

Pursuer Coordination Against a Fast Evader via Coverage Control

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Abstract—Scalable pursuer coordination for reach-avoid games against a fast evader are developed leveraging coverage control over manifolds. The maintenance of a manifold, termed defense surface, prevents the evader and its target from occupying the same half-space and shown sufficient as a cooperative capture strategy. Nonlinear control synthesis continually reconfigures the pursuers to enable a defense surface via coverage. Simulation results empirically validate that the proposed condition serves as a surrogate objective for pursuer team coordination.

Index Terms—Cooperative Control, Nonlinear Systems, Game Theory

I. INTRODUCTION

Reach-Avoid (RA) games are a particular type of pursuit-evasion game in which one player, the evader, tries to reach a goal set while actively avoiding other bad sets [19]. The generalization of this type of dynamic game has applications ranging from safety in robotics to surveillance and defense [19]. Differential game theory (DGT) provides the most complete framework for studying pursuit-evasion [8]. However, optimal solutions relying on Hamilton-Jacobi-Isaacs methods commonly found in the dynamic game literature are not real-time enabled and do not scale well, especially when considering multiple agents in arbitrary dimensions with nonlinear dynamics. Furthermore, most pursuit-evasion strategies found in the literature are still limited to either two dimensions or simple dynamics [4], [5], [18] as well as assume the kinematic superiority of the pursuer team [1], [6], [14], [16].

The proposed approach, introduced in [12], tackles the fast-evader problem where pursuer coordination is needed to successfully trade off kinematic inferiority with numbers. The design of pursuer strategies is developed under the framework of swarm control. The objective is the design of local level rules for a team of pursuers that results in the desired global behavior (evader capture). This was achieved through the construction of a sufficient condition for pursuer win, then enforcing its satisfaction via coverage-control, that has been shown real-time enabled [17]. Then, tools from the multi-input, multi-output feedback linearization literature are used to synthesize node-level coverage control laws for players with control-affine non-

linear dynamics, which preserve the same exponential tracking properties as previously proposed coverage control laws [3]. Thus, the advantages of the proposed approach are scalability to any arbitrary number of agents, generalizability for games in any dimension, and tracking performance guarantees for a class of nonlinear systems. This is achieved at the expense of optimality provided by the DGT framework.

The contributions of this work are: First, a surrogate objective for pursuer team coordination is provided that relies on the existence of any boundary separating the evader from its target set. Second, a coordination strategy that meets the surrogate objective through the use of coverage-control is outlined. Lastly, an approach for enforcing coverage-control on arbitrary dimensions for a class of nonlinear control-affine agents is provided.

The remainder of this paper is organized as follows. The RA game definitions along with the sufficient condition for pursuer win are provided Section II. This section also provides the coverage-control preliminaries along with the elements of a pursuer coordination strategy. Section III provides a controller synthesis strategy for nonlinear plants enforcing coverage-control dynamics on an arbitrary domain and its tracking performance analysis. Two realizations of pursuer coordination strategies and their empirical evaluation are included in Section IV. Concluding remarks are provided in Section V.

The following notation will be used for the remainder of the paper. Letter superscripts denote game-related group membership, where superscript (e) denotes evader, and (i) is the i^{th} member of the pursuer team. There are N pursuers, and the pursuer team index is provided by the set $\mathcal{N} = \{1, \dots, N\}$. Subscripts are used for elements of a vector or set, where x_i denotes the i^{th} element of vector x . Sets are written in script font, e.g., $\mathcal{A}$, and the symbol (∂) denotes the set boundary.

II. RA GAMES FOR NONLINEAR SYSTEMS

The considered dynamics for all agents in this work are provided by the following definition:

Definition II.1 (System Dynamics).

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u(t), & x \in \mathbb{R}^n, u \in \mathcal{U} \subset \mathbb{R}^m, \\ y = h(x), & y \in \mathbb{R}^m, \end{cases} \quad (1)$$

where x is the state variable, u is the input variable, y is the output variable, and $h(x)$ is assumed at least C^1 . It is further assumed that the dynamics are Lipschitz continuous, and $u(t) \in \mathcal{U}$ is a measurable function. The focus of this work will be on square systems where both $y, u \in \mathbb{R}^m$. System (1) will be referred to as Σ .

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A common type of nonlinear dynamics of class Σ is given by a unicycle dynamics with bounded controls and position output. For this case, $x \in \mathbb{R}^3$, x_1, x_2 are position states, x_3 is the angular orientation state, $u \in \mathbb{R}^2$, u_1 is linear velocity, u_2 is angular velocity, $\mathcal{U} \subset \mathbb{R}^2$, and the dynamics Σ are given by

$$f(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, g(x) = \begin{bmatrix} \cos(x_3) & 0 \\ \sin(x_3) & 0 \\ 0 & 1 \end{bmatrix}, y(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (2)$$

Next, the specific class of pursuit-evasion game considered in this work is introduced, along with conditions for pursuer win.

A. RA Games

Definition II.2 (ε -Capture). Consider pursuers and evaders with dynamics of class Σ , and output function $h(x)$. ε -Capture is defined as $\min_{i \in \mathcal{N}} \|h(x^i(t)) - h(x^e(t))\| \leq \varepsilon$.

Definition II.3 (RA Game [19]). Consider a team of N pursuers and a single evader, all with dynamics of class Σ . The RA Game is defined by the tuple $\{\Sigma, \mathcal{D}, \mathcal{P}\}$. Σ corresponds to the nonlinear dynamic's class (1). $\mathcal{D} \subseteq \mathbb{R}^p$ is a compact game domain such that $h(x) \in \mathcal{D}$. $\mathcal{P} \subset \mathbb{R}^p$ is a compact target set for the evader. The evader only wins if $h(x^e) \in \mathcal{P}$ while avoiding ε -Capture.

Remark 1. Note that the RA Game evolves over the output of the dynamical system Σ . These outputs can be defined over a domain of any dimension.

In what follows, a sufficient condition for ε -Capture is provided through domain partitioning. Throughout this section, the reader is referred to Fig. 1 for conceptual clarity.

The minimum time to reach a δ -ball around the point $m \in \mathcal{D}$ for the dynamical system Σ is defined as the solution to the problem $T(m, \delta, \Sigma) = \min t$, s.t. $\|h(x(t)) - m\| \leq \delta$ given Σ .

Definition II.4 (Capture Set). Given a pursuer with dynamics and output (Σ^i, y^i) , and an evader with dynamics and output (Σ^e, y^e) , the pursuer capture set is given by $\mathcal{Y}_\delta^i = \{y \subseteq \mathbb{R}^p \mid T(y, \delta, \Sigma^i) = T(y, \delta, \Sigma^e)\}$. A capture set for the pursuer team is defined as $\mathcal{Y}_\delta := \bigcup_{i \in \mathcal{N}} \mathcal{Y}_\delta^i$.

Definition II.5 (Defense Surface). For the RA Game $\{\Sigma, \mathcal{D}, \mathcal{P}\}$, a defense surface $\mathcal{M}(t)$ at time t is defined as a boundary that partitions the game domain $\mathcal{D}$ into two nonoverlapping sets $\mathcal{D}_{\{1,2\}}$, i.e., $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ and $\mathcal{D}_1 \cap \mathcal{D}_2 = \mathcal{M}$, such that $\mathcal{P} \subset \mathcal{D}_2$ and $\mathcal{P} \cap \mathcal{D}_1 = \emptyset$.

Definition II.6 (Pursuer Defendable). Consider a pursuer team capture set $\mathcal{Y}_\delta$. A defense surface $\mathcal{M}(t)$ is pursuer-defendable if $\exists t_m$ such that $\forall t \in [t_m, \infty)$, $\mathcal{M} \subseteq \mathcal{Y}_\delta$, and $h(x^e(t_m)) \in \mathcal{D}_1$.

The following lemma states that if an evader trajectory crosses the defense surface at a time when it is pursuer-defendable, then there is at least one pursuer who can get within δ of that location at the same time.

Lemma II.1. Recalling t_m from Definition II.6, for a pursuer-defendable surface $\mathcal{M}(t)$, at a time $t_e \geq t_m$, if $\exists m \in \mathcal{M}(t_e)$ s.t. $\|h(x^e(t_e)) - m\| = 0$, then $\exists i \in \mathcal{N}$ and $\exists u^i(t) \in [t_m, \infty)$, s.t. $\|h(x^i(t_e)) - h(x^e(t_e))\| \leq 2\delta$.

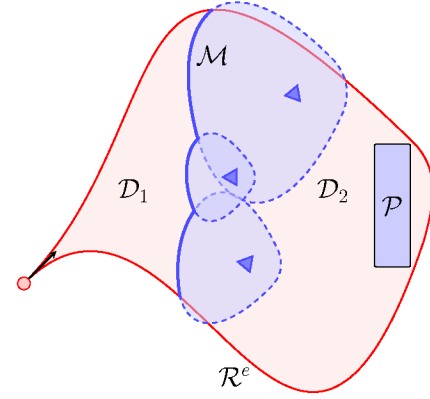


Fig. 1. Set Relationships for RA Games in Finite Time. The red circle corresponds to the position of the evader, the blue triangles the position of the pursuers. The shaded red region corresponds to the evader's constrained reachable set $\mathcal{R}^e$, the shaded blue region correspond to the pursuers' capture set, and the thick blue line is a pursuer-defendable defense surface $\mathcal{M}$.

Proof. By the definition of the pursuer-defendable defense surface, $\exists i \in \mathcal{N}$ such that $m \in \mathcal{Y}_\delta^i$. By the definition of the Capture Set $\mathcal{Y}_\delta^i$, this implies $\exists u^i(t)$ for $t \in [t_m, \infty)$ such that $\|h(x^i(t_e)) - m\| \leq \delta$. Thus, if at $t_e = t_m$, $\|h(x^e(t_e)) - m\|$, by the pursuer-defendable property of the defense surface that would imply $\|h(x^i(t_e)) - h(x^e(t_e))\| \leq \delta$. If $\|h(x^i(t_m)) - h(x^e(t_m))\| > 2\delta$, then for $t_e > t_m$ there exists a $u^i(t)$ such that $\|h(x^i(t_e)) - h(x^e(t_e))\| \leq 2\delta$ by the triangle inequality. $\square$

Proposition II.2 (ε -Capture Sufficient Condition). For an RA Game $\{\Sigma, \mathcal{D}, \mathcal{P}\}$, assume at a time t_m there exists a defense surface $\mathcal{M}$ that is pursuer-defendable with $\delta = \varepsilon/2$. Then, for any trajectory of the evader $h(x^e(t))$, $t \geq t_m$ for which $h(x^e(t)) \in \mathcal{P}$, $\exists i \in \mathcal{N}$ and $\exists u^i(t)$ that leads to ε -Capture.

Proof. Given the dynamics Σ^e are Lipschitz continuous and $h(x^e(t)) \in \mathcal{P}$, by the definition of the defense surface $\mathcal{M}(t)$, $\exists t_e \geq t_m$ such that $h(x^e(t_e)) \cap \mathcal{M}(t_e) = m$. Applying Lemma II.1 leads to the existence of $i \in \mathcal{N}$ for which $u^i(t)$ for $t \geq t_m$ leads to $\|h(x^i(t_e)) - h(x^e(t_e))\| \leq 2\delta = \varepsilon$. $\square$

Remark 2. Proposition II.2 is consistent with pursuer coordination strategies that aim at encircling an evader [4].

It has been shown that it is not possible to achieve point capture using a team of pursuers with inferior kinematic capability than the evader on open domains [7]. However, by imposing a maximum time on the game, we can construct a bounded set of the evader's output while reaching the target set. This avails teams of pursuers, with inferior kinematic capability than the evader, to satisfy the conditions in Proposition II.2.

B. RA Games in Finite Time

Definition II.7 (RA Game in Finite Time). The RA Game in Finite Time is defined as the RA Game $\{\Sigma, \mathcal{D}, \mathcal{P}, t_f\}$ in which the evader only wins if it reaches a desired target set $\mathcal{P}$ before a predefined final time t_f .

Definition II.8 (Constrained Reachable Set). *Given the dynamical system Σ with output $y = h(x)$, the constrained reachable set is defined as*

$$\mathcal{R}(t_0, \mathcal{X}_0, \mathcal{X}_f) = \{y \in \mathbb{R}^p \mid \dot{x} = f(x) + g(x)u(t), \quad h(x(t_0)) \in \mathcal{X}_0, h(x(t_f)) \in \mathcal{X}_f, u(t) \in \mathcal{U}, t \in [t_0, t_f]\}. \quad (3)$$

Corollary 1. *Assuming a pursuer-defendable defense surface $\mathcal{M}(t)$ exists at a time $t_m \leq t_f$, then the ε -Capture sufficient condition holds for the RA game in Finite Time given by $\{\Sigma, \mathcal{R}^e, \mathcal{P}, t_f\}$.*

Proof. Trivially follows from changing the domain definition in Proposition (II.2). $\square$

Theorem II.3. *For an RA Game in Finite Time $\{\Sigma, \mathcal{R}^e, \mathcal{P}, t_f\}$, the existence of a defense surface $\mathcal{M}$ that is pursuer-defendable with $\delta = \varepsilon/2$ for a time $t \in [t_m, t_f]$ provides a sufficient condition for a winning pursuer strategy.*

Proof. Verifying the conditions in Proposition II.2 and Corollary 1 leads to the existence of a pursuer-defendable defense surface for $t \in [t_m, \infty)$ for the RA Game, and $t \in [t_m, t_f]$ for the RA Game in Finite Time. This implies the evader $h(x^e(t))$ trajectory needs to pass through $\mathcal{M}$ for the evader to win. If it does, this leads to a pursuer achieving ε -Capture as stated in Lemma II.1. If the evader $h(x^e(t))$ trajectory does not pass through $\mathcal{M}$, it cannot reach $\mathcal{P}$ by t_f and loses. Both cases lead to a winning pursuer strategy. $\square$

The significance of the Theorem II.3 is that knowledge of the actual $u^i(t)$ trajectory that leads to ε -Capture or an explicit construction of the pursuer-defendable defense surface are not needed for a pursuer winning strategy. Only the existence of a pursuer-defendable defense surface needs to be verified. The sufficient condition in Theorem II.3 can be used as a surrogate design objective for feedback controllers of the pursuer team.

C. Pursuer Coordination for RA Games in Finite Time

The proposed coordination strategy ensures scalability in satisfying a global objective through the design of local-level rules. The global objective is to verify the sufficient conditions in Theorem II.3, that can be reduced to ensuring the existence of a pursuer-defendable defense surface. From Definition II.6, this can be interpreted as the pursuer team capture set being able to cover a defense surface, motivating the use of coverage control, a framework from the multi-agent robotics community with demonstrated scalability [17], introduced next.

The proposed coverage approach is based on *locational cost* minimization as described in [2]. The performance metric to be minimized by the the multi-agent team is given by

$$\mathcal{H}(p, t) = \sum_{i=1}^N \int_{\Omega^i(p, t)} \|q - p^i\|^2 \rho(q, t) dq, \quad (4)$$

where $p^i(t) \in \mathbb{R}^m$ is the location of the agents on the coverage domain, and $\Omega^i(p, t)$ forms a proper partition of the coverage convex domain $\Omega(t) \subset \mathbb{R}^m$. It is assumed in this work that the density function $\rho(q, t) = 1$, but instead the coverage domain $\Omega(t)$ is of time varying nature [17]. The centers of mass $c^i(p, t) = \int_{\Omega^i(p, t)} q \rho(q, t) dq / \int_{\Omega^i(p, t)} \rho(q, t) dq \in \Omega^i(p, t)$

provide the optimal coverage configuration of positions $p^i(t)$ within the domain in terms of minimizing (4). Exponential convergence to this configuration can be achieved for dynamics $\dot{p}(t)$ as described in Lemma II.4.

Lemma II.4 (from [17]). *For the aggregated location state $p \in \mathbb{R}^{m \cdot N}$, exponential convergence $\|c(p, t) - p(t)\| = e^{-\kappa t} \|c(p, 0) - p(0)\|$ can be achieved by*

$$\dot{p} = \left(I - \frac{\partial c(p, t)}{\partial p} \right)^{-1} \left(\kappa (c(p, t) - p) + \frac{\partial c(p, t)}{\partial t} \right), \quad (5)$$

for some gain $\kappa > 0$, as long as $p(t)$ is initialized sufficiently close to $c(p, t)$.

Remark 3. *Previous work has shown how the dynamics in (5) can be derived analytically in a fully distributed fashion over one-dimensional domains [12], and distributed approximations can be obtained over fixed domains [3] and time-varying domains [17] in any dimension.*

The link between the sufficient conditions for a winning pursuer strategy outlined in Theorem II.3 and coverage control can be made explicit from the following topological observation. Based on Definition II.6, conditions for a potentially simply connected (i.e., no holes) defense surface $\mathcal{M}$ to be pursuer-defendable are for the pursuer team capture set $\mathcal{Y}_\delta$ to also be simply connected while sufficiently spanning the domain. By sufficiently spanning the domain, it is meant that the a boundary that partitions the game domain needs to exists inside $\mathcal{Y}_\delta$. In what follows, this topological observation is leveraged to provide necessary conditions for the existence of a pursuer-defendable defense surface.

First, we provide two lemmas used in the construction of the next proposition.

Lemma II.5. *Let us consider the projection operator $Proj_\Omega(y) = \arg \min_{p \in \Omega} \|p - y\|$, where $\Omega \subset \mathbb{R}^m$ is a compact convex set. We further assume Ω is a convex polytope that can be written in the form $Ap + b \geq 0$, where $\text{rank}(A) = m$. Then, a necessary condition for a set $\mathcal{A}$ to be simply connected is for the image $Proj_\Omega(\mathcal{A})$ to be simply connected.*

Proof. First, note that the minimization of the projection operator is equivalent to minimizing $f(p) = \frac{1}{2} \|p - y\|^2$ that is strongly convex; thus, a solution always exists and is unique. Now, the solution follows from the constrained optimization problem $\mathcal{L} = \frac{1}{2} \|p - y\|^2 + \lambda^\top (Ap + b)$. The solution to this problem (p^*, λ^*) , yields

$$\begin{bmatrix} \frac{\partial}{\partial p} \mathcal{L}(p^*, \lambda^*) \\ \frac{\partial}{\partial \lambda} \mathcal{L}(p^*, \lambda^*) \end{bmatrix} = 0.$$

Differentiating this expression with respect to y , yields

$$\begin{bmatrix} \frac{\partial^2}{\partial p^2} \mathcal{L} & \frac{\partial}{\partial p \partial \lambda} \mathcal{L} \\ \frac{\partial}{\partial p \partial \lambda} \mathcal{L} & 0 \end{bmatrix} \begin{bmatrix} \frac{dp^*}{dy} \\ \frac{d\lambda^*}{dy} \end{bmatrix} + \begin{bmatrix} \frac{\partial}{\partial p \partial y} \mathcal{L} \\ \frac{\partial}{\partial \lambda \partial y} \mathcal{L} \end{bmatrix} = 0.$$

Note that the matrix constitutes the KKT conditions for a strongly convex, affine optimization problem; thus it is invertible [10, Section 15.1]. This implies the solution (p^*, λ^*) is differentiable with respect to y , which means it is continuous [11, Proposition 4.7.2]. Now that we have established the

projection operator is well defined and continuous, from [11, Th 4.2.1] we have that the image of a path-connected set through a continuous function is path-connected. The claim follows. $\square$

Lemma II.6. *Consider sets $\mathcal{A}, \mathcal{B} \in X$ and a continuous mapping $f : X \rightarrow Y$. Then, $f(\mathcal{A}) \cup f(\mathcal{B}) = f(\mathcal{A} \cup \mathcal{B})$.*

The following motivates the use of coverage control to enable the existence of a pursuer-defendable defense surface.

Proposition II.7. *Consider the locational cost (4), and the individual pursuer capture set provided by $\mathcal{Y}_\delta^i$. Assume the coverage domain $\Omega(t)$ is a convex defense surface. If $\exists t_m$ such that $\forall t \in [t_m, t_f]$ the partitions $\Omega^i(p, t) = \text{Proj}_\Omega(\mathcal{Y}_\delta^i)$, then $\text{Proj}_\Omega(\mathcal{Y}_\delta^i)$ satisfies the necessary conditions to be simply connected.*

Proof. From the theorem statement, we have $\bigcup_{i \in \mathcal{N}} \Omega_i(p, t) = \bigcup_{i \in \mathcal{N}} \text{Proj}_\Omega(\mathcal{Y}_\delta^i)$, from Lemma II.6, $\bigcup_{i \in \mathcal{N}} \text{Proj}_\Omega(\mathcal{Y}_\delta^i) = \text{Proj}_\Omega(\bigcup_{i \in \mathcal{N}} \mathcal{Y}_\delta^i)$; thus, $\Omega(t) = \text{Proj}_\Omega(\mathcal{Y}_\delta)$. Furthermore, given $\Omega_i(p, t)$ constitute a proper partition of the domain, $\Omega(t)$ is simply connected. The claim follows from Lemma II.5. $\square$

The significance of Proposition II.7 is that through coverage we can enable the topological conditions to make a defense surface be pursuer-defendable. Based on this proposition, the elements comprising the proposed coordination strategy are provided for the pursuer team.

Definition II.9 (Coordination Strategy). *For an RA Game in Finite Time $\{\Sigma, \mathcal{R}^e, \mathcal{P}, t_f\}$, a pursuer coordination strategy is given by the tuple $\{\Omega, \phi, p_d\}$, where*

- $\Omega(t) : \mathbb{R}_{\geq 0} \rightarrow \mathcal{R}^e \subset \mathbb{R}^m$ is an defense surface that will serve as the coverage domain.
- $\phi(x, t) : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \Omega$ is an output map from a pursuer state to the coordination domain Ω .
- $p_d^i(t) : \mathbb{R}_{\geq 0} \rightarrow \Omega$ is the desired output dynamics of x^i on the domain Ω .

In what follows, Section III characterizes conditions for which square nonlinear systems can maintain the exponential tracking performance of the controller in (5) given output maps $\phi(x^i, t)$. Since Proposition II.7 only provides necessary condition for maintaining capture set overlap, Section IV empirically analyzes the performance of two coordination strategy realizations $\{\Sigma, \mathcal{R}^e, \mathcal{P}, t_f\}$ for an increasing number of pursuers under suitably selected metrics. It is shown how performance monotonically improves with the number of pursuers, and how the strategy that better satisfies the conditions outlined in Theorem II.3 leads to the best capture performance.

III. COORDINATION TRACKING PERFORMANCE

In this section, we show how tools from MIMO feedback linearization can be used to enforce coverage dynamics on a manifold by a team of agents with nonlinear dynamics in the form of (1) while retaining exponential tracking of the coverage dynamics on the coordination domain. Output tracking is used to synthesize nonlinear controllers, which enforces the first-order coverage control dynamics on manifold $\Omega(t)$ as provided in Lemma II.4. This problem is addressed using the framework

of MIMO feedback linearization as presented in [15]. Let us define the Lie derivative of a function $h(x)$ with respect to $f(x)$ as $L_f h := \frac{\partial h}{\partial x} f$. For a system of the form (1), define the relative degree γ as the number of differentiations of the output mapping $y = h(x)$ required until $L_g L_f^{\gamma-1} h(x) \neq 0$, where $L_f^2 h = L_f(L_f h)$. In the MIMO case, define the total relative degree γ_T as the sum of the relative degrees for all outputs $\gamma_T = \sum_{j=1}^p \gamma_j$, where p is the dimensionality of the output map. For the remainder of this section, assume there exists a $\mathcal{C}^1$ output mapping $y = \phi(x, t) : \mathbb{R}^n \times [t_0, t_f] \rightarrow \Omega(t)$, where $\Omega(t)$ is an arbitrary coverage manifold. Note that for the i^{th} agent, we wish to enforce $\phi(x^i, t) = p_d^i(t)$.

Using the notation $x^{(n)} = \frac{d^n}{dt^n} x(t)$, the output mapping for a single agent

$$\begin{aligned} y &= [\phi_1(x, t) \quad \cdots \quad \phi_m(x, t)]^\top, \\ y_j^{(\gamma_j)} &= L_f^{\gamma_j} \phi_j(x, t) + \phi_j^{(\gamma_j)}(x, t) \\ &\quad + \sum_{k=1}^m L_{g_k} L_f^{\gamma_j-1} \phi_j(x, t) u_k, \end{aligned} \quad (6)$$

where γ_j is the relative degree for output j . Repeating this procedure for all outputs, we have

$$\begin{bmatrix} y^{(\gamma_1)} \\ \vdots \\ y^{(\gamma_m)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_f^{\gamma_1} \phi_1(x, t) + \phi_1^{(\gamma_1)}(x, t) \\ \vdots \\ L_f^{\gamma_m} \phi_m(x, t) + \phi_m^{(\gamma_m)}(x, t) \end{bmatrix}}_{L(x, t)} + E(x)u,$$

where $E(x) \in \mathbb{R}^{m \times m}$ and whose elements are obtained from the differentiation in (6). The first-order nature of the coverage control law in Lemma II.4 implies $\gamma_T \geq m$. Thus, two strategies will be provided for the cases where $\gamma_T = m$ and $\gamma_T > m$. Next, let us denote $y_d \in \mathbb{R}^m$ as the desired output signal, and $f_\phi(y_d, t)$ as the desired dynamics given by (5).

First, assume $\gamma_T = m$, and the decoupling matrix $E(x)$ in (6) is invertible. Then, for the desired first-order dynamics $\dot{y}_d = f_\phi(y_d, t)$, an output tracking controller is given by

$$u(x, t) = E(x)^{-1}(f_\phi(\phi, t) - L(x, t)), \quad (7)$$

where $\gamma_j = 1, \forall j \in \{1, \dots, m\}$. This follows from direct substitution of (7) in (6), which yields $\dot{y} = f_\phi(y, t)$.

In the case where the total relative degree $\gamma_T > m$, separation of timescale can be leveraged to synthesize a tracking controller that does not require higher order derivatives of the desired output dynamics. Assume system (1) can be partitioned into states $x = [x_1 \ x_2]^\top$ so that it has the structure

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2) + g(x)u(t), \\ y &= \phi(x_1, t), \end{aligned} \quad (8)$$

with the first derivative of the output given by

$$\dot{y} = L_{f_1} \phi(x_1, t) + \frac{\partial}{\partial t} \phi(x_1, t) = h(x_1, x_2, t). \quad (9)$$

Proposition III.1. *For system (8) with output derivative (9), assume $g(x)$ is invertible, and that there exists a mapping $x_2 = r(x_1, t)$ s.t. $L_{f_1} \phi(x_1, t) + \frac{\partial}{\partial t} \phi(x_1, t) = h(x_1, r(x_1, t), t)$, where*

$$\begin{aligned} h(x_1, r(x_1, t), t) &= f_\phi(\phi(x_1, t), t), \\ u(x, t) &= g(x)^{-1} [\bar{\kappa}(r(x_1, t) - x_2) - f_2(x_1, x_2)]. \end{aligned} \quad (10)$$

Then, if $\bar{\kappa} \gg 1$, system (8) tracks the desired output dynamics $\dot{y}_d = f_\phi(y_d, t)$ with tracking error $y(t) - y_d(t) \leq O(1/\bar{\kappa})$.

Proof. This follows by applying Tikhonov's theorem of singular perturbation [9] on the singularly perturbed system $\dot{y} = L_{f_1}\phi(x_1, t) + \frac{\partial}{\partial t}\phi(x_1, t)$, $\dot{x}_1 = f_1(x_1, x_2)$, $\varepsilon\dot{x}_2 = (r(x_1, t) - x_2)$, with output solution trajectory $y(t, \varepsilon)$, where $\varepsilon = 1/\bar{\kappa}$. Because $\phi(x_1, t)$ is $\mathcal{C}^1$, $(L_{f_1}\phi(x_1, t), \frac{\partial}{\partial t}\phi(x_1, t))$ are continuous, the reduced problem $\dot{\bar{y}} = f_\phi(\bar{y}, t)$, $\dot{\bar{x}}_1 = f_1(\bar{x}_1, r(\bar{x}_1, t))$ yields the desired output tracking trajectory $\bar{y}(t)$. Defining the boundary layer variable $\xi := x_2 - r(x_1, t)$, and scaled time $\tau := t/\varepsilon$, the boundary layer problem $\frac{d\xi}{d\tau} = \bar{\kappa}[r(x_1, t) - (\xi + r(x_1, t))] = -\bar{\kappa}\xi$, has the origin as an exponentially stable equilibrium point uniformly in (t, x_1) . Thus, the solution $y(t, \varepsilon) - \bar{y}(t) = O(\varepsilon)$ by Theorem 11.1 in [9]. $\square$

Corollary 2. *The coverage dynamics for the controller in Proposition III.1 exponentially converges to a ball of radius $O((1/\bar{\kappa})^2)$ with convergence rate κ for $\bar{\kappa} \gg 1$.*

Proof. The desired configuration dynamics have convergence rate of $\|c(\bar{y}, t) - \bar{y}(t)\| \leq e^{-\kappa t} \|c(\bar{y}, 0) - \bar{y}(0)\|$ by Lemma II.4. It follows that $\|c(\bar{y}, t) - \bar{y}(t)\|^2 = \sum_{i=1}^N \|c_i(\bar{y}, t) - \bar{y}_i(t)\|^2 \leq e^{-2\kappa t} \|c(\bar{y}, 0) - \bar{y}(0)\|^2$. From Proposition III.1, the output trajectories of the real system $y_i(t)$ track the desired dynamics $\bar{y}_i(t)$ by $y_i(t) - \bar{y}_i(t) = O(\varepsilon)$ for $\varepsilon := 1/\bar{\kappa}$. It now follows that

$$\begin{aligned} \sum_{i=1}^N \|c_i(y, t) - y_i(t)\|^2 &= \sum_{i=1}^N \|c_i(y, t) - \bar{y}_i(t) - O(\varepsilon)\|^2 \\ &\leq \sum_{i=1}^N [\|c_i(y, t) - \bar{y}_i(t)\| + \|O(\varepsilon)\|]^2 \\ &\leq \sum_{i=1}^N [\|c_i(y, t) - \bar{y}_i(t)\|^2 + \|c_i(y, t) - \bar{y}_i(t)\|O(\varepsilon) + O(\varepsilon^2)] \\ &\leq e^{-2\kappa t} \|c(\bar{y}, 0) - \bar{y}(0)\|^2 + Ne^{-\kappa t} \|c(\bar{y}, 0) - \bar{y}(0)\|O(\varepsilon) + O(\varepsilon^2) \\ &\leq \alpha e^{-\kappa t} + O(\varepsilon^2), \end{aligned}$$

where $\alpha = \|c(\bar{y}, 0) - \bar{y}(0)\|^2 + N\|c(\bar{y}, 0) - \bar{y}(0)\|O(\varepsilon)$. $\square$

For a coordination strategy $\{\Omega, \phi, p_d\}$, we can conclude that for square systems, if the total relative degree $\gamma_T = m$, exponential tracking of the desired output dynamics $p_d^i(t)$ can be achieved by (7). If the total relative degree $\gamma_T > m$, exponential tracking of the desired output dynamics $p_d^i(t)$ can be achieved by (10) as shown in Corollary 2.

IV. COORDINATION STRATEGY EVALUATION

We now provide two coordination strategies (called Type-I and Type-II) for RA Games in Finite Time. The objective of this section is to quantify the performance of the proposed coordination strategies, and to show that the best performance is attained by the strategy that best satisfies the conditions of Theorem II.3.

The Type-I coordination strategy consists of selecting a pursuer-defendable defense surface as the coordination domain, and the projection of the heading direction as the output map. The Type-II coordination strategy consists of selecting a line

segment that is a defense surface as the coordination domain, and the projection of the center of the pursuers capture sets as the output map. Expressions for the output mapping $\phi(x)$ and the pursuer inputs $u_i^j(t) \forall i \in \mathcal{N}$ for the Type-I and Type-II coordination strategies can be found in [13]. A pure-pursuit strategy (optimal pursuit strategy when capture time is minimized [8]) where each agent tracks an instantaneous angle $r(x) = \tan^{-1} \left(\frac{x_2 - x_2^e}{x_1 - x_1^e} \right)$ with a control law, $u_2^i = \kappa(r(x) - x_3^i)$ was also implemented as an uncoordinated approach to ensure performance improvements with increasing number of pursuers is not due to spatial diversity, but rather coordination.

The RA Game in Finite Time for $N \in \{2, 3, 4\}$ was implemented in simulation enforcing unicycle dynamics (2) on all players. It was assumed $\|u_1\| = 2\|u_2\|$, $\max \|u_2^e\| = \max \|u_2^i\|$, where $\|u_1^i\| = 0.1$, $\|u_2^i\| \leq 2\pi \forall t, i \in \mathcal{N}$. The final time of the engagement was varied from $t_f \in \{230, 265, 300\}$. The evader is initialized at x-position $x_1^e = -10$, and random y-position $x_2^e \sim U_d(-5, 5)$, where U_d is the uniform distribution. The evader employs random maneuvers while trying to reach a rectangular target set $\mathcal{P}$ with dimensions 1×10 , and center $(10, 0)$. Pursuers were initialized at equidistant locations inside $\mathcal{P}$, at random locations inside $\mathcal{P}$, and at random locations outside $\mathcal{P}$. Results are provided for 100 runs under each initial location assumption, for each pursuit strategy, and for each considered final time.

Two performance metrics were considered: miss distance $\min_{i \in \mathcal{N}, t \in [t_0, t_f]} \|y^i(t) - y^e(t)\|$ and a defense surface index. The defense surface index is defined as $\frac{1}{\Delta t} \sum_{t=t_m}^{t_f} I(t)$, where t_m is the first time the defense surface is pursuer defendable, t_f is the final time of the engagement, $\Delta t = t_f - t_m$ is the span of time the defense surface should be defended. $I(t) \in \{0, 1\}$ is an indicator function that has a value of 1 if the defense surface is pursuer-defendable, 0 if it is not, and t was discretely sampled in post-processing. This metric quantifies the ability of the coordination strategy to maintain the existence of a pursuer-defendable defense surface.

The simulation results under the aforementioned metrics are summarized in Fig. 2. For all simulation both the worst case and median miss distance improve under coordination (Type-I & II) over pure-pursuit, and the results were proven statistically significant under a matched-pair *t-test* with worst case p-value $< 4 \times 10^{-2}$. The miss distance is monotonically reduced with increasing number of pursuers in all scenarios. Coordination also lead to an improvement in the worst case, median, and lower quantile defense surface index for agents initialized inside the domain. For agents initialized outside the domain, the worst case index was zero for all considered pursuer strategies, but the median and lower quantile were improved under coordination.

The Type-II coordination strategy provides the highest median defense surface index. This is consistent with the Type-II coordination strategy also achieving the smallest miss distance over uncoordinated approaches. Also note that the Type-II median defense surface index monotonically increases with the number of pursuers, and leads to the most wins for an ε -Capture distance $\varepsilon = 0.3$. This validates the design objective for this approach that aims at improving the conditions for the existence of a pursuer-defendable defense as prescribed in

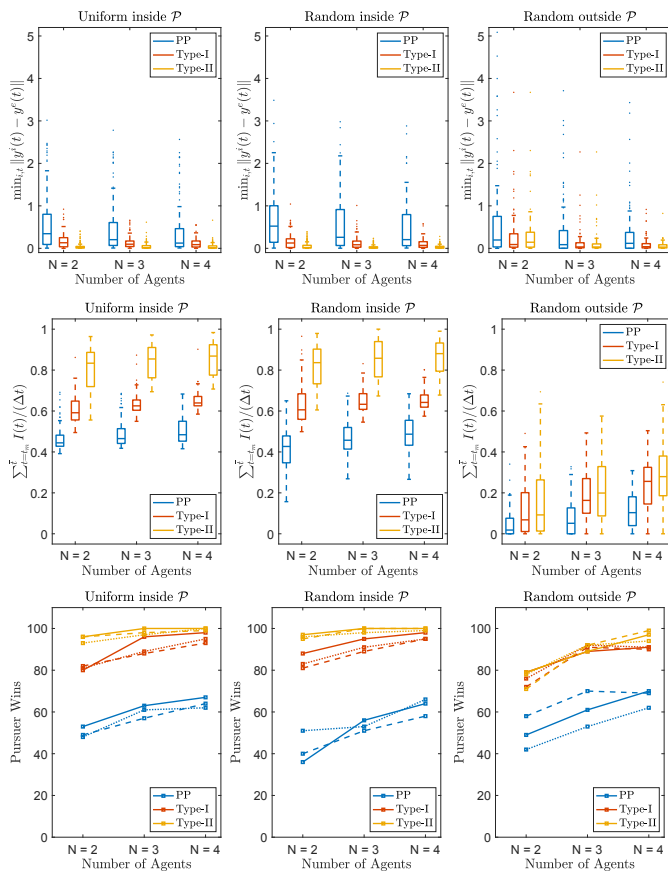


Fig. 2. Miss distance (first row), defense surface index (second row), and number of pursuer wins (third row) for increasing final time values (Solid line) $t_f = 230[s]$, (Dotted line) $t_f = 265[s]$, (Dashed line) $t_f = 300[s]$. The box correspond to the 25 and 75 percentile range of the data, whiskers extend to the minimum and maximum values while the dots represent outliers. The horizontal line inside each box represents median value. Pursuer initial location is the same column-wise and stated in the title of each plot.

Proposition II.7.

V. CONCLUSION

A class of scalable pursuer coordination strategies was introduced for RA Games in Finite Time. A sufficient condition for a pursuer team win was developed. It relies on the existence of a defense surface that partitions the game domain, separating the evader from the target set, and ensures pursuers can always reach the surface (pursuer defendability). One of the main contributions of this work is the notion that only the existence of the defense surface is needed, the defense surface does not need to be explicitly constructed. Pursuer coordination strategies that rely on coverage control, from the multi-agent literature, are proposed to preserve the existence of such surface given its ease of extendability to any arbitrary pursuer team size. Two coordination strategies are presented and evaluated through simulation for a single evader with twice the maximum speed as any player in the pursuer team. Both coordination strategies lead to increase in time a pursuer-defendable defense surface existed in the game (called defense surface index), and higher number of wins over the pure-pursuit strategy. The coordination strategy with the highest defense surface index led to the closest capture distances

and most number of pursuer wins. Thus, we conclude the presented analysis empirically validates the sufficient condition for pursuer team win against a fast evader introduced in this work.

REFERENCES

- [1] M. Chen, Z. Zhou, and C. J. Tomlin, "Multiplayer reach-avoid games via pairwise outcomes," *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1451–1457, 2016.
- [2] J. Cortes, S. Martinez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Transactions on Robotics and Automation*, vol. 20, no. 2, pp. 243–255, 2004.
- [3] Y. Diaz-Mercado, S. G. Lee, and M. Egerstedt, "Human–swarm interactions via coverage of time-varying densities," in *Trends in Control and Decision-Making for Human–Robot Collaboration Systems*. Springer, 2017, pp. 357–385.
- [4] E. Garcia and S. D. Bopardikar, "Cooperative containment of a high-speed evader," in *2021 American Control Conference (ACC)*. IEEE, 2021, pp. 4698–4703.
- [5] E. Garcia, D. W. Casbeer, and M. Pachter, "Optimal strategies for a class of multi-player reach-avoid differential games in 3d space," *IEEE Robotics and Automation Letters*, vol. 5, no. 3, pp. 4257–4264, 2020.
- [6] E. Garcia, D. W. Casbeer, A. Von Moll, and M. Pachter, "Multiple pursuer multiple evader differential games," *IEEE Transactions on Automatic Control*, 2020.
- [7] S. Y. Hayoun and T. Shima, "On guaranteeing point capture in linear n-on-1 endgame interception engagements with bounded controls," *Automatica*, vol. 85, pp. 122–128, 2017.
- [8] R. Isaacs, *Differential games: a mathematical theory with applications to warfare and pursuit, control and optimization*. Courier Corporation, 1999.
- [9] H. K. Khalil and J. Grizzle, *Nonlinear systems*. Prentice Hall Upper Saddle River, NJ, 2002, vol. 3.
- [10] D. G. Luenberger, Y. Ye *et al.*, *Linear and nonlinear programming*. Springer, 1984, vol. 2.
- [11] J. E. Marsden, M. J. Hoffman *et al.*, *Elementary classical analysis*. Macmillan, 1993.
- [12] P. Rivera-Ortiz and Y. Diaz-Mercado, "On guaranteed capture in multi-player reach-avoid games via coverage control," *IEEE Control Systems Letters*, vol. 2, no. 4, pp. 767–772, 2018.
- [13] P. Rivera-Ortiz, Y. Diaz-Mercado, and M. Kobilarov, "Multi-player pursuer coordination for nonlinear reach-avoid games in arbitrary dimensions via coverage control," in *2020 American Control Conference (ACC)*. IEEE, 2020, pp. 2747–2753.
- [14] D. Shishika and V. Kumar, "Local-game decomposition for multiplayer perimeter-defense problem," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 2093–2100.
- [15] J.-J. E. Slotine and W. e. a. Li, *Applied nonlinear control*. Prentice hall Englewood Cliffs, NJ, 1991, vol. 199, no. 1.
- [16] I. E. Weintraub, M. Pachter, and E. Garcia, "An introduction to pursuit-evasion differential games," *2020 American Control Conference (ACC)*, pp. 1049–1066, 2020.
- [17] X. Xu and Y. Diaz-Mercado, "Multi-agent control using coverage over time-varying domains," in *2020 American Control Conference (ACC)*. IEEE, 2020, pp. 2030–2035.
- [18] R. Yan, Z. Shi, and Y. Zhong, "Task assignment for multiplayer reach-avoid games in convex domains via analytical barriers," *IEEE Transactions on Robotics*, vol. 36, no. 1, pp. 107–124, 2019.
- [19] Z. Zhou, R. Takei, H. Huang, and C. J. Tomlin, "A general, open-loop formulation for reach-avoid games," in *2012 IEEE Conference on Decision and Control (CDC)*. IEEE, 2012, pp. 6501–6506.