t)^2 \right) \right)$$

$$\text{leading part of } -r \int_0^r \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2} \right) v_{ell,2,0,main}(t, \frac{s}{\lambda(t)}) ds$$
$$:= (u_{w,2,ell} - u_{w,2,ell,0})_{princ,2}(t,r)$$
$$= \frac{1}{3} r \lambda(t)^2 \left(-24j_1(t) \log(\lambda(t)) + 24j_1(t) \log(r) - 12j_1(t) - 12j_2(t) \log^2(\lambda(t)) \right.$$
$$\left. -12j_2(t) \log(\lambda(t)) + 12j_2(t) \log^2(r) - \pi^2 j_2(t) \right)$$

where

$$j_1(t) = \frac{3}{32} \lambda''''(t) + \frac{1}{8} \partial_t^2 \left(\lambda''(t) + \lambda''(t) \log(\lambda(t)) + \frac{\lambda'(t)^2}{2\lambda(t)} \right)$$

$$j_2(t) = \frac{-1}{8} \lambda''''(t)$$

and we used

$$v_{ell,2,0,main}(t, \frac{r}{\lambda(t)}) = r^3 j_1(t) + r^3 \log(r) j_2(t)$$

We now estimate the difference between $u_{w,2,ell} - u_{w,2,ell,0}$ and its principal part in the matching region, which will help us study other contributions to the third order matching.

Lemma 15 *Let*

$$(u_{w,2,ell}(t,r) - u_{w,2,ell,0}(t,r))_{princ}$$
$$:= -\frac{1}{2} \left((u_{w,2,ell} - u_{w,2,ell,0})_{princ,1}(t,r) + (u_{w,2,ell} - u_{w,2,ell,0})_{princ,2}(t,r) \right)$$
$$- \frac{r}{2} \lambda(t) \int_t^\infty W_3(\frac{s-t}{\lambda(t)}) \lambda''''(s) ds - \frac{r}{2} \int_0^\infty \left(\frac{-8}{\xi \lambda(t)^2} + 4\xi K_2(\xi \lambda(t)) + 2\xi \right) \sin(t\xi) \widehat{v_{2,0}}(\xi) d\xi$$

Then, for $0 \leqslant k \leqslant 2$, and $0 \leqslant j \leqslant 8$ we have

$$|\partial_t^j \partial_r^k \left(u_{w,2,ell}(t,r) - u_{w,2,ell,0}(t,r) - (u_{w,2,ell}(t,r) - u_{w,2,ell,0}(t,r))_{princ} \right)|$$
$$\leqslant \frac{C}{r^{1+k}} \frac{\lambda(t)^3 \log^3(t)}{t^{4+j}} \left(\frac{r^4}{t^2} + \lambda(t)^2 \right), \quad \lambda(t) \leqslant r \leqslant \frac{t}{2}$$

Moreover, for $0 \leqslant m + k \leqslant 1$, $m, k \geqslant 0$,

$$|\partial_t^{2+k} \partial_r^m \left(u_{w,2,ell}(t,r) - u_{w,2,ell,0}(t,r) \right)| \leqslant \frac{C\lambda(t)^2 \log(t) \sup_{y \in [100,t]} \left(\lambda(y) \log(y) \right)}{t^{9/2} \langle t - r \rangle^{1/2+k+m}}, \quad \frac{t}{2} \leqslant r \leqslant 2t \qquad (4.112)$$

In addition, for $s \geqslant t$, and $0 \leqslant k \leqslant 2$,

$$|\partial_s^{4+k} \left(u_{w,2,ell} - u_{w,2,ell,0} \right)(s,y)| \leqslant C\lambda(s)^2 \log^3(s) \sup_{x \in [100,s]} \left(\lambda(x) \log(x) \right) \begin{cases} \frac{y}{s^{8+k}}, & y \leqslant \frac{s}{2} \\ \frac{1}{s^{9/2} t^{5/2+k}}, & \frac{s}{2} \leqslant y \leqslant s - t + 2g(t)\lambda(t) \end{cases}$$
$$(4.113)$$

Proof We first note that

$$u_{w,2,ell}(t,r) - u_{w,2,ell,0}(t,r) - (u_{w,2,ell}(t,r) - u_{w,2,ell,0}(t,r))_{princ}$$

$$= -\frac{1}{2r} \left(\int_0^r s^2 \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2} \right) \left(w_{1,sub}(t,s) + v_{2,sub}(t,s) - \left(v_{2,cubic,main}(t,s) + w_{1,cubic,main}(t,s) \right) \right) ds$$

$$+ \int_0^r s^2 \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2} \right) v_{ell,2,0,main}(t, \frac{s}{\lambda(t)}) ds - r(u_{w,2,ell} - u_{w,2,ell,0})_{princ,1}(t,r) \right)$$

$$+ \frac{r}{2} \left(\int_0^r \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2} \right) \left(w_{1,sub}(t,s) + v_{2,sub}(t,s) - \left(v_{2,cubic,main}(t,s) + w_{1,cubic,main}(t,s) \right) \right) ds$$

$$+ \int_0^r \left(\frac{\cos(2Q_1(\frac{s}{\lambda(t)})) - 1}{s^2} \right) v_{ell,2,0,main}(t, \frac{s}{\lambda(t)}) ds + \frac{1}{r} (u_{w,2,ell} - u_{w,2,ell,0})_{princ,2}(t,r) \right)$$
$$(4.114)$$

where we used

$$v_{2,cubic,main}(t,s) + w_{1,cubic,main}(t,s) = v_{ell,2,0,main}(t, \frac{s}{\lambda(t)})$$

which follows from the second order matching. Next, we use Lemmas 1 and 6, noting that $s \leqslant r \leqslant \frac{t}{2}$ in the integrals which compose the first and third terms of (4.114).

The estimate (4.112) follows directly from Lemma 10 and (4.61), and (4.113) is proven with the same procedure as in Lemma 10. □

Next, we need to understand $w_3(t,r) := u_{w,2}(t,r) - u_{w,2,ell}(t,r) - u_{w,2,sub,0}$, since w_3 will turn out to contribute terms which are logarithmically smaller than the largest contributions of $u_{w,2,sub,0}(t,r)$ in the matching region, but not quite perturbative. Recalling the equations that these functions solve, (4.84) and (4.85), we see that w_3 solves the following equation with 0 Cauchy data at infinity.

$$-\partial_t^2 w_3 + \partial_r^2 w_3 + \frac{1}{r} \partial_r w_3 - \frac{w_3}{r^2} = \partial_t^2 \left(u_{w,2,ell} - u_{w,2,ell,0,cont} \right)$$

Therefore,

$$w_3(t,r) = \int_t^\infty w_{3,s}(t,r) ds, \quad w_{3,s} \text{ solves } \begin{cases} -\partial_t^2 w_{3,s} + \partial_r^2 w_{3,s} + \frac{1}{r} \partial_r w_{3,s} - \frac{w_{3,s}}{r^2} = 0 \\ w_{3,s}(s,r) = 0 \\ \partial_1 w_{3,s}(s,r) = \partial_1^2 (u_{w,2,ell} - u_{w,2,ell,0,cont})(s,r) \end{cases}$$

By the finite speed of propagation, in the region $r \leqslant t$, we have

$$w_{3,s}(t,r) = v_{3,s}(t,r), \quad v_{3,s} \text{ solves } \begin{cases} -\partial_t^2 v_{3,s} + \partial_r^2 v_{3,s} + \frac{1}{r} \partial_r v_{3,s} - \frac{v_{3,s}}{r^2} = 0 \\ v_{3,s}(s,r) = 0 \\ \partial_1 v_{3,s}(s,r) = \psi_{\leqslant 1}(r - s) \partial_1^2 (u_{w,2,ell} - u_{w,2,ell,0,cont})(s,r) \end{cases}$$

where

$$\psi_{\leqslant 1} \in C^\infty(\mathbb{R}), \text{ and } \psi_{\leqslant 1}(x) = \begin{cases} 1, & x \leqslant 0 \\ 0, & x \geqslant 1 \end{cases}$$

Since we will only be interested in estimating $w_3(t,r)$ in the matching region $g(t)\lambda(t) \leqslant r \leqslant 2g(t)\lambda(t)$, it will suffice to estimate v_3 given below

$$v_3(t,r) := \int_t^\infty v_{3,s}(t,r)ds$$

I.e., v_3 solves

$$-\partial_t^2 v_3 + \partial_r^2 v_3 + \frac{1}{r}\partial_r v_3 - \frac{v_3}{r^2} = \psi_{\leqslant 1}(r-t)\partial_t^2(u_{w,2,ell} - u_{w,2,ell,0,cont})(t,r) := RHS_3(t,r) \qquad (4.115)$$

with 0 Cauchy data at infinity. Let

$$ellsoln = \frac{-1}{2}\left(\frac{1}{r}\int_0^r s^2 RHS_3(t,s)ds + r\int_r^\infty RHS_3(t,s)ds\right)$$

and

$$Q(t) = \frac{-1}{2}\int_0^\infty \psi_{\leqslant 1}(s-t)\partial_t^2\left(u_{w,2,ell} - u_{w,2,ell,0}\right)(t,s)ds$$

Note that $ellsoln$ satisfies

$$\partial_r^2 ellsoln + \frac{1}{r}\partial_r ellsoln - \frac{ellsoln}{r^2} = RHS_3(t,r)$$

Lemma 16 *We have the following estimates. For $0 \leqslant j,k \leqslant 1$,*

$$|r^k\partial_r^k\partial_t^j\left(ellsoln - rQ(t)\right)| \leqslant \begin{cases} \frac{Cr\lambda(t)^3\left(|\log^3(r)| + \log^3(t)\right)}{t^{4+j}}, & r \leqslant \lambda(t) \\ \frac{Cr^3\lambda(t)^2\log^3(t)\sup_{x\in[100,t]}(\lambda(x)\log(x))}{t^{6+j}} + \frac{C}{r}\frac{\lambda(t)^5\log^2(t)}{t^{4+j}}, & \lambda(t) < r < \frac{t}{2} \\ \frac{C\lambda(t)^2\log^3(t)\sup_{x\in[100,t]}(\lambda(x)\log(x))}{t^{3+\frac{j}{2}}}, & \frac{t}{2} \leqslant r \leqslant 2t \end{cases}$$

$$|\partial_r^2\left(ellsoln - rQ(t)\right)| \leqslant \frac{C\lambda(t)^5\log^2(t)}{r^3 t^4} + \frac{Cr\lambda(t)^2\sup_{x\in[100,t]}(\lambda(x)\log(x))\log^3(t)}{t^6}, \quad g(t)\lambda(t) \leqslant r \leqslant 2g(t)\lambda(t)$$

Proof We have

$$ellsoln - rQ(t)$$
$$= \frac{-1}{2r}\int_0^r s^2\psi_{\leqslant 1}(s-t)\partial_t^2\left(u_{w,2,ell} - u_{w,2,ell,0,cont}\right)(t,s)ds + \frac{r}{2}\int_0^r \psi_{\leqslant 1}(s-t)\partial_t^2\left(u_{w,2,ell} - u_{w,2,ell,0}\right)(t,s)ds$$
$$- \frac{r}{2}\int_r^\infty \psi_{\leqslant 1}(s-t)\partial_t^2\left(u_{w,2,ell,0} - u_{w,2,ell,0,cont}\right)(t,s)ds$$

We then estimate directly, using Lemmas 15 and 10 and the definitions of $u_{w,2,ell,0}$ and $u_{w,2,ell,0,cont}$ from (4.58), (4.61), and (4.62). We use the same procedure to estimate the higher derivatives, except for the last estimate in the lemma statement, which is obtained by using

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\left(ellsoln - rQ(t)\right) = \psi_{\leqslant 1}(r-t)\partial_t^2\left(u_{w,2,ell} - u_{w,2,ell,0,cont}\right)$$

and Lemmas 15 and 10. □

We return to v_3, which we recall solves (4.115), with 0 Cauchy data at infinity. If we define

$$q_3(t,r) := v_3(t,r) - \left(ellsoln - rQ(t)\right)\psi_{\leqslant 1}(r-t)$$

then, q_3 solves the following equation. Moreover, our estimates from Lemmas 16 and 10 show that $q_3(t,r)$ also has 0 Cauchy data at infinity.

$$-\partial_t^2 q_3 + \partial_r^2 q_3 + \frac{1}{r}\partial_r q_3 - \frac{q_3}{r^2} = \psi_{\leqslant 1}(r-t)\left(\partial_t^2\left(ellsoln - rQ(t)\right) + \partial_t^2\left(u_{w,2,ell} - u_{w,2,ell,0,cont}\right)\cdot\left(1 - \psi_{\leqslant 1}(r-t)\right)\right)$$
$$+ \psi_{\leqslant 1}'(r-t)\left(-2\left(\partial_t + \partial_r\right)\left(ellsoln - rQ(t)\right) - \frac{\left(ellsoln - rQ(t)\right)}{r}\right)$$
$$:= RHS_4(t,r)$$

In other words, we have

$$q_3(t,r) = \int_t^\infty q_{3,s}(t,r)ds, \quad q_{3,s} \text{ solves } \begin{cases} \left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) q_{3,s} = 0 \\ q_{3,s}(s,r) = 0 \\ \partial_1 q_{3,s}(s,r) = RHS_4(s,r) \end{cases}$$

We recall that we only need to estimate $q_3(t,r)$ for $r \leqslant 2g(t)\lambda(t) < t - 2$, and

$$\psi_{\leqslant 1}(x) = 1, \quad \psi'_{\leqslant 1}(x) = 0, \quad x \leqslant 0$$

Therefore, by the finite speed of propagation, for $r \leqslant 2g(t)\lambda(t)$, we have

$$q_{3,s}(t,r) = q_{4,s}(t,r), \quad q_{4,s} \text{ solves } \begin{cases} \left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) q_{4,s} = 0 \\ q_{4,s}(s,r) = 0 \\ \partial_1 q_{4,s}(s,r) = \psi_{\leqslant 1}(r-s)\partial_s^2 \left(ellsoln - rQ(s)\right) \end{cases}$$

So, it suffices to estimate

$$q_4(t,r) := \int_t^\infty q_{4,s}(t,r)ds$$

We recall that

$$\psi_{\leqslant 1}(r-t)\partial_t^2 \left(ellsoln(t,r) - rQ(t)\right)$$

$$= \psi_{\leqslant 1}(r-t)\partial_t^2 \left(\frac{-1}{2r}\int_0^r s^2\partial_t^2 \left(u_{w,2,ell} - u_{w,2,ell,0}\right)(t,s)\psi_{\leqslant 1}(s-t)ds\right.$$

$$- \frac{1}{2r}\int_0^r s^2\partial_t^2 \left(u_{w,2,ell,0} - u_{w,2,ell,0,cont}\right)(t,s)\psi_{\leqslant 1}(s-t)ds$$

$$+ \frac{r}{2}\int_0^r \partial_t^2 \left(u_{w,2,ell} - u_{w,2,ell,0}\right)(t,s)\psi_{\leqslant 1}(s-t)ds$$

$$\left. - \frac{r}{2}\int_r^\infty \partial_t^2 \left(u_{w,2,ell,0} - u_{w,2,ell,0,cont}\right)(t,s)\psi_{\leqslant 1}(s-t)ds\right)$$

We correspondingly decompose q_4 into

$$q_4(t,r) := q_{4,1}(t,r) + q_{4,2}(t,r)$$

where $q_{4,2}$ solves the following equation with 0 Cauchy data at infinity (and $q_{4,1} = q_4 - q_{4,2}$)

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) q_{4,2}(t,r)$$

$$= \psi_{\leqslant 1}(r-t)\partial_t^2 \left(\frac{-1}{2r}\int_0^r s^2\partial_t^2 \left(u_{w,2,ell,0} - u_{w,2,ell,0,cont}\right)(t,s)\psi_{\leqslant 1}(s-t)ds\right)$$

$$- \psi_{\leqslant 1}(r-t)\partial_t^2 \left(\frac{r}{2}\int_r^\infty \partial_t^2 \left(u_{w,2,ell,0} - u_{w,2,ell,0,cont}\right)(t,s)\psi_{\leqslant 1}(s-t)ds\right) \qquad (4.116)$$

$$:= RHS_{4,2}(t,r)$$

To estimate $q_{4,2}$, it will suffice to use energy estimates.

Lemma 17 *We have the following estimates, for $k = 0, 1$, and $r > 0$:*

$$|\partial_t^k q_{4,2}(t,r)| + \sqrt{E(\partial_t^k q_{4,2}, \partial_t^{k+1} q_{4,2})} + ||\partial_r^2 q_{4,2}||_{L^2((\lambda(t)g(t), 2\lambda(t)g(t)), rdr)} \leqslant \frac{C\lambda(t)^5 \log^3(t)}{t^5}$$

Proof We first note that the right-hand side of (4.116) includes the term

$$\psi_{\leqslant 1}(r-t)\left(\frac{-1}{2r}\int_0^r s^2\psi''_{\leqslant 1}(s-t)\partial_t^2(u_{w,2,ell,0} - u_{w,2,ell,0,cont})ds - \frac{r}{2}\int_r^\infty \psi''_{\leqslant 1}(s-t)\partial_t^2(u_{w,2,ell,0} - u_{w,2,ell,0,cont})ds\right)$$

We integrate by parts in each integral, integrating $\psi''_{\leqslant 1}(s-t)$, and note that the boundary contributions from each integral at $s = r$ cancel. Then, we directly insert the estimates from Lemma 10 into the other terms in (4.116), and use the same procedure used for (4.69). Finally, the symbol type estimates on $u_{w,2,ell,0} - u_{w,2,ell,0,cont}$ from Lemma 10 show that $\partial_t q_{4,2}$ solves

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right) \partial_t q_{4,2}(t,r) = \partial_t RHS_{4,2}(t,r)$$

also with 0 Cauchy data at infinity. Then, we use the same procedure used to estimate $q_{4,2}$. Finally, we estimate

$$||\partial_r^2 q_{4,2}||_{L^2((\lambda(t)g(t), 2\lambda(t)g(t)), rdr)}$$

using the equation solved by $q_{4,2}$ and our earlier estimates from the proof of this lemma. □

Now, we study $q_{4,1}$. Let

$$RHS_{4,1}(t,r) = \psi_{\leq 1}(r-t)\partial_t^2 \left(\frac{-1}{2r} \int_0^r s^2 \partial_t^2 \left(u_{w,2,ell} - u_{w,2,ell,0} \right)(t,s)\psi_{\leq 1}(s-t)ds \right.$$
$$\left. + \frac{r}{2} \int_0^r \partial_t^2 \left(u_{w,2,ell} - u_{w,2,ell,0} \right)(t,s)\psi_{\leq 1}(s-t)ds \right) \tag{4.117}$$

so that we have

$$q_{4,1}(t,r) = \int_t^\infty q_{4,1,s}(t,r)ds, \quad q_{4,1,s} \text{ solves } \begin{cases} \left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} \right) q_{4,1,s} = 0 \\ q_{4,1,s}(s,r) = 0 \\ \partial_1 q_{4,1,s}(s,r) = RHS_{4,1}(s,r) \end{cases}$$

Since we only need to estimate $q_{4,1}(t,r)$ (and hence $q_{4,1,s}(t,r)$) in the region $r \leq 2g(t)\lambda(t) < \frac{t}{2}$, the finite speed of propagation shows that $q_{4,1,s}(t,r)$ only depends on $RHS_{4,1}(s,y)$ for $y \leq s - t + 2g(t)\lambda(t) < s - \frac{t}{2}$. Given the limits of the integrals in the terms defining $RHS_{4,1}$, this means that we may replace $\psi_{\leq 1}(s-t)$ and $\psi_{\leq 1}(r-t)$ by 1 when considering $q_{4,1}(t,r)$ in the region $r \leq 2g(t)\lambda(t)$. Now, we estimate $RHS_{4,1}$.

Lemma 18 *For $0 \leq k, j \leq 2$, $j = 3, 0 \leq k \leq 1$, and $s \geq t$, we have*

$$y^k |\partial_y^k \partial_s^j RHS_{4,1}(s,y)| \leq C\lambda(s)^2 \log^3(s) \sup_{x \in [100,s]} (\lambda(x)\log(x)) \begin{cases} \frac{y^3}{s^{8+j}}, & y \leq \frac{s}{2} \\ \frac{1}{s^{5/2}t^{5/2+j}}, & \frac{s}{2} < y < s - t + 2g(t)\lambda(t) \end{cases}$$

Proof We re-write (4.117) as

$$RHS_{4,1}(s,y) = \int_0^y F(w,y)\partial_s^4 \left(RHS_2 - RHS_{2,0} \right)(s,w)dw - \frac{y^3}{16} \int_0^\infty \partial_s^4 \left(RHS_2 - RHS_{2,0} \right)(s,w)dw \tag{4.118}$$

where

$$F(w,y) = \frac{-\left(w^4 - y^4 + 4w^2 y^2 \log(y/w) \right)}{16y}$$

This is of the same form as (4.70), and is treated in the same way, noting that $\partial_1^j F(y,y) = 0$, $j = 0,1,2$. In addition, we use $s - t + 2g(t)\lambda(t) < s - \frac{t}{2}$, which implies that

$$\frac{1}{\langle s - y \rangle} \leq \frac{C}{t}, \quad \frac{s}{2} < y < s - t + 2g(t)\lambda(t)$$

We then directly differentiate (4.118) to treat higher derivatives of $RHS_{4,1}$. □

Let

$$q_{4,1,0}(t,r) := \frac{-r}{2} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \left(\partial_2 RHS_{4,1}(s,\rho) + \frac{RHS_{4,1}(s,\rho)}{\rho} \right) \tag{4.119}$$

Now, we can estimate $q_{4,1}$.

Lemma 19 *For $0 \leq k \leq 1$, and $0 \leq j \leq 2$ or $k = 2, j = 0$, we have*

$$|r^k \partial_r^k \partial_t^j \left(q_{4,1}(t,r) - q_{4,1,0}(t,r) \right)| \leq \frac{Cr^2 \sup_{x \in [100,t]} (\lambda(x)\log(x)) \lambda(t)^2 \log^3(t)}{t^{5+j}}, \quad \lambda(t)g(t) \leq r \leq 2\lambda(t)g(t) \tag{4.120}$$

Also, for $0 \leq j \leq 3$,

$$|\partial_t^j q_{4,1,0}(t,r)| \leq \frac{Cr \sup_{x \in [100,t]} (\lambda(x)\log(x)) \lambda(t)^2 \log^3(t)}{t^{4+j}}$$

Proof The spherical means representation formula gives (see also (4.99), pg. 24 of [25])

$$q_{4,1}(t,r) = -\frac{1}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^{2\pi} \frac{RHS_{4,1}(s,\sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)})}{\sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}} \left(r + \rho\cos(\theta) \right)d\theta \tag{4.121}$$

If

$$G(s,r,\rho) = \int_0^{2\pi} d\theta \frac{(r + \rho\cos(\theta))}{\sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}} RHS_{4,1}(s,\sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}), \quad \rho > 0$$

Then, by the dominated convergence theorem and Lemma 18, $\partial_r G(s,r,\rho)$ can be computed by differentiation under the integral sign, and

$$G(s,0,\rho) = 0 \implies G(s,r,\rho) = r \int_0^1 \partial_2 G(s,\beta r,\rho)d\beta$$

Therefore, we have

$$q_{4,1}(t,r) = \frac{-r}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \int_0^{2\pi} I_{RHS_{4,1}}(s, r\beta, \rho, \theta) d\theta \qquad (4.122)$$

where we recall the notation (2.2). Recalling (4.119), we get

$$q_{4,1}(t,r) - q_{4,1,0}(t,r) = \frac{-r}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \int_0^{2\pi} d\theta \, \text{integrand}_{4,1,2}(s, r\beta, \rho, \theta) \qquad (4.123)$$

where

$$\text{integrand}_{4,1,2}(s, y, \rho, \theta) = I_{RHS_{4,1}}(s, y, \rho, \theta) - \left(\partial_2 RHS_{4,1}(s, \rho) \cos^2(\theta) + \frac{RHS_{4,1}(s, \rho)}{\rho} \sin^2(\theta) \right)$$

We note that

$$\text{integrand}_{4,1,2}(s, y, \rho, \theta) = I_{RHS_{4,1}}(s, y, \rho, \theta) - I_{RHS_{4,1}}(s, 0, \rho, \theta)$$

Also, for all $r \geqslant 0, \rho > 3r, s \geqslant T_0, \theta \in [0, 2\pi]$, and $\beta \in [0, 1]$,

$$y \mapsto I_{RHS_{4,1}}(s, y, \rho, \theta) \in C^1([0, r\beta]), \quad \text{(note that, in this setting, } \sqrt{y^2 + \rho^2 + 2y\rho \cos(\theta)} \geqslant C\rho > 0 \text{)}$$

Therefore, when $s - t \geqslant 3r$, and $\rho > 3r$ in the integral in (4.123), we use

$$\text{integrand}_{4,1,2}(s, r\beta, \rho, \theta) = \int_0^{r\beta} \partial_2 I_{RHS_{4,1}}(s, y, \rho, \theta) dy$$

and we directly substitute our estimates from Lemma 18 into (4.123) for the other regions, to get (4.120) for $k = 0, j = 0$. Next, we use (4.123) to get

$$\partial_r \left(q_{4,1} - q_{4,1,0} \right)(t,r) = -\frac{1}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^{2\pi} d\theta \, \text{integrand}_{4,1,2}(s, r, \rho, \theta)$$

which is treated with the same argument used for (4.123). This gives (4.120) for $k = 1, j = 0$. To estimate $\partial_t^j \left(q_{4,1} - q_{4,1,0} \right)(t,r)$, we return to (4.123), and let $w = s - t$. Then, the dominated convergence theorem, along with the estimates of Lemma 18 allow us to differentiate under the resulting integral signs, and we get, for $j = 1, 2$,

$$\partial_t^j \left(q_{4,1} - q_{4,1,0} \right)(t,r) = \frac{-r}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \int_0^{2\pi} d\theta \, \partial_s^j \text{integrand}_{4,1,2}(s, r\beta, \rho, \theta)$$

We then repeat the same procedure used for $q_{4,1} - q_{4,1,0}$. To estimate $q_{4,1,0}$, we use

$$q_{4,1,0}(t,r) = \frac{-r}{2} \int_0^\infty dw \int_0^w \frac{\rho d\rho}{\sqrt{w^2 - \rho^2}} \left(\partial_2 RHS_{4,1}(t + w, \rho) + \frac{RHS_{4,1}(t + w, \rho)}{\rho} \right)$$

differentiate under the integral sign, then substitute the estimates from Lemma 18. We prove (4.120) for $j = 0, k = 2$ by using the equation solved by $q_{4,1}$ and our previous estimates from this lemma. $\qquad \square$

Next, we study the analogous quantities related to $v_{ex,sub}$ which we recall is defined in (4.83). In particular, we start by studying

$$w_5(t,r) := v_{ex,sub}(t,r) - v_{ex,sub,0}(t,r)$$

The function w_5 satisfies the following equation with 0 Cauchy data at infinity

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} \right) w_5(t,r) = \partial_t^2 \left(v_{ex,ell} - v_{ex,cont} \right) = \partial_t^2 v_{ex,ell,1}(t,r)$$

where we define

$$v_{ex,ell,1}(t,r) := v_{ex}(t,r) - v_{ex,sub}(t,r) - v_{ex,cont}(t,r) = v_{ex,ell}(t,r) - v_{ex,cont}(t,r)$$

and recall that $v_{ex,ell}$ and $v_{ex,cont}$ are explicitly given in (4.36) and (4.80), respectively. We have the following lemma

Lemma 20 *For $0 \leqslant k, j \leqslant 2$, we have*

$$|\partial_t^j \partial_r^k v_{ex,ell,1}(t,r)| \leqslant \frac{C\lambda(t)^{2-k}}{t^{2+j} g(t)^{3+k}}, \quad g(t)\lambda(t) \leqslant r \leqslant 2g(t)\lambda(t)$$

Proof We directly estimate $v_{ex,ell,1}$ from the formulae (4.36) and (4.80). $\quad\square$

Next, we define $ellsoln_2$ by

$$ellsoln_2(t,r) := \frac{-1}{2}\left(\frac{1}{r}\int_0^r s^2 \partial_t^2 v_{ex,ell,1}(t,s)ds + r\int_r^\infty \partial_t^2 v_{ex,ell,1}(t,s)ds\right)$$

Then, we have the following lemma.

Lemma 21 *For all $j, k \geqslant 0$, there exists $C_{j,k} > 0$ such that*

$$r^k t^j |\partial_r^k \partial_t^j ellsoln_2(t,r)| \leqslant C_{j,k} \cdot \begin{cases} \frac{1}{r}\frac{\lambda(t)^5(1+|\log(\frac{r}{\lambda(t)})|)}{t^4}, & r \geqslant \lambda(t) \\ r\frac{\lambda(t)^3(1+\log^2(\frac{r}{\lambda(t)}))}{t^4}, & r \leqslant \lambda(t) \end{cases} \tag{4.124}$$

In addition, $\partial_t^2 ellsoln_2(t,r) = F_{ellsoln_2}(t,r) + \log^2(r)rg_1''(t) + r\log(r)g_2''(t)$, where, for each $t \geqslant T_0$,

$$r \mapsto F_{ellsoln_2}(t,r) \text{ admits a } C^2 \text{ extension to } [0,\infty)$$

For $0 \leqslant j + k \leqslant 2$,

$$|\partial_r^k \partial_t^j F_{ellsoln_2}(t,r)| \leqslant C \begin{cases} \frac{r^{1-k}\lambda(t)^3 \log^2(t)}{t^{6+j}}, & 0 \leqslant k \leqslant 1 \\ \frac{r\lambda(t)(1+|\log(\frac{r}{\lambda(t)})|)}{t^{6+j}}, & k = 2 \end{cases}, \quad r \leqslant \lambda(t)$$

$$|g_1^{(k)}(t)| + |g_2^{(k)}(t)| \leqslant \frac{C_k \log(t)\lambda(t)^3}{t^{4+k}}, \quad k \geqslant 0$$

Proof We have

$$ellsoln_2(t,r) = \frac{-1}{2}\left(\frac{1}{r}\int_0^r s^2 \partial_t^2 v_{ex,ell,1}(t,s)ds + r\int_r^\infty \partial_t^2 v_{ex,ell,1}(t,s)ds\right) = -\frac{1}{2}\left(\frac{1}{r}f_7(t,\frac{r}{\lambda(t)}) + rf_8(t,\frac{r}{\lambda(t)})\right)$$

where

$$\begin{aligned} f_7(t,R) = \frac{\lambda(t)}{8}\Bigg(& \frac{-8(-2R^2 + 3(1+R^2)\log(1+R^2))\lambda'(t)^4}{1+R^2} \\ & -24\lambda(t)\left(R^2\log(1+\frac{1}{R^2}) + 3\log(R^2+1)\right)\lambda'(t)^2\lambda''(t) \\ & -4\lambda(t)^2\left(R^2\log(1+\frac{1}{R^2}) + \log(1+R^2)\right)\left(3\lambda''(t)^2 + 4\lambda'(t)\lambda'''(t)\right) \\ & -\lambda(t)^3\left(-R^2 + R^2(2+R^2)\log(1+\frac{1}{R^2}) + \log(1+R^2)\right)\lambda''''(t)\Bigg) \end{aligned}$$

and

$$\begin{aligned} f_8(t,R) = \frac{1}{4}\Bigg(& \frac{-4(2+(1+R^2)\log(1+\frac{1}{R^2}))\lambda'(t)^4}{(1+R^2)\lambda(t)} \\ & + 12\left(-2\log(1+\frac{1}{R^2}) + \text{Li}_2(-\frac{1}{R^2})\right)\lambda'(t)^2\lambda''(t) + 2\lambda(t)\text{Li}_2(\frac{-1}{R^2})\left(3\lambda''(t)^2 + 4\lambda'(t)\lambda'''(t)\right) \\ & + \lambda(t)^2(-1+(1+R^2)\log(1+\frac{1}{R^2}) + \text{Li}_2(\frac{-1}{R^2}))\lambda''''(t)\Bigg) \end{aligned}$$

The lemma statement now follows from inspection. $\quad\square$

We define

$$q_5(t,r) := w_5(t,r) - ellsoln_2(t,r)\psi_{\leqslant 1}(r-t)$$

so that q_5 solves the following equation with 0 Cauchy data at infinity.

$$\begin{aligned} -\partial_t^2 q_5 + \partial_r^2 q_5(t,r) + \frac{1}{r}\partial_r q_5 - \frac{q_5}{r^2} &= \psi_{\leqslant 1}(r-t)\partial_t^2 ellsoln_2 + (1-\psi_{\leqslant 1}(r-t))\partial_t^2 v_{ex,ell,1}(t,r) \\ &\quad - \psi_{\leqslant 1}'(r-t)\left(2\left(\partial_t + \partial_r\right)ellsoln_2 + \frac{1}{r}ellsoln_2\right) \end{aligned} \tag{4.125}$$

We define $q_{5,0}$ by

$$q_{5,0}(t,r) = \frac{-r}{2}\int_t^\infty ds\int_0^{s-t}\frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}}\left(\partial_{112}ellsoln_2(s,\rho) + \frac{\partial_{11}ellsoln_2(s,\rho)}{\rho}\right) \tag{4.126}$$

Then, we have the following lemma.

Lemma 22 *For $0 \leqslant k \leqslant 1$ and $0 \leqslant j \leqslant 2$,*

$$|t^j r^k \partial_t^j \partial_r^k (q_5 - q_{5,0})(t,r)| \leqslant \frac{Cr^2 \lambda(t)^3 \left(\log^3(t) + |\log(r)|^3\right)}{t^5}, \quad r \leqslant 2g(t)\lambda(t)$$

For all $j \geqslant 0$, there exists $C_j > 0$ such that

$$|\partial_t^j q_{5,0}(t,r)| \leqslant \frac{C_j r \log^2(t) \lambda(t)^3}{t^{4+j}}$$

$$|\partial_r^2 (q_5 - q_{5,0})(t,r)| \leqslant \frac{C\lambda(t)^3 \log^3(t)}{t^5}, \quad g(t)\lambda(t) \leqslant r \leqslant 2g(t)\lambda(t)$$

Proof First, we claim that in the region $r \leqslant 2g(t)\lambda(t)$,

$$q_5(t,r) = \frac{-1}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^{2\pi} d\theta \partial_1^2 ellsoln_2(s, \sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}) \frac{(r + \rho\cos(\theta))}{\sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}} \tag{4.127}$$

To verify this, we recall (4.125), and use the finite speed of propagation. The only item remaining is to show that the integral on the right-hand side of (4.127) solves (4.125) with $\partial_t^2 ellsoln_2$ on the right-hand side, and zero Cauchy data at infinity. For this purpose, we note that

$$\frac{-1}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^{2\pi} d\theta \partial_1^2 ellsoln_2(s, \sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}) \frac{(r + \rho\cos(\theta))}{\sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}}$$

$$= \frac{-1}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^{2\pi} d\theta F_{ellsoln_2}(s, \sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}) \frac{(r + \rho\cos(\theta))}{\sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}}$$

$$+ \frac{-r^3}{2\pi} \int_0^\infty dy \int_0^y \frac{w dw}{\sqrt{y^2 - w^2}} \int_0^{2\pi} d\theta (1 + w\cos(\theta)) \left(\log^2(r) g_1''(t + ry)\right.$$

$$+ \log(r)\left(\log(1 + w^2 + 2w\cos(\theta)) g_1''(t + ry) + g_2''(t + ry)\right)$$

$$+ \log^2(\sqrt{1 + w^2 + 2w\cos(\theta)}) g_1''(t + ry)$$

$$+ \left. \log(\sqrt{1 + w^2 + 2w\cos(\theta)}) g_2''(t + ry)\right)$$

The point of this splitting is that, by Lemma 21 and the dominated convergence theorem, we can differentiate up to two times in (t,r) under the integral sign. Then, we can proceed as in any standard verification of the spherical means formula. Then, we return to (4.127) and use (4.124) and the same procedure used to establish Lemma 19. □

Now, we compute the leading behavior of $u_{ell,2}(t,r) - v_{ell,2,0,main}(t, \frac{r}{\lambda(t)})$ in the matching region. We first recall that $v_{ell,2,0}$ and $v_{ell,2,0,main}$ are explicitly given in (4.12) and (4.14), respectively. So, it suffices to explain how to compute the leading behavior of $u_{ell,2}(t,r) - v_{ell,2,0}(t, \frac{r}{\lambda(t)})$. From (4.16), we have

$$u_{ell,2}(t,r) - v_{ell,2,0}(t, \frac{r}{\lambda(t)}) = \frac{-\phi_0(\frac{r}{\lambda(t)})}{2} \int_0^{\frac{r}{\lambda(t)}} err_1(t, s\lambda(t)) se_2(s) ds + e_2(\frac{r}{\lambda(t)}) \int_0^{\frac{r}{\lambda(t)}} err_1(t, s\lambda(t)) \frac{s\phi_0(s)}{2} ds$$

$$= \frac{-\phi_0(\frac{r}{\lambda(t)})}{2} \int_0^{\frac{r}{\lambda(t)}} err_{1,0}(t, s\lambda(t)) se_2(s) ds + \frac{-\phi_0(\frac{r}{\lambda(t)})}{2} \int_0^{\frac{r}{\lambda(t)}} (err_1 - err_{1,0})(t, s\lambda(t)) se_2(s) ds$$

$$+ \frac{r}{2\lambda(t)} \int_0^{\frac{r}{\lambda(t)}} err_1(t, s\lambda(t)) \frac{s\phi_0(s)}{2} ds + \left(e_2(\frac{r}{\lambda(t)}) - \frac{r}{2\lambda(t)}\right) \int_0^{\frac{r}{\lambda(t)}} err_1(t, s\lambda(t)) \frac{s\phi_0(s)}{2} ds$$

where we recall that $err_{1,0}(t,r)$ is defined in (4.18). Using the definition of $v_{ell,2,0,main}$ given in (4.14), we can write down the leading behavior of $u_{ell,2}(t,r) - v_{ell,2,0,main}(t, \frac{r}{\lambda(t)})$ in the region $r \sim g(t)\lambda(t)$, which we denote by $u_{e,3}(t,r)$.

$$u_{e,3}(t,r) = \frac{-2\lambda(t)}{2r} \int_0^{\frac{r}{\lambda(t)}} err_{1,0}(t, s\lambda(t)) \frac{s}{2} s ds + \frac{r}{2\lambda(t)} \int_0^\infty (err_1(t, s\lambda(t)) - err_{1,0}(t, s\lambda(t))) s \frac{\phi_0(s)}{2} ds$$

$$+ \frac{r}{2\lambda(t)} \int_0^{\frac{r}{\lambda(t)}} err_{1,0}(t, s\lambda(t)) \frac{\phi_0(s) s ds}{2} + \frac{r\lambda(t)^2 \lambda''''(t)}{96} \left(111 - 2\pi^2 - 108\log(\frac{r}{\lambda(t)}) + 24\log^2(\frac{r}{\lambda(t)})\right)$$

$$+ \frac{r}{16}(5 - 4\log(\frac{r}{\lambda(t)})) \left(\partial_t^2 \left(\frac{\lambda'(t)^2}{2\lambda(t)} + \lambda''(t)(1 + \log(\lambda(t)))\right)\right) 2\lambda(t)^2 - 2\lambda(t)^2 \lambda''''(t)(1 + \log(\lambda(t)))\right)$$

$$\tag{4.128}$$

By computing the integrals, we get

$$
\begin{aligned}
&u_{e,3}(t, R\lambda(t)) \\
&= \frac{\left(2\lambda'(t)^4 - 7\lambda(t)\lambda'(t)^2\lambda''(t) + 4\lambda(t)^2\lambda''(t)^2 + 6\lambda(t)^2\lambda'(t)\lambda'''(t)\right) R \left(5 - 4\log(R)\right)}{16} \\
&+ \frac{1}{96}\left(111 - 2\pi^2 - 108\log(R) + 24\log^2(R)\right) R\lambda(t)^3\lambda''''(t) \\
&+ \frac{3}{4}R\lambda'(t)^4 + \frac{3}{8}R\left(-2\lambda'(t)^4 + 5\lambda(t)\lambda'(t)^2\lambda''(t)\right) + \frac{3}{4}R\lambda(t)^2\partial_t\left(\lambda'(t)\lambda''(t)\right) \\
&+ \frac{1}{24}R\lambda(t)^2\left(21 + \pi^2 - 18\log(R) + 12\log^2(R)\right)\partial_t^2\left(\lambda(t)\lambda''(t)\right) \\
&+ \frac{1}{24}R\lambda(t)^2\left(39 + \pi^2 - 42\log(R) + 12\log^2(R)\right)\left(\partial_t^2\left(\lambda(t)\lambda''(t)\right) - \lambda(t)\lambda''''(t)\right) \\
&+ \frac{1}{12}R\lambda(t)\left(60 + \pi^2 - 54\log(R) + 12\log^2(R)\right)\lambda'(t)^2\lambda''(t) \\
&- \frac{1}{4}R\lambda(t)^2 c_1''(t) - \frac{R}{2}\left(\lambda(t)c_1'(t)\lambda'(t) + \frac{c_1(t)}{2}\left(-2\lambda'(t)^2 + \lambda(t)\lambda''(t)\right)\right) - \frac{R}{2}c_1(t)\lambda'(t)^2 \\
&+ \frac{R}{8}\lambda'(t)^4\left(17 - 12\log(R)\right) - \frac{R}{4}\left(5\lambda(t)\lambda'(t)^2\lambda''(t) - 2\lambda'(t)^4\right)\left(-3 + 2\log(R)\right) \\
&+ \frac{R}{4}\left(3 - 8\log(R)\right)\lambda(t)^2\partial_t\left(\lambda''(t)\lambda'(t)\right) \\
&+ \frac{R\lambda(t)^2\partial_t^2\left(\lambda(t)\lambda''(t)\right)}{24}\left(6 + \pi^2 + 12\zeta(3) - \left(36 + 2\pi^2\right)\log(R) + 12\log^2(R) - 8\log^3(R)\right) \\
&- \frac{R\lambda(t)^2}{24}\left(\partial_t^2\left(\lambda(t)\lambda''(t)\right) - \lambda(t)\lambda''''(t)\right)\Big(-27 - 2\pi^2 - 12\zeta(3) + (36 + 2\pi^2)\log(R) \\
&\hspace{6cm} -24\log^2(R) + 8\log^3(R)\Big) \\
&+ R\frac{\lambda(t)\lambda''(t)\lambda'(t)^2}{72}\left(255 + 17\pi^2 + 72\zeta(3) - (360 + 12\pi^2)\log(R) + 252\log^2(R) - 48\log^3(R)\right) \\
&+ R\frac{c_1''(t)\lambda(t)^2}{4}\left(-1 + 2\log(R)\right) + R\left(\lambda(t)\lambda'(t)c_1'(t) + \frac{c_1(t)}{2}\left(\lambda(t)\lambda''(t) - 2\lambda'(t)^2\right)\right)\left(-1 + \log(R)\right) \\
&+ R\frac{c_1(t)}{12}\lambda'(t)^2\left(-17 + 12\log(R)\right) + R\frac{\lambda(t)^3\lambda''''(t)}{48}\left(\pi^2 - 12\log^2(R)\right) \\
&+ \frac{R}{2}\log(R)\lambda(t)^3\left(\partial_t^2\left(\frac{\lambda'(t)^2}{2\lambda(t)} + \lambda''(t)(1 + \log(\lambda(t)))\right) - \log(\lambda(t))\lambda''''(t)\right)
\end{aligned} \tag{4.129}
$$

Now, we estimate the difference between $u_{ell,2}(t, r) - v_{ell,2,0,main}(t, \frac{r}{\lambda(t)})$ and $u_{e,3}(t, r)$.

Lemma 23 *For $u_{ell,2}$ defined in (4.9), $v_{ell,2,0,main}$ defined in (4.14), and $u_{e,3}$ defined in (4.128) we have, for $0 \leqslant j + k \leqslant 2$,*

$$
\left|\partial_t^j\partial_r^k\left(u_{ell,2}(t, r) - v_{ell,2,0,main}(t, \frac{r}{\lambda(t)}) - u_{e,3}(t, r)\right)\right| \leqslant \frac{C\lambda(t)^5(1 + \log^4(\frac{r}{\lambda(t)}))}{r^{1+k}t^{4+j}}, \quad r > \lambda(t)
$$

Proof We estimate each term in the difference between (4.15) plus (4.16) and (4.128) directly, using the symbol type estimates on $\lambda(t)$, and the fact that $err_1(t, r)$ and $err_{1,0}(t, r)$ are symbols in (t, r). □

Combining our work, we get

$$
\begin{aligned}
&\text{leading part of } \left(u_{w,2} - u_{w,2,ell,0} + v_{ex,sub}\right)(t, r) := u_{w,3}(t, r) \\
&= u_{w,2,sub,0,cont}(t, r) + v_{ex,sub,0,cont}(t, r) \\
&\quad - \frac{1}{2}\left((u_{w,2,ell} - u_{w,2,ell,0})_{princ,1}(t, r) + (u_{w,2,ell} - u_{w,2,ell,0})_{princ,2}(t, r)\right) \\
&\quad - \frac{r}{2}\lambda(t)\int_t^\infty W_3\left(\frac{s-t}{\lambda(t)}\right)\lambda''''(s)\,ds - \frac{r}{2}\int_0^\infty d\xi\left(\frac{-8}{\xi\lambda(t)^2} + 4\xi K_2(\xi\lambda(t)) + 2\xi\right)\sin(t\xi)\widehat{v_{2,0}}(\xi) \\
&\quad - \frac{r}{2}\int_t^\infty ds\int_0^{s-t}\frac{\rho\,d\rho}{\sqrt{(s-t)^2 - \rho^2}}\left(\partial_{112}ellsoln_2(s, \rho) + \frac{\partial_{11}ellsoln_2(s, \rho)}{\rho}\right) \\
&\quad - \frac{r}{2}\int_0^\infty dw\int_0^w\frac{\rho\,d\rho}{\sqrt{w^2 - \rho^2}}\left(\partial_2 RHS_{4,1}(t + w, \rho) + \frac{RHS_{4,1}(t + w, \rho)}{\rho}\right)
\end{aligned} \tag{4.130}
$$

A careful inspection of all of the terms shows that the $r\log^k(r)$ terms from $u_{e,3}$ and $u_{w,3}$ all exactly match, for $k = 1, 2, 3$. In other words, we have

$$
\begin{aligned}
&u_{e,3}(t,r) - u_{w,3}(t,r) \\
&= rF_2(t) + \frac{r\lambda(t)}{2}\int_t^\infty W_3\big(\frac{s-t}{\lambda(t)}\big)\lambda''''(s)\,ds \\
&\quad + \frac{r}{2}\int_t^{2t}\left(\frac{(g_0+h_0)''(s) - (g_0+h_0)''(t)}{(s-t)} + \frac{\big((g_1+2f_1)''(s) - (g_1+2f_1)''(t)\big)\log(2(s-t))}{(s-t)}\right. \\
&\quad\qquad \left. - \frac{\big(g_2''(s) - g_2''(t)\big)\Big(-12\log(s-t)\log(4(s-t)) + \pi^2 - 12\log^2(2)\Big)}{12(s-t)}\right)ds \\
&\quad + \frac{r}{2}\int_{2t}^\infty\left((g_0+h_0)''(s) + (g_1+2f_1)''(s)\log(2(s-t))\right. \\
&\quad\qquad \left. - \frac{g_2''(s)\Big(-12\log(s-t)\log(4(s-t)) + \pi^2 - 12\log^2(2)\Big)}{12}\right)\frac{ds}{(s-t)} \\
&\quad + \frac{r}{2}\int_0^\infty d\xi\left(\frac{-8}{\xi\lambda(t)^2} + 4\xi K_2(\xi\lambda(t)) + 2\xi\right)\sin(t\xi)\widehat{v_{2,0}}(\xi) \\
&\quad + \frac{r}{2}\int_t^\infty ds\int_0^{s-t}\frac{\rho\,d\rho}{\sqrt{(s-t)^2 - \rho^2}}\left(\partial_{112}ellsoln_2(s,\rho) + \frac{\partial_{11}ellsoln_2(s,\rho)}{\rho}\right) \\
&\quad + \frac{r}{2}\int_0^\infty dw\int_0^w\frac{\rho\,d\rho}{\sqrt{w^2 - \rho^2}}\left(\partial_2 RHS_{4,1}(t+w,\rho) + \frac{RHS_{4,1}(t+w,\rho)}{\rho}\right) \\
&:= rF_3(t)
\end{aligned}
\tag{4.131}
$$

where $F_2(t) = F_{2,0}(t) + F_{2,1}(t)$, and

$$
\begin{aligned}
F_{2,0}(t) =&\ \frac{3\lambda'(t)^4}{4\lambda(t)} + \frac{(12\log(\lambda(t)) + 17)\lambda'(t)^4}{8\lambda(t)} - \frac{c_1(t)\lambda'(t)^2}{2\lambda(t)} + \frac{c_1(t)(-12\log(\lambda(t)) - 17)\lambda'(t)^2}{12\lambda(t)} \\
&+ \frac{1}{12}\Big(12\log^2(\lambda(t)) + 54\log(\lambda(t)) + \pi^2 + 60\Big)\lambda''(t)\lambda'(t)^2 + \frac{3}{4}\partial_t\Big(\lambda'(t)\lambda''(t)\Big)\lambda(t) \\
&+ \frac{1}{4}\partial_t\Big(\lambda''(t)\lambda'(t)\Big)\lambda(t)(8\log(\lambda(t)) + 3) - \frac{\big(5\lambda(t)\lambda'(t)^2\lambda''(t) - 2\lambda'(t)^4\big)(-2\log(\lambda(t)) - 3)}{4\lambda(t)} \\
&+ \frac{1}{24}\partial_t^2\Big(\lambda(t)\lambda''(t)\Big)\lambda(t)\Big(12\log^2(\lambda(t)) + 18\log(\lambda(t)) + \pi^2 + 21\Big) \\
&+ \frac{1}{24}\partial_t^2\Big(\lambda(t)\lambda''(t)\Big)\lambda(t)\Big(8\log^3(\lambda(t)) + 12\log^2(\lambda(t)) + \big(36 + 2\pi^2\big)\log(\lambda(t)) + \pi^2 + 12\zeta(3) + 6\Big) \\
&- \frac{1}{24}\lambda(t)\Big(\partial_t^2\Big(\lambda(t)\lambda''(t)\Big) - \lambda(t)\lambda''''(t)\Big)\Big(-8\log^3(\lambda(t)) - 24\log^2(\lambda(t)) - \big(36 + 2\pi^2\big)\log(\lambda(t)) \\
&\qquad\qquad\qquad -2\pi^2 - 12\zeta(3) - 27\Big) \\
&+ \frac{(-\log(\lambda(t)) - 1)\big(\lambda(t)c_1'(t)\lambda'(t) + \frac{1}{2}c_1(t)\big(\lambda(t)\lambda''(t) - 2\lambda'(t)^2\big)\big)}{\lambda(t)} - \frac{1}{4}\lambda(t)c_1''(t) \\
&+ \frac{1}{4}\lambda(t)(-2\log(\lambda(t)) - 1)c_1''(t) \\
&+ \frac{\lambda''(t)}{72}\lambda'(t)^2\Big(48\log^3(\lambda(t)) + 252\log^2(\lambda(t)) + \big(360 + 12\pi^2\big)\log(\lambda(t)) + 17\pi^2 + 72\zeta(3) + 255\Big) \\
&+ \frac{3\big(5\lambda(t)\lambda'(t)^2\lambda''(t) - 2\lambda'(t)^4\big)}{8\lambda(t)} - \frac{\big(\lambda(t)c_1'(t)\lambda'(t) + \frac{1}{2}c_1(t)\big(\lambda(t)\lambda''(t) - 2\lambda'(t)^2\big)\big)}{2\lambda(t)} \\
&+ \frac{(4\log(\lambda(t)) + 5)\big(2\lambda'(t)^4 - 7\lambda(t)\lambda''(t)\lambda'(t)^2 + 6\lambda(t)^2\lambda'''(t)\lambda'(t) + 4\lambda(t)^2\lambda''(t)^2\big)}{16\lambda(t)} \\
&+ \frac{1}{96}\lambda(t)^2\Big(24\log^2(\lambda(t)) + 108\log(\lambda(t)) - 2\pi^2 + 111\Big)\lambda''''(t) + \frac{1}{48}\lambda(t)^2\Big(\pi^2 - 12\log^2(\lambda(t))\Big)\lambda''''(t) \\
&+ \frac{1}{24}\lambda(t)\Big(12\log^2(\lambda(t)) + 42\log(\lambda(t)) + \pi^2 + 39\Big)\Big(\partial_t^2\Big(\lambda(t)\lambda''(t)\Big) - \lambda(t)\lambda''''(t)\Big)
\end{aligned}
$$

$$F_{2,1}(t) = -\frac{1}{2}\log(\lambda(t))\lambda(t)^2\left(\partial_t^2\left(\frac{\lambda'(t)^2}{2\lambda(t)} + (\log(\lambda(t))+1)\lambda''(t)\right) - \log(\lambda(t))\lambda''''(t)\right)$$

$$-\left(c_0 g_2''(t) - \frac{1}{2}g_0''(t) - \frac{1}{4}(\log(4)-1)g_0''(t) - \frac{1}{2}\log(t)g_0''(t) - \frac{1}{16}\left(\pi^2 - 12\right)g_1''(t)\right)$$

$$-\left(-\left(\frac{\log^2(t)}{4} + \frac{1}{2}\log(2)\log(t)\right)g_1''(t) - \frac{1}{24}\left(-2\pi^2 + 15 + 6\log^2(2)\right)g_1''(t)\right)$$

$$-\left(-\frac{1}{16}\left(-28\zeta(3) + \pi^2 + 28\right)g_2''(t) - \frac{1}{24}\log(t)\left(4\log(t)(\log(t)+\log(8)) - \pi^2 + 12\log^2(2)\right)g_2''(t)\right)$$

$$-\left(-\frac{1}{8}\left(\pi^2 - 12\right)f_1''(t) - 2f_1''(t)\left(\frac{1}{4}\log^2(t) + \frac{1}{2}\log(2)\log(t)\right) - \frac{1}{2}h_0''(t)\right)$$

$$-\left(-\frac{1}{4}(\log(4)-1)h_0''(t) - \frac{1}{2}\log(t)h_0''(t) - \frac{1}{12}\left(-2\pi^2 + 15 + 6\log^2(2)\right)f_1''(t)\right)$$

$$+\frac{1}{2}\left(\frac{1}{3}\lambda(t)^2\left(-24j_1\log(\lambda(t)) - 12j_1 - 12j_2\log^2(\lambda(t)) - 12j_2\log(\lambda(t)) - \pi^2 j_2\right)\right)$$

$$+\frac{1}{2}\left(-2\left(2j_1\lambda(t)^2 - j_2\lambda(t)^2\right)\right)$$

We quickly record some symbol-type estimates on $F_3(t)$.

Lemma 24 *There exists $C > 0$ such that, for $0 \leqslant k \leqslant 3$,*

$$t^k|F_3^{(k)}(t)| \leqslant \frac{C\lambda(t)^2}{t^4}\sup_{x\in[100,t]}(\lambda(x)\log(x))\log^3(t) \tag{4.132}$$

Proof Using the symbol type estimates on λ, Lemmas 19 and 22, and integration by parts in the region $\xi t \geqslant 1$ to treat the fifth term in (4.131), we get

$$|F_3(t)| \leqslant \frac{C\lambda(t)^2}{t^4}\sup_{x\in[100,t]}(\lambda(x)\log(x))\log^3(t)$$

We prove (4.132) for $k > 0$ as follows. Using the symbol-type estimates on λ, we see that c_1, f_n, h_n, g_n, and j_n are all symbols, with estimates implying that their contributions to $F_3^{(k)}(t)$ are bounded above by the right-hand side of (4.132). Moreover,

$$\int_t^\infty W_3\left(\frac{s-t}{\lambda(t)}\right)\lambda''''(s)ds = \int_0^\infty W_3(w)\lambda''''(\lambda(t)w + t)\lambda(t)dw$$

is a symbol in t because $\lambda(t)$ is. We similarly note that the sum of the third and fourth terms on the right-hand side of (4.131) is a symbol in t. Finally, we study the following integral

$$I_7(t) = \int_0^\infty d\xi\left(\frac{-8}{\xi\lambda(t)^2} + 4\xi K_2(\xi\lambda(t)) + 2\xi\right)\sin(t\xi)\widehat{v_{2,0}}(\xi)$$

Letting

$$H(x) = \frac{-8}{x} + 4xK_2(x) + 2x, \quad x > 0$$

we get, for $0 \leqslant k \leqslant 7$,

$$|H^{(k)}(x)| \leqslant C\begin{cases} x^{3-k}(1 + |\log(x)|), & x \leqslant \frac{1}{2} \\ \begin{cases} x^{1-k}, & 0 \leqslant k \leqslant 1 \\ \frac{1}{x^{1+k}}, & 2 \leqslant k \end{cases}, & x > \frac{1}{2} \end{cases}$$

Then, by Lemma (4) and inspection of the following formula, $I_7(t)$ is a symbol in t with $|t^k I_7^{(k)}(t)|$ bounded above by the right-hand side of (4.132) (for $0 \leqslant k \leqslant 3$).

$$I_7(t) = \frac{1}{\lambda(t)}\int_0^\infty \frac{d\omega}{t}H\left(\frac{\omega\lambda(t)}{t}\right)\sin(\omega)\widehat{v_{2,0}}(\frac{\omega}{t})$$

$\square$

Now, we can choose the free wave, $v_{2,2}$ from (4.50), so that the leading part of $v_{2,2}(t,r)$ in the region $r \sim g(t)\lambda(t)$ is exactly equal to $u_{e,3} - u_{w,3}$. As in (4.37), the leading part of $v_{2,2}(t,r)$ in the matching region is given by

$$\frac{-r}{4}\left(-2\int_0^\infty \xi \sin(t\xi)\widehat{v_{2,3}}(\xi)d\xi\right) := \frac{-r}{4}G_3(t) \tag{4.133}$$

and we choose the initial velocity, $v_{2,3}$ by the following, which is the analog of (4.41).

$$\widehat{v_{2,3}}(\xi) = \frac{-1}{\pi\xi}\int_0^\infty (-4F_3(t))\psi_2(t)\sin(t\xi)dt, \quad \psi_2(x) = \begin{cases} 0, & x \leqslant T_2 \\ 1, & x \geqslant 2T_2 \end{cases}, \quad 0 \leqslant \psi_2(x) \leqslant 1 \tag{4.134}$$

(We recall that T_2 is defined in (4.1)). Note that the sine transform inversion formula implies that

$$G_3(t) = -4F_3(t)\psi_2(t), \quad t > 0 \tag{4.135}$$

Up to this point, all of our computations and estimates were valid for all $t \geqslant T_0$ for any $T_0 \geqslant T_2$. In particular, they are valid for all $t \geqslant T_2$. At this stage, we restrict T_0 so that $T_0 \geqslant 2T_2$ but is otherwise arbitrary. Thus, we have $G_3(t) = -4F_3(t)$ for $t \geqslant T_0$. We start with some estimates on $\widehat{v_{2,3}}(\xi)$:

Lemma 25 *For* $0 \leqslant k \leqslant 3$,

$$|\xi^k \partial_\xi^k \widehat{v_{2,3}}(\xi)| \leqslant \begin{cases} C, & \xi < \frac{1}{100} \\ \frac{C\xi^k}{\xi^4}, & \xi \geqslant \frac{1}{100} \end{cases}$$

Proof We use the same procedure as in Lemma 4. For instance, if $\xi < \frac{1}{100}$, then,

$$\widehat{v_{2,3}}(\xi) = \frac{-1}{\pi\xi}\int_0^{\frac{1}{\xi}} (-4F_3(t))\psi_2(t)\sin(t\xi)dt - \frac{1}{\pi\xi}\int_{\frac{1}{\xi}}^\infty (-4F_3(t))\psi_2(t)\sin(t\xi)dt$$

So,

$$|\widehat{v_{2,3}}(\xi)|$$

$$\leqslant C\int_0^{\frac{1}{\xi}} \frac{\lambda(t)^2 \sup_{x\in[100,t]}(\lambda(x)\log(x))\log^3(t)\mathbb{1}_{\{t\geqslant T_2\}}dt}{t^3} + \frac{C}{\xi}\int_{\frac{1}{\xi}}^\infty \frac{\lambda(t)^2 \sup_{x\in[100,t]}(\lambda(x)\log(x))\log^3(t)\mathbb{1}_{\{t\geqslant T_2\}}dt}{t^4}$$

$$\leqslant C$$

where we used (4.3). Also, the estimates established in this lemma are not rapidly decaying because we only estimated up to 3 derivatives of F_3 in Lemma 24. This was due to the fact that the seventh term of (4.131) was estimated using Lemma 19. $\square$

Now, we can estimate $v_{2,2}$

Lemma 26 *We have the following estimates. For* $0 \leqslant j \leqslant 2$, $0 \leqslant k \leqslant 1$,

$$|\partial_t^j \partial_r^k v_{2,2}(t,r)| \leqslant \begin{cases} Cr^{1-k}\frac{\lambda(t)^2}{t^{4+j}}\sup_{x\in[100,t]}(\lambda(x)\log(x))\log^3(t), & r \leqslant \frac{t}{2} \\ \frac{C}{\langle t-r\rangle^{5/2+j+k}\sqrt{t}}\sup_{x\in[100,t]}(\lambda(x)\log(x))\sup_{x\in[100,t]}(\lambda(x)^2)\log^3(t), & t > r > \frac{t}{2} \\ \frac{C}{\sqrt{r}\langle t-r\rangle^{\frac{1}{2}+j+k}}, & r > t, \quad j+k \leqslant 2 \end{cases}$$

$$|\partial_r^2 v_{2,2}(t,r)| \leqslant \begin{cases} \frac{Cr\lambda(t)^2}{t^6}\sup_{x\in[100,t]}(\lambda(x)\log(x))\log^3(t), & r \leqslant \frac{t}{2} \\ \frac{C}{\langle t-r\rangle^{9/2}\sqrt{t}}\sup_{x\in[100,t]}(\lambda(x)\log(x))\sup_{x\in[100,t]}(\lambda(x)^2)\log^3(t), & t > r > \frac{t}{2} \\ \frac{C}{\sqrt{r}\langle t-r\rangle^{\frac{5}{2}}}, & r > t \end{cases}$$

For all $0 \leqslant j+k \leqslant 3$,

$$|\partial_t^j \partial_r^k v_{2,2}(t,r)| \leqslant \frac{C}{\sqrt{r}}, \quad r \geqslant \frac{t}{2}$$

Finally, for $0 \leqslant j,k \leqslant 1$ *or* $k = 2, j = 0$,

$$|\partial_t^j \partial_r^k (v_{2,2}(t,r) - rF_3(t))| \leqslant \frac{Cr^{3-k}\lambda(t)^2 \sup_{x\in[100,t]}(\lambda(x)\log(x))\log^3(t)}{t^{6+j}}, \quad r \leqslant \frac{t}{2} \tag{4.136}$$

Proof In the region $r \leqslant \frac{t}{2}$, we use (4.3), Lemma 24, and the same procedure used to estimate v_2. Next, we treat the region $t > r > \frac{t}{2}$. Here, we use

$$v_{2,2}(t,r) = \frac{-r}{4\pi} \int_0^\pi \sin^2(\theta) \left(G_3(t + r\cos(\theta)) + G_3(t - r\cos(\theta)) \right) d\theta \tag{4.137}$$

Note that $t \pm r\cos(\theta) > 0$ since $t > r$, so $G_3(t \pm r\cos(\theta)) = -4 \left(F_3 \cdot \psi_2 \right)(t \pm r\cos(\theta))$, by (4.135). It suffices to treat the following integral

$$|\frac{-r}{4\pi} \int_0^\pi \sin^2(\theta) G_3(t + r\cos(\theta)) d\theta| \leqslant Cr \sup_{x \in [100, t]} \left(\lambda(x) \log(x) \right) \sup_{x \in [100, t]} \left(\lambda(x)^2 \right) \log^3(t) \int_0^\pi \frac{\sin^2(\theta) d\theta}{(t + r\cos(\theta))^4}$$

where we use the fact that $\psi_2(x) = 0, \quad x < 100$. We then use Cauchy's residue theorem, recalling that $t > r$, to get

$$\int_0^\pi \frac{\sin^2(\theta) d\theta}{(t + r\cos(\theta))^4} = \frac{1}{2} \int_0^{2\pi} \frac{\sin^2(\theta) d\theta}{(t + r\cos(\theta))^4} = \frac{\pi t}{2 \left(t^2 - r^2 \right)^{5/2}} \tag{4.138}$$

and this gives

$$|v_{2,2}(t,r)| \leqslant \frac{C}{(t-r)^{5/2}\sqrt{t}} \sup_{x \in [100, t]} \left(\lambda(x) \log(x) \right) \sup_{x \in [100, t]} \left(\lambda(x)^2 \right) \log^3(t), \quad t > r > \frac{t}{2}$$

The higher derivatives are treated similarly. Note that $G_3(t) = -2 \int_0^\infty \xi \sin(t\xi) \widehat{v_{2,3}}(\xi) d\xi$ is an odd function of t, and

$$G_3(t) = -4F_3(t)\psi_2(t), \quad t > 0$$

Also, if $r > t$, then, the argument of G_3 is negative in a region of the θ integral in (4.137). So, the procedure of estimating $v_{2,2}(t,r)$ in the region $r \geqslant t$ using (4.137) is more involved than in the region $r < t$. Instead, we simply use

$$v_{2,2}(t,r) = \int_0^\infty \widehat{v_{2,3}}(\xi) J_1(r\xi) \sin(t\xi) d\xi$$

and proceed exactly as in the proof of Lemma 5. We also do the same procedure as in (4.46) to finish the proof of all of the estimates in the lemma statement except for (4.136). The estimate (4.136) follows from

$$v_{2,2}(t,r) - rF_3(t) = \frac{-r}{4\pi} \int_0^\pi \sin^2(\theta) \left(G_3(t_+) - G_3(t) - r\cos(\theta) G_3'(t) + G_3(t_-) - G_3(t) + r\cos(\theta) G_3'(t) \right) d\theta$$

where

$$t_\pm = t \pm r\cos(\theta)$$

$\square$

Finally, we are ready to obtain the main result of this section, Proposition 2. We start with the decomposition

$$u_{ell,2}(t,r) - v_{ell,2,0,main}(t, \frac{r}{\lambda(t)}) - \left(v_{ex}(t,r) - v_{ex,cont}(t,r) + u_{w,2}(t,r) - u_{w,2,ell,0,cont}(t,r) + v_{2,2}(t,r) \right)$$

$$= u_{ell,2}(t,r) - v_{ell,2,0,main}(t, \frac{r}{\lambda(t)}) - u_{e,3}(t,r)$$

$$- \left(v_{ex,sub,0} - v_{ex,sub,0,cont} + q_5 - q_{5,0} + v_{ex,ell} - v_{ex,cont} + u_{w,2,sub,0} - u_{w,2,sub,0,cont} \right)$$

$$- \left(q_{4,2} + (ellsoln - rQ(t))\psi_{\leqslant 1}(r - t) + u_{w,2,ell} - u_{w,2,ell,0} - (u_{w,2,ell}(t,r) - u_{w,2,ell,0}(t,r))_{princ} \right)$$

$$- \left(q_{4,1} - q_{4,1,0} + u_{w,2,ell,0} - u_{w,2,ell,0,cont} + v_{2,2} - rF_3(t) + ellsoln_2(t,r)\psi_{\leqslant 1}(r - t) \right)$$

and use Lemmas 10, 12, 13, 15 - 17, 19 - 23, 26. $\square$

4.8 Joining the small r and large r solutions

Define
$$u_e(t,r) = u_{ell}(t,r) + u_{ell,2}(t,r), \quad u_{wave}(t,r) = u_w(t,r) + u_{w,2}(t,r) + v_{2,2}(t,r) \tag{4.139}$$
where we recall that u_w is defined in (4.20). The point of the matching done in the previous sections is that we can transition between $u_e(t,r)$, which is accurate for small r and $u_{wave}(t,r)$, which is accurate for large r with an expression of the form
$$\chi_{\leq 1}\left(\frac{r}{g(t)\lambda(t)}\right) u_e(t,r) + \left(1 - \chi_{\leq 1}\left(\frac{r}{g(t)\lambda(t)}\right)\right) u_{wave}(t,r)$$

for an appropriate choice of $\chi_{\leq 1}$ (which will be defined later), and not incur large error terms when derivatives act on $\chi_{\leq 1}$, upon substitution of the above expression into the left-hand side of (4.5). As mentioned previously, the basic idea of this procedure is inspired by matched asymptotic expansions. (The books [3], [23] have more information about matched asymptotic expansions for ODEs). In addition, the idea is motivated by the fact that the correction denoted by v_1, defined in (4.12), pg. 11 of [25], has a leading order cancellation with v_2, defined in (4.63), pg. 18 of [25] near the origin, and simultaneously a leading order cancellation with $Q_1(\frac{r}{\lambda(t)}) - \pi$ for large r, reminiscent of procedures used to match asymptotic expansions in various regions.

We now define a cutoff $\chi_{\geq 1} \in C^\infty([0,\infty))$, with the following properties.

Lemma 27 *[Properties of $\chi_{\geq 1}$] There exists a function $\chi_{\geq 1} \in C^\infty([0,\infty))$ satisfying*
$$\chi_{\geq 1}(x) = \begin{cases} 0, & x \leq 1 \\ 1, & x \geq 2 \end{cases}, \quad \int_0^\infty \frac{\chi_{\geq 1}(x)}{x^3}dx = \int_0^\infty x^3(1-\chi_{\geq 1}(x))\,dx = \int_0^\infty x^3\log(x)(1-\chi_{\geq 1}(x))\,dx = 0 \tag{4.140}$$
In particular, this implies that, for all $k \geq 0$, there exists $C_k > 0$ such that
$$|\widehat{\frac{\chi_{\geq 1}(\cdot)}{(\cdot)^5}}(\eta)| \leq \begin{cases} C\eta^3\langle\log(\eta)\rangle, & \eta < 1 \\ \frac{C_k}{\eta^k}, & \eta \geq 1 \end{cases}$$

Proof A direct computation shows that
$$\chi_{\geq 1}(x) = \left(\int_{-\infty}^\infty \alpha(y)dy\right)^{-1}\int_{-\infty}^x \alpha(s)ds$$
where
$$\alpha(s) = \frac{1}{s^4}\frac{d}{ds}\left(s\frac{d}{ds}\left(s^7\phi'(s)\right)\right), \quad \phi(x) = \begin{cases} e^{-\frac{1}{1-(2x-3)^2}}, & |x-\frac{3}{2}| < \frac{1}{2} \\ 0, & |x-\frac{3}{2}| \geq \frac{1}{2} \end{cases}$$
satisfies (4.140), given that
$$\alpha \in C_c^\infty([1,2]), \quad \int_0^\infty \alpha(s)\frac{ds}{s^2} = \int_0^\infty \alpha(s)s^4 ds = \int_0^\infty \alpha(s)s^4\log(s)ds = 0, \quad \int_{-\infty}^\infty \alpha(s)ds \neq 0$$
To verify the stated estimates on $\widehat{\frac{\chi_{\geq 1}(\cdot)}{(\cdot)^5}}(\eta)$, we start with the case of $\eta > 1$. We have
$$\widehat{\frac{\chi_{\geq 1}(\cdot)}{(\cdot)^5}}(\eta) = \int_0^\infty \frac{\chi_{\geq 1}(x)}{x^5}J_1(\eta x)x dx = \int_0^\infty \frac{\chi_{\geq 1}(x)}{x^5}\frac{H_1(J_1(\eta x))}{\eta^2}x dx$$
where
$$H_1(f)(x) = -f''(x) - \frac{1}{x}f'(x) + \frac{f(x)}{x^2}$$
Then,
$$|\widehat{\frac{\chi_{\geq 1}(\cdot)}{(\cdot)^5}}(\eta)| \leq \frac{C_k}{\eta^k}, \quad \eta > 1$$
follows from repeated integration by parts, noting that there are no non-zero boundary terms obtained in the process. For $\eta < 1$, we use the integral condition on $\chi_{\geq 1}$. In particular, we have
$$\widehat{\frac{\chi_{\geq 1}(\cdot)}{(\cdot)^5}}(\eta) = \frac{\eta}{2}\int_0^{\frac{1}{\eta}}\frac{\chi_{\geq 1}(x)}{x^3}dx + O\left(\eta^3\int_0^{\frac{1}{\eta}}\frac{|\chi_{\geq 1}(x)|}{x}dx\right) + O\left(\int_{\frac{1}{\eta}}^\infty \frac{|\chi_{\geq 1}(x)|}{x^4}dx\right)$$
which implies the estimate in the lemma statement. □

We recall that u_e and u_{wave} are defined in (4.139), and define $\chi_{\leqslant 1}(x) = 1 - \chi_{\geqslant 1}(x)$, and

$$u_c(t,r) = \chi_{\leqslant 1}(\frac{r}{\lambda(t)g(t)})u_e(t,r) + \left(1 - \chi_{\leqslant 1}(\frac{r}{\lambda(t)g(t)})\right)u_{wave}(t,r) \qquad (4.141)$$

Let $h(t) = \lambda(t)g(t)$. The error term of u_c in solving the linear PDE (4.5) is

$$-\left(\left(-\partial_{tt} + \partial_{rr} + \frac{1}{r}\partial_r - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}\right)u_c - \partial_t^2 Q_1(\frac{r}{\lambda(t)})\right)$$

$$= \chi_{\leqslant 1}(\frac{r}{h(t)})\partial_{tt}u_{ell,2}(t,r) + \left(1 - \chi_{\leqslant 1}(\frac{r}{h(t)})\right)\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2}\right)(v_{ex}(t,r) + u_{w,2}(t,r) + v_{2,2}(t,r))$$

$$- (u_e - u_{wave})\left(-\chi_{\leqslant 1}''(\frac{r}{h(t)})\frac{r^2 h'(t)^2}{h(t)^4} - \chi_{\leqslant 1}'(\frac{r}{h(t)})\left(-\frac{rh''(t)}{(h(t))^2} + \frac{2rh'(t)^2}{h(t)^3}\right) + \frac{1}{r}\frac{\chi_{\leqslant 1}'(\frac{r}{h(t)})}{h(t)} + \frac{\chi_{\leqslant 1}''(\frac{r}{h(t)})}{h(t)^2}\right)$$

$$- 2\chi_{\leqslant 1}'(\frac{r}{h(t)})\frac{rh'(t)}{h(t)^2}\partial_t(u_e - u_{wave}) - \frac{2}{h(t)}\chi_{\leqslant 1}'(\frac{r}{h(t)})\partial_r(u_e - u_{wave})$$

$$(4.142)$$

Some of the error terms in (4.142) are already perturbative, while some will need additional corrections. We first record estimates on those terms which are already perturbative. We recall the definitions of $v_{ex,ell}$ in (4.36), and $u_{w,2}$ in (4.56), and start with the following lemma.

Lemma 28 *If*

$$e_{ex,ell}(t,r) = \left(1 - \chi_{\leqslant 1}(\frac{r}{\lambda(t)g(t)})\right)\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2}\right)v_{ex,ell}(t,r)$$

and

$$e_{w,2}(t,r) = \left(1 - \chi_{\leqslant 1}(\frac{r}{\lambda(t)g(t)})\right)\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2}\right)u_{w,2}(t,r)$$

then, for $k = 0, 1$,

$$\|L_{\frac{1}{\lambda(t)}}^k(e_{ex,ell})(t,r)\|_{L^2(rdr)} \leqslant \frac{C\lambda(t)^{1-k}\log(t)}{t^2 g(t)^{4+k}} \qquad (4.143)$$

$$\|L_{\frac{1}{\lambda(t)}}^k(e_{w,2})(t,r)\|_{L^2(rdr)} \leqslant \frac{C\lambda(t)^{2-k}}{t^4}\frac{(1+\lambda(t))\log^5(t)\sup_{x\in[100,t]}(\lambda(x)\log(x))}{g(t)^2} + \frac{C\lambda(t)^{1-k}\log^2(t)}{g(t)^{4+k}t^2} \quad (4.144)$$

Proof The estimate in (4.143) follows from straightforward estimation of the expression (4.36) for $v_{ex,ell}$. To establish (4.144), we write $u_{w,2} = u_{w,2} - u_{w,2,ell} + u_{w,2,ell} - u_{w,2,ell,0} + u_{w,2,ell,0}$, and use Lemma 10 as well as (4.61). □

We recall the definitions of $v_{ell,sub}$ from (4.78) and $v_{ell,sub,cont}$ from (4.79), and define

$$v_{ell,sub,1}(t,R) = v_{ell,sub}(t,R) - v_{ell,sub,cont}(t,R\lambda(t)) \qquad (4.145)$$

We will now describe the difference $u_e - u_{wave}$. We have

$$u_e(t,r) = u_{ell}(t,r) + u_{ell,2}(t,r), \quad u_{wave}(t,r) = w_1(t,r) + v_{ex}(t,r) + v_2(t,r) + u_{w,2}(t,r) + v_{2,2}(t,r)$$

We recall the first order matching (4.82), which says that

$$u_{ell}(t,r) - v_{ell,sub}(t,\frac{r}{\lambda(t)}) = v_2(t,r) - v_{2,sub}(t,r) + w_1(t,r) - w_{1,sub}(t,r)$$

From the second order matching, we have

$$v_{ell,2,0,main}(t,\frac{r}{\lambda(t)}) = w_{1,cubic,main}(t,r) + v_{2,cubic,main}(t,r)$$

and

$$v_{ell,sub,cont}(t,r) = v_{ex,cont}(t,r) + u_{w,2,ell,0,cont}(t,r)$$

From the third order matching, we have the estimates from Proposition 2. These give

$$
\begin{aligned}
&u_e(t,r) - u_{wave}(t,r)\\
&= u_{ell,2}(t,r) - v_{ell,2,0,main}(t,\tfrac{r}{\lambda(t)}) - \left(u_{w,2}(t,r) - u_{w,2,ell,0,cont}(t,r) + v_{2,2}(t,r) + v_{ex} - v_{ex,cont} - q_{4,2}\right)\\
&\quad - q_{4,2}(t,r) + v_{ell,sub,1}(t,\tfrac{r}{\lambda(t)})\\
&\quad - \left(w_{1,sub}(t,r) - w_{1,cubic,main}(t,r) - \frac{r^5\lambda^{(6)}(t)}{576}(5 + \log(8))\right.\\
&\quad\quad \left. -\frac{r^5}{192}\left(\int_t^{2t}\frac{\left(\lambda^{(6)}(s) - \lambda^{(6)}(t)\right)ds}{s-t} + \lambda^{(6)}(t)\log(\tfrac{t}{r}) + \int_{2t}^{\infty}\frac{\lambda^{(6)}(s)ds}{s-t}\right)\right)\\
&\quad -\frac{r^5}{192}\left(\int_t^{2t}\frac{\left(\lambda^{(6)}(s) - \lambda^{(6)}(t)\right)ds}{s-t} + \lambda^{(6)}(t)\log(\tfrac{t}{r}) + \int_{2t}^{\infty}\frac{\lambda^{(6)}(s)ds}{s-t}\right)\\
&\quad -\frac{r^5\lambda^{(6)}(t)}{576}(5 + \log(8)) - \left(v_{2,sub}(t,r) - v_{2,cubic,main}(t,r) + \frac{r^5}{768}F^{(4)}(t)\right) + \frac{r^5}{768}F^{(4)}(t)
\end{aligned}
$$

where any function appearing without arguments is evaluated at the point (t,r). We also recall that F is defined in (4.38) (see also (4.41)). Using

$$
\begin{aligned}
F^{(4)}(t) = 4\left(\left(\log(2) - \frac{1}{2}\right)\lambda^{(6)}(t) + \int_t^{2t}\frac{\left(\lambda^{(6)}(s) - \lambda^{(6)}(t)\right)}{s-t}ds + \lambda^{(6)}(t)\log(t)\right.\\
\left. -\partial_t^4\left(\lambda''(t)\log(\lambda(t))\right) - \partial_t^4\left(\frac{\lambda'(t)^2}{2\lambda(t)}\right) + \int_{2t}^{\infty}\frac{\lambda^{(6)}(s)}{s-t}ds\right)
\end{aligned}
$$

we get

$$
\begin{aligned}
&u_e(t,r) - u_{wave}(t,r)\\
&= u_{ell,2}(t,r) - v_{ell,2,0,main}(t,\tfrac{r}{\lambda(t)}) - \left(u_{w,2}(t,r) - u_{w,2,ell,0,cont}(t,r) + v_{2,2}(t,r) + v_{ex} - v_{ex,cont} - q_{4,2}\right)\\
&\quad - q_{4,2}(t,r) + v_{ell,sub,1}(t,\tfrac{r}{\lambda(t)})\\
&\quad - \left(w_{1,sub}(t,r) - w_{1,cubic,main}(t,r) - \frac{r^5\lambda^{(6)}(t)}{576}(5 + \log(8))\right.\\
&\quad\quad \left. -\frac{r^5}{192}\left(\int_t^{2t}\frac{\left(\lambda^{(6)}(s) - \lambda^{(6)}(t)\right)ds}{s-t} + \lambda^{(6)}(t)\log(\tfrac{t}{r}) + \int_{2t}^{\infty}\frac{\lambda^{(6)}(s)ds}{s-t}\right)\right)\\
&\quad - \left(v_{2,sub}(t,r) - v_{2,cubic,main}(t,r) + \frac{r^5}{768}F^{(4)}(t)\right)\\
&\quad -\frac{r^5}{1152}\left(6\left(\partial_t^4\left(\lambda''(t)\log(\lambda(t))\right) + \partial_t^4\left(\frac{\lambda'(t)^2}{2\lambda(t)}\right)\right) + (13 - 6\log(r))\lambda^{(6)}(t)\right)
\end{aligned}
$$

All of the terms (and sufficiently many of their derivatives) appearing in this expression have been estimated already, except for the last term (which will be eliminated with another correction, utilizing the properties of $\chi_{\geqslant 1}$) and $v_{ell,sub,1}(t,\tfrac{r}{\lambda(t)})$. We now record estimates on $v_{ell,sub,1}(t,\tfrac{r}{\lambda(t)})$, and another estimate on $v_{ex,sub}$ which will be useful later on.

Lemma 29 *For $j = 0,1$ and $k = 0,1,2$, the following estimate is true.*

$$
|\partial_t^j\partial_r^k\left(v_{ell,sub,1}(t,\tfrac{r}{\lambda(t)})\right)| \leqslant C\frac{\lambda(t)^{2-k}\log^2(t)}{t^{2+j}g(t)^{3+k}}, \quad g(t)\lambda(t) \leqslant r \leqslant 2g(t)\lambda(t)
$$

In addition, we have

$$
|v_{ex}(t,r)| \leqslant \frac{C\log^2(t+r)\lambda(t)^3}{\sqrt{r}t^{5/2}}, \quad r \geqslant \frac{t}{2} \tag{4.146}
$$

$$
|\partial_t^k v_{ex,sub}(t,r)| \leqslant \frac{Cr\lambda(t)^3\log^3(t+r)}{t^{4+k}}, \quad g(t)\lambda(t) \leqslant r, \quad k = 0,1,2 \tag{4.147}
$$

$$|\partial_r \partial_t^k v_{ex,sub}(t,r)| \leqslant \frac{C\lambda(t)^3 \log^3(t+r)}{t^{4+k}}, \quad g(t)\lambda(t) \leqslant r, \quad k = 0,1$$

$$|\partial_r^2 v_{ex,sub}(t,r)| \leqslant \frac{C\lambda(t)^3 \log^3(t)}{rt^4}, \quad g(t)\lambda(t) \leqslant r \leqslant t$$

Finally, we also have the following (non-sharp, but sufficient) estimates

$$|\partial_r v_{ex}(t,r)| \leqslant \frac{C\lambda(t)^{\frac{5}{2}}}{r^{\frac{3}{2}} t^2}, \quad |\partial_t v_{ex}(t,r)| \leqslant \frac{C\lambda(t)^{3/2}}{tr^{3/2}}, \quad r \geqslant g(t)\lambda(t) \tag{4.148}$$

Proof We directly estimate $v_{ell,sub,1}$ using its definition, (4.145), and the explicit formulae for $v_{ell,sub}$ and $v_{ell,sub,cont}$ from (4.78) and (4.79), respectively. To obtain (4.147), we start with the definition (4.83), use Fubini's theorem to switch the order of s and ξ integrals, and (for $r \geqslant g(t)\lambda(t)$) divide the s integral into three regions.

$$\mathbf{a} : t \leqslant s \leqslant t + \lambda(t), \quad \mathbf{b} : t + \lambda(t) \leqslant s \leqslant t + r, \quad \mathbf{c} : t + r \leqslant s < \infty \tag{4.149}$$

In each above region of s integration, we then divide the ξ region of integration into four subintervals, based on the scales $\frac{1}{r}, \frac{1}{\lambda(t)}, \frac{1}{s-t}$. For example, in region $\mathbf{a}$ of the s integration, we have the four regions of the ξ variable

$$0 < \xi \leqslant \frac{1}{r}, \quad \frac{1}{r} < \xi \leqslant \frac{1}{\lambda(t)}, \quad \frac{1}{\lambda(t)} < \xi \leqslant \frac{1}{s-t}, \quad \frac{1}{s-t} < \xi < \infty$$

In the regions where $\xi \leqslant \frac{1}{s-t}$, we use

$$|\sin((t-s)\xi)| \leqslant C(s-t)\xi$$

On the other hand, when $\xi > \frac{1}{s-t}$, we integrate by parts in the ξ variable. To be clear, we show how the s integral over the region $\mathbf{c}$ is treated.

$$-\int_{t+r}^{\infty} ds \int_0^{\infty} d\xi \frac{\sin((t-s)\xi)}{\xi^2} J_1(r\xi) \partial_s^2 \widehat{RHS}(s,\xi)$$

$$= -\int_{t+r}^{\infty} ds \int_0^{\frac{1}{s-t}} d\xi \frac{\sin((t-s)\xi)}{\xi^2} J_1(r\xi) \partial_s^2 \widehat{RHS}(s,\xi) - \int_{t+r}^{\infty} ds \int_{\frac{1}{s-t}}^{\infty} d\xi \frac{\sin((t-s)\xi)}{\xi^2} J_1(r\xi) \partial_s^2 \widehat{RHS}(s,\xi)$$

The integral over the region $\xi \geqslant \frac{1}{s-t}$ is then further treated as

$$-\int_{t+r}^{\infty} ds \int_{\frac{1}{s-t}}^{\infty} d\xi \frac{\sin((t-s)\xi)}{\xi^2} J_1(r\xi) \partial_s^2 \widehat{RHS}(s,\xi)$$

$$= -\int_{t+r}^{\infty} ds J_1(\frac{r}{s-t}) \partial_1^2 \widehat{RHS}(s, \frac{1}{s-t})(s-t)^2 \frac{\cos(1)}{(t-s)} - \int_{t+r}^{\infty} ds \int_{\frac{1}{s-t}}^{\frac{1}{r}} \frac{\cos((t-s)\xi)}{(t-s)} \partial_\xi \left(\frac{J_1(r\xi) \partial_s^2 \widehat{RHS}(s,\xi)}{\xi^2} \right) d\xi$$

$$- \int_{t+r}^{\infty} ds \int_{\frac{1}{r}}^{\frac{1}{\lambda(t)}} \frac{\cos((t-s)\xi)}{(t-s)} \partial_\xi \left(\frac{J_1(r\xi) \partial_s^2 \widehat{RHS}(s,\xi)}{\xi^2} \right) d\xi - \int_{t+r}^{\infty} ds \int_{\frac{1}{\lambda(t)}}^{\infty} \frac{\cos((t-s)\xi)}{(t-s)} \partial_\xi \left(\frac{J_1(r\xi) \partial_s^2 \widehat{RHS}(s,\xi)}{\xi^2} \right) d\xi$$

Finally, we use the formula for $\widehat{RHS}(s,\xi)$, namely (4.33), to get, for $0 \leqslant k \leqslant 5$

$$\xi s^k |\partial_\xi \partial_s^k \widehat{RHS}(s,\xi)| + s^k |\partial_s^k \widehat{RHS}(s,\xi)| \leqslant \begin{cases} \frac{C\xi\lambda(s)^3 (1 + |\log(\xi\lambda(s))|)}{s^2}, & \xi\lambda(s) < \frac{1}{2} \\ \frac{C\lambda(s)^2}{s^2 \xi\lambda(s)}, & \xi\lambda(s) > \frac{1}{2} \end{cases} \tag{4.150}$$

Then, we use

$$|J_1(x)| \leqslant C \begin{cases} x, & x \leqslant 1 \\ \frac{1}{\sqrt{x}}, & x > 1 \end{cases}$$

as appropriate to estimate each integral above. We get (4.146) by using the same procedure as above, except that we don't integrate by parts when $\xi \geqslant \frac{1}{s-t}$ and $s \leqslant t + r$, combined with (4.36). For (4.147), we use

$$\partial_t^k v_{ex,sub}(t,r) = -\int_0^{\infty} d\xi J_1(r\xi) \int_t^{\infty} ds \frac{\sin((t-s)\xi)}{\xi^2} \partial_s^{2+k} \widehat{RHS}(s,\xi), \quad k = 1,2$$

and the same procedure used for $k = 0$. Finally, to estimate $\partial_r v_{ex,sub}(t,r)$, we start with

$$\partial_r v_{ex,sub}(t,r) = -\int_0^{\infty} d\xi J_1'(r\xi)\xi \int_t^{\infty} ds \frac{\sin((t-s)\xi)}{\xi^2} \partial_s^2 \widehat{RHS}(s,\xi)$$

We make the same decomposition as described in (4.149). This time, we integrate by parts in ξ (integrating $J_1'(r\xi)$) when $t \leqslant s \leqslant t + \lambda(t)$, and $\frac{1}{r} \leqslant \xi \leqslant \frac{1}{s-t}$. We also integrate by parts in s (integrating $\sin((t-s)\xi)$) when $\xi \geqslant \frac{1}{s-t}$ and $t + \lambda(t) \leqslant s$. Next, using

$$\partial_{tr}v_{ex,sub}(t,r) = -\int_0^\infty d\xi J_1'(r\xi)\xi \int_t^\infty ds \frac{\sin((t-s)\xi)}{\xi^2}\partial_s^3\widehat{RHS}(s,\xi)$$

and the same argument above, we obtain the estimates on $\partial_{tr}v_{ex,sub}(t,r)$ in the lemma statement. Finally, we use the fact that $v_{ex,sub}$ satisfies

$$-\partial_{tt}v_{ex,sub} + \partial_{rr}v_{ex,sub} + \frac{1}{r}\partial_r v_{ex,sub} - \frac{v_{ex,sub}}{r^2} = \partial_t^2 v_{ex,ell}(t,r)$$

to estimate $\partial_r^2 v_{ex,sub}(t,r)$. To obtain the estimate (4.148), we start with

$$\begin{aligned}\partial_r v_{ex,sub}(t,r) = &-\int_0^\infty d\xi m_{\leqslant 1}(r\xi)J_1'(r\xi)\int_t^\infty ds \frac{\sin((t-s)\xi)}{\xi}\partial_s^2\widehat{RHS}(s,\xi)\\ &-\int_0^\infty d\xi(1-m_{\leqslant 1}(r\xi))J_1'(r\xi)\int_t^\infty ds \frac{\sin((t-s)\xi)}{\xi}\partial_s^2\widehat{RHS}(s,\xi)\end{aligned} \qquad (4.151)$$

where $m_{\leqslant 1}$ is defined in (2.1). For the first term on the right-hand side of (4.151), we simply directly insert (4.150) into the integral, and estimate. For the second term, we integrate by parts, to get

$$\begin{aligned}&-\int_0^\infty d\xi(1-m_{\leqslant 1}(r\xi))J_1'(r\xi)\int_t^\infty ds \frac{\sin((t-s)\xi)}{\xi}\partial_s^2\widehat{RHS}(s,\xi)\\ &=\frac{1}{r}\int_t^\infty ds\int_0^\infty d\xi J_1(r\xi)\partial_\xi\left((1-m_{\leqslant 1}(r\xi))\frac{\sin((t-s)\xi)}{\xi}\partial_s^2\widehat{RHS}(s,\xi)\right)d\xi\end{aligned}$$

and then, we directly estimate using (4.150). Next, using (4.36) and $v_{ex}(t,r) = v_{ex,ell}(t,r) + v_{ex,sub}(t,r)$, we get

$$|\partial_r v_{ex}(t,r)| \leqslant \frac{C\lambda(t)^{\frac{5}{2}}}{r^{\frac{3}{2}}t^2}, \quad r \geqslant g(t)\lambda(t)$$

Finally, the dominated convergence theorem and the formula for v_{ex}, namely (4.34), give

$$\partial_t v_{ex}(t,r) = -\int_0^\infty dw \int_0^\infty d\xi J_1(r\xi)\sin(w\xi)\partial_1\widehat{RHS}(w+t,\xi)$$

Then, we use the same procedure used in (4.151) to finish the proof of (4.148). $\quad\square$

Finally, we let

$$(u_e - u_{wave})_0(t,r) = u_e - u_{wave} + \frac{r^5}{1152}\left(6\left(\partial_t^4\left(\lambda''(t)\log(\lambda(t))\right) + \partial_t^4\left(\frac{\lambda'(t)^2}{2\lambda(t)}\right)\right) + (13 - 6\log(r))\lambda^{(6)}(t)\right)$$

and we note the following lemma.

Lemma 30 *We have the following estimate, for $0 \leqslant j,k \leqslant 1$ or $k = 2, j = 0$, and $g(t)\lambda(t) \leqslant r \leqslant 2g(t)\lambda(t)$*

$$\begin{aligned}&|\partial_t^j\partial_r^k\left((u_e - u_{wave})_0 + q_{4,2}\right)|\\ &\leqslant \frac{C\lambda(t)^{2-k}\log^2(t)}{t^{2+j}g(t)^{3+k}} + \frac{Cg(t)^{2-k}\lambda(t)^{4-k}\log^4(t)\sup_{x\in[100,t]}(\lambda(x)\log(x))}{t^{5+j}} + \frac{C\lambda(t)^{8-k}g(t)^{7-k}\log(t)}{t^{8+j}}\end{aligned}$$

Proof We directly combine the results of Proposition 2, Lemma 1, Lemma 6, and Lemma 29 $\quad\square$

We now can estimate the $(u_e - u_{wave})_0$ contribution to the error terms of our ansatz (4.141) which involve at least one derivative of $\chi_{\leqslant 1}$. Let

$$\begin{aligned}&e_{match,0}(t,r)\\ &= -(u_e - u_{wave})_0\left(-\chi_{\leqslant 1}''(\frac{r}{h(t)})\frac{r^2h'(t)^2}{h(t)^4} - \chi_{\leqslant 1}'(\frac{r}{h(t)})\left(-\frac{rh''(t)}{(h(t))^2} + \frac{2rh'(t)^2}{h(t)^3}\right) + \frac{1}{r}\frac{\chi_{\leqslant 1}'(\frac{r}{h(t)})}{h(t)} + \frac{\chi_{\leqslant 1}''(\frac{r}{h(t)})}{h(t)^2}\right)\\ &\quad - 2\chi_{\leqslant 1}'(\frac{r}{h(t)})\frac{rh'(t)}{h(t)^2}\partial_t(u_e - u_{wave})_0 - \frac{2}{h(t)}\chi_{\leqslant 1}'(\frac{r}{h(t)})\partial_r(u_e - u_{wave})_0\end{aligned}$$

Then, we have the following estimates.

Lemma 31 *[Estimates on the matching-induced error terms] For $0 \leqslant k \leqslant 1$,*

$$||L^k_{\frac{1}{\lambda(t)}}(e_{match,0})(t,r)||_{L^2(rdr)} \leqslant \frac{C\lambda(t)^5 \log^2(t)}{t^2 h(t)^{4+k}} + \frac{Ch(t)^{1-k}\lambda(t)^2 \log^4(t) \sup_{x \in [100,t]}(\lambda(x)\log(x))}{t^5} + \frac{C\lambda(t)h(t)^{6-k}\log(t)}{t^8}$$

Proof We directly use Lemma 30 and the estimates on $q_{4,2}$ from Proposition 2. □

Next, we consider the error terms of u_c which involve derivatives of $\chi_{\leqslant 1}$, and which result from replacing $u_e - u_{wave}$ with $(u_e - u_{wave})_0$. In particular, let

$$e_5(t,r) = -f_5(t,r)\left(-\chi''_{\leqslant 1}(\frac{r}{h(t)})\frac{r^2 h'(t)^2}{h(t)^4} - \chi'_{\leqslant 1}(\frac{r}{h(t)})\left(-\frac{rh''(t)}{(h(t))^2} + \frac{2rh'(t)^2}{h(t)^3}\right) + \frac{1}{r}\frac{\chi'_{\leqslant 1}(\frac{r}{h(t)})}{h(t)} + \frac{\chi''_{\leqslant 1}(\frac{r}{h(t)})}{h(t)^2}\right)$$
$$- 2\chi'_{\leqslant 1}(\frac{r}{h(t)})\frac{rh'(t)}{h(t)^2}\partial_t f_5(t,r) - \frac{2}{h(t)}\chi'_{\leqslant 1}(\frac{r}{h(t)})\partial_r f_5(t,r)$$

where

$$f_5(t,r) = -\frac{r^5}{1152}\left(6\left(\partial_t^4\left(\lambda''(t)\log(\lambda(t))\right) + \partial_t^4\left(\frac{\lambda'(t)^2}{2\lambda(t)}\right)\right) + (13 - 6\log(r))\lambda^{(6)}(t)\right)$$

The following piece of e_5 turns out to be perturbative. Let

$$e_{5,1}(t,r) = -f_5(t,r)\left(-\chi''_{\leqslant 1}(\frac{r}{h(t)})\frac{r^2 h'(t)^2}{h(t)^4} - \chi'_{\leqslant 1}(\frac{r}{h(t)})\left(-\frac{rh''(t)}{(h(t))^2} + \frac{2rh'(t)^2}{h(t)^3}\right)\right) - 2\chi'_{\leqslant 1}(\frac{r}{h(t)})\frac{rh'(t)}{h(t)^2}\partial_t f_5(t,r)$$

Lemma 32 *We have the following estimates for $k = 0, 1$.*

$$||L^k_{\frac{1}{\lambda(t)}}(e_{5,1})(t,r)||_{L^2(rdr)} \leqslant \frac{Ch(t)^{6-k}\lambda(t)\log(t)}{t^8}$$

Proof This follows from a straightforward and direct computation □

Next, we need to consider $e_{5,0}$, which is given by

$$e_{5,0}(t,r) := e_5(t,r) - e_{5,1}(t,r) = -f_5(t,r)\left(\frac{1}{r}\frac{\chi'_{\leqslant 1}(\frac{r}{h(t)})}{h(t)} + \frac{\chi''_{\leqslant 1}(\frac{r}{h(t)})}{h(t)^2}\right) - \frac{2}{h(t)}\chi'_{\leqslant 1}(\frac{r}{h(t)})\partial_r f_5(t,r) \quad (4.152)$$

The point is that, although $e_{5,0}(t,r)$ does not decay fast enough (in L^2, for example) to be perturbative, it is orthogonal to $\phi_0(\frac{r}{\lambda(t)})$ to leading order. This is because $e_{5,0}(t,r)$ is supported in the region $\lambda(t) \ll h(t) \leqslant r \leqslant 2h(t)$. So, the leading order behavior of $(2\lambda(t))^{-1}\langle\phi_0(\frac{\cdot}{\lambda(t)}), e_{5,0}(t,\cdot)\rangle_{L^2(rdr)}$ is

$$\int_0^\infty e_{5,0}(t,r)dr = \int_0^\infty \chi_{\leqslant 1}(x)\left((24p_0(t) + 10p_1(t))h(t)^3 x^3 + 24p_1(t)h(t)^3 x^3(\log(h(t)) + \log(x))\right)h(t)dx = 0$$
$$(4.153)$$

where we integrated by parts, defined $p_j(t)$ by

$$f_5(t,r) = (p_0(t) + p_1(t)\log(r))r^5$$

and used Lemma 27. Therefore, we will add a term to u_c which will be an appropriate truncation of a solution to the ODE

$$\partial_{RR}w(t,R) + \frac{1}{R}\partial_R w(t,R) - \frac{\cos(2Q_1(R))}{R^2}w(t,R) = F(t,R) \quad (4.154)$$

for an appropriate choice of F. (We leave F general here, since we will use a correction of this form to eliminate error terms of u_c other than just $e_{5,0}$). For the class of F which we will need to consider, we will use the following particular solution to (4.154) (recall the notation (2.3))

$$w(t,R) = -\frac{R}{1+R^2}\int_0^R F(t,s)se_2(s)ds + e_2(R)\int_0^R \frac{F(t,s)s^2}{1+s^2}ds \quad (4.155)$$

Returning to the correction associated to $e_{5,0}$, we establish the following lemma

Lemma 33 *Let $w_{5,0}$ denote the function defined in (4.155) where*

$$F(t, R) = \lambda(t)^2 e_{5,0}(t, R\lambda(t))$$

Then, we have the following estimate, for $0 \leqslant j + k \leqslant 2$, $j = 2, k = 1$, $j = 0, k = 3$, and $j = 1, k = 2$.

$$|R^k t^j \partial_t^j \partial_R^k w_{5,0}(t, R)| \leqslant \begin{cases} 0, & R \leqslant \frac{h(t)}{\lambda(t)} \\ C \frac{\lambda(t) h(t)^5 \log(t)}{t^6}, & \frac{h(t)}{\lambda(t)} \leqslant R \leqslant \frac{2h(t)}{\lambda(t)} \\ \frac{C}{R} \frac{h(t)^6 \log(t)}{t^6} + \frac{CRh(t)^2 \lambda(t)^4 \log(t)}{t^6}, & R \geqslant \frac{2h(t)}{\lambda(t)} \end{cases}$$

Proof By a straightforward insertion of the definition of $e_{5,0}$, (4.152), into the following integral, we have

$$\left| \int_0^R \lambda(t)^2 e_{5,0}(t, s\lambda(t)) s e_2(s) ds \right| \leqslant \begin{cases} 0, & R \leqslant \frac{h(t)}{\lambda(t)} \\ \frac{Ch(t)^6 \log(t)}{t^6}, & R \geqslant \frac{h(t)}{\lambda(t)} \end{cases}$$

On the other hand, we use (4.153) to get

$$\lambda(t)^2 \int_0^R e_{5,0}(t, s\lambda(t)) \frac{s^2 ds}{1 + s^2} = -\lambda(t)^2 \int_R^\infty e_{5,0}(t, s\lambda(t)) ds + \lambda(t)^2 \int_0^R e_{5,0}(t, s\lambda(t)) \left(\frac{-1}{1 + s^2} \right) ds$$

(Note that the support properties of $\chi_{\leqslant 1}$ imply that the left-hand side of the above equation vanishes when $R \leqslant \frac{h(t)}{\lambda(t)}$). Then, we get

$$\left| \lambda(t)^2 \int_0^R e_{5,0}(t, s\lambda(t)) \frac{s^2 ds}{1 + s^2} \right| \leqslant \begin{cases} 0, & R \leqslant \frac{h(t)}{\lambda(t)} \\ \frac{Ch(t)^4 \lambda(t)^2 \log(t)}{t^6}, & \frac{h(t)}{\lambda(t)} \leqslant R \leqslant \frac{2h(t)}{\lambda(t)} \\ \frac{Ch(t)^2 \lambda(t)^4 \log(t)}{t^6}, & R \geqslant \frac{2h(t)}{\lambda(t)} \end{cases}$$

This gives the estimate of the lemma statement for $j = k = 0$. Next, we have

$$\partial_R w_{5,0}(t, R) = -\partial_R \left(\frac{R}{1 + R^2} \right) \int_0^R \lambda(t)^2 e_{5,0}(t, s\lambda(t)) s e_2(s) ds + e_2'(R) \int_0^R \frac{\lambda(t)^2 e_{5,0}(t, s\lambda(t)) s^2}{1 + s^2} ds$$

So, $\partial_R w_{5,0}(t, R)$ is the same expression as $w_{5,0}(t, R)$, except with an extra derivative on the coefficients of each integral term. Our proof of the lemma for $j = 0, k = 0$ therefore immediately implies the lemma statement is true for $j = 0, k = 1$. We prove the $j = 0, k = 2$ case of the lemma statement by noting that

$$\partial_R^2 w_{5,0}(t, R) = \lambda(t)^2 e_{5,0}(t, R\lambda(t)) - \frac{1}{R} \partial_R w_{5,0}(t, R) + \frac{\cos(2Q_1(R))}{R^2} w_{5,0}(t, R)$$

Finally, the symbol-type estimates on $\lambda(t)$, definition of $e_{5,0}$, and the fact that

$$\int_0^\infty \lambda(t)^2 e_{5,0}(t, s\lambda(t)) ds = 0 \text{ for all t implies } \partial_t^j \left(\int_0^\infty \lambda(t)^2 e_{5,0}(t, s\lambda(t)) ds \right) = 0$$

finishes the proof of the lemma. $\square$

We define $f_{5,0}(t, r)$, the function to be added to our ansatz, by

$$f_{5,0}(t, r) = m_{\leqslant 1}(\frac{r}{t}) w_{5,0}(t, \frac{r}{\lambda(t)}) \tag{4.156}$$

where we recall the definition of $m_{\leqslant 1}$ in (2.1). Then, we define the error term of $f_{5,0}$ by

$$e_{f_{5,0}}(t, r) := - \left(-\partial_{tt} f_{5,0} + \partial_{rr} f_{5,0} + \frac{1}{r} \partial_r f_{5,0} - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2} f_{5,0} \right) + e_{5,0}(t, r)$$

$$= -m_{\leqslant 1}''(\frac{r}{t}) \left(\frac{-r^2}{t^4} + \frac{1}{t^2} \right) w_{5,0}(t, \frac{r}{\lambda(t)})$$

$$- m_{\leqslant 1}'(\frac{r}{t}) \left(\left(\frac{-2r}{t^3} + \frac{1}{rt} \right) w_{5,0}(t, \frac{r}{\lambda(t)}) + \frac{2r}{t^2} \partial_t \left(w_{5,0}(t, \frac{r}{\lambda(t)}) \right) + \frac{2}{t} \partial_r \left(w_{5,0}(t, \frac{r}{\lambda(t)}) \right) \right)$$

$$+ m_{\leqslant 1}(\frac{r}{t}) \partial_t^2 \left(w_{5,0}(t, \frac{r}{\lambda(t)}) \right) + e_{5,0}(t, r) \left(1 - m_{\leqslant 1}(\frac{r}{t}) \right)$$

Then, an insertion of our estimates from Lemma 33 into the above expression for $e_{f_{5,0}}$ gives the following lemma.

Lemma 34 *For $k = 0, 1$,*

$$||L^k_{\frac{1}{\lambda(t)}}(e_{f_{5,0}})(t, r)||_{L^2(rdr)} \leqslant \frac{C\lambda(t)}{t^2}\left(\frac{h(t)^{6-k}\log^2(t)}{t^6} + \frac{h(t)^{2-k}\lambda(t)^2\log^2(t)}{t^4}\right)$$

Now, we will consider the error term in (4.142) involving $\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)}))-1}{r^2}\right)v_{ex,sub}(t, r)$.

Lemma 35 *Let $w_{ex,sub}(t, R)$ be defined by the expression (4.155) for*

$$F(t, s) = \left(\frac{\cos(2Q_1(s)) - 1}{s^2}\right)v_{ex,sub}(t, s\lambda(t))\chi_{\geqslant 1}(\frac{s}{g(t)})$$

Then, we have

$$w_{ex,sub}(t, \frac{r}{\lambda(t)}) = 0, \quad r \leqslant \lambda(t)g(t)$$

For $0 \leqslant k + j \leqslant 2$, $j = 1, k = 2$, and $j = 2, k = 1$,

$$|\partial_t^k\partial_r^j\left(w_{ex,sub}(t, \frac{r}{\lambda(t)})\right)| \leqslant \frac{C\lambda(t)^5\log^3(t)}{r^{1+j}t^{4+k}} + \frac{Cr^{1-j}\lambda(t)^3\log^2(t)}{t^{4+k}g(t)^2}, \quad g(t)\lambda(t) \leqslant r \leqslant t$$

Proof We define $int_{ex,sub}$ by the integral

$$int_{ex,sub}(t, R) = \int_0^R \frac{F(t, s)s^2}{1 + s^2}ds = -\int_0^\infty d\xi \int_t^\infty dx \frac{\sin((t-x)\xi)}{\xi^2}\partial_x^2\widehat{RHS}(x, \xi)I_1(R, \xi, t)$$

where

$$I_1(R, \xi, t) = \int_0^R \left(\frac{\cos(2Q_1(s)) - 1}{s^2}\right)J_1(s\lambda(t)\xi)\frac{\chi_{\geqslant 1}(\frac{s}{g(t)})s^2}{1 + s^2}ds$$

and we used Fubini's theorem. Note that $I_1(R, \xi, t) = 0$ for $R \leqslant g(t)$. For $R \geqslant g(t)$, we decompose I_1 as

$$I_1(R, \xi, t) = I_{100}(\xi, t) + I_{101}(\xi, t) + I_{11}(R, \xi, t)$$

where

$$I_{100}(\xi, t) = \frac{-8}{g(t)^3}\frac{\widehat{\chi_{\geqslant 1}(\cdot)}}{(\cdot)^5}(\xi g(t)\lambda(t))$$

$$I_{101}(\xi, t) = \int_0^\infty \left(\frac{(\cos(2Q_1(s)) - 1)}{s^2}\frac{s^2}{1 + s^2} + \frac{8}{s^4}\right)J_1(s\lambda(t)\xi)\chi_{\geqslant 1}(\frac{s}{g(t)})ds$$

and

$$I_{11}(R, \xi, t) = -\int_R^\infty \left(\frac{\cos(2Q_1(s)) - 1}{s^2}\right)J_1(s\lambda(t)\xi)\chi_{\geqslant 1}(\frac{s}{g(t)})\frac{s^2ds}{1 + s^2}$$

We claim that, for $0 \leqslant j \leqslant 2$,

$$|\partial_t^j I_{100}(\xi, t)| \leqslant \frac{C}{g(t)^3t^j}\begin{cases}\xi^3g(t)^3\lambda(t)^3\langle\log(\xi g(t)\lambda(t))\rangle, & \xi \leqslant \frac{1}{g(t)\lambda(t)} \\ \frac{C_k}{(\xi g(t)\lambda(t))^k}, & \xi \geqslant \frac{1}{g(t)\lambda(t)}\end{cases}$$

$$|\partial_t^j I_{101}(\xi, t)| \leqslant \frac{C}{t^j}\begin{cases}\frac{\xi\lambda(t)}{g(t)^4}, & \xi\lambda(t) \leqslant \frac{1}{g(t)} \\ \frac{C_k}{g(t)^5(\xi\lambda(t)g(t))^{2k}}, & k \geqslant 5, \quad \xi\lambda(t) \geqslant \frac{1}{g(t)}\end{cases}$$

$$|\partial_t^j\left(I_{11}(\frac{r}{\lambda(t)}, \xi, t)\right)| \leqslant \frac{C}{t^j}\begin{cases}\frac{\lambda(t)^3}{\xi^{3/2}r^{9/2}}, & \xi \geqslant \frac{1}{\lambda(t)g(t)} \\ \frac{\xi\lambda(t)^3}{r^2}, & \xi \leqslant \frac{1}{r} \\ \frac{\lambda(t)^3}{r^{7/2}\sqrt{\xi}}, & \frac{1}{r} \leqslant \xi \leqslant \frac{1}{g(t)\lambda(t)}\end{cases}$$

For $j = 0$, I_{100} is estimated by using Lemma 27, while, for all $0 \leqslant j \leqslant 2$, I_{101} and I_{11} are estimated by integrating by parts when $\xi\lambda(t) \geqslant \frac{1}{g(t)}$, and directly estimating otherwise (as in Lemma 27). For $j > 0$, the most delicate estimate is on $\partial_t^j I_{100}(\xi, t)$ in the region $\xi \leqslant \frac{1}{\lambda(t)g(t)}$. We write

$$I_{100}(\xi, t) = -8\int_0^\infty \frac{J_1(g(t)\lambda(t)y\xi)}{g(t)^3y^4}\chi_{\geqslant 1}(y)dy$$

which gives

$$\partial_t I_{100}(\xi,t) = \frac{-3g'(t)}{g(t)} I_{100}(\xi,t) - 8\int_0^\infty \frac{J_1'(g(t)\lambda(t)y\xi)}{g(t)^3 y^4}(g(t)\lambda(t))'y\xi\chi_{\geqslant 1}(y)dy$$

Then,

$$-8\int_0^\infty \frac{J_1'(g(t)\lambda(t)y\xi)}{g(t)^3 y^4}(g(t)\lambda(t))'y\xi\chi_{\geqslant 1}(y)dy$$

$$= -8\int_0^{\frac{1}{g(t)\lambda(t)\xi}} \frac{1}{2}\frac{(g(t)\lambda(t))'}{g(t)^3 y^4}y\xi\chi_{\geqslant 1}(y)dy - 8\int_0^{\frac{1}{g(t)\lambda(t)\xi}} \frac{\left(J_1'(g(t)\lambda(t)y\xi) - \frac{1}{2}\right)(g(t)\lambda(t))'}{g(t)^3 y^4}y\xi\chi_{\geqslant 1}(y)dy$$

$$- 8\int_{\frac{1}{g(t)\lambda(t)\xi}}^\infty \frac{J_1'(g(t)\lambda(t)y\xi)}{g(t)^3 y^4}(g(t)\lambda(t))'y\xi\chi_{\geqslant 1}(y)dy$$

and to estimate the first term on the right hand side of the above expression, we use Lemma 27. We use a similar procedure for $\partial_t^2 I_{100}(\xi,t)$.

$$\partial_t^2\left(-int_{ex,sub}(t,\frac{r}{\lambda(t)})\right) = \partial_t^2\left(\int_0^\infty d\xi\int_t^\infty dx\frac{\sin((t-x)\xi)}{\xi^2}\partial_x^2\widehat{RHS}(x,\xi)I_1(\frac{r}{\lambda(t)},\xi,t)\right)$$

$$= \partial_t^2\left(-\int_0^\infty d\xi\int_0^\infty dw\frac{\sin(w\xi)}{\xi^2}\partial_1^2\widehat{RHS}(t+w,\xi)\left(I_{100}(\xi,t) + I_{101}(\xi,t) + I_{11}(\frac{r}{\lambda(t)},\xi,t)\right)\right)$$

After differentiating under the integral sign in the last integral, we let $x = w + t$, and split the x integration into two regions: $t \leqslant x \leqslant t + \frac{1}{\xi}$ and $x > t + \frac{1}{\xi}$. In the latter region, we integrate by parts in the x variable, integrating $\sin((t-x)\xi)$. Then, we make a similar decomposition of the ξ integral as was made while proving Lemma 29. This gives, for $k = 0,1,2$,

$$\left|\partial_t^k\left(int_{ex,sub}(t,\frac{r}{\lambda(t)})\right)\right| \leqslant \frac{C\lambda(t)^4\log^2(t)}{t^{4+k}g(t)^2}, \quad t \geqslant r \geqslant g(t)\lambda(t)$$

We recall the definition of e_2 in (2.3). The next integral to consider is

$$int_{ex,sub,2}(t,R) = \int_0^R \left(\frac{\cos(2Q_1(s)) - 1}{s^2}\right)v_{ex,sub}(t,s\lambda(t))\chi_{\geqslant 1}(\frac{s}{g(t)})se_2(s)ds$$

$$= -\int_0^\infty d\xi\int_t^\infty dx\frac{\sin((t-x)\xi)}{\xi^2}\partial_x^2\widehat{RHS}(x,\xi)I_2(R,\xi,t)$$

where

$$I_2(R,\xi,t) = \int_0^R \left(\frac{\cos(2Q_1(s)) - 1}{s^2}\right)J_1(s\lambda(t)\xi)\chi_{\geqslant 1}(\frac{s}{g(t)})se_2(s)ds$$

As with I_1 above, $I_2(R,\xi,t) = 0$ for $R \leqslant g(t)$. A direct estimation gives, for $0 \leqslant k \leqslant 2$,

$$\left|\partial_t^k\left(I_2(\frac{r}{\lambda(t)},\xi,t)\right)\right| \leqslant \frac{C}{t^k}\begin{cases}\log(\frac{r}{\lambda(t)g(t)})\lambda(t)\xi, & \xi \leqslant \frac{1}{r}\\ \xi\lambda(t)\langle\log(\xi\lambda(t)g(t))\rangle, & \frac{1}{r} \leqslant \xi \leqslant \frac{1}{\lambda(t)g(t)}\\ \frac{C}{g(t)^{3/2}\sqrt{\lambda(t)\xi}}, & \xi \geqslant \frac{1}{\lambda(t)g(t)}\end{cases}$$

Then, as above, we get, for $k = 0,1,2$,

$$\left|\partial_t^k\left(int_{ex,sub,2}(t,\frac{r}{\lambda(t)})\right)\right| \leqslant \frac{C\lambda(t)^4\log^3(t)}{t^{4+k}}, \quad g(t) \leqslant \frac{r}{\lambda(t)} \leqslant \frac{t}{\lambda(t)}$$

Recalling (4.155), we have

$$w_{ex,sub}(t,\frac{r}{\lambda(t)}) = -\frac{\phi_0(\frac{r}{\lambda(t)})}{2}int_{ex,sub,2}(t,\frac{r}{\lambda(t)}) + e_2(\frac{r}{\lambda(t)})int_{ex,sub}(t,\frac{r}{\lambda(t)})$$

which gives, for $0 \leqslant k \leqslant 2$,

$$\left|\partial_t^k\left(w_{ex,sub}(t,\frac{r}{\lambda(t)})\right)\right| \leqslant \frac{C\lambda(t)^5\log^3(t)}{rt^{4+k}} + \frac{Cr\lambda(t)^3\log^2(t)}{t^{4+k}g(t)^2}, \quad g(t)\lambda(t) \leqslant r \leqslant t$$

Next, we note that

$$\partial_r\left(w_{ex,sub}(t,\frac{r}{\lambda(t)})\right) = \frac{-\phi_0'(\frac{r}{\lambda(t)})}{2\lambda(t)}int_{ex,sub,2}(t,\frac{r}{\lambda(t)}) + e_2'(\frac{r}{\lambda(t)})\frac{1}{\lambda(t)}int_{ex,sub}(t,\frac{r}{\lambda(t)})$$

Finally, we estimate $\partial_r^2\left(w_{ex,sub}(t,\frac{r}{\lambda(t)})\right)$ using the equation solved by $w_{ex,sub}$, Lemma 29, and our previous estimates from the proof of this lemma. $\quad\square$

The truncation of $w_{ex,sub}$ which we will add to our ansatz is

$$f_{ex,sub}(t,r) = m_{\leqslant 1}(\frac{r}{t}) w_{ex,sub}(t, \frac{r}{\lambda(t)}) \tag{4.157}$$

where we recall that $m_{\leqslant 1}$ was defined in (2.1). We define the error term of $f_{ex,sub}(t,r)$ as

$$e_{ex,sub}(t,r) = -\left(-\partial_{tt} f_{ex,sub} + \partial_{rr} f_{ex,sub} + \frac{1}{r}\partial_r f_{ex,sub} - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2} f_{ex,sub}\right)$$
$$+ \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2}\right) v_{ex,sub}(t,r)\chi_{\geqslant 1}(\frac{r}{\lambda(t)g(t)})$$

We get

$$e_{ex,sub}(t,r) = -m''_{\leqslant 1}(\frac{r}{t})\left(\frac{-r^2}{t^4} + \frac{1}{t^2}\right) w_{ex,sub}(t, \frac{r}{\lambda(t)})$$
$$- m'_{\leqslant 1}(\frac{r}{t})\left(\left(\frac{-2r}{t^3} + \frac{1}{rt}\right) w_{ex,sub}(t, \frac{r}{\lambda(t)}) + \frac{2r}{t^2}\partial_t\left(w_{ex,sub}(t, \frac{r}{\lambda(t)})\right) + \frac{2}{t}\partial_r\left(w_{ex,sub}(t, \frac{r}{\lambda(t)})\right)\right)$$
$$+ m_{\leqslant 1}(\frac{r}{t})\partial_t^2\left(w_{ex,sub}(t, \frac{r}{\lambda(t)})\right)$$
$$+ \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2}\right) v_{ex,sub}(t,r)\chi_{\geqslant 1}(\frac{r}{\lambda(t)g(t)})\left(1 - m_{\leqslant 1}(\frac{r}{t})\right)$$

Lemma 35 directly gives

Lemma 36 *For $k = 0, 1$,*

$$\|L_{\frac{1}{\lambda(t)}}^k(e_{ex,sub})(t,r)\|_{L^2(rdr)} \leqslant \frac{C\lambda(t)^{3-k}\log^2(t)}{t^4 g(t)^{2+k}}$$

Lemma 37 *Let $g_{ell,2}(t,R)$ be defined by the expression (4.155) for the choice*

$$F(t,s) = \lambda(t)^2 \chi_{\leqslant 1}(\frac{s}{g(t)})\partial_1^2 u_{ell,2}(t, s\lambda(t))$$

and let

$$f_{ell,2}(t,r) = g_{ell,2}(t, \frac{r}{\lambda(t)})m_{\leqslant 1}(\frac{r}{t})$$

Then, for $0 \leqslant j \leqslant 2$, $0 \leqslant k \leqslant 3$,

$$|R^k t^j \partial_t^j \partial_R^k g_{ell,2}(t,R)| \leqslant \frac{C\lambda(t)^6}{t^6}\begin{cases} R^5\left(1 + \log^2(R)\right), & R \leqslant 1 \\ R^5\log(t), & 1 \leqslant R \leqslant 2g(t) \\ Rg(t)^2\log^3(t) + \frac{g(t)^6\log(t)}{R}, & R > 2g(t) \end{cases}$$

Also, letting $e_{ell,2}$ denote the error term of $f_{ell,2}$:

$$e_{ell,2}(t,r) = -\left(-\partial_{tt} f_{ell,2} + \partial_{rr} f_{ell,2} + \frac{1}{r}\partial_r f_{ell,2} - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2} f_{ell,2}\right) + \chi_{\leqslant 1}(\frac{r}{g(t)\lambda(t)})\partial_1^2 u_{ell,2}(t,r)$$

we have, for $k = 0, 1$,

$$\|L_{\frac{1}{\lambda(t)}}^k(e_{ell,2})(t,r)\|_{L^2(rdr)} \leqslant \frac{C\lambda(t)^{7-k}\log^3(t)g(t)^{6-k}}{t^8}$$

Proof We will take advantage of two of the orthogonality conditions in Lemma 27 when studying $g_{ell,2}$. We recall that $u_{ell,2}$ is defined in (4.9), and we use (4.13) to get

$$g_{ell,2}(t,R) = \frac{e_2(R)}{2}\lambda(t)^2 \int_0^R \chi_{\leqslant 1}(\frac{s}{g(t)})\left(\partial_t^2\left(v_{ell,2,0,main}(t, \frac{r}{\lambda(t)})\right)\Big|_{r=s\lambda(t)} + \partial_t^2\left(soln_1(t, \frac{r}{\lambda(t)})\right)\Big|_{r=s\lambda(t)}\right.$$
$$\left. + \partial_t^2\left(soln_2(t, \frac{r}{\lambda(t)})\right)\Big|_{r=s\lambda(t)}\right) s\phi_0(s)ds$$
$$- \frac{\phi_0(R)}{2}\lambda(t)^2 \int_0^R \chi_{\leqslant 1}(\frac{s}{g(t)})\partial_1^2 u_{ell,2}(t, s\lambda(t))se_2(s)ds$$

where, for the reader's convenience, we recall that

$$v_{ell,2,0,main}(t,s) = \frac{s^3 \lambda(t)^3}{8} \left(\partial_t^2 \left(\frac{\lambda'(t)^2}{2\lambda(t)} + \lambda''(t) + \lambda''(t) \log(\lambda(t)) \right) - \lambda''''(t) \log(s\lambda(t)) + \frac{3}{4} \lambda''''(t) \right)$$

We then estimate $g_{ell,2}$ by using the orthogonality conditions of Lemma 27 to treat the term in the above expression for $g_{ell,2}$ which involves $v_{ell,2,0,main}$, and estimate the rest of $g_{ell,2}$ directly. We have the following two estimates for $0 \leqslant k \leqslant 1$ and $0 \leqslant j \leqslant 2$,

$$|\partial_t^j \partial_s^k \left(\partial_t^2 \left(soln_1(t, \frac{r}{\lambda(t)}) \right) \Big|_{r=s\lambda(t)} \right)| \leqslant \frac{C\lambda(t)^4}{t^{6+j}} \begin{cases} s^{5-k}(1 + |\log(s)|), & s \leqslant 1 \\ s^{1-k}(1 + \log^2(s)), & s \geqslant 1 \end{cases} \tag{4.158}$$

$$|\partial_t^j \partial_s^k \left(\partial_t^2 \left(soln_2(t, \frac{r}{\lambda(t)}) \right) \Big|_{r=s\lambda(t)} \right)| \leqslant \frac{C\lambda(t)^4}{t^{6+j}} \begin{cases} s^{3-k}(1 + \log^2(s)), & s \leqslant 1 \\ s^{1-k}(1 + \log^3(s)), & s \geqslant 1 \end{cases} \tag{4.159}$$

This results in the estimates on $g_{ell,2}$ and $e_{ell,2}$ in the lemma statement. □

Finally, we treat the linear error term associated to $v_{2,2}$ which we recall is a free wave added to $u_{w,2}$, and is chosen so as to allow for the third order matching. This error term is

$$e_{2,2}(t,r) = \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) v_{2,2}(t,r) \left(1 - \chi_{\leqslant 1}(\frac{r}{h(t)}) \right)$$

We define $u_{2,2}$ to be the solution to the following equation, with 0 Cauchy data at infinity.

$$-\partial_t^2 u_{2,2} + \partial_r^2 u_{2,2} + \frac{1}{r} \partial_r u_{2,2} - \frac{u_{2,2}}{r^2} = e_{2,2}(t,r)$$

Then, we have the following lemma.

Lemma 38 *For* $\frac{h(t)}{2} \leqslant r \leqslant \frac{t}{2}$, *and* $0 \leqslant k \leqslant 1$, $0 \leqslant j \leqslant 2$,

$$t^j r^k |\partial_t^j \partial_r^k u_{2,2}(t,r)| \leqslant \frac{Cr\lambda(t)^4 \sup_{x \in [100,t]} (\lambda(x) \log(x)) \log^5(t)}{t^4 h(t)^2}$$
$$+ \frac{Cr\lambda(t)^2 \sup_{x \in [100,t]} (\lambda(x) \log(x)) \sup_{x \in [100,t]} (\lambda(x)^2) \log^4(t)}{t^6} \tag{4.160}$$

Also, for $\frac{h(t)}{2} \leqslant r \leqslant \frac{t}{2}$,

$$|\partial_r^2 u_{2,2}(t,r)| \leqslant \frac{C\lambda(t)^4 \sup_{x \in [100,t]} (\lambda(x) \log(x)) \log^5(t)}{t^4 r h(t)^2}$$
$$+ \frac{C\lambda(t)^2 \sup_{x \in [100,t]} (\lambda(x) \log(x)) \sup_{x \in [100,t]} (\lambda(x)^2) \log^4(t)}{r t^6}$$

In addition, for all $r > 0$,

$$\sqrt{E(u_{2,2}, \partial_t u_{2,2})} + |u_{2,2}(t,r)| \leqslant \frac{C\lambda(t)^2 \sup_{x \in [100,t]} (\lambda(x) \log(x)) \sup_{x \in [100,t]} (\lambda(x)^2) \log^3(t)}{t^3} \tag{4.161}$$

Finally, for $0 \leqslant j \leqslant 2$,

$$|\partial_t^j \partial_r u_{2,2}(t,r)| \leqslant \frac{C\lambda(t)^4 \sup_{x \in [100,t]} (\lambda(x) \log(x)) \log^3(t)}{\langle t-r \rangle^{1+j} h(t)^2 t^3}$$
$$+ \frac{C\lambda(t)^2 \sup_{x \in [100,t]} (\lambda(x) \log(x)) \sup_{x \in [100,t]} (\lambda(x)^2) \log^3(t)}{t^{5/2} \langle t-r \rangle^{7/2+j}}, \quad \frac{t}{2} < r < t \tag{4.162}$$

$$|\partial_r u_{2,2}(t,r)| \leqslant \frac{C \log^4(t)}{t^{\frac{5}{2} - 5C_u}}, \quad r > 0 \tag{4.163}$$

Proof As in (4.122), we have

$$u_{2,2}(t,r) = \frac{-r}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \int_0^{2\pi} d\theta I_{e_{2,2}}(s, r\beta, \rho, \theta)$$

where we recall the notation (2.2). We start with the estimate of the lemma statement in the region $h(t) \leqslant r \leqslant 2h(t) \leqslant \frac{t}{2}$. For all $\rho \leqslant s - t$, $r \leqslant \frac{t}{2}$, and $0 \leqslant \beta \leqslant 1$, we have

$$\sqrt{\beta^2 r^2 + \rho^2 + 2\beta r\rho \cos(\theta)} \leqslant \beta r + \rho \leqslant s - t + \frac{t}{2} < s$$

From Lemma 26, we get

$$|I_{e_{2,2}}(s, r\beta, \rho, \theta)| \leqslant C \left(\frac{\lambda(s)^2}{s^4} \sup_{x \in [100, s]} (\lambda(x)\log(x)) \frac{\log^3(s)\lambda(s)^2}{(h(s)^2 + \beta^2 r^2 + \rho^2 + 2\beta r\rho\cos(\theta))^2} \right.$$
$$\left. + \frac{\lambda(s)^2 \sup_{x \in [100,s]} (\lambda(x)\log(x)) \sup_{x \in [100,s]} (\lambda(x)^2) \log^3(s)}{\langle s - \sqrt{r^2\beta^2 + \rho^2 + 2\beta r\rho\cos(\theta)}\rangle^{7/2} s^{9/2}} \right)$$

We then make the analogous decomposition as in (4.123), and use

$$s - \sqrt{\beta^2 r^2 + \rho^2 + 2\beta r\rho\cos(\theta)} \geqslant s - (r + \rho) \geqslant t - r \geqslant Ct$$

which is true for $\rho \leqslant s - t$ and all $r \leqslant \frac{t}{2}$. This gives (4.160) for $k = 0, j = 0$, and $\frac{h(t)}{2} \leqslant r \leqslant 2h(t)$. For $k = 1, j = 0$, we note that

$$\partial_r u_{2,2}(t,r) = \frac{-1}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^{2\pi} d\theta I_{e_{2,2}}(s, r, \rho, \theta)$$

which, when combined with the same procedure used for $u_{2,2}$, completes the proof of (4.160) for $j = 0$ and $0 \leqslant k \leqslant 1$, in the region $\frac{h(t)}{2} \leqslant r \leqslant 2h(t)$. To treat higher j, we simply note that

$$\partial_t^j u_{2,2}(t,r) = \frac{-r}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \int_0^{2\pi} d\theta \partial_1^j I_{e_{2,2}}(s, r\beta, \rho, \theta)$$

and use the same procedure used for $j = 0$.

In the region $2h(t) \leqslant r \leqslant \frac{t}{2}$, a slightly more complicated argument is needed because factors of $\frac{r}{h(t)}$ are no longer controlled by a constant in this region. We have

$$|u_{2,2}(t,r)|$$
$$\leqslant Cr \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \int_0^{2\pi} d\theta \left(\frac{\lambda(s)^4 \sup_{x \in [100,s]} (\lambda(x)\log(x)) \log^3(s)}{s^4 (r^2\beta^2 + \rho^2 + 2r\beta\rho\cos(\theta) + h(s)^2)^2} \right.$$
$$\left. + \frac{\lambda(s)^2 \sup_{x \in [100,s]} (\lambda(x)\log(x)) \sup_{x \in [100,s]} (\lambda(x)^2) \log^3(s)}{s^{9/2} t^{7/2}} \right) \tag{4.164}$$

The second term of the integrand of the expression (4.164) is estimated with the following simple procedure.

$$|r \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \int_0^{2\pi} d\theta \frac{\lambda(s)^2 \sup_{x \in [100,s]} (\lambda(x)\log(x)) \sup_{x \in [100,s]} (\lambda(x)^2) \log^3(s)}{s^{9/2} t^{7/2}}|$$
$$\leqslant Cr \int_t^\infty ds \frac{\lambda(s)^2 \sup_{x \in [100,s]} (\lambda(x)\log(x)) \sup_{x \in [100,s]} (\lambda(x)^2) \log^3(s)}{s^{7/2} t^{7/2}}$$
$$\leqslant \frac{Cr\lambda(t)^2 \sup_{x \in [100,t]} (\lambda(x)\log(x)) \sup_{x \in [100,t]} (\lambda(x)^2) \log^3(t)}{t^6}, \quad r \leqslant \frac{t}{2}$$

Using Cauchy's residue theorem appropriately, we get the following, for $a, \rho, y > 0$.

$$\int_0^{2\pi} \frac{d\theta}{(y^2 + \rho^2 + 2\rho y\cos(\theta) + a^2)^2} = \frac{2\pi(a^2 + \rho^2 + y^2)}{((a^2 + (\rho - y)^2)(a^2 + (\rho + y)^2))^{3/2}} \tag{4.165}$$

So, for the first term of the integrand in (4.164), we get

$$
\left| r \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \int_0^{2\pi} d\theta \frac{\lambda(s)^4 \sup_{x \in [100,s]} (\lambda(x) \log(x)) \log^3(s)}{s^4 (r^2 \beta^2 + \rho^2 + 2r\beta\rho \cos(\theta) + h(s)^2)^2} \right|
$$

$$
\leqslant Cr \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \frac{\lambda(s)^4 \sup_{x \in [100,s]} (\lambda(x) \log(x)) \log^3(s)}{s^4 \sqrt{h(s)^2 + \rho^2 + r^2\beta^2} (h(s)^2 + (\rho - r\beta)^2)^{3/2}}
$$

$$
\leqslant Cr \int_t^{t+3h(t)} ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \frac{\lambda(s)^4 \sup_{x \in [100,s]} (\lambda(x) \log(x)) \log^3(s)}{s^4 h(s)^4}
$$

$$
+ Cr \int_{t+3h(t)}^{t+3r} ds \int_0^{3h(t)} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \frac{\lambda(s)^4 \sup_{x \in [100,s]} (\lambda(x) \log(x)) \log^3(s)}{s^4 h(s)^4}
$$

$$
+ Cr \int_{t+3h(t)}^{t+3r} ds \int_{3h(t)}^{s-t} \frac{d\rho}{\sqrt{(s-t)^2 - \rho^2}} \frac{\lambda(s)^4 \sup_{x \in [100,s]} (\lambda(x) \log(x)) \log^3(s)}{s^4} \int_0^1 \frac{d\beta}{(h(s)^2 + (\rho - r\beta)^2)^{3/2}}
$$

$$
+ Cr \int_{t+3r}^\infty ds \int_0^{3h(t)} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \frac{\lambda(s)^4 \sup_{x \in [100,s]} (\lambda(x) \log(x)) \log^3(s)}{s^4 h(s)^4}
$$

$$
+ Cr \int_{t+3r}^\infty ds \int_{3h(t)}^{3r} \frac{d\rho}{\sqrt{(s-t)^2 - \rho^2}} \frac{\lambda(s)^4 \sup_{x \in [100,s]} (\lambda(x) \log(x)) \log^3(s)}{s^4} \int_0^1 \frac{d\beta}{(h(s)^2 + (\rho - r\beta)^2)^{3/2}}
$$

$$
+ Cr \int_{t+3r}^\infty ds \int_{3r}^{s-t} \frac{d\rho}{\sqrt{(s-t)^2 - \rho^2}} \frac{\lambda(s)^4 \sup_{x \in [100,s]} (\lambda(x) \log(x)) \log^3(s)}{s^4 (r + \rho)^3}
$$

$$
\leqslant \frac{Cr\lambda(t)^4 \sup_{x \in [100,t]} (\lambda(x) \log(x)) \log^5(t)}{t^4 h(t)^2}, \quad 2h(t) \leqslant r \leqslant \frac{t}{2}
$$

The derivatives of $u_{2,2}$ in the region $2h(t) \leqslant r \leqslant \frac{t}{2}$ are treated in a similar way. A similar procedure establishes (4.162). Finally, the estimate on $\partial_r^2 u_{2,2}$ is obtained by noting that

$$
\partial_r^2 u_{2,2}(t,r) = e_{2,2}(t,r) + \frac{u_{2,2}(t,r)}{r^2} - \frac{1}{r} \partial_r u_{2,2}(t,r) + \partial_t^2 u_{2,2}(t,r)
$$

For (4.161), we again use Lemma 26 to get that

$$
|e_{2,2}(t,r)| \leqslant \mathbb{1}_{\{r \geqslant h(t)\}} \begin{cases} \frac{C\lambda(t)^2}{r^4} \frac{r\lambda(t)^2}{t^4} \sup_{x \in [100,t]} (\lambda(x) \log(x)) \log^3(t), & h(t) \leqslant r \leqslant \frac{t}{2} \\ \frac{C\lambda(t)^2}{r^4} \frac{\sup_{x \in [100,t]} (\lambda(x) \log(x)) \sup_{x \in [100,t]} (\lambda(x)^2) \log^3(t)}{\langle t-r \rangle^{5/2} \sqrt{t}}, & \frac{t}{2} \leqslant r \leqslant t \\ \frac{C\lambda(t)^2}{r^{9/2} \sqrt{\langle t-r \rangle}}, & r \geqslant t \end{cases}
$$

which gives

$$
\|e_{2,2}(t,r)\|_{L^2(rdr)} \leqslant \frac{C\lambda(t)^2 \sup_{x \in [100,t]} (\lambda(x) \log(x)) \sup_{x \in [100,t]} (\lambda(x)^2) \log^3(t)}{t^4}
$$

Then, the same procedure used in (4.69) (energy estimate) finishes the proof of (4.161). Finally, we prove (4.163) by using Lemma 26 to get

$$
|I_{e_{2,2}}(s,r,\rho,\theta)| \leqslant \frac{C \log^4(s)}{s^{\frac{9}{2} - 5C_u}}
$$

which implies

$$
|\partial_r u_{2,2}(t,r)| \leqslant C \int_t^\infty ds \frac{(s-t) \log^4(s)}{s^{\frac{9}{2} - 5C_u}} \leqslant \frac{C \log^4(t)}{t^{\frac{5}{2} - 5C_u}}
$$

□

Note that

$$
\left(1 - \chi_{\leqslant 1}\left(\frac{2r}{h(t)}\right)\right)\left(1 - \chi_{\leqslant 1}\left(\frac{r}{h(t)}\right)\right) = \left(1 - \chi_{\leqslant 1}\left(\frac{r}{h(t)}\right)\right)
$$

So, we will add the following truncation of $u_{2,2}$ into our ansatz:

$$
f_{2,2}(t,r) := u_{2,2}(t,r)\left(1 - \chi_{\leqslant 1}\left(\frac{2r}{h(t)}\right)\right) \tag{4.166}
$$

We define the error term associated to $f_{2,2}$ by

$$
e_{f_{2,2}}(t,r) := \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) v_{2,2}(t,r) \left(1 - \chi_{\leqslant 1}(\frac{r}{h(t)}) \right) - \left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} \right) f_{2,2}(t,r)
$$

$$
+ \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) f_{2,2}
$$

$$
= u_{2,2}(t,r) \left(\frac{2}{rh(t)} \chi'_{\leqslant 1}(\frac{2r}{h(t)}) - \frac{4r}{h(t)^3} \chi'_{\leqslant 1}(\frac{2r}{h(t)}) h'(t)^2 - \frac{4r^2 h'(t)^2}{h(t)^4} \chi''_{\leqslant 1}(\frac{2r}{h(t)}) \right.
$$

$$
\left. + \frac{4\chi''_{\leqslant 1}(\frac{2r}{h(t)})}{h(t)^2} + \frac{2rh''(t)}{h(t)^2} \chi'_{\leqslant 1}(\frac{2r}{h(t)}) \right)
$$

$$
+ \frac{4\chi'_{\leqslant 1}(\frac{2r}{h(t)})}{h(t)^2} \left(h(t) \partial_r u_{2,2} + rh'(t) \partial_t u_{2,2} \right) + \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) u_{2,2}(t,r) \left(1 - \chi_{\leqslant 1}(\frac{2r}{h(t)}) \right)
$$

Then, a direct estimation gives the following lemma.

Lemma 39 *For* $k = 0, 1$,

$$
||L^k_{\frac{1}{\lambda(t)}}(e_{f_{2,2}})(t,r)||_{L^2(rdr)} \leqslant \frac{C\lambda(t)^4 \sup_{x\in[100,t]} (\lambda(x)\log(x)) \log^5(t)}{t^4 h(t)^{2+k}}
$$

Now, we let

$$
u_a(t,r) = u_c(t,r) + f_{5,0}(t,r) + f_{ex,sub}(t,r) + f_{ell,2}(t,r) + f_{2,2}(t,r) \tag{4.167}
$$

$$
u_{corr}(t,r) = u_a(t,r) + u_1(t,r)
$$

where we recall that u_c is defined by (4.141). The equation for u_1 which results from substituting $Q_{\frac{1}{\lambda(t)}}(r) + u_{corr}$ into (1.1) is

$$
-\partial_t^2 u_1 + \partial_r^2 u_1 + \frac{1}{r}\partial_r u_1 - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2} u_1
$$

$$
= e_a(t,r) + \left(\frac{\cos(2u_{corr}) - 1}{2r^2} \right) \sin(2Q_1(\frac{r}{\lambda(t)})) + \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} (\sin(2u_{corr}) - 2u_{corr})
$$

where

$$
e_a(t,r) = \left(e_{ex,ell} + e_{w,2} + e_{match,0} + e_{5,1} + e_{f_{5,0}} + e_{ex,sub} + e_{ell,2} + e_{f_{2,2}} \right)(t,r)
$$

Combining Lemmas 39, 37, 36, 34, 32, 31, 28 and using (1.7) and (4.2) gives the following lemma.

Lemma 40

$$
\frac{||e_a||_{L^2(rdr)}}{\lambda(t)^2} \leqslant \frac{C\log^6(t)}{t^{4+min\{4\alpha-C_l-2, 1-\alpha-2C_u, 4-5C_u-6\alpha\}}}
$$

$$
||L_{\frac{1}{\lambda(t)}}(e_a)||_{L^2(rdr)} \leqslant \frac{C\log^6(t)}{t^{4+2\alpha-3C_u}} + \frac{C\log^2(t)}{t^{2+5\alpha}} + \frac{C\log^5(t)}{t^{5-3C_u}} + \frac{C\log^3(t)}{t^{8-6C_u-5\alpha}}
$$

4.9 First set of nonlinear interactions

It only remains to treat the terms which are nonlinear in $u_{corr} = u_a + u_1$. For this, we define

$$
NL(t,r) = \left(\frac{\cos(2u_{corr}) - 1}{2r^2} \right) \sin(2Q_1(\frac{r}{\lambda(t)})) + \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} (\sin(2u_{corr}) - 2u_{corr})
$$

$$
= \left(\frac{\cos(2u_1) - 1}{2r^2} \right) \sin(2Q_1(\frac{r}{\lambda(t)}) + 2u_a)
$$

$$
+ \left(\frac{\sin(2u_1) - 2u_1}{2r^2} \right) \cos(2Q_1(\frac{r}{\lambda(t)}) + 2u_a) + \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} (\sin(2u_a) - 2u_a)
$$

$$
+ u_1 \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)}) + 2u_a) - \cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2} \right) + \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} (\cos(2u_a) - 1)
$$

We focus on the u_1-independent terms of the expression above. In particular, we define

$$N_a(t,r) = \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\sin(2u_a) - 2u_a) + \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\cos(2u_a) - 1)$$

We recall that

$$u_a = \chi_{\leqslant 1}(\frac{r}{h(t)})u_e + \left(1 - \chi_{\leqslant 1}(\frac{r}{h(t)})\right)u_{wave} + f_{5,0} + f_{ex,sub} + f_{ell,2} + f_{2,2}$$

We write

$$u_a = u_{a,0} + u_{a,1} \tag{4.168}$$

with

$$u_{a,0}(t,r) = \chi_{\leqslant 1}(\frac{r}{h(t)})u_{ell}(t,r) + \left(1 - \chi_{\leqslant 1}(\frac{r}{h(t)})\right)(w_1(t,r) + v_2(t,r) + v_{2,2}(t,r))$$

$$u_{a,1} = \chi_{\leqslant 1}(\frac{r}{h(t)})(u_e - u_{ell}) + \left(1 - \chi_{\leqslant 1}(\frac{r}{h(t)})\right)(v_{ex} + u_{w,2}) + f_{5,0} + f_{ex,sub} + f_{ell,2} + f_{2,2}$$

Then, we get

$$N_a(t,r) = \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\sin(2u_{a,0}) - 2u_{a,0}) + \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\cos(2u_{a,0}) - 1)$$

$$+ \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\sin(2u_{a,0})(\cos(2u_{a,1}) - 1) + (\sin(2u_{a,1}) - 2u_{a,1})\cos(2u_{a,0}) + (\cos(2u_{a,0}) - 1)2u_{a,1})$$

$$+ \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}((\cos(2u_{a,1}) - 1)\cos(2u_{a,0}) - 2\sin(2u_{a,0})u_{a,1} - \sin(2u_{a,0})(\sin(2u_{a,1}) - 2u_{a,1}))$$

$$:= N_0 + N_1$$

where

$$N_1(t,r)$$
$$= \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\sin(2u_{a,0})(\cos(2u_{a,1}) - 1) + (\sin(2u_{a,1}) - 2u_{a,1})\cos(2u_{a,0}) + (\cos(2u_{a,0}) - 1)2u_{a,1})$$
$$+ \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}((\cos(2u_{a,1}) - 1)\cos(2u_{a,0}) - 2\sin(2u_{a,0})u_{a,1} - \sin(2u_{a,0})(\sin(2u_{a,1}) - 2u_{a,1}))$$
$$\tag{4.169}$$

We proceed to estimate the term N_1, which turns out to be perturbative.

Lemma 41 *We have the following estimates on N_1.*

$$\frac{||N_1(t,r)||_{L^2(rdr)}}{\lambda(t)^2} \leqslant \frac{C\log^3(t)}{t^{6-3C_u-2\alpha}} + \frac{C\log^{13}(t)}{t^{5-\frac{3}{2}C_l-9C_u}}, \quad ||L_{\frac{1}{\lambda(t)}}(N_1)(t,r)||_{L^2(rdr)} \leqslant \frac{C\log^3(t)}{t^{6-5C_u-2\alpha-C_l}} + \frac{C\log^{13}(t)}{t^{5-\frac{19}{2}C_u-C_l}}$$

Proof We estimate N_1 by combining the following estimates on various terms of $u_{a,0}$ and $u_{a,1}$. We use the explicit formulae following (4.7) and (4.81) to estimate u_{ell}. We use the decomposition (4.13), along with (4.14) and the analogous estimates to (4.158) and (4.159) to estimate $u_{ell,2}$. We use (4.146) in the region $r \geqslant \frac{t}{2}$, and (4.147) and the expression for $v_{ex,ell}$, namely (4.36) in the region $g(t)\lambda(t) \leqslant r \leqslant \frac{t}{2}$, to estimate v_{ex}. We use Lemma 10 to estimate $u_{w,2}$. (In particular, we use (4.63) and (4.67) in the region $g(t)\lambda(t) \leqslant r \leqslant \frac{t}{2}$, and (4.64) in the region $r \geqslant \frac{t}{2}$). Next, we use Lemma (2), Lemma (5), and Lemma 26 to estimate w_1, v_2, and $v_{2,2}$, respectively. Finally, we use Lemma 38, and (4.166) (for $f_{2,2}$), Lemma 37 (for $f_{ell,2}$), Lemma 33 and (4.156) (for $f_{5,0}$), and Lemma 35 and (4.157) (for $f_{ex,sub}$). This gives rise to

$$|u_{a,0}(t,r)|$$
$$\leqslant C \begin{cases} \frac{r\lambda(t)\log(t)}{t^2}, & r \leqslant \frac{t}{2} \\ \frac{\sup_{x \in [100,r]}(\lambda(x)\log(x))\log(r)}{\sqrt{r}\sqrt{\langle t-r \rangle}} + \frac{\sup_{x \in [100,t]}(\lambda(x)\log(x))\sup_{x \in [100,t]}(\lambda(x)^2)\log^3(t)\mathbb{1}_{\{t > r > \frac{t}{2}\}}}{\langle t-r \rangle^{5/2}\sqrt{t}}, & r > \frac{t}{2} \end{cases} \tag{4.170}$$

and

$$|u_{a,1}(t,r)| \leqslant C \begin{cases} \frac{r^3(\log^2(r) + \log^2(t))}{t^{4-C_u}}, & r \leqslant 2h(t) \\ \frac{\lambda(t)\log^3(t)}{t^{2-2C_u}r} + \frac{Cr\lambda(t)\log^6(t)}{t^{4-3C_u}}, & 2h(t) \leqslant r \leqslant \frac{t}{2} \\ \frac{\log^5(t)}{t^{\frac{5}{2}-\frac{7}{2}C_u}} \cdot \min\{1, \frac{\lambda(t)}{t^{\frac{C_u-C_l}{2}}}\}, & r \geqslant \frac{t}{2} \end{cases} \tag{4.171}$$

Then, a straightforward, but slightly long computation gives the estimate on $||N_1(t,r)||_{L^2(rdr)}$ from the lemma statement. Recall that

$$L_{\frac{1}{\lambda(t)}} N_1(t,r) = \partial_r N_1(t,r) - \frac{\cos(Q_1(\frac{r}{\lambda(t)}))}{r} N_1(t,r)$$

To estimate $||L_{\frac{1}{\lambda(t)}} N_1(t,r)||_{L^2(rdr)}$, we use the procedure outlined at the beginning of the proof to estimate $\partial_r u_{a,0}$ and $\partial_r u_{a,1}$. The only difference here is the following. After differentiating $N_1(t,r)$ with respect to r, we use the pointwise estimates on $\partial_r u_{2,2}(t,r)$ given in (4.160) for the region $r \leqslant \frac{t}{2}$. We use Holder's inequality and (4.161) to estimate the terms involving $\partial_r u_{2,2}(t,r)$ in the region $r \geqslant \frac{t}{2}$. $\quad\square$

It remains to treat N_0, which we recall is defined by

$$N_0(t,r) = \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} (\sin(2u_{a,0}) - 2u_{a,0}) + \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} (\cos(2u_{a,0}) - 1) \qquad (4.172)$$

where

$$u_{a,0}(t,r) = \chi_{\leqslant 1}(\frac{r}{h(t)}) u_{ell}(t,r) + \left(1 - \chi_{\leqslant 1}(\frac{r}{h(t)})\right) (w_1(t,r) + v_2(t,r) + v_{2,2}(t,r))$$

We define u_{N_0} to be the solution to the following equation with 0 Cauchy data at infinity.

$$-\partial_t^2 u_{N_0} + \partial_r^2 u_{N_0} + \frac{1}{r} \partial_r u_{N_0} - \frac{u_{N_0}}{r^2} = N_0(t,r) \qquad (4.173)$$

Then, we have

Lemma 42 *We have the following estimates. For $0 \leqslant k \leqslant 2$ and $0 \leqslant j \leqslant 1$,*

$$|\partial_t^k \partial_r^j u_{N_0}(t,r)| \leqslant \frac{C r^{1-j} \left(\sup_{x \in [100,t]} (\lambda(x) \log(x))\right)^3 \log^3(t)}{t^{4+k}}, \quad r \leqslant \frac{t}{2} \qquad (4.174)$$

$$|u_{N_0}(t,r)| + \sqrt{E(u_{N_0}, \partial_t u_{N_0})} \leqslant \frac{C \left(\sup_{x \in [100,t]} (\lambda(x) \log(x))\right)^3 \left(\sup_{x \in [100,t]} (\lambda(x)^2)\right)^3 \log^9(t)}{t^2} \qquad (4.175)$$

Finally,

$|\partial_r u_{N_0}(t,r)|$

$$\leqslant \frac{C \left(\sup_{x \in [100,t]} (\lambda(x) \log(x))\right)^3 \log^3(t)}{t^{3/2} \langle t-r \rangle^{5/2}} + \frac{C \left(\sup_{x \in [100,t]} (\lambda(x) \log(x))\right)^3 \left(\sup_{x \in [100,t]} (\lambda(x)^2)\right)^3 \log^9(t)}{t^{3/2} \langle t-r \rangle^{17/2}}$$

$$+ \frac{C \lambda(t) \left(\sup_{x \in [100,t]} (\lambda(x) \log(x))\right)^2 \left(\sup_{x \in [100,t]} (\lambda(x)^2)\right)^2 \log^6(t)}{t^2 \langle t-r \rangle^6}, \quad \frac{t}{2} < r < t \qquad (4.176)$$

$$||\partial_r u_{N_0}(t,r)||_{L_r^\infty} \leqslant \frac{C \log^{12}(t)}{t^{\frac{3}{2} - 9 C_u}} \qquad (4.177)$$

Proof As in (4.121), we have

$$u_{N_0}(t,r) = -\frac{r}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \int_0^{2\pi} d\theta I_{N_0}(s, r\beta, \rho, \theta)$$

where we use the notation defined in (2.2). We then estimate $u_{a,0}$ and its derivatives using the procedure described in the proof of Lemma 41. This results in the following estimate for $j = 0,1,2$ and $k = 0,1$.

$|\partial_t^j \partial_r^k N_0(t,r)|$

$$\leqslant C \left(\frac{1}{r} + \frac{1}{\langle t-r \rangle}\right)^k \left(\frac{1}{t} + \frac{1}{\langle t-r \rangle}\right)^j \begin{cases} \frac{r \lambda(t)^3 \log^3(t)}{t^4 (r^2 + \lambda(t)^2)}, & r \leqslant \frac{t}{2} \\ \frac{\lambda(t)}{t^4} \frac{(\sup_{x \in [100,t]}(\lambda(x) \log(x)))^2}{\langle t-r \rangle^5} \left(\sup_{x \in [100,t]} (\lambda(x)^2)\right)^2 \log^6(t) \\ + \frac{(\sup_{x \in [100,t]}(\lambda(x) \log(x)))^3 \log^3(t)}{r^{7/2} \langle t-r \rangle^{3/2}} \\ + \frac{(\sup_{x \in [100,t]}(\lambda(x) \log(x)))^3 (\sup_{x \in [100,t]}(\lambda(x)^2))^3 \log^9(t)}{\langle t-r \rangle^{15/2} t^{7/2}}, & \frac{t}{2} < r < t \end{cases}$$

$$(4.178)$$

We start with the region $r \leqslant \frac{t}{2}$. Using an analog of (4.165), we get

$$|u_{N_0}(t,r)|$$
$$\leqslant Cr \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \left(\frac{\lambda(s)^3 \log^3(s)}{s^4 \sqrt{(\lambda(s)^2 + (\rho + r\beta)^2)(\lambda(s)^2 + (\rho - r\beta)^2)}} \right.$$
$$+ \frac{\lambda(s) \left(\sup_{x \in [100,s]} (\lambda(x) \log(x)) \right)^2 \left(\sup_{x \in [100,s]} (\lambda(x)^2) \right)^2 \log^6(s)}{s^4(s - (\rho + r))^6}$$
$$+ \frac{\left(\sup_{x \in [100,s]} (\lambda(x) \log(x)) \right)^3 \log^3(s)}{s^{7/2}(s - (\rho + r))^{5/2}}$$
$$\left. + \frac{\left(\sup_{x \in [100,s]} (\lambda(x) \log(x)) \right)^3 \left(\sup_{x \in [100,s]} (\lambda(x)^2) \right)^3 \log^9(s)}{s^{7/2}(s - (\rho + r))^{17/2}} \right)$$
$$(4.179)$$

where we used the fact that, if $0 \leqslant \beta \leqslant 1$, then,

$$\langle s - \sqrt{\rho^2 + r^2 \beta^2 + 2r\beta\rho \cos(\theta)} \rangle \geqslant C (s - (\rho + r)) .$$

By a direct estimation, we get

$$\left| \int_0^1 \frac{d\beta}{\sqrt{\lambda(s)^2 + (r\beta - \rho)^2}} \frac{1}{(\lambda(s) + r\beta + \rho)} \right| \leqslant \frac{C \log(s)}{(r + \rho)(\lambda(s) + \rho)}$$

and therefore,

$$r \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \frac{\lambda(s)^3 \log^3(s)}{s^4 \log^{2b}(s) \sqrt{(\lambda(s)^2 + (\rho + r\beta)^2)(\lambda(s)^2 + (\rho - r\beta)^2)}} \leqslant \frac{Cr\lambda(t)^3 \log^6(t)}{t^4}$$

The other terms of (4.179) are treated with a similar argument, using $s - (r + \rho) \geqslant s - r - (s - t) \geqslant Ct$ (since $r \leqslant \frac{t}{2}$). This gives (4.174), for $k = 0, j = 0$. To obtain (4.174) for $k = 1, 2$ and $j = 0$, we first note that

$$\partial_t^k u_{N_0}(t,r) = -\frac{r}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^1 d\beta \int_0^{2\pi} d\theta \partial_1^k I_{N_0}(s, r\beta, \rho, \theta)$$

Then, after applying (4.178), we obtain an extra factor of

$$\left(\frac{1}{s} + \frac{1}{\langle s - \sqrt{r^2\beta^2 + \rho^2 + 2r\beta\rho \cos(\theta)} \rangle} \right)^k \leqslant \left(\frac{C}{s} + \frac{C}{s - (r + s - t)} \right)^k \leqslant \frac{C}{t^k}$$

in the integrand of the expression for $\partial_t^k u_{N_0}(t,r)$, relative to that for u_{N_0}, which we just estimated. This immediately gives (4.174) for $k = 1, 2$. Next, we use the same procedure as in (4.77), to get (4.175). Finally, we use

$$\partial_r u_{N_0}(t,r) = \frac{-1}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_0^{2\pi} d\theta I_{N_0}(s, r, \rho, \theta)$$

and a similar argument used to estimate u_{N_0}, to get (4.174) for $j = 1, k = 0$. For completeness, we show how to estimate the most delicate integral, which is

$$\int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \frac{\lambda(s)^3 \log^3(s)}{s^4 \sqrt{\lambda(s)^2 + (\rho + r)^2} \sqrt{\lambda(s)^2 + (\rho - r)^2}}$$

Here, we use

$$\frac{d}{d\rho} \left(\mathrm{arctanh}(\frac{\rho - r}{\sqrt{a^2 + (\rho - r)^2}}) \right) = \frac{1}{\sqrt{a^2 + (r - \rho)^2}}$$

and estimate the integral directly. A similar argument establishes (4.176). Finally, to obtain (4.177), we again estimate $u_{a,0}$ and $\partial_r u_{a,0}$ using the procedure described in the proof of Lemma 41 to get, for all $r > 0, \theta \in [0, 2\pi], \rho \leqslant s - t$,

$$|I_{N_0}(s, r, \rho, \theta)| \leqslant \frac{C \log^{12}(s)}{s^{\frac{7}{2} - 9C_u}}$$

This gives

$$|\partial_r u_{N_0}(t,r)| \leqslant C \int_t^\infty ds (s - t) \frac{C \log^{12}(s)}{s^{\frac{7}{2} - 9C_u}} \leqslant \frac{C \log^{12}(t)}{t^{\frac{3}{2} - 9C_u}}$$

$\square$

The linear error term of u_{N_0} is e_{N_0} defined by

$$e_{N_0}(t,r) := \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) u_{N_0}(t,r) \tag{4.180}$$

Then, a straightforward estimation, using Lemma 42 gives the following (recall (2.1)).

Lemma 43 *We have the following estimates for $k = 0, 1$.*

$$||L^k_{\frac{1}{\lambda(t)}}(e_{N_0}(t,r)(1 - m_{\leqslant 1}(\frac{r}{2h(t)})))||_{L^2(rdr)}$$

$$\leqslant \frac{C\lambda(t)^2 \left(\sup_{x \in [100,t]} (\lambda(x) \log(x)) \right)^3 \log^3(t)}{t^4} \left(\frac{1}{h(t)^{2+k}} + \frac{\left(\sup_{x \in [100,t]} (\lambda(x)^2) \right)^3 \log^6(t)}{t^{1+k}} \right)$$

It remains to treat $m_{\leqslant 1}(\frac{r}{2h(t)})e_{N_0}(t,r)$. For the reader's convenience, we repeat the outline (that was given in Section 3, just below (3.5)) of the procedure to be used. We start with solving the equation

$$\partial_{rr}u_{N_0,ell} + \frac{1}{r}\partial_r u_{N_0,ell} - \frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))}{r^2}u_{N_0,ell} = m_{\leqslant 1}(\frac{r}{2h(t)})e_{N_0}(t,r) \tag{4.181}$$

and then inserting the following function into the ansatz

$$u_{N_0,corr}(t,r) := \left(u_{N_0,ell}(t,r) - \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t,R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)} \right) m_{\leqslant 1}(\frac{2r}{t}) + v_{2,4}(t,r) \tag{4.182}$$

where $v_{2,4}$ solves

$$-\partial_{tt}v_{2,4} + \partial_{rr}v_{2,4} + \frac{1}{r}\partial_r v_{2,4} - \frac{v_{2,4}}{r^2} = 0 \tag{4.183}$$

and $v_{2,4}(t,r)$ matches $\frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t,R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}$ for small r. (Recall that $m_{\leqslant 1}$ was defined in (2.1)). In particular, we choose the initial velocity, say $v_{2,5}$ of $v_{2,4}$ by requiring, for all t sufficiently large,

$$\frac{-r}{4} \cdot -2\int_0^\infty \xi \sin(t\xi)\widehat{v_{2,5}}(\xi)d\xi = \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t,R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}$$

We refer the reader to the discussion in Section 3, just below (3.5), for some intuition regarding this procedure.

We start by estimating $u_{N_0,ell}(t,r) - \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t,R\lambda(t)), \phi_0(R)\rangle$.

Lemma 44 *We have the following estimates. For $0 \leqslant k \leqslant 1$ and $0 \leqslant j \leqslant 2$,*

$$t^j r^k |\partial_t^j \partial_r^k \left(u_{N_0,ell}(t,r) - \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t,R\lambda(t)), \phi_0(R)\rangle \right)|$$

$$\leqslant \begin{cases} \frac{Cr\log^3(t)(\sup_{x \in [100,t]}(\lambda(x)\log(x)))^3}{t^4}, & r \leqslant \lambda(t) \\ \frac{C(\log(t) + \log(\frac{r}{\lambda(t)}))\lambda(t)^2(\sup_{x \in [100,t]}(\lambda(x)\log(x)))^3 \log^3(t)}{rt^4}, & \lambda(t) < r \end{cases} \tag{4.184}$$

In addition, we have

$$|\partial_r^2 \left(u_{N_0,ell}(t,r) - \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t,R\lambda(t)), \phi_0(R)\rangle \right)|$$

$$\leqslant C \begin{cases} \frac{r}{\lambda(t)^2}\frac{(\sup_{x \in [100,t]}(\lambda(x)\log(x)))^3 \log^3(t)}{t^4}, & r \leqslant \lambda(t) \\ \frac{\lambda(t)^2(\log(t) + \log(\frac{r}{\lambda(t)}))(\sup_{x \in [100,t]}(\lambda(x)\log(x)))^3 \log^3(t)}{r^3 t^4}, & r \geqslant \lambda(t) \end{cases} \tag{4.185}$$

Finally, for $0 \leqslant k \leqslant 2$,

$$|\partial_t^k \langle e_{N_0}(t,R\lambda(t))m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)}), \phi_0(R)\rangle| \leqslant \frac{C\log^3(t) \left(\sup_{x \in [100,t]} (\lambda(x)\log(x)) \right)^3}{t^{4+k}\lambda(t)} \tag{4.186}$$

Proof We consider the particular solution given by

$$u_{N_0,ell}(t,r) = v_{N_0,ell}(t, \frac{r}{\lambda(t)})$$

where

$$v_{N_0,ell}(t,R) = e_2(R) \int_0^R \lambda(t)^2 m_{\leqslant 1}(\frac{s}{2g(t)}) e_{N_0}(t, s\lambda(t)) \phi_0(s) \frac{sds}{2} - \phi_0(R) \int_0^R \lambda(t)^2 m_{\leqslant 1}(\frac{s}{2g(t)}) e_{N_0}(t, s\lambda(t)) e_2(s) \frac{sds}{2}$$

We remind the reader that e_{N_0} is defined in (4.180). Let $int_{N_0,i}(t,r)$ denote the ith term on the right-hand side of the above equation, evaluated at $R = \frac{r}{\lambda(t)}$. In the region $r \leqslant \lambda(t)$, we separately estimate $int_{N_0,1}$, $int_{N_0,2}$, and $\frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)}) e_{N_0}(t, R\lambda(t)), \phi_0(R)\rangle$, using Lemma 42. This gives (4.184), in the region $r \leqslant \lambda(t)$, and for $k = j = 0$. When $r \geqslant \lambda(t)$, we take advantage of the fact that

$$|e_2(R) - \frac{R}{2}| \leqslant C \frac{(1 + \log(R))}{R}, \quad R \geqslant 1$$

In particular, we estimate $int_{N_0,2}(t,r)$ directly, using Lemma 42 and write

$$int_{N_0,1}(t,r) - \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)}) e_{N_0}(t, R\lambda(t)), \phi_0(R)\rangle$$

$$= \left(\frac{e_2(\frac{r}{\lambda(t)})}{2} - \frac{r}{4\lambda(t)}\right) \int_0^{\frac{r}{\lambda(t)}} \lambda(t)^2 m_{\leqslant 1}(\frac{s}{2g(t)}) e_{N_0}(t, s\lambda(t)) \phi_0(s) sds$$

$$- \frac{r\lambda(t)}{4} \int_{\frac{r}{\lambda(t)}}^\infty m_{\leqslant 1}(\frac{s}{2g(t)}) e_{N_0}(t, s\lambda(t)) \phi_0(s) sds$$

Directly estimating the integrals using Lemma 42, we obtain (4.184), in the region $r \geqslant \lambda(t)$, and $k = j = 0$. To obtain (4.184) for $k = 1$, we use

$$\partial_r u_{N_0,ell}(t,r) = \frac{-\phi_0'(\frac{r}{\lambda(t)})}{\lambda(t)} \int_0^{\frac{r}{\lambda(t)}} \lambda(t)^2 m_{\leqslant 1}(\frac{s}{2g(t)}) e_{N_0}(t, s\lambda(t)) e_2(s) \frac{sds}{2}$$

$$+ \frac{e_2'(\frac{r}{\lambda(t)})}{\lambda(t)} \int_0^{\frac{r}{\lambda(t)}} \lambda(t)^2 m_{\leqslant 1}(\frac{s}{2g(t)}) e_{N_0}(t, s\lambda(t)) \phi_0(s) \frac{sds}{2}$$

and then repeat the same procedure used to obtain (4.184) for $k = 0$. For the estimate (4.184) for $j > 0$, we first re-write $u_{N_0,ell}$ as follows, and then differentiate in t directly.

$$u_{N_0,ell}(t,r) = -\phi_0(\frac{r}{\lambda(t)}) \int_0^r m_{\leqslant 1}(\frac{x}{2h(t)}) e_{N_0}(t,x) e_2(\frac{x}{\lambda(t)}) \frac{xdx}{2} + e_2(\frac{r}{\lambda(t)}) \int_0^r m_{\leqslant 1}(\frac{x}{2h(t)}) e_{N_0}(t,x) \phi_0(\frac{x}{\lambda(t)}) \frac{xdx}{2}$$

Using the same procedure as for (4.184) with $j = 0$, and noting the symbol-type nature of the estimates in Lemma 42, we finish the proof of (4.184) for $j > 0$. Finally, to get (4.185), we use the equation solved by $u_{N_0,ell}$. The estimates in (4.186) follow directly from Lemma 42. $\square$

Just after (4.135), we restricted T_0 to satisfy $T_0 \geqslant 2T_2$, and all of our computations and estimates are valid for all $t \geqslant T_0$ for any $T_0 \geqslant 2T_2$. At this stage, we restrict T_0 so that $T_0 \geqslant 4T_2$ but is otherwise arbitrary. Then, recalling the cutoff ψ_2 defined in (4.134), we define a function $v_{2,5}$ by

$$\widehat{v_{2,5}}(\xi) = \frac{1}{\pi\xi} \int_0^\infty \lambda(t) \langle m_{\leqslant 1}(\frac{R}{2g(t)}) e_{N_0}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)} \psi_2(\frac{t}{2}) \sin(t\xi) dt \qquad (4.187)$$

Letting

$$F_{2,4}(t) = -2 \int_0^\infty \xi \sin(t\xi) \widehat{v_{2,5}}(\xi) d\xi$$

the inversion of the sine transform gives

$$F_{2,4}(t) = -\lambda(t) \psi_2(\frac{t}{2}) \langle m_{\leqslant 1}(\frac{R}{2g(t)}) e_{N_0}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}, \quad t > 0$$

Therefore, (4.186) gives the following estimate for $0 \leqslant k \leqslant 2$ and $t > 0$:

$$|\partial_t^k F_{2,4}(t)| \leqslant C \mathbb{1}_{\{t > 2T_2\}} \frac{\left(\sup_{x \in [100,t]} (\lambda(x) \log(x))\right)^3 \log^3(t)}{t^{4+k}}$$

We have the following estimates on $\widehat{v_{2,5}}$.

Lemma 45 *For $0 \leqslant k \leqslant 2$,*

$$|\xi^k \partial_\xi^k \widehat{v_{2,5}}(\xi)| \leqslant \begin{cases} C, & \xi \leqslant \frac{1}{100} \\ \frac{C}{\xi^{3-k}}, & \xi > \frac{1}{100} \end{cases} \tag{4.188}$$

Proof We recall

$$\widehat{v_{2,5}}(\xi) = \frac{-1}{\pi\xi} \int_0^\infty F_{2,4}(t) \sin(t\xi)dt$$

In the region $\xi \leqslant \frac{1}{100}$, we simply directly estimate as follows

$$|\widehat{v_{2,5}}(\xi)| \leqslant \frac{C}{\xi} \int_0^{\frac{1}{\xi}} |F_{2,4}(t)|t\xi dt + \frac{C}{\xi} \int_{\frac{1}{\xi}}^\infty |F_{2,4}(t)|dt$$

If $\xi > \frac{1}{100}$, then, we have

$$\widehat{v_{2,5}}(\xi) = \frac{1}{\pi\xi^3} \int_0^\infty \sin(t\xi) F_{2,4}''(t)dt$$

where we note that we can integrate by parts only twice (simply because of the estimates we have on u_{N_0}, which gave rise to estimates on up to two derivatives of $F_{2,4}$) which is why the decay for large ξ in the estimate (4.188) is not as strong as the analogous estimate for $\widehat{v_{2,0}}$ in Lemma 4. To estimate the derivatives of $\widehat{v_{2,5}}(\xi)$, we first re-write its formula as

$$\widehat{v_{2,5}}(\xi) = \frac{-1}{\pi\xi^2} \int_0^\infty F_{2,4}\left(\frac{\sigma}{\xi}\right) \sin(\sigma)d\sigma$$

and then differentiate in ξ. Then, we estimate as above.   □

We define $v_{2,4}$ to be the solution to the following Cauchy problem.

$$\begin{cases} -\partial_t^2 v_{2,4} + \partial_r^2 v_{2,4} + \frac{1}{r}\partial_r v_{2,4} - \frac{v_{2,4}}{r^2} = 0 \\ v_{2,4}(0,r) = 0 \\ \partial_t v_{2,4}(0,r) = v_{2,5}(r) \end{cases} \tag{4.189}$$

Now, we obtain estimates on $v_{2,4}(t,r) - \left(\frac{-r}{4}F_{2,4}(t)\right)$

Lemma 46 *For $0 \leqslant k \leqslant 2$,*

$$\left|\partial_r^k \left(v_{2,4}(t,r) + \frac{r}{4}F_{2,4}(t)\right)\right| \leqslant \frac{Cr^{3-k}\log^3(t)\left(\sup_{x\in[100,t]}(\lambda(x)\log(x))\right)^3}{t^6}, \quad r \leqslant \frac{t}{2} \tag{4.190}$$

For $0 \leqslant k \leqslant 2$, $0 \leqslant j \leqslant 1$ and $j + k \leqslant 2$.

$$|\partial_t^k \partial_r^j v_{2,4}(t,r)| \leqslant \frac{Cr^{1-j}\log^3(t)\left(\sup_{x\in[100,t]}(\lambda(x)\log(x))\right)^3}{t^{4+k}}, \quad r \leqslant \frac{t}{2}$$

For $0 \leqslant k + j \leqslant 2$, we have the following estimate.

$$|\partial_r^k \partial_t^j v_{2,4}(t,r)| \leqslant \frac{C\log^3(t)\left(\sup_{x\in[100,t]}(\lambda(x)\log(x))\right)^3}{\sqrt{t}\langle t-r\rangle^{5/2+k+j}}, \quad \frac{t}{2} \leqslant r < t \tag{4.191}$$

Finally, for $0 \leqslant k \leqslant 1$,

$$|\partial_r^k v_{2,4}(t,r)| \leqslant \frac{C}{\sqrt{r}\langle t-r\rangle^{\frac{1}{2}+k}}, \quad r > t \tag{4.192}$$

Proof The proof of this lemma uses the same procedure as the proof of Lemma 26. In particular, we have

$$v_{2,4}(t,r) = \frac{-r}{2\pi} \int_0^\pi \sin^2(\theta) F_{2,4}(t + r\cos(\theta))d\theta \tag{4.193}$$

In the region $r \leqslant \frac{t}{2}$, we have

$$v_{2,4}(t,r) + \frac{r}{4}F_{2,4}(t) = \frac{-r}{2\pi} \int_0^\pi \sin^2(\theta) \left(F_{2,4}(t + r\cos(\theta)) - F_{2,4}(t)\right) d\theta$$

and this immediately gives rise to (4.190).

Next, we note that

$$v_{2,4}(t,r) = \int_0^\infty d\xi\, J_1(r\xi)\sin(t\xi)\widehat{v_{2,5}}(\xi)$$

which leads to the following estimate, for all $r \geqslant \frac{t}{2}$.

$$|v_{2,4}(t,r)| \leqslant C\int_0^{\frac{2}{r}} d\xi\, r\xi + C\int_{\frac{1}{r}}^{\frac{1}{100}} \frac{d\xi}{\sqrt{r\xi}} + C\int_{\frac{1}{100}}^\infty \frac{d\xi}{\sqrt{r\xi}}\frac{1}{\xi^3} \leqslant \frac{C}{\sqrt{r}} \qquad (4.194)$$

The same computation leads to

$$|\partial_r^k \partial_t^j v_{2,4}(t,r)| \leqslant \frac{C}{\sqrt{r}}, \quad r \geqslant \frac{t}{2}, \quad 0 \leqslant j+k \leqslant 2$$

On the other hand, if $t > r > \frac{t}{2}$, we have (using (4.193))

$$|v_{2,4}(t,r)| \leqslant Cr\log^3(t)\left(\sup_{x\in[100,t]}(\lambda(x)\log(x))\right)^3 \int_0^\pi \frac{\sin^2(\theta)\,d\theta}{(t+r\cos(\theta))^4}$$

Recalling (4.138), the above estimate, combined with (4.194), gives (4.191). (The same procedure is used to estimate derivatives of $v_{2,4}(t,r)$ in the region $\frac{t}{2} < r < t$). Finally, to establish (4.192), we use the identical procedure used to establish the analogous estimates in Lemma 5. In particular, we do the same procedure which starts with (4.47). (The amount of high frequency decay in the estimate (4.188) is sufficient for this). $\square$

We can now define and estimate the linear error term associated to $u_{N_0,corr}$. For ease of notation, let

$$h_2(t) := \lambda(t)\langle m_{\leqslant 1}(\frac{R\lambda(t)}{2h(t)})e_{N_0}(t,R\lambda(t)),\phi_0(R)\rangle_{L^2(RdR)}$$

We first recall the definition of $u_{N_0,corr}$ in (4.182):

$$u_{N_0,corr}(t,r) = \left(u_{N_0,ell}(t,r) - \frac{rh_2(t)}{4}\right)m_{\leqslant 1}(\frac{2r}{t}) + v_{2,4}(t,r)$$

Then, we define the linear error term associated to $u_{N_0,corr}$ by

$$e_{N_0,corr}(t,r) = -\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}\right)u_{N_0,corr}(t,r) + m_{\leqslant 1}(\frac{r}{2h(t)})e_{N_0}(t,r)$$

We therefore get

$$e_{N_0,corr}(t,r) = \partial_t^2\left(u_{N_0,ell}(t,r) - \frac{r}{4}h_2(t)\right)m_{\leqslant 1}(\frac{2r}{t}) - 2\partial_t\left(u_{N_0,ell}(t,r) - \frac{r}{4}h_2(t)\right)m'_{\leqslant 1}(\frac{2r}{t})\cdot\frac{2r}{t^2}$$

$$+ \left(u_{N_0,ell}(t,r) - \frac{r}{4}h_2(t)\right)\left(m''_{\leqslant 1}(\frac{2r}{t})\frac{4r^2}{t^4} + \frac{4r}{t^3}m'_{\leqslant 1}(\frac{2r}{t}) - \frac{4m''_{\leqslant 1}(\frac{2r}{t})}{t^2}\right)$$

$$- 4\partial_r\left(u_{N_0,ell}(t,r) - \frac{r}{4}h_2(t)\right)\frac{m'_{\leqslant 1}(\frac{2r}{t})}{t} - \frac{1}{r}m'_{\leqslant 1}(\frac{2r}{t})\frac{2}{t}\left(u_{N_0,ell} - \frac{r}{4}h_2(t)\right)$$

$$+ \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2}\right)\left(v_{2,4}(t,r) - \frac{r}{4}h_2(t)\right)m_{\leqslant 1}(\frac{2r}{t})$$

$$+ \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2}\right)v_{2,4}(t,r)\left(1 - m_{\leqslant 1}(\frac{2r}{t})\right)$$

where we used the fact that

$$(m_{\leqslant 1}(\frac{2r}{t}) - 1)\cdot m_{\leqslant 1}(\frac{r}{2h(t)}) = 0$$

We write

$$e_{N_0,corr}(t,r) = e_{N_0,corr,1}(t,r) + e_{N_0,corr,2}(t,r)$$

where

$$e_{N_0,corr,2}(t,r) = \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2}\right)v_{2,4}(t,r)\left(1 - m_{\leqslant 1}(\frac{2r}{t})\right) \qquad (4.195)$$

Then, it turns out that $e_{N_0,corr,1}$ is perturbative, as per the following lemma.

Lemma 47 *For $k = 0, 1$,*

$$||L^k_{\frac{1}{\lambda(t)}}(e_{N_0,corr,1})(t,r)||_{L^2(rdr)} \leqslant \frac{C\lambda(t)^{2-k}\log^{9/2}(t)}{t^6}\left(\sup_{x \in [100,t]}(\lambda(x)\log(x))\right)^3$$

Proof This is a direct consequence of Lemma 44 and 46   □

We need to add one more term to our ansatz in order to eliminate $e_{N_0,corr,2}$, given in (4.195). In particular, we define $u_{N_0,corr,2}$ to be the solution to the following equation with 0 Cauchy data at infinity.

$$-\partial_t^2 u_{N_0,corr,2} + \partial_r^2 u_{N_0,corr,2} + \frac{1}{r}\partial_r u_{N_0,corr,2} - \frac{1}{r^2}u_{N_0,corr,2} = e_{N_0,corr,2}(t,r) \tag{4.196}$$

Then, we prove the following lemma.

Lemma 48 *We have the following estimates on $u_{N_0,corr,2}$. For $0 \leqslant k \leqslant 1$ and $0 \leqslant j \leqslant 1$,*

$$|\partial_t^j \partial_r^k u_{N_0,corr,2}(t,r)| \leqslant \frac{Cr^{1-k}\lambda(t)^2 \log^3(t)\left(\sup_{x \in [100,t]}(\lambda(x)\log(x))\right)^3}{t^{5/2}\langle t-r\rangle^{7/2+j}}, \quad r \leqslant t \tag{4.197}$$

In addition,

$$|\partial_t^2 u_{N_0,corr,2}(t,r)| + |\partial_r^2 u_{N_0,corr,2}(t,r)| \leqslant \frac{C\lambda(t)^2 \log^3(t)\left(\sup_{x \in [100,t]}(\lambda(x)\log(x))\right)^3}{\langle t-r\rangle^{9/2}t^{5/2}}, \quad r \leqslant t \tag{4.198}$$

For all $r > 0$,

$$|u_{N_0,corr,2}(t,r)| + \sqrt{E(u_{N_0,corr,2}, \partial_t u_{N_0,corr,2})} \leqslant \frac{C\lambda(t)^2 \log^3(t)\left(\sup_{x \in [100,t]}(\lambda(x)\log(x))\right)^3}{t^3} \tag{4.199}$$

Finally,

$$||\partial_r u_{N_0,corr,2}(t,r)||_{L_r^\infty} \leqslant \frac{C\log^6(t)}{t^{\frac{5}{2}-5C_u}} \tag{4.200}$$

Proof As in Lemma 42, we have

$$u_{N_0,corr,2}(t,r) = \frac{-r}{2\pi}\int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}}\int_0^1 d\beta \int_0^{2\pi} I_{e_{N_0,corr,2}}(s,r\beta,\rho,\theta)d\theta$$

where we recall the notation (2.2). From Lemma 46, we get the following estimate.

$$|I_{e_{N_0,corr,2}}(s,r\beta,\rho,\theta)| \leqslant C\frac{\lambda(s)^2 \log^3(s)\left(\sup_{x \in [100,s]}(\lambda(x)\log(x))\right)^3}{s^{9/2}\langle t-r\rangle^{7/2}}, \quad s \geqslant t \geqslant r, \quad \rho \leqslant s-t, \quad 0 \leqslant \beta \leqslant 1$$

Using (4.3), we get (4.197) for $k = j = 0$. The formulae

$$\partial_r u_{N_0,corr,2}(t,r) = \frac{-1}{2\pi}\int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}}\int_0^{2\pi} d\theta I_{e_{N_0,corr,2}}(s,r,\rho,\theta) \tag{4.201}$$

and

$$\partial_t u_{N_0,corr,2}(t,r) = \frac{-r}{2\pi}\int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}}\int_0^1 d\beta \int_0^{2\pi} \partial_1 I_{e_{N_0,corr,2}}(s,r\beta,\rho,\theta)d\theta,$$

along with the same procedure used to establish (4.197) for $k = j = 0$, implies (4.197) for all larger k, j in the lemma statement. Next, we note that

$$u_{N_0,corr,2}(t,r) = \frac{-1}{2\pi}\int_0^\infty dw \int_0^w \frac{\rho d\rho}{\sqrt{w^2 - \rho^2}}\int_0^{2\pi} \frac{e_{N_0,corr,2}(t+w, \sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)})}{\sqrt{r^2 + \rho^2 + 2r\rho\cos(\theta)}}(r + \rho\cos(\theta))\,d\theta.$$

Differentiation under the integral sign, combined with the procedure used to prove (4.197) gives the estimate on $\partial_t^2 u_{N_0,corr,2}$ in (4.198). We estimate $\partial_r^2 u_{N_0,corr,2}$ by directly differentiating (4.201). Next,

$$||e_{N_0,corr,2}(t,r)||_{L^2(rdr)} \leqslant \frac{C\lambda(t)^2 \log^3(t)\left(\sup_{x \in [100,t]}(\lambda(x)\log(x))\right)^3}{t^4}$$

and this implies (4.199). Finally, using Lemma 46, we get

$$|I_{e_{N_0,corr,2}}(s,r,\rho,\theta)| \leqslant \frac{C\lambda(s)^2 \log^6(s)}{s^{\frac{9}{2}-3C_u}}, \quad s \geqslant t, \quad r > 0, \rho \leqslant s - t, \theta \in [0, 2\pi]$$

and this gives

$$|\partial_r u_{N_0,corr,2}(t,r)| \leqslant C \int_t^\infty ds(s-t) \frac{\log^6(s)}{s^{\frac{9}{2}-5C_u}} \leqslant \frac{C \log^6(t)}{t^{\frac{5}{2}-5C_u}}$$

□

Let $m_{\geqslant 1} : \mathbb{R} \to \mathbb{R}$ be any function satisfying

$$m_{\geqslant 1}(x) = \begin{cases} 1, & x \geqslant \frac{1}{4} \\ 0, & x \leqslant \frac{1}{8} \end{cases} \quad , \quad m_{\geqslant 1} \in C^\infty(\mathbb{R}) \tag{4.202}$$

Then, we will add $m_{\geqslant 1}(\frac{r}{t})u_{N_0,corr,2}(t,r)$ to our ansatz in order to eliminate the error term $e_{N_0,corr,2}$. Accordingly, we define the error term, $err_{N_0,corr,2}$, associated to $m_{\geqslant 1}(\frac{r}{t})u_{N_0,corr,2}(t,r)$, by

$$err_{N_0,corr,2}(t,r) = -\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}\right)\left(m_{\geqslant 1}(\frac{r}{t})u_{N_0,corr,2}(t,r)\right) + e_{N_0,corr,2}(t,r)$$

$$= -\left(\frac{m''_{\geqslant 1}(\frac{r}{t})}{t^2}\left(1 - \frac{r^2}{t^2}\right) + \frac{m'_{\geqslant 1}(\frac{r}{t})}{rt}\left(1 - \frac{2r^2}{t^2}\right)\right)u_{N_0,corr,2}(t,r)$$

$$- \frac{2m'_{\geqslant 1}(\frac{r}{t})}{t}\left(\partial_r u_{N_0,corr,2}(t,r) + \frac{r}{t}\partial_t u_{N_0,corr,2}(t,r)\right)$$

$$- \left(\frac{1 - \cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}\right)m_{\geqslant 1}(\frac{r}{t})u_{N_0,corr,2}(t,r)$$

where we used the fact that

$$(1 - m_{\geqslant 1}(\frac{r}{t}))e_{N_0,corr,2}(t,r) = 0$$

which follows from the definition of $m_{\geqslant 1}$, and $e_{N_0,corr,2}$, which we recall is given in (4.195). The following lemma shows that $err_{N_0,corr,2}$ is small enough to be treated with our final, perturbative argument.

Lemma 49 *We have the following estimates for $k = 0, 1$.*

$$\|L_{\frac{1}{\lambda(t)}}^k(err_{N_0,corr,2})(t,r)\|_{L^2(rdr)} \leqslant \frac{C\lambda(t)^2(1 + \lambda(t)^2)\log^3(t)\left(\sup_{x\in[100,t]}(\lambda(x)\log(x))\right)^3}{t^{6+k}}$$

Proof We directly apply Lemma 48. The only point to note is that, to estimate $\left(\frac{1-\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}\right)m_{\geqslant 1}(\frac{r}{t})u_{N_0,corr,2}(t,r)$, we use (4.197) in the region $r \leqslant t - t^{3/7}$ and (4.199) for the region $r \geqslant t - t^{3/7}$. □

4.10 Second set of nonlinear interactions

The next step is to treat the nonlinear interactions between the newest addition to our ansatz, namely

$$u_n(t,r) := u_{N_0}(t,r) + u_{N_0,corr}(t,r) + m_{\geqslant 1}(\frac{r}{t})u_{N_0,corr,2}(t,r), \tag{4.203}$$

and u_a. The interactions between $v_{2,4}$ (part of $u_{N_0,corr}$) and itself, along with the interactions between $v_{2,4}$ and $u_{a,0}$ are not quite perturbative, but the rest of the nonlinear interactions are. In order to show this, we note that

$$\sin(2Q_1(\frac{r}{\lambda(t)}) + 2u_a + 2u_n) = 2r^2(N_0 + N_1 + N_2 + N_3) + \cos(2Q_1(\frac{r}{\lambda(t)})) \cdot 2(u_a + u_n) + \sin(2Q_1(\frac{r}{\lambda(t)}))$$

where N_0 is defined in (4.172), N_1 is defined in (4.169), and we define

$$N_2(t,r) := \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(-2(u_{a0} + v_{2,4}) + \sin(2(u_{a,0} + v_{2,4})) - (\sin(2u_{a,0}) - 2u_{a,0}))$$

$$+ \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2}(\cos(2u_{a,0} + 2v_{2,4}) - \cos(2u_{a,0})) \tag{4.204}$$

We therefore have

$$N_3(t,r) = \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} \left(\cos(2u_a + 2u_n) - \cos(2u_a) - (\cos(2u_{a,0} + 2v_{2,4}) - \cos(2u_{a,0})) \right)$$
$$+ \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} \left(\sin(2u_a + 2u_n) - 2(u_a + u_n) - (\sin(2u_a) - 2u_a) \right.$$
$$\left. - (\sin(2u_{a,0} + 2v_{2,4}) - 2(u_{a,0} + v_{2,4})) + \sin(2u_{a,0}) - 2u_{a,0} \right)$$

where we recall the definition of u_a, from (4.168). We define u_{new} by

$$u_{new}(t,r) := u_n(t,r) - v_{2,4}(t,r)$$

We remark that N_2 contains the interactions between $v_{2,4}$ and $u_{a,0}$, except for the $u_{a,0}$ self-interactions, which were already contained in N_0. To give the reader an idea of how we estimate N_3, we re-write its expression as follows.

$$N_3(t,r)$$
$$= \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} \left((\cos(2u_a) - 1)((\cos(2u_{new}) - 1)\sin(2v_{2,4})) \right.$$
$$+ (\cos(2u_{new} + 2v_{2,4}) - 1)(\sin(2u_{a,0})(\cos(2u_{a,1}) - 1))$$
$$+ \sin(2v_{2,4})(\cos(2u_{a,0})(\cos(2u_{a,1}) - 1)) + \sin(2u_{a,0})((\cos(2u_{new}) - 1)\cos(2v_{2,4}))$$
$$\left. + (\cos(2u_{new}) - 1)\sin(2v_{2,4}) - 2u_{new} + \sin(2u_{new}) \right)$$
$$+ \frac{\sin(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} \left(\cos(2(u_a + u_n)) - \cos(2u_a) - \cos(2u_{a,0} + 2v_{2,4}) + \cos(2u_{a,0}) \right)$$
$$+ \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{2r^2} \left((\cos(2u_a) - 1)\sin(2u_{new})\cos(2v_{2,4}) + (\cos(2(u_{new} + v_{2,4})) - 1)\cos(2u_{a,0})\sin(2u_{a,1}) \right.$$
$$- \sin(2v_{2,4})\sin(2u_{a,0})\sin(2u_{a,1}) - \sin(2u_{a,0})\sin(2u_{new})\sin(2v_{2,4})$$
$$\left. + \sin(2u_{new})(\cos(2v_{2,4}) - 1) \right)$$
$$\tag{4.205}$$

Lemma 50 *We have the following estimates.*

$$\left\| \frac{N_3(t,r)}{\lambda(t)^2} \right\|_{L^2(rdr)} \leqslant \frac{C\log^{24}(t)}{t^{9/2 - 15C_u - 2C_l}}, \quad \left\| L_{\frac{1}{\lambda(t)}}(N_3(t,r)) \right\|_{L^2(rdr)} \leqslant \frac{C\log^{24}(t)}{t^{9/2 - 15C_u - C_l}}$$

Proof We note that, as per (4.202),

$$1 - m_{\geqslant 1}(\frac{r}{2t}) = 0, \quad r \geqslant \frac{t}{2}$$

Using Lemma 42, Lemma 44, and Lemma 48, we get

$$|u_{new}(t,r)| \leqslant \begin{cases} \frac{Cr\log^4(t)\left(\sup_{x \in [100,t]}(\lambda(x)\log(x))\right)^3}{t^4}, & r \leqslant \frac{t}{2} \\ \frac{\left(\sup_{x \in [100,t]}(\lambda(x)\log(x))\right)^3 \left(\sup_{x \in [100,t]}(\lambda(x)^2)\right)^3 \log^9(t)}{t^2}, & \frac{t}{2} \leqslant r \end{cases} \tag{4.206}$$

We then straightforwardly estimate the expression (4.205), for N_3 (as well as its r derivative), using (4.171), (4.170), and Lemma 46 (to estimate $u_{a,1}$, $u_{a,0}$, and $v_{2,4}$, respectively). □

Next, we add a correction, u_{N_2} to improve the error term N_2, defined in (4.204). In particular, we define u_{N_2} to be the solution to the following equation with 0 Cauchy data at infinity.

$$-\partial_t^2 u_{N_2} + \partial_r^2 u_{N_2} + \frac{1}{r}\partial_r u_{N_2} - \frac{u_{N_2}}{r^2} = N_2(t,r) \tag{4.207}$$

Then, we have the following lemma.

Lemma 51 *For $0 \leqslant k \leqslant 1$, and $r \leqslant t$*

$$r^k |\partial_r^k u_{N_2}(t,r)|$$
$$\leqslant Cr(r + \lambda(t))\left(\frac{\log^7(t)}{t^{5-3C_u}}\right) + \frac{Cr\lambda(t)}{t^2\langle t-r\rangle^4}\left(t^{4C_u}\log^8(t) + \frac{t^{6C_u}\log^{12}(t)}{\langle t-r\rangle^2}\right) + \frac{Cr\log^{10}(t)t^{5C_u}}{\langle t-r\rangle^{9/2}t^{3/2}}\left(1 + \frac{t^{4C_u}\log^8(t)}{\langle t-r\rangle^4}\right)$$

In addition, for all $r > 0$,

$$|u_{N_2}(t,r)| + \sqrt{E(u_{N_2}, \partial_t u_{N_2})} \leqslant \frac{C \log^{18}(t)}{t^{2-9C_u}}$$

Finally,

$$||\partial_r u_{N_2}(t,r)||_{L_r^\infty} \leqslant \frac{C \log^{18}(t)}{t^{\frac{3}{2}-9C_u}} \tag{4.208}$$

Proof The lemma is proven with the same procedure as in Lemma 42, using the estimates from Lemma 46 and (4.170). $\square$

It only remains to estimate the linear error term associated to $u_{N_2}(t,r)$ as well as its nonlinear interactions with all of the previous terms added into our ansatz. We start with the linear error term, which we denote by e_{N_2}:

$$e_{N_2}(t,r) := \left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})) - 1}{r^2} \right) u_{N_2}(t,r)$$

By directly estimating e_{N_2} using the estimates from Lemmas 51, we get the following lemma.

Lemma 52 *For $k = 0, 1$,*

$$||L_{\frac{1}{\lambda(t)}}^k(e_{N_2})(t,r)||_{L^2(rdr)} \leqslant \frac{C \log^{18}(t)}{t^{5-11C_u-kC_l}}$$

4.11 Third set of nonlinear interactions

Finally, it remains to treat the nonlinear interactions between u_{N_2} and all of the previous terms of our ansatz (the sum of which is equal to $u_a + u_n$). I.e., it suffices to estimate N_4 given by

$$\sin(2Q_1(\frac{r}{\lambda(t)}) + 2u_a + 2u_n + 2u_{N_2})$$
$$= 2r^2 (N_0 + N_1 + N_2 + N_3) + \cos(2Q_1(\frac{r}{\lambda(t)})) \cdot 2(u_a + u_n + u_{N_2}) + \sin(2Q_1(\frac{r}{\lambda(t)})) + 2r^2 N_4(t,r)$$

So,

$$N_4(t,r) = \frac{1}{2r^2} \left(\sin(2Q_1(\frac{r}{\lambda(t)}) + 2u_a + 2u_n)(\cos(2u_{N_2}) - 1) + \cos(2Q_1(\frac{r}{\lambda(t)}))(\sin(2u_{N_2}) - 2u_{N_2}) \right.$$
$$\left. + \left(\cos(2Q_1(\frac{r}{\lambda(t)}) + 2u_a + 2u_n) - \cos(2Q_1(\frac{r}{\lambda(t)})) \right) \sin(2u_{N_2}) \right)$$

Lemma 53
$$\lambda(t)^{-2}||N_4(t,r)||_{L^2(rdr)} \leqslant \frac{C \log^{30}(t)}{t^{\frac{9}{2}-2C_l-15C_u}}, \quad ||L_{\frac{1}{\lambda(t)}}(N_4)(t,r)||_{L^2(rdr)} \leqslant \frac{C \log^{30}(t)}{t^{\frac{9}{2}-15C_u-C_l}}$$

Proof We have

$$|N_4(t,r)| \leqslant \frac{C}{r^2} \left(\left(\frac{r\lambda(t)}{(r^2 + \lambda(t)^2)}|u_{N_2}| + u_{N_2}^2 \right) (|u_{N_2}| + |u_a| + |u_n|) + (u_a^2 + u_n^2)|u_{N_2}| \right)$$

where we recall the definitions of u_a and u_n in (4.168), and (4.203), respectively. We then use Lemmas 51 and 46, and estimates (4.170), (4.171), and (4.206), to estimate u_{N_2} and the various terms in u_a and u_n, and then straightforwardly estimate N_4, and $L_{\frac{1}{\lambda(t)}} N_4$. $\square$

We recall that u_a is defined in (4.167), u_n is defined in (4.203), and u_{N_2} is defined in (4.207), and define our final ansatz by the following.

$$u_{ansatz}(t,r) := u_a(t,r) + u_n(t,r) + u_{N_2}(t,r) \tag{4.209}$$

The error term of u_{ansatz}, denoted by F_5, is equal to the following.

$$F_5(t,r) = e_{N_0} \left(1 - m_{\leqslant 1}(\frac{r}{2h(t)}) \right) + e_{N_0,corr,1} + err_{N_0,corr,2} + N_1 + N_3 + e_{N_2} + N_4 + e_a \tag{4.210}$$

Lemma 54 *[Estimates on the error of u_{ansatz}] There exists $C_5 > 0$ such that, if*

$$\delta := \frac{1}{2} min\{4\alpha - C_l - 2, 1 - 2C_u - \alpha, 4 - 5C_u - 6\alpha, \frac{1}{2} - 2C_l - 15C_u\}$$

then, $2\delta > C_l$, and

$$\frac{||F_5(t,r)||_{L^2(rdr)}}{\lambda(t)^2} \leqslant \frac{C_5 \log^{30}(t)}{t^{4+2\delta}}, \quad ||L_{\frac{1}{\lambda(t)}} F_5(t,r)||_{L^2(rdr)} \leqslant \frac{C \log^2(t)}{t^{2+5\alpha}} + \frac{C \log^6(t)}{t^{4+2\alpha-3C_u}} + \frac{C \log^{30}(t)}{t^{9/2-C_l-15C_u}}$$

Proof The inequality $2\delta > C_l$ follows from (1.7) and (4.2). The *i*th term in (4.210) is estimated in the *ith* Lemma in the following list: Lemma 43, 47, 49, 41, 50, 52, 53, 40. □

We finish this section with some estimates on various quantities involving u_{ansatz}.

Lemma 55 *We have the following estimates.*

$$||\frac{8\lambda(t)u_{ansatz}(t,r)}{r(r^2+\lambda(t)^2)}||_{L_r^\infty} \leqslant \frac{4}{t^2}\left(M^2 + \frac{C_2}{3}\left(3+2\pi^2\right)\right) + \frac{C\log(t)}{t^{2+2\alpha}} + \frac{C\log^6(t)}{t^{\frac{7}{2}-4C_u}} \qquad (4.211)$$

$$||\frac{u_{ansatz}(t,r)}{r}||_{L_r^\infty} \leqslant \frac{C\log^6(t)}{t^{\frac{3}{2}-3C_u}}$$

$$||\partial_r u_{ansatz}(t,r)||_{L_r^\infty} \leqslant \frac{C\log^6(t)}{t^{\frac{1}{2}-3C_u}} \qquad (4.212)$$

$$||\frac{u_{ansatz}(t,r)\partial_r u_{ansatz}(t,r)}{r}||_{L_r^\infty(\{r\leqslant\lambda(t)\})} \leqslant \frac{C\log^7(t)}{t^{\frac{5}{2}-4C_u}} \qquad (4.213)$$

$$||\frac{u_{ansatz}(t,r)\partial_r u_{ansatz}(t,r)}{r^2}||_{L_r^\infty(\{r\geqslant\lambda(t)\})} \leqslant \frac{C\log^7(t)}{t^{\frac{5}{2}-3C_u}} \qquad (4.214)$$

$$\lambda(t)||\frac{\partial_r u_{ansatz}(t,r)}{r^2+\lambda(t)^2}||_{L_r^\infty} \leqslant \frac{1}{2t^2}\left(M^2 + \frac{C_2}{3}(3+2\pi^2)\right) + \frac{C\log^6(t)}{t^{\frac{5}{2}-4C_u}}$$

Proof We recall (4.7), and note that

$$\frac{8\lambda(t)u_{ell}(t,R\lambda(t))}{R\lambda(t)^3(R^2+1)} = \frac{4}{(1+R^2)}\left(\frac{\lambda'(t)}{\lambda(t)}\right)^2 + \frac{\lambda''(t)}{\lambda(t)}m_1(R)$$

where

$$m_1(R) = \frac{8\text{Li}_2\left(-R^2\right) + \frac{8}{3}\left(3R^2+\pi^2\right) - \frac{4(R^4-1)\log(R^2+1)}{R^2}}{(1+R^2)^2} := \frac{num_1(R)}{(1+R^2)^2}$$

We start with

$$num_1'(R) = \frac{8(1+R^2)(R^2-(1+R^2)\log(1+R^2))}{R^3}$$

Then, since

$$\partial_R\left(R^2-(1+R^2)\log(1+R^2)\right) = -2R\log(1+R^2) \leqslant 0$$

and

$$\left(R^2-(1+R^2)\log(1+R^2)\right)\Big|_{R=0} = 0$$

we have

$$R^2-(1+R^2)\log(1+R^2) \leqslant 0, \quad R \geqslant 0$$

Therefore,

$$num_1'(R) \leqslant 0, \quad R > 0$$

So,

$$num_1(R) \geqslant num_1(1) = 2\pi^2 + 8 > 0, \quad 0 < R \leqslant 1$$

Therefore,

$$R \mapsto \frac{num_1(R)}{(1+R^2)^2} = m_1(R) \text{ is non-increasing on } (0,1]$$

Hence, for $0 \leqslant R \leqslant 1$,

$$|m_1(R)| \leqslant \lim_{R\to 0^+} m_1(R) = \frac{4(3+2\pi^2)}{3}$$

If $R \geqslant 1$, then,

$$|num_1(R)| \leqslant |num_1(1)| + 8 \int_1^R \frac{(1+s^2)(s^2 + (1+s^2)\log(1+s^2))}{s^3} ds$$

$$\leqslant 2\left(2 + \pi^2 + 2R^2 + 2\log(1+R^2) + 2(R^2 + 2\log(1+R^2))\log(1+R^2)\right) := m_2(R)$$

If $1 \leqslant R \leqslant 2$, then,

$$\frac{d}{dR}\left(\frac{m_2(R)}{(1+R^2)^2}\right) = \frac{-8R(\pi^2 + \log(1+R^2)(-3 + R^2 + 4\log(1+R^2)))}{(1+R^2)^3} \leqslant \frac{R}{(1+R^2)^3}\left(-8\pi^2 + 24\log(5)\right) < 0$$

Therefore, for $1 \leqslant R \leqslant 2$,

$$|m_1(R)| \leqslant \frac{|num_1(R)|}{(1+R^2)^2} \leqslant \frac{m_2(R)}{(1+R^2)^2} \leqslant \frac{m_2(1)}{2^2} = 2 + \frac{\pi^2}{2} + 2\log^2(2) + \log(4)$$

Finally, if $R \geqslant 2$, then,

$$|m_1(R)| \leqslant \frac{m_2(R)}{(1+R^2)^2} \leqslant \frac{4}{5} + \frac{2}{25}\left(2 + \pi^2\right) + \frac{8\log^2(5)}{25} + \frac{4\log(5)}{5} + \frac{4\log(5)}{25} \tag{4.215}$$

This gives (4.211). To prove (4.212), we estimate $\partial_r u_a(t,r)$ using the procedure detailed in the proof of Lemma 41. We then estimate $\partial_r u_n(t,r)$ using (4.177), (4.200), (4.206) and Lemma 46. Finally, we estimate $\partial_r u_{N_2}$ using (4.208). With the same procedure, we also get (4.213) and (4.214). Next,

$$\frac{\partial_r u_{ell}(t,r)}{r^2 + \lambda(t)^2}\Big|_{r=R\lambda(t)} = \frac{\lambda'(t)^2}{2(1+R^2)\lambda(t)^3} + \frac{\lambda''(t)}{\lambda(t)^2}m_3(R)$$

for

$$m_3(R) = \frac{1}{(1+R^2)^3}\left(1 + 3R^2 + \frac{\pi^2}{3}(1-R^2) - \frac{(1+7R^2+7R^4+R^6)\log(1+R^2)}{2R^2} + (1-R^2)\mathrm{Li}_2(-R^2)\right)$$

With a similar procedure as that used to obtain (4.215), and its analogs for other regions of R, we get

$$|m_3(R)| \leqslant \frac{3 + 2\pi^2}{6}, \quad R > 0$$

□

The proof of our main proposition of this section, Proposition 1, is finished, once we establish (4.4). Let

$$v_{rad}(t,r) = v_2(t,r) + v_{2,2}(t,r) + v_{2,4}(t,r) \tag{4.216}$$

We first note that

$$u_{ansatz}(t,r) - v_{rad} = \chi_{\leqslant 1}\left(\frac{r}{\lambda(t)g(t)}\right)(u_{ell} + u_{ell,2}) + (1 - \chi_{\leqslant 1}\left(\frac{r}{\lambda(t)g(t)}\right))(w_1 + v_{ex} + u_{w,2})$$

$$- \chi_{\leqslant 1}\left(\frac{r}{\lambda(t)g(t)}\right)(v_2 + v_{2,2}) + f_{5,0} + f_{ex,sub} + f_{ell,2} + f_{2,2} + u_{N_0}$$

$$+ \left(u_{N_0,ell} - \frac{r\lambda(t)}{4}\langle m_{\leqslant 1}\left(\frac{R\lambda(t)}{2h(t)}\right)e_{N_0}(t,R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}\right)m_{\leqslant 1}\left(\frac{2r}{t}\right)$$

$$+ m_{\geqslant 1}\left(\frac{r}{t}\right)u_{N_0,corr,2} + u_{N_2}$$

Since, for non-constant λ, $\partial_t Q_{\frac{1}{\lambda(t)}}(r), \partial_t\left(\chi_{\geqslant 1}\left(\frac{r}{g(t)\lambda(t)}\right)w_1(t,r)\right) \notin L^2(rdr)$, the most delicate term to estimate will be the following, which *is* finite.

$$\left\|\partial_t\left(Q_{\frac{1}{\lambda(t)}}(r) + \chi_{\geqslant 1}\left(\frac{r}{g(t)\lambda(t)}\right)w_1(t,r)\right)\right\|_{L^2(rdr)} \tag{4.217}$$

We note that

$$\chi_{\geqslant 1}\left(\frac{r}{\lambda(t)g(t)}\right) = 1 - \chi_{\leqslant 1}\left(\frac{r}{\lambda(t)g(t)}\right) = 1, \quad r \geqslant 2\lambda(t)g(t)$$

Therefore, we will show in detail how to estimate (4.217), when the L^2 norm is taken over the region $r \geqslant 2g(t)\lambda(t)$, since the norm in the region $r \leqslant 2g(t)\lambda(t)$ is a direct estimation, using Lemma 2. From (4.24),

$$w_1(t,r) = -\frac{2}{r}\left(\lambda(t+r) - \lambda(t) - r\lambda'(t+r)\right) + 2r\int_1^\infty \lambda''(t+ry)(y - \sqrt{y^2-1})dy \tag{4.218}$$

and

$$Q_1(\frac{r}{\lambda(t)}) = \pi - 2\arctan(\frac{\lambda(t)}{r})$$

Therefore,

$$Q_1(\frac{r}{\lambda(t)}) + w_1(t,r) = \pi - 2\arctan(\frac{\lambda(t)}{r}) + 2\frac{\lambda(t)}{r}$$
$$- \frac{2}{r}\left(\lambda(t+r) - r\lambda'(t+r)\right) + 2r\int_1^\infty \lambda''(t+ry)(y - \sqrt{y^2-1})dy$$

Therefore,

$$\partial_t\left(Q_1(\frac{r}{\lambda(t)}) + w_1(t,r)\right) = \frac{2\lambda'(t)}{r}\left(\frac{\lambda(t)^2}{r^2 + \lambda(t)^2}\right) - \frac{2}{r}\left(\lambda'(t+r) - r\lambda''(t+r)\right)$$
$$+ 2r\int_1^\infty \lambda'''(t+ry)(y - \sqrt{y^2-1})dy$$

and this gives

$$||\partial_t\left(Q_1(\frac{r}{\lambda(t)}) + \chi_{\geqslant 1}(\frac{r}{\lambda(t)g(t)})w_1(t,r)\right)||_{L^2(r\geqslant 2\lambda(t)g(t),rdr)}^2 \leqslant \frac{C\log^2(t)}{t^{2-2C_u}}$$

We complete the proof of (4.4) by using (4.7) and (4.81) (for u_{ell}), (4.9), and the formulae following it, for $u_{ell,2}$, Lemma 2 (for w_1), Lemma 29, for v_{ex}, Lemma 10, for $u_{w,2}$, Lemma 5 for v_2, Lemma 26 for $v_{2,2}$, Lemma 33, for $f_{5,0}$, Lemma 35, for $f_{ex,sub}$, Lemma 37, for $f_{ell,2}$, Lemma 38, for $f_{2,2}$, Lemma 42, for u_{N_0}, Lemma 44, Lemma 48 for $u_{N_0,corr,2}$, Lemma 51 for u_{N_2}.

5 Constructing the exact solution

If we substitute

$$u(t,r) = Q_1(\frac{r}{\lambda(t)}) + u_{ansatz}(t,r) + v_6(t,r)$$

into (1.1), we get

$$-\partial_t^2 v_6 + \partial_r^2 v_6 + \frac{1}{r}\partial_r v_6 - \frac{\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}v_6 = F_5 + F_3(v_6) \tag{5.1}$$

where (note that the following expressions are essentially the same as F_3 on pg. 144 of [25])

$$F_3(f) = N(f) + L_1(f) \tag{5.2}$$

$$N(f) = \frac{\sin\left(2Q_1(\frac{r}{\lambda(t)}) + 2u_{ansatz}\right)}{2r^2}(\cos(2f) - 1) + \frac{\cos\left(2Q_1(\frac{r}{\lambda(t)})\right)}{2r^2}(\sin(2f) - 2f)$$

$$L_1(f) = -\frac{\sin\left(2Q_1(\frac{r}{\lambda(t)})\right)}{2r^2}\sin(2u_{ansatz})\sin(2f) + \frac{\cos\left(2Q_1(\frac{r}{\lambda(t)})\right)}{2r^2}(\cos(2u_{ansatz}) - 1)\sin(2f) \tag{5.3}$$

and F_5 is as in Proposition 1. We will solve (5.1) by first formally deriving the equation for y (namely (5.5)) given by

$$y(t,\xi) = \mathcal{F}(\sqrt{\cdot}v_6(t,\cdot\lambda(t)))(\xi\lambda(t)^2)$$

where $\mathcal{F}$ denotes the distorted Fourier transform of [16] (which is defined in section 5 of [16]). Then, we will prove that (5.5) admits a solution, say y_0 (with 0 Cauchy data at infinity) which has enough regularity to rigorously justify the statement that if v_6 given by the following expression, with $y = y_0$

$$v_6(t,r) = \sqrt{\frac{\lambda(t)}{r}}\mathcal{F}^{-1}\left(y(t,\frac{\cdot}{\lambda(t)^2})\right)\left(\frac{r}{\lambda(t)}\right), \tag{5.4}$$

then, v_6 is a solution to (5.1). We have (see also (5.4), (5.5), pg. 145 of [25])

$$\partial_{tt}y + \omega y = -\mathcal{F}(\sqrt{\cdot}F_5(t,\cdot\lambda(t)))(\omega\lambda(t)^2) + F_2(y)(t,\omega) - \mathcal{F}(\sqrt{\cdot}F_3(v_6(y))(t,\cdot\lambda(t)))(\omega\lambda(t)^2) \tag{5.5}$$

where $v_6(y)$, which appears in the argument of F_3, is the expression given in (5.4), and

$$F_2(y)(t,\omega) = -\frac{\lambda'(t)}{\lambda(t)}\partial_t y(t,\omega) + \frac{2\lambda'(t)}{\lambda(t)}\mathcal{K}\left(\partial_1 y(t,\frac{\cdot}{\lambda(t)^2})\right)(\omega\lambda(t)^2) + \left(\frac{-\lambda''(t)}{2\lambda(t)} + \frac{\lambda'(t)^2}{4\lambda(t)^2}\right)y(t,\omega)$$
$$+ \frac{\lambda''(t)}{\lambda(t)}\mathcal{K}\left(y(t,\frac{\cdot}{\lambda(t)^2})\right)(\omega\lambda(t)^2) + 2\frac{\lambda'(t)^2}{\lambda(t)^2}\left([\xi\partial_\xi,\mathcal{K}](y(t,\frac{\cdot}{\lambda(t)^2}))\right)(\omega\lambda(t)^2) \qquad (5.6)$$
$$- \frac{\lambda'(t)^2}{\lambda(t)^2}\mathcal{K}\left(\mathcal{K}(y(t,\frac{\cdot}{\lambda(t)^2}))\right)(\omega\lambda(t)^2)$$

where $\mathcal{K}$ is the transference operator of [16] (which is defined in section 6 of [16]). Next, we note that, by Proposition 5.7 b of [16], there exists $C_\rho > 0$ such that

$$\frac{\rho(\omega\lambda(t)^2)}{\rho(\omega\lambda(x)^2)} \leqslant C_\rho\left(\frac{\lambda(x)^2}{\lambda(t)^2} + \frac{\lambda(t)^2}{\lambda(x)^2}\right) \qquad (5.7)$$

Recall Proposition 1, and let $C_5' = C_5\sqrt{C_\rho}$. Now, we define the space Z in which we will solve (5.5). Let Z be the set of (equivalence classes) of measurable functions $y : [T_0,\infty) \times (0,\infty) \to \mathbb{R}$ such that

$$y(t,\omega)t^{2+2\delta}\sqrt{\rho(\omega\lambda(t)^2)}\langle\omega\lambda(t)^2\rangle \in C_t^0([T_0,\infty), L^2(d\omega))$$
$$\partial_t y(t,\omega)t^{3+2\delta}\left(1 + \sqrt{\omega}\lambda(t)\right)\sqrt{\rho(\omega\lambda(t)^2)} \in C_t^0([T_0,\infty), L^2(d\omega))$$

and $\|y\|_Z < \infty$ where

$$\|y\|_Z := \sup_{t\geqslant T_0}\left(t^{2+2\delta}\left(C_5'^{-1}\beta^{-1}\log^{-30}(t)\|y(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + \|\sqrt{\omega\lambda(t)^2}\sqrt{\langle\omega\lambda(t)^2\rangle}y(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2))}\right)\right.$$
$$\left. + t^{3+2\delta}\log^{-30}(t)\left(C_5'^{-1}\beta^{-1}\|\partial_t y(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + C_Z^{-1}\|\sqrt{\omega\lambda(t)^2}\partial_t y(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)\right) \qquad (5.8)$$

and β and C_Z are positive, but otherwise arbitrary, and will be further constrained later on. We start with the following estimates on F_2.

Lemma 56 *For all $y \in Z$, and $k = 0,1$, we have*

$$\|(\omega\lambda(t)^2)^{\frac{k}{2}}F_2(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\leqslant \|\langle\omega\lambda(t)^2\rangle^{k/2}\partial_1 y(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\frac{|\lambda'(t)|}{\lambda(t)}\left(1 + 2\|\mathcal{K}\|_{\mathcal{L}(L_\rho^{2,\frac{k}{2}})}\right)$$
$$+ \|\langle\omega\lambda(t)^2\rangle^{k/2}y(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\left(\left(\frac{\lambda'(t)}{\lambda(t)}\right)^2\left(\frac{1}{4} + 2\|[\xi\partial_\xi,\mathcal{K}]\|_{\mathcal{L}(L_\rho^{2,\frac{k}{2}})} + \|\mathcal{K}\|_{\mathcal{L}(L_\rho^{2,\frac{k}{2}})}^2\right) + \frac{|\lambda''(t)|}{\lambda(t)}\left(\frac{1}{2} + \|\mathcal{K}\|_{\mathcal{L}(L_\rho^{2,\frac{k}{2}})}\right)\right)$$

Proof This follows directly from (5.6), and Theorem 6.1, Proposition 6.2, and Proposition 5.7 of [16]. □

For $i = 1,2$, let $y_i \in Z$, and define f_i by

$$f_i(t,r) = \sqrt{\frac{\lambda(t)}{r}}\mathcal{F}^{-1}\left(y_i(t,\frac{\cdot}{\lambda(t)^2})\right)\left(\frac{r}{\lambda(t)}\right).$$

We now estimate $F_3(f_2) - F_3(f_1)$.

Lemma 57 *There exists $C > 0$ such that, for all $y_1, y_2 \in Z$,*

$$\|\mathcal{F}\left(\sqrt{\cdot}(L_1(f_2) - L_1(f_1))(t,\cdot\lambda(t))\right)(\omega\lambda(t)^2)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\leqslant \|(y_2 - y_1)(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\left(8\lambda(t)\|\frac{u_{ansatz}(t,r)}{r(r^2 + \lambda(t)^2)}\|_{L_r^\infty} + 2\|\frac{u_{ansatz}(t,r)}{r}\|_{L_r^\infty}^2\right)$$

$$\|\sqrt{\omega\lambda(t)^2}\mathcal{F}\left(\sqrt{\cdot}(L_1(f_2) - L_1(f_1))(t,\cdot\lambda(t))\right)(\omega\lambda(t)^2)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\leqslant C\|\sqrt{\langle\omega\lambda(t)^2\rangle}(y_2 - y_1)(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\cdot\left(\lambda(t)\|\frac{\partial_r u_{ansatz}(t,r)}{r^2 + \lambda(t)^2}\|_{L_r^\infty} + \|\frac{u_{ansatz}(t,r)\partial_r u_{ansatz}(t,r)}{r}\|_{L_r^\infty(r\leqslant\lambda(t))} + \lambda(t)\|\frac{u_{ansatz}(t,r)\partial_r u_{ansatz}(t,r)}{r^2}\|_{L_r^\infty(r\geqslant\lambda(t))}\right.$$
$$+ \left(\lambda(t)\|\frac{u_{ansatz}(t,r)}{r(r^2 + \lambda(t)^2)}\|_{L_r^\infty} + \|\frac{u_{ansatz}(t,r)}{r}\|_{L_r^\infty}^2\right)$$
$$\left.\cdot\left(\lambda(t)^2(\|y_2(t,\omega)\sqrt{\langle\omega\lambda(t)^2\rangle}\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}^2 + \|y_1(t,\omega)\sqrt{\langle\omega\lambda(t)^2\rangle}\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}^2) + 1\right)\right)$$

$$||\mathcal{F}\left(\sqrt{\cdot}\left(N(f_1)-N(f_2)\right)(t,\cdot\lambda(t))\right)(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

$$\leqslant C||\sqrt{\langle\omega\lambda(t)^2\rangle}(y_1-y_2)(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

$$\cdot\left(||\langle\omega\lambda(t)^2\rangle y_1(t,\omega)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\langle\omega\lambda(t)^2\rangle y_2(t,\omega)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right.$$

$$\left.+\left(||\frac{u_{ansatz}(t,r)}{r}||_{L^\infty_r}+\frac{1}{\lambda(t)}\right)\left(||\sqrt{\langle\omega\lambda(t)^2\rangle}y_1(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\sqrt{\langle\omega\lambda(t)^2\rangle}y_2(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)\right)$$

$$||\sqrt{\omega\lambda(t)^2}\mathcal{F}\left(\sqrt{\cdot}\left(N(f_2)-N(f_1)\right)(t,\cdot\lambda(t))\right)(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

$$\leqslant C||\langle\omega\lambda(t)^2\rangle(y_2-y_1)(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\left(||\langle\omega\lambda(t)^2\rangle y_2(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\langle\omega\lambda(t)^2\rangle y_1(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$

$$\cdot\left(\frac{1}{\lambda(t)}+||\partial_r u_{ansatz}(t,r)||_{L^\infty_r}+||\frac{u_{ansatz}(t,r)}{r}||_{L^\infty_r}\right)$$

$$+C||\langle\omega\lambda(t)^2\rangle(y_2-y_1)(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\left(||\langle\omega\lambda(t)^2\rangle y_2(t,\omega)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\langle\omega\lambda(t)^2\rangle y_1(t,\omega)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$

Proof We can read off most of the required estimates on L_1 and N from (5.83), (5.84), pg. 155 of [25], along with pgs. 180-187 of [25]. The only extra information we need is an estimate on

$$||\mathcal{F}\left(\sqrt{\cdot}\left(L_1(f_2)-L_1(f_1)\right)(t,\cdot\lambda(t))\right)(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

with quantitative constants, which is obtained by direct estimation of (5.3). $\square$

Now, we are ready to solve (5.5). For $y\in Z$, define $T(y)$ by

$$T(y)(t,\omega):=\int_t^\infty\frac{\sin((x-t)\sqrt{\omega})}{\sqrt{\omega}}\left(-\mathcal{F}(\sqrt{\cdot}\left(F_5+F_3(v_6(y))\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)+F_2(y)(x,\omega)\right)dx,\quad t\geqslant T_0 \tag{5.9}$$

Then, we have

Proposition 3 *There exists $\beta_0>0$ such that, for all $\beta>\beta_0$, there exists $C_9>0$ such that, for all $C_Z>C_9$, there exists $T_1>0$ such that, if $T_0>T_1$, then, T is a strict contraction on $\overline{B}_1(0)\subset Z$.*

Proof Let $y\in\overline{B}_1(0)\subset Z$. Any constant C appearing in this proof is *independent* of y and T_0. First,

$$||\partial_t^k T(y)(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

$$\leqslant\sqrt{C_\rho}\int_t^\infty(x-t)^{1-k}\left(\left(\frac{x}{t}\right)^{C_u}+\left(\frac{x}{t}\right)^{C_l}\right)\left(\frac{||F_5(x,r)||_{L^2(rdr)}}{\lambda(x)^2}+||\mathcal{F}(\sqrt{\cdot}F_3(x,\cdot\lambda(x)))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right.$$

$$\left.+||F_2(x,\omega)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right)dx,\quad k=0,1$$

where we used (5.7) and

$$\frac{|\partial_t^k\left(\sin((x-t)\sqrt{\omega})\right)|}{\sqrt{\omega}}\leqslant(x-t)^{1-k}$$

Since the inequality in (1.8) is strict, there exists a positive constant $\epsilon_4<\frac{1}{100}$ such that (1.8) is true with $\frac{179}{267}$ replaced by $\frac{179}{267}+4\epsilon_4$ on the left-hand side. Then, we directly apply Proposition 1 and Lemmas 57 and 56, to get

$$||\partial_t^k T(y)(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

$$\leqslant\frac{C_5'\log^{30}(t)}{t^{2+2\delta+k}}\left(\frac{1}{\frac{5}{2}-C_u}+\frac{1}{\frac{5}{2}}\right)+\frac{\beta C_5'\sqrt{C_\rho}\log^{30}(t)}{t^{2+k+2\delta}}\left(M(1+2||\mathcal{K}||_{\mathcal{L}(L^2_\rho)})+M^2\left(\frac{1}{4}+2||[\xi\partial_\xi,\mathcal{K}]||_{\mathcal{L}(L^2_\rho)}+||\mathcal{K}||^2_{\mathcal{L}(L^2_\rho)}\right)\right.$$

$$\left.+C_2\left(\frac{1}{2}+||\mathcal{K}||_{\mathcal{L}(L^2_\rho)}\right)+4\left(M^2+\frac{C_2}{3}(3+2\pi^2)\right)\right)$$

$$\cdot\left(\frac{1}{(3-C_u+2\delta-4\epsilon_4)}+\frac{1}{(3-C_l+2\delta-4\epsilon_4)}\right)+\frac{C(1+\beta^3)\log^{60}(t)}{t^{2+k+2\delta+\delta_3}}$$

$$\leqslant\frac{\beta C_5'\log^{30}(t)}{t^{2+k+2\delta}}\left(\frac{1}{3}-\epsilon_3\right)+\frac{C(1+\beta^3)\log^{60}(t)}{t^{2+k+2\delta+\delta_3}} \tag{5.10}$$

for some $\epsilon_3 > 0$, β sufficiently large, depending on ϵ_3, and if T_0 is further constrained to satisfy $T_0 > e^{\frac{15}{2\epsilon_4}}$. Here, $\delta_3 > 0$ is given by

$$\delta_3 = \min\{2\alpha,\ 1 - 6C_u,\ 2\delta - C_l\}$$

The ϵ_4 terms in (5.10) arise from noting that $x \mapsto \frac{\log^{30}(x)}{x^{4\epsilon_4}}$ is decreasing on $(e^{\frac{15}{2\epsilon_4}}, \infty)$. The last inequality in (5.10) is true by (1.8) and $\frac{1}{(3-C_u+2\delta-4\epsilon_4)} + \frac{1}{(3-C_l+2\delta-4\epsilon_4)} < \frac{179}{267} + 2\epsilon_4$. Next,

$$\|\omega\lambda(t)^2 T(y)(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

$$\leqslant C\lambda(t) \int_t^\infty \left(\frac{x}{t}\right)^{M+C_l} \left(\frac{\|L_{\frac{1}{\lambda(x)}} F_5(x,r)\|_{L^2(rdr)}}{\lambda(x)} + \|\sqrt{\omega\lambda(x)^2}\mathcal{F}(\sqrt{\cdot}F_3(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\|_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right.$$

$$\left. + \|\sqrt{\omega\lambda(x)^2}F_2(x,\omega)\|_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right) dx$$

$$\leqslant \frac{C}{t^{2+2\delta}} \left(\frac{\log^2(t)}{t^{-1-2\delta+5\alpha}} + \frac{\log^6(t)}{t^{1-2\delta+2\alpha-3C_u}} + \frac{\log^{30}(t)}{t^{\frac{3}{2}-2\delta-C_l-15C_u}} + \frac{(\beta^3 + 1 + C_Z)\log^{30}(t)}{t^{1-C_u}}\right)$$

$$(5.11)$$

where we used $|\sin((x-t)\sqrt{\omega})| \leqslant 1$. Similarly, there exist $C_7 > 0$ and $0 < \epsilon_1 < \frac{1}{3}$, both *independent* of C_Z, y and T_0, such that

$$\|\sqrt{\omega\lambda(t)^2}\partial_t T(y)(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

$$\leqslant C\int_t^\infty \left(\frac{x}{t}\right)^{M+C_l} \|\sqrt{\omega\lambda(x)^2}\mathcal{F}(\sqrt{\cdot}F_5(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\|_{L^2(\rho(\omega\lambda(x)^2)d\omega)}dx$$

$$+ \sqrt{C_\rho}\int_t^\infty \left(\left(\frac{x}{t}\right)^{2C_l} + \left(\frac{x}{t}\right)^{C_u+C_l}\right)\|\sqrt{\omega\lambda(x)^2}F_2(y)(x,\omega)\|_{L^2(\rho(\omega\lambda(x)^2)d\omega)}dx$$

$$+ \sqrt{C_\rho}\int_t^\infty \left(\left(\frac{x}{t}\right)^{2C_l} + \left(\frac{x}{t}\right)^{C_u+C_l}\right)\|\sqrt{\omega\lambda(x)^2}\mathcal{F}(\sqrt{\cdot}L_1(v_6(y))(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\|_{L^2(\rho(\omega\lambda(x)^2)d\omega)}dx$$

$$+ C\int_t^\infty \left(\frac{x}{t}\right)^{M+C_l}\|\sqrt{\omega\lambda(x)^2}\mathcal{F}(\sqrt{\cdot}N(v_6(y))(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\|_{L^2(\rho(\omega\lambda(x)^2)d\omega)}dx$$

$$(5.12)$$

$$\leqslant \frac{C_7(\beta^3 + 1)\log^{30}(t)}{t^{3+2\delta}} + \frac{C_Z(\frac{1}{3}-\epsilon_1)\log^{30}(t)}{t^{3+2\delta}} \leqslant \frac{C_Z\log^{30}(t)}{t^{3+2\delta}}\left(\frac{1}{3} - \frac{\epsilon_1}{2}\right)$$

where the second to last inequality in (5.12) follows from the strictness of the inequality in (1.9) and is true for T_0 sufficiently large, depending on absolute constants (not C_Z), with a similar argument used in (5.10). The last inequality in (5.12) is true, as long as $C_Z > \frac{2C_7(\beta^3+1)}{\epsilon_1}$, which we can enforce, recalling that C_7 and ϵ_1 are independent of C_Z. With the same argument used in (5.10), we get

$$\|\sqrt{\omega\lambda(t)^2}T(y)(t,\omega)\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C(1+\beta^3)\log^{30}(t)}{t^{3+2\delta-C_u}}$$

Thus, by (5.10), (5.11), and (5.12), there exists $\beta' > 0$ such that, for all $\beta > \beta'$, there exists $C_6 > 0$ such that, for all $C_Z > C_6$, there exists $T_2 > 0$ such that if $T_0 > T_2$ then, for all $y \in \overline{B}_1(0) \subset Z$, $\|Ty\|_Z < 1$. So, T maps $\overline{B}_1(0)$ into itself, and it remains to show that T is a strict contraction. If $y_2, y_1 \in \overline{B}_1(0)$, then, since the expression for F_2 ((5.6)) depends linearly on y, we have

$$T(y_2) - T(y_1) = \int_t^\infty \frac{\sin((x-t)\sqrt{\omega})}{\sqrt{\omega}}\left(F_2(y_2 - y_1)(x,\omega) - \mathcal{F}(\sqrt{\cdot}(N(v_6(y_2)) - N(v_6(y_1)))(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right.$$

$$\left. - \mathcal{F}(\sqrt{\cdot}(L_1(v_6(y_2)) - L_1(v_6(y_1)))(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right) dx$$

Then, by the same procedure used in (5.10), (5.11), and (5.12), we get, for β and C_Z sufficiently large, and some $\epsilon_2 > 0$,

$$\|\partial_t^k (T(y_2) - T(y_1))\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \|y_1 - y_2\|_Z \left(\frac{\beta C_5'\log^{30}(t)}{t^{2+k+2\delta}}\left(\frac{1}{3} - \epsilon_2\right) + \frac{C(1+\beta^3)\log^{60}(t)}{t^{2+k+2\delta+\delta_3}}\right), \quad k = 0, 1$$

$$\|\sqrt{\omega\lambda(t)^2}(T(y_2) - T(y_1))\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C(1+\beta^3)\|y_2 - y_1\|_Z \log^{30}(t)}{t^{3+2\delta-C_u}}$$

$$\|\omega\lambda(t)^2(T(y_2) - T(y_1))\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C\|y_1 - y_2\|_Z C_Z \log^{30}(t)}{t^{3+2\delta-C_u}}$$

$$\|\sqrt{\omega\lambda(t)^2}\partial_t (T(y_2) - T(y_1))\|_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C_Z \log^{30}(t)\|y_1 - y_2\|_Z}{t^{3+2\delta}}\left(\frac{1}{3} - \epsilon_2\right)$$

This completes the proof of the Proposition. $\quad\square$

By Proposition 3, completeness of $(\overline{B}_1(0), ||\cdot||_Z)$, and the Banach fixed point theorem, there exists $\beta > 0$ and $C_Z > 0$ such that, for all T_0 sufficiently large, there exists $y_0 \in \overline{B}_1(0)$ such that $T(y_0) = y_0$. By inspection of (5.9), and (5.5), y_0 is a solution to (5.5). By the derivation of (5.5) from (5.1), the function $v_{6,0}$, defined by (5.4) with $y = y_0$, for $r > 0$, is a solution to (5.1). We also note that, $v_{6,0}(t, \cdot)$ admits a continuous extension to $[0, \infty)$, by defining $v_{6,0}(t, 0) := 0$, by Lemma 5.1 of [25].

6 Decomposition of the solution as in 1.11

Let
$$u(t,r) := Q_1\left(\frac{r}{\lambda(t)}\right) + u_{ansatz}(t,r) + v_{6,0}(t,r)$$

In this section, we prove the following lemma, which finishes the proof of Theorem 1.

Lemma 58
$$E\left(u - Q_{\frac{1}{\lambda(t)}} - v_{rad}, \partial_t\left(u - v_{rad}\right)\right) \leqslant \frac{C\log^2(t)}{t^{2-2C_u}}$$

Proof By Lemma 5.1 of [25],

$$||v_{6,0}(t,r)||_{\dot{H}^1_e} = ||v_{6,0}(t,r\lambda(t))||_{\dot{H}^1_e} \leqslant C\left(||L(v_{6,0}(t,r\lambda(t)))||_{L^2(rdr)} + ||v_{6,0}(t,r\lambda(t))||_{L^2(rdr)}\right)$$

$$\leqslant C\lambda(t)||\langle\omega\lambda(t)^2\rangle y_0(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C\log^{30}(t)}{t^{2+2\delta-C_u}}$$

By the transference identity,

$$||\partial_t v_{6,0}(t,r)||_{L^2(rdr)} \leqslant C\frac{|\lambda'(t)|}{\lambda(t)}||\sqrt{\frac{\lambda(t)}{r}}\mathcal{F}^{-1}(y_0(t,\frac{\cdot}{\lambda(t)^2}))(\frac{r}{\lambda(t)})||_{L^2(rdr)} + C||\sqrt{\frac{\lambda(t)}{r}}\mathcal{F}^{-1}(\partial_1 y_0(t,\frac{\cdot}{\lambda(t)^2}))(\frac{r}{\lambda(t)})||_{L^2(rdr)}$$

$$+ C\frac{|\lambda'(t)|}{\lambda(t)}||\sqrt{\frac{\lambda(t)}{r}}\mathcal{F}^{-1}(\mathcal{K}(y_0(t,\frac{\cdot}{\lambda(t)^2})))(\frac{r}{\lambda(t)})||_{L^2(rdr)}$$

$$\leqslant \frac{C\log^{30}(t)}{t^{3+2\delta-2C_u}}$$

The lemma now follows from (4.4).   □

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