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Well-posedness, ill-posedness, and traveling waves for models of pulsatile flow in viscoelastic vessels

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Abstract. We study dispersive models of fluid flow in viscoelastic vessels, derived in the study of blood flow. The unknowns in the models are the velocity of the fluid in the axial direction and the displacement of the vessel wall from rest. We prove that one such model has a well-posed initial value problem, while we argue that a related model instead has an ill-posed initial value problem; in the second case, we still prove the existence of solutions in analytic function spaces. Finally, we prove the existence of some periodic traveling waves.

Mathematics Subject Classification. 76Z05, 35Q35, 35A01, 35C07, 35B32.

1. Introduction

We consider a fluid-structure interaction problem with a fluid flowing within a viscoelastic vessel, motivated by hemodynamics. The specific models to be studied have been derived in [23], based on the prior work [22]. As shown in Fig. 1, we consider an axisymmetric flow. The model equations begin from the Navier–Stokes equations for incompressible flow, making a number of assumptions, such as laminar flow with small viscosity.

The vessel containing the fluid is taken to have a given undisturbed radius $r_0(x)$, and we study the displacement, $\eta(x,t)$, of this; we call the total radius of the vessel, then, $r^w(x,t) = r_0(x) + \eta(x,t)$. The horizontal component of the fluid velocity is $u(x,t)$; this is taken to be the horizontal velocity at a particular distance between the centerline of the vessel and the outer wall. A classical Boussinesq system of equations is derived, making assumptions on the scaling of the various velocities; this system is

$$\eta_t + \frac{1}{2}(r_0+\eta)u_x + (r_0+\eta)_x u = 0, \tag{1.1}$$

$$[1-\bar{\alpha}r_{0xx}]u_t + (\bar{\beta}\eta)_x + uu_x - \frac{(4\bar{\alpha}+r_0)r_0}{8}u_{xxt} + \frac{(3\bar{\alpha}+r_0)r_{0x}}{2}(\bar{\beta}\eta)_{xx} + \kappa u - \gamma\left(\bar{\beta}(r_{0x}u + \frac{r_0}{2}u_x)_x\right) = 0. \tag{1.2}$$

There are a number of parameters here which must be described. First, ρ is the density of the fluid while ρ^w is the density of the wall material, and h is the thickness of the wall. These are combined in the parameter $\bar{\alpha} = \frac{\rho^w h}{\rho}$, measuring the relative densities of the wall and the fluid. The parameter E measures elasticity of the wall, and then $\bar{\beta}$ (which is a function of x rather than being constant) is given by $\bar{\beta}(x) = \frac{Eh}{\rho r_0^2(x)}$. The parameters κ and γ are both viscosities, with κ being the fluid viscous frequency parameter (i.e., the Rayleigh damping coefficient). We have said that the wall of the vessel is taken to be viscoelastic, and γ measures the viscous properties of the wall.

Prior models have considered the vessel wall to be elastic, rather than viscoelastic [11,20,22]. However, accurate modeling of the anatomy of blood vessels requires the more detailed (viscoelastic) description. Specifically, as described in [8], there are three layers of a blood vessel, the tunica intima (inner layer), tunica media (middle layer), and tunica externa (outer layer), and the smooth muscle cells in the tunica

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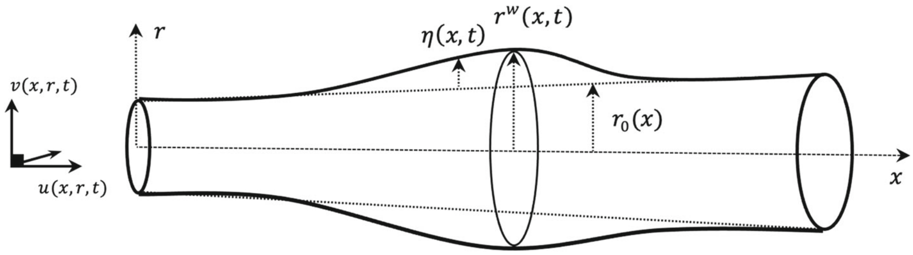

Fig. 1. Sketch of a vessel segment

media exhibit viscoelastic properties [28]. Furthermore, in some regimes, these viscoelastic properties are dominant as compared to purely elastic effects [3]. Nevertheless, if one wished to consider a purely elastic wall (as in the above references), our existence theorems are valid even without accounting for the viscous properties of the wall, i.e., with $\gamma = 0$.

In the case that r_0 is constant, the model simplifies considerably; notice that not only do derivatives of r_0 now vanish, but also $\bar{\beta}$ becomes constant so that its derivatives now also vanish. The result is

$$\eta_t + \frac{1}{2}(r_0 + \eta)u_x + \eta_x u = 0, \tag{1.3}$$

$$u_t + \bar{\beta}\eta_x + uu_x - \frac{(4\bar{\alpha} + r_0)r_0}{8}u_{xxt} + \kappa u - \gamma\left(\bar{\beta}\frac{r_0}{2}u_{xx}\right) = 0. \tag{1.4}$$

We prove three main results in the present work. First, for the model (1.3), (1.4) with constant r_0, we demonstrate well-posedness of the initial value problem in Sobolev spaces. Notably, by contrast, we provide evidence that the more general model (1.1), (1.2) instead has an ill-posed initial value problem. That an initial value problem is ill-posed does not imply that there are no solutions, however. An example of this is the classical vortex sheet initial value problem, which is known to be ill-posed in Sobolev spaces [10]. Existence of solutions for the vortex sheet problem may be established in analytic function spaces [13,27]. Similarly to [27], we prove existence of solutions for the initial value problem for (1.1), (1.2) in analytic spaces based on the Wiener algebra, making use of an abstract Cauchy–Kowalevski theorem [16]. The interested reader might also see [7,15] for other examples of model equations in free-surface fluid dynamics for which solutions have been proved to exist in analytic function spaces, when the well-posedness in spaces of finite regularity is in question.

The model (1.1), (1.2) is bidirectional, in that waves may propagate either to the left or the right. The authors of [23] also derive unidirectional models, related to the Korteweg–de Vries equation and the Benjamin–Bona–Mahony equation. These models are simpler and reduce to a single equation for η. We consider the contrast in the bidirectional case between well-posedness when r_0 is constant and likely ill-posedeness when r_0 is non-constant to be an interesting feature of the present work; this contrast is not present in the unidirectional models, as (relying on results such as those of [1,2,6], or [12]) the unidirectional models can be shown to be well-posed in either case. As the bidirectional models are therefore more interesting, we restrict our studies to them.

In addition to developing the models we study here, the authors of [23] also studied properties of traveling waves, including the case $\gamma = \kappa = 0$. For our third main result, then, we prove existence of such waves. Specifically, we prove existence of periodic traveling waves of the system (1.3), (1.4) in the case that $\gamma = \kappa = 0$. This is the doubly inviscid case, meaning that for the existence of traveling waves, we neglect the viscous properties of the fluid and of the vessel wall. We prove this by a "bifurcation from a simple eigenvalue" method, after studying the kernel of the linearized operator associated to (1.3), (1.4). In general, this operator has a two-dimensional kernel, but when $\gamma = \kappa = 0$, we may enforce symmetry, reducing the dimension of the kernel to one. Analytical studies of the traveling waves in the more general case, with the two-dimensional kernel, will be the subject of future work.

The plan of the paper is as follows. In Sect. 2, we prove well-posedness in Sobolev spaces of the initial value problem for the system (1.3), (1.4); the main theorems of this section are Theorem 2.7 demonstrating existence, and Theorem 2.8 demonstrating uniqueness and continuous dependence on the data. In Sect. 3, we give a calculation suggesting ill-posedness of the more general system (1.1), (1.2), and then prove existence of solutions for this system in analytic function spaces by application of an abstract Cauchy–Kowalevski theorem. The main theorem of Sect. 3 is Theorem 3.6. In Sect. 4, we prove existence of periodic traveling waves for the system (1.3), (1.4) when $\kappa = \gamma = 0$; this is the content of Theorem 4.2. We make some concluding remarks in Sect. 5.

2. Well-posedness in Sobolev spaces when $r(x)$ is constant

In this section, we use the energy method to prove well-posedness in Sobolev spaces of the spatially periodic initial value problem for the system (1.3), (1.4). We argue along the same lines as the second other used for a toy model for the vortex sheet with surface tension in [5].

We recall the model (1.3), (1.4), and we rearrange terms as follows:

$$\eta_t = -\frac{1}{2}r_0 u_x - \frac{1}{2}\eta u_x - \eta_x u,$$
$$u_t = \left(1 - \frac{(4\overline{\alpha} + r_0)r_0}{8}\partial_{xx}\right)^{-1}\left(-\overline{\beta}\eta_x - uu_x - \kappa u + \frac{\gamma\overline{\beta}r_0}{2}u_{xx}\right).$$

We introduce an approximate system, giving equations for η_t^ϵ and u_t^ϵ using mollifier operators $\mathcal{J}_\epsilon$ for any approximation parameter $\epsilon > 0$. (For a detailed description of mollifier operators and their properties, the interested reader could consult Chapter 3 of [21]; it is enough to say that they are self-adjoint smoothing operators and could be taken specifically to be truncation of the Fourier series at level $1/\varepsilon$.) Our approximate system is:

$$\eta_t^\epsilon = -\frac{1}{2}r_0 u_x^\epsilon - \frac{1}{2}\eta^\epsilon u_x^\epsilon - \mathcal{J}_\epsilon\left((\mathcal{J}_\epsilon \eta_x^\epsilon)u^\epsilon\right), \tag{2.1}$$

$$u_t^\epsilon = A^{-1}\left(-\overline{\beta}\eta_x^\epsilon - u^\epsilon u_x^\epsilon - \kappa u^\epsilon + \frac{\gamma\overline{\beta}r_0}{2}u_{xx}^\epsilon\right), \tag{2.2}$$

where $A^{-1} = \left[1 - \frac{(4\overline{\alpha}+r_0)r_0}{8}\partial_{xx}\right]^{-1}$. The system (2.1), (2.2) is taken with initial conditions, namely

$$\eta^\epsilon(\cdot, 0) = \eta_0 \in H^s, \quad u^\epsilon(\cdot, 0) = u_0 \in H^{s+1}. \tag{2.3}$$

Here, $s \in \mathbb{N}$ with $s \geq 2$, and $H^s = H^s(\mathbb{T})$ and $H^{s+1} = H^{s+1}(\mathbb{T})$ are the standard spatially periodic L^2-based Sobolev spaces, equipped with the usual norms.

We will show that given initial data η_0 and u_0, there exists a time interval $[0, T]$ (depending only on the size of the data) such that there exists a solution (η, u) solving our initial value problem over the time interval $[0, T]$. Our first step is to apply the Picard Theorem on Banach spaces, which we now state [21].

Theorem 2.1. (Picard Theorem). *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{O} \subseteq \mathcal{B}$ be an open set. Let $F : \mathcal{O} \to \mathcal{B}$ such that F is locally Lipschitz: $\forall X \in \mathcal{O}, \exists \lambda > 0$ and an open set $U \subseteq \mathcal{O}$ such that $\forall Y, Z \in U$,*

$$\|F(Y) - F(Z)\|_{\mathcal{B}} \leq \lambda \|Y - Z\|_{\mathcal{B}}.$$

Then, $\forall X_0 \in \mathcal{O}, \exists\, T > 0$ and a unique $X \in C^1([-T, T]; \mathcal{O})$ such that X solves the initial value problem

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X), \quad X(0) = X_0.$$

We will take $\mathcal{O} = \mathcal{B} = H^s \times H^{s+1}$ and introduce the following lemma:

Lemma 2.2. *Let $(\eta_0, u_0) \in \mathcal{O}$ be given. For any $\epsilon > 0$, there exists $T_\epsilon > 0$ and $(\eta^\epsilon, u^\epsilon) \in C^1([0, T_\epsilon]; \mathcal{O})$ such that $(\eta^\epsilon, u^\epsilon)$ satisfies (2.1), (2.2) and the initial conditions (2.3).*

We omit the proof of Lemma 2.2; it follows immediately from the Picard Theorem and from properties of mollifiers. Note that we only introduced two mollifier operators on the right-hand side of (2.1), and none on the right-hand side of (2.2). For (2.1), this is because when solving (1.3) for η_t, if we consider $(\eta, u) \in H^s \times H^{s+1}$, then the only unbounded term is $\eta_x u$. (We have included two instances of $\mathcal{J}_\epsilon$ to be able to achieve a balance when integrating by parts in the energy estimates to follow.) For (2.2), when solving (1.4) for u_t and again considering $(\eta, u) \in H^s \times H^{s+1}$, there are no unbounded terms (because of the presence of the operator A^{-1}).

2.1. Energy estimate

Next, we will show that there exists $T > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the solutions $(\eta^\epsilon, u^\epsilon)$ are elements of $C([0, T]; \mathcal{O})$. In order to complete the proof, we will use the following ODE theorem [21]:

Theorem 2.3. (Continuation Theorem for ODEs). *Let $\mathcal{B}$ be a Banach space and $\Omega \subseteq \mathcal{B}$ be an open set and $F : \Omega \to \mathcal{B}$ be locally Lipschitz continuous. Let $X_0 = (\eta_0, u_0) \in \Omega$ and $X = (\eta, u)$ be the solution of initial value problem:*

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X), \quad X(0) = X_0,$$

and let $T > 0$ be the maximal time such that $X \in C^1([0, T]; \Omega)$. Then either $T = \infty$ or $T < \infty$ with $X(t)$ leaving the set Ω as $t \to T$.

In order to use Theorem 2.3, we need to prove that the norm of $(\eta^\epsilon, u^\epsilon)$ may be controlled uniformly with respect to ϵ. We establish this in the following lemma using the energy method.

Lemma 2.4. *Let $(\eta_0, u_0) \in \mathcal{O}$. There exists $T > 0$ such that for all $\epsilon \in (0, 1]$, the initial value problem (2.1), (2.2), (2.3) has a solution $(\eta^\epsilon, u^\epsilon) \in C([0, T], \mathcal{O})$.*

Proof. Let $\epsilon \in (0, 1]$ be given. We know there exists $T_\epsilon > 0$ and $(\eta^\epsilon, u^\epsilon) \in C^1([-T_\epsilon, T_\epsilon]; \mathcal{O})$, which solves the regularized initial value problem. Now, we will show that these solutions can be continued until a time T, with T being independent of ϵ.

We define an energy $E(t) = E_0(t) + E_1(t) + E_2(t)$ to be

$$E_0(t) = \frac{1}{2} \int_0^{2\pi} (\eta^\epsilon)^2 + (u^\epsilon)^2 \mathrm{d}x,$$

$$E_1(t) = \frac{1}{2} \int_0^{2\pi} (\partial_x^s \eta^\epsilon)^2 \mathrm{d}x,$$

$$E_2(t) = \frac{1}{2} \int_0^{2\pi} (\partial_x^{s+1} u^\epsilon)^2 \mathrm{d}x.$$

Of course, this energy is equivalent to the square of the H^s-norm of η^ϵ plus the square of the H^{s+1}-norm of u^ϵ. We will show that the time derivative of the energy is bounded in terms of the energy, as long as $s \geq 2$.

We begin with showing $\frac{dE_0}{dt}$ is bounded appropriately, so we calculate

$$\frac{\mathrm{d}E_0}{\mathrm{d}t} = \int_0^{2\pi} \eta^\epsilon(\eta_t^\epsilon) + u^\epsilon(u_t^\epsilon)\ \mathrm{d}x. \tag{2.4}$$

Substituting (2.1) and (2.2) into (2.4), we have

$$\begin{aligned}\frac{\mathrm{d}E_0}{\mathrm{d}t} &= \int_0^{2\pi} \eta^\epsilon \left(-\frac{1}{2}r_0 u_x^\epsilon - \frac{1}{2}\eta^\epsilon u_x^\epsilon - \mathcal{J}_\epsilon\left((\mathcal{J}_\epsilon \eta_x^\epsilon)u^\epsilon\right)\right)\ \mathrm{d}x \\ &+ \int_0^{2\pi} u^\epsilon \left(1 - \frac{(4\overline{\alpha}+r_0)r_0}{8}\partial_{xx}\right)^{-1}\left(-\overline{\beta}\eta_x - uu_x - \kappa u + \frac{\gamma\overline{\beta}r_0}{2}u_{xx}\right)\ \mathrm{d}x.\end{aligned}$$

We may then immediately bound this as

$$\begin{aligned}\frac{\mathrm{d}E_0}{\mathrm{d}t} \le c\Big(&\|\eta^\epsilon\|_{H^0}\|u^\epsilon\|_{H^1} + \|\eta^\epsilon\|_{H^0}\|\eta^\epsilon\|_{H^0}\|u^\epsilon\|_{H^2} \\ &+ \|\eta^\epsilon\|_{H^0}\|\eta^\epsilon\|_{H^2}\|u^\epsilon\|_{H^0} + \|u^\epsilon\|_{H^0}\|\eta^\epsilon\|_{H^0} + \|u^\epsilon\|_{H^0}^3 + \|u^\epsilon\|_{H^0}^2 + \|u^\epsilon\|_{H^0}^2\Big).\end{aligned}$$

Therefore, $\frac{\mathrm{d}E_0}{\mathrm{d}t}$ satisfies the following energy estimate as long as $s \ge 2$:

$$\frac{\mathrm{d}E_0}{\mathrm{d}t} \le c(E + E^{\frac{3}{2}}).$$

Now, we turn to E_1; taking its time derivative, we have

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = \int_0^{2\pi} (\partial_x^s \eta^\epsilon)(\partial_x^s \eta_t^\epsilon)\ \mathrm{d}x. \tag{2.5}$$

Substituting (2.1) into (2.5), we have

$$\begin{aligned}\frac{\mathrm{d}E_1}{\mathrm{d}t} &= -\frac{r_0}{2}\int_0^{2\pi}(\partial_x^s\eta^\epsilon)(\partial_x^{s+1}u^\epsilon)\ \mathrm{d}x - \frac{1}{2}\int_0^{2\pi}(\partial_x^s\eta^\epsilon)\partial_x^s[\eta^\epsilon u_x^\epsilon]\ \mathrm{d}x \\ &+ \int_0^{2\pi}(\partial_x^s\mathcal{J}_\epsilon\eta^\epsilon)\partial_x^s[(\mathcal{J}_\epsilon\eta_x^\epsilon)u^\epsilon]\ \mathrm{d}x = \sum_{k=1}^{3}\Psi_k.\end{aligned} \tag{2.6}$$

In the formula for Ψ_3, we have already used that the mollifier operator $\mathcal{J}_\epsilon$ is self-adjoint. We will show each Ψ_k in (2.6) is bounded in terms of the energy, E.

Since the energy is equivalent to the sum of the square of the H^s-norm of η^ϵ and the square of the H^{s+1}-norm of u^ϵ, the bound

$$\Psi_1 \le cE \tag{2.7}$$

is immediate. For Ψ_2, we immediately may bound it as

$$\Psi_2 \le \|\partial_x^s\eta^\epsilon\|_0\|\partial_x^s(\eta^\epsilon u_x^\epsilon)\|_0 \le \|\eta^\epsilon\|_s\|\eta^\epsilon u_x^\epsilon\|_s.$$

Since $s \ge 1$, we may use the Sobolev algebra property, finding

$$\Psi_2 \le c\|\eta^\epsilon\|_s^2\|u^\epsilon\|_{s+1} \le cE^{\frac{3}{2}}. \tag{2.8}$$

Now, we turn to the third term, Ψ_3, on the right-hand side of (2.6). Using the product rule to expand derivatives, Ψ_3 can be rewritten as follows:

$$\Psi_3 = -\int_0^{2\pi} (\partial_x^s \mathcal{J}_\epsilon \eta^\epsilon) \sum_{k=0}^{s} \binom{s}{k} (\partial_x^{k+1} \mathcal{J}_\epsilon \eta^\epsilon)(\partial_x^{s-k} u^\epsilon)\ \mathrm{d}x. \tag{2.9}$$

The most singular term on the right-hand side of (2.9) is the $k = s$ term, for which all derivatives fall on η^ϵ. Thus, we decompose (2.9) as

$$\begin{aligned}\Psi_3 = &-\int_0^{2\pi} (\partial_x^s \mathcal{J}_\epsilon \eta^\epsilon)(\partial_x^{s+1} \mathcal{J}_\epsilon \eta^\epsilon)(\partial_x u^\epsilon)\ \mathrm{d}x \\ &-\int_0^{2\pi} (\partial_x^s \mathcal{J}_\epsilon \eta^\epsilon) \sum_{k=0}^{s-1} \binom{s}{k} (\partial_x^{k+1} \mathcal{J}_\epsilon \eta^\epsilon)(\partial_x^{s-k} u^\epsilon)\ \mathrm{d}x.\end{aligned} \tag{2.10}$$

The first term on the right-hand side of (2.10) can be integrated by parts, arriving at

$$\begin{aligned}\Psi_3 = &\ \frac{1}{2}\int_0^{2\pi} (\partial_x^s \mathcal{J}_\epsilon \eta^\epsilon)^2(\partial_x^2 u^\epsilon)\ \mathrm{d}x \\ &-\int_0^{2\pi} (\partial_x^s \mathcal{J}_\epsilon \eta^\epsilon) \sum_{k=0}^{s-1} \binom{s}{k} (\partial_x^{k+1} \mathcal{J}_\epsilon \eta^\epsilon)(\partial_x^{s-k} u^\epsilon)\ \mathrm{d}x.\end{aligned} \tag{2.11}$$

We see then that the right-hand side of (2.11) involves at most s derivatives of η^ε and at most $s+1$ derivatives of u^ε; this implies

$$\Psi_3 \leq cE^{3/2}. \tag{2.12}$$

Combining (2.7), (2.8), and (2.12), we have

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} \leq c(E + E^{3/2}).$$

Just as $\frac{\mathrm{d}E_0}{\mathrm{d}t}$ and $\frac{\mathrm{d}E_1}{\mathrm{d}t}$ are bounded by the energy, we will also show $\frac{\mathrm{d}E_2}{\mathrm{d}t}$ is bounded by E. Taking the derivative of E_2 with respect to time, we have

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} = \frac{1}{2}\int_0^{2\pi} (\partial_x^{s+1} u^\epsilon)(\partial_x^{s+1} u_t^\epsilon)\mathrm{d}x. \tag{2.13}$$

Substituting (2.2) into (2.13) leads to the following sum:

$$\begin{aligned}\frac{\mathrm{d}E_2}{\mathrm{d}t} = &-\frac{1}{2}\int_0^{2\pi} \left(\partial_x^{s+1} u^\epsilon\right)\left(\partial_x^{s+1} A^{-1}\overline{\beta}\eta_x^\epsilon\right)\ \mathrm{d}x \\ &-\frac{1}{2}\int_0^{2\pi} \left(\partial_x^{s+1} u^\epsilon\right)\left(\partial_x^{s+1} A^{-1}(u^\epsilon u_x^\epsilon)\right)\ \mathrm{d}x - \frac{1}{2}\int_0^{2\pi} \left(\partial_x^{s+1} u^\epsilon\right)\left(\partial_x^{s+1} A^{-1}\kappa u^\epsilon\right)\ dx \\ &+\frac{1}{2}\int_0^{2\pi} \left(\partial_x^{s+1} u^\epsilon\right)\left(\partial_x^{s+1} A^{-1}\frac{\gamma\overline{\beta}r_0}{2}u_{xx}^\epsilon\right)\ \mathrm{d}x = \sum_{k=1}^{4}\Omega_k.\end{aligned} \tag{2.14}$$

We begin to estimate the first term in the summation in (2.14). We can bound both factors in L^2 :

$$\Omega_1 \leq c \left\| \partial_x^{s+1} u^\epsilon \right\|_{L^2} \left\| \partial_x^{s+2} A^{-1} \eta^\epsilon \right\|_{L^2}.$$

We recall that A^{-1} smoothes by two derivatives, leading us to find

$$\Omega_1 \leq c \left\| u^\epsilon \right\|_{H^{s+1}} \left\| \eta^\epsilon \right\|_{H^s}.$$

Thus, we have Ω_1 bounded by the energy:

$$\Omega_1 \leq c E_2^{\frac{1}{2}} E_1^{\frac{1}{2}} \leq cE.$$

Next, we turn to the second summand on the right-hand side of (2.14), Ω_2. We again bound each of the two factors in L^2 :

$$\Omega_2 \leq c \left\| \partial_x^{s+1} u^\epsilon \right\|_{L^2} \left\| \partial_x^{s+1} A^{-1} (u^\epsilon u_x^\epsilon) \right\|_{L^2}.$$

Again using that A^{-1} smoothes by two derivatives, we have

$$\Omega_2 \leq c \left\| u^\epsilon \right\|_{H^{s+1}} \left\| u^\epsilon u_x^\epsilon \right\|_{H^{s-1}}.$$

Using the Sobolev algebra property, this yields the desired bound, namely

$$\Omega_2 \leq c E^{\frac{3}{2}}.$$

We move on to Ω_3, and estimate it similarly, finding

$$\Omega_3 \leq c \left\| \partial_x^{s} u^\epsilon \right\|_{L^2} \left\| \partial_x^{s} u^\epsilon \right\|_{L^2},$$

which implies

$$\Omega_3 \leq cE.$$

Lastly, we estimate Ω_4. For the second factor in Ω_4, we use again that A^{-1} is smoothing by two derivatives. These considerations yield the bound

$$\Omega_4 \leq c \left\| \partial_x^{s+1} u^\epsilon \right\|_{L^2} \left\| \partial_x^{s+1} u^\epsilon \right\|_{L^2} \leq cE.$$

We have now established $\sum_{k=1}^{4} \Omega_k \leq c(E + E^{\frac{3}{2}})$. Thus, we arrive at the corresponding bound for $\frac{dE_2}{dt}$,

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} \leq c \left(E + E^{\frac{3}{2}} \right),$$

and also for $\frac{dE}{dt}$,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E_0}{\mathrm{d}t} + \frac{\mathrm{d}E_1}{\mathrm{d}t} + \frac{\mathrm{d}E_2}{\mathrm{d}t} \leq c \left(E + E^{\frac{3}{2}} \right). \tag{2.15}$$

We let $\bar{d} > 0$ be such that $E(0) \leq \bar{d}$. We ask on what interval of values of t we may guarantee that $E(t) \leq 2\bar{d}$; for such values of t, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq c \left(E + E^{\frac{3}{2}} \right) \leq c \left(2\bar{d} + (2\bar{d})^{\frac{3}{2}} \right).$$

This implies that on an interval on which $E \leq 2\bar{d}$,

$$E \leq c \left(2\bar{d} + (2\bar{d})^{\frac{3}{2}} \right) t + \bar{d}.$$

Thus, we can conclude that $E(t) \leq 2\bar{d}$ for all t satisfying

$$t \in \left[0, \frac{\bar{d}}{c \left(2\bar{d} + (2\bar{d})^{\frac{3}{2}} \right)} \right].$$

As this time interval is independent of ϵ, this completes the proof. □

Remark 2.5. We used several times above that the operator A^{-1} is smoothing by two derivatives. To be more precise, since we are in the spatially periodic case we may use the Fourier series to see that A^{-1} is a bounded linear operator between any space H^{ℓ} and $H^{\ell+2}$. This is immediate because the A operator here has constant coefficients. In Sect. 3, we will need to use an analogous operator, but in the more general case of non-constant coefficients. This will be more involved, and understanding this inverse on certain function spaces (exponentially weighted Wiener algebras) will be a significant focus of Sect. 3.

2.2. Well-posedness of the initial value problem

In this section, we establish the three elements of well-posedness (existence, uniqueness, and continuous dependence upon the initial data) for the initial value problem for the non-mollified system (1.3), (1.4). We begin with existence and will at the same time establish regularity of the solution. In demonstrating the highest regularity (that the solution is continuous in time with values in $H^s \times H^{s+1}$), we rely on the following elementary interpolation inequality; the proof of this may be found many places, one of which is [4].

Lemma 2.6. (Interpolation Inequality) *Let $s' \geq 0$ and $s \geq s'$ be given. There exists $c > 0$ such that for every $f \in H^s$, the following inequality holds:*

$$\|f\|_{H^{s'}} \leq c \|f\|_{H^0}^{1-\frac{s'}{s}} \|f\|_{H^s}^{\frac{s'}{s}}.$$

The following is our existence theorem.

Theorem 2.7. *Let $s \in \mathbb{N}$ such that $s \geq 2$ be given. Let $\eta_0 \in H^s$ and $u_0 \in H^{s+1}$ be given. Let $T > 0$ be as in Lemma 2.4. Then there exists $(\eta, u) \in C([0,T]; H^s \times H^{s+1})$ which solves the initial value problem (1.3), (1.4) with data $\eta(\cdot,0) = \eta_0$, $u(\cdot,0) = u_0$.*

Proof. The energy estimate we have established shows that $(\eta^\epsilon, u^\epsilon)$ is uniformly bounded in $C([0,T] : H^s \times H^{s+1})$, with this T being independent of ϵ. This implies that $(\eta^\epsilon_t, u^\epsilon_t)$ is uniformly bounded with respect to ϵ in $L^\infty \times L^\infty$ as well, when $s \geq 2$. This implies that the sequence $(\eta^\epsilon, u^\epsilon)$ is an equicontinuous family, and thus by the Arzela–Ascoli theorem there exists a subsequence (which we do not relabel) $(\eta^\epsilon, u^\epsilon)$ which converges uniformly to some $(\eta, u) \in (C([0,2\pi] \times [0,T]))^2$. We now establish regularity of this (η, u) and that (η, u) is a solution of the non-regularized initial value problem.

Since the sequence $(\eta^\epsilon, u^\epsilon)$ is uniformly bounded with respect to both ϵ and t in $H^s \times H^{s+1}$, and since the unit ball of a Hilbert space is weakly compact, for any $t \in [0,T]$ we may find a weak limit in $H^s \times H^{s+1}$. Clearly this limit must again equal (η, u), and thus we conclude that for every t, $(\eta(\cdot,t), u(\cdot,t)) \in H^s \times H^{s+1}$, and that $(\eta, u) \in L^\infty([0,T]; H^s \times H^{s+1})$.

Since $(\eta^\epsilon, u^\epsilon)$ converges to (η, u) in $(C([0,2\pi] \times [0,T]))^2$, the convergence also holds in $C([0,T]; H^0 \times H^0)$. Then using the uniform bound on $(\eta^\epsilon, u^\epsilon)$ provided by the proof of Lemma 2.4, and also using Lemma 2.6, we see that the convergence also holds in $C([0,T]; H^{s'} \times H^{s'+1})$, for any $0 \leq s' < s$.

We have concluded so far that the limit $(\eta, u) \in C([0,T]; H^{s'} \times H^{s'+1}) \cap L^\infty([0,T]; H^s \times H^{s+1})$. We can in fact show that $(\eta, u) \in C([0,T]; H^s \times H^{s+1})$, but we will delay this until after showing that (η, u) solves the unregularized initial value problem.

To show that (η, u) satisfies the appropriate system, we use the fundamental theorem of calculus on the approximate solutions,

$$\eta^\epsilon(\cdot,t) = \eta_0 + \int_0^t -\frac{r_0}{2}u^\epsilon_x - \frac{1}{2}\eta^\epsilon u^\epsilon_x - \mathcal{J}_\epsilon\left((\mathcal{J}_\epsilon \eta^\epsilon_x)u^\epsilon\right)\, \mathrm{d}\tau,$$

$$u^\epsilon(\cdot,t) = u_0 + \int_0^t A^{-1}\left[-\overline{\beta}\eta^\epsilon_x - u^\epsilon u^\epsilon_x - \kappa u^\epsilon + \frac{\gamma\overline{\beta}r_0}{2}u^\epsilon_{xx}\right]\, \mathrm{d}\tau.$$

We have established sufficient regularity to pass to the limit under the integrals, and thus we have

$$\eta(\cdot,t) = \eta_0 + \int_0^t \left[-\frac{r_0}{2}u_x - \frac{1}{2}\eta u_x - \eta_x u\right]\,\mathrm{d}\tau,$$

$$u(\cdot,t) = u_0 + \int_0^t A^{-1}\left[-\overline{\beta}\eta_x - uu_x - \kappa u + \frac{\gamma\overline{\beta}r_0}{2}u_{xx}\right]\,\mathrm{d}\tau.$$

Taking the derivative of these equations with respect to time, we see that (η, u) does indeed satisfy the unregularized initial value problem.

We now may demonstrate $(\eta, u) \in C([0,T]; H^s \times H^{s+1})$. By a standard argument (see, for example, the proof of Theorem 3.4 of [21]), the uniform bound on solutions and the continuity in time in $H^{s'} \times H^s$ for all $0 \le s' < s$ implies weak continuity in time, i.e., $(\eta, u) \in C_W([0,T]; H^s \times H^{s+1})$. Since weak convergence plus convergence of the norm implies convergence in a Hilbert space, all that remains to show, then, is continuity of the $H^s \times H^{s+1}$ norm with respect to time. To establish continuity of the norm, it is enough to establish right-continuity at the initial time, $t = 0$. The general case (i.e., continuity of the norm at times other than the initial time) follows by considering any other time to be a new initial time; by uniqueness of solutions, which is part of the content of Theorem 2.8, the solution starting from some time $t_* \in [0,T)$ is the same as the solution we have already found starting from $t = 0$. In this way, establishing right-continuity of the norm at the initial time demonstrates right-continuity of the norm at any time in $[0,T)$. Left-continuity of the norm follows from time-reversibility of the equations.

So, as we have said, all that remains to establish is right-continuity of the $H^s \times H^{s+1}$ norm of the solution at $t = 0$. Weak continuity implies

$$\liminf_{t\to 0^+} \|(\eta,u)\|_{H^s\times H^{s+1}} \ge \|(\eta_0,u_0)\|_{H^s\times H^{s+1}}. \tag{2.16}$$

Similarly, for any $t \in [0,T]$, we have

$$\limsup_{\epsilon\to 0^+} \|(\eta^\epsilon(\cdot,t), u^\epsilon(\cdot,t))\|_{H^s\times H^{s+1}} \ge \|(\eta(\cdot,t), u(\cdot,t))\|_{H^s\times H^{s+1}}.$$

Then, the energy estimate (2.15) implies

$$\begin{aligned}\|(\eta_0,u_0)\|_{H^s\times H^{s+1}} &\ge \limsup_{t\to 0^+}\limsup_{\epsilon\to 0^+} \|(\eta^\epsilon(\cdot,t), u^\epsilon(\cdot,t))\|_{H^s\times H^{s+1}} \\ &\ge \limsup_{t\to 0^+} \|(\eta(\cdot,t), u(\cdot,t))\|_{H^s\times H^{s+1}}.\end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17), we have the conclusion. This completes the proof of the theorem. □

Now, we will seek to establish the uniqueness of solutions (η, u), and continuous dependence on the initial data.

Theorem 2.8. *Let* $(\eta_0, u_0) \in H^s \times H^{s+1}$ *and* $(\eta_0^*, u_0^*) \in H^s \times H^{s+1}$ *be given. Let* $K > 0$ *be given such that*

$$\|(\eta_0,u_0)\|_{H^s\times H^{s+1}} < K, \quad \|(\eta_0^*,u_0^*)\|_{H^s\times H^{s+1}} < K.$$

Let $T > 0$ *be such that there is a solution* $(\eta, u) \in C([0,T]; H^s \times H^{s+1})$ *solving* (1.3), (1.4) *with initial data* $\eta(\cdot,0) = \eta_0$, $u(\cdot,0) = u_0$ *and such that there is a solution* $(\eta^*, u^*) \in C([0,T]; H^s \times H^{s+1})$ *solving* (1.3), (1.4) *with initial data* $\eta(\cdot,0) = \eta_0^*$, $u(\cdot,0) = u_0^*$, *and such that*

$$\sup_{t\in[0,T]} \|(\eta(\cdot,t), u(\cdot,t))\|_{H^s\times H^{s+1}} \le K, \qquad \sup_{t\in[0,T]} \|(\eta^*(\cdot,t), u^*(\cdot,t))\|_{H^s\times H^{s+1}} \le K.$$

Then there exists $c > 0$, *depending only on* K *and* s, *such that for any* $s' \in [0,s)$,

$$\sup_{t\in[0,T]} \|(\eta(\cdot,t)-\eta^*(\cdot,t), u(\cdot,t)-u^*(\cdot,t))\|_{H^{s'}\times H^{s'+1}} \leq c\left(\|(\eta_0-\eta_0^*, u_0-u_0^*)\|_{H^0\times H^1}\right)^{1-s'/s}. \tag{2.18}$$

In particular, solutions of the initial value problem for (1.3), (1.4) *are unique.*

Proof. We define an energy for the difference of the two solutions, Z, as

$$Z(t) = \int_0^t (\eta-\eta^*)^2 + (u-u^*)^2 + (u_x-u_x^*)^2 \, \mathrm{d}x.$$

We will estimate the growth of Z. Its time derivative is

$$\frac{\mathrm{d}Z(t)}{\mathrm{d}t} = 2\int_0^t (\eta-\eta^*)(\eta_t-\eta_t^*) + (u-u^*)(u_t-u_t^*) + (u_x-u_x^*)(u_x-u_x^*)_t \, \mathrm{d}x.$$

We expand the time-derivatives as follows:

$$(\eta_t-\eta_t^*) = -\frac{r_0}{2}(u_x-u_x^*) - \frac{1}{2}(\eta u_x - \eta^* u_x^*) - (\eta_x u - \eta_x^* u^*) = \sum_{k=1}^{3} z_k,$$

$$(u_t-u_t^*) = A^{-1}\left[\overline{\beta}(\eta_x-\eta_x^*) + (uu_x - u^*u_x^*) + \kappa(u-u^*) - \frac{\gamma\overline{\beta}r_0}{2}(u_{xx}-u_{xx}^*)\right] = \sum_{k=4}^{7} z_k,$$

$$(u_x-u_x^*)_t = (u_t-u_t^*)_x = \sum_{k=4}^{7}(z_k)_x.$$

Thus, we have

$$\frac{\mathrm{d}Z(t)}{\mathrm{d}t} = 2\int_0^t \left[(\eta-\eta^*)\sum_{k=1}^{3} z_k + (u-u^*)\sum_{k=4}^{7} z_k + (u_x-u_x^*)\sum_{k=4}^{7}(z_k)_x\right] \mathrm{d}x. \tag{2.19}$$

We begin by estimating the first term. Bounding each factor in L^2, we have

$$2\int_0^t (\eta-\eta^*)(z_1)dx \leq c\,\|\eta-\eta^*\|_{L^2}\,\|z_1\|_{L^2}.$$

Since $\|z_1\|_{L^2} \leq c\,\|u-u^*\|_{H^1}$, we also have $\|z_1\|_{L^2} \leq cZ(t)^{\frac{1}{2}}$. This then implies

$$2\int_0^t (\eta-\eta^*)(z_1)\mathrm{d}x \leq cZ(t).$$

Next, we look at the term involving z_2. We again bound each factor in L^2 :

$$2\int_0^t (\eta-\eta^*)(z_2)\mathrm{d}x \leq c\,\|\eta-\eta^*\|_{L^2}\,\|z_2\|_{L^2}.$$

Here we may bound $\|z_2\|_{L^2}$ as follows:

$$\|z_2\|_{L^2} \leq \frac{1}{2}\,\|(\eta u_x + (\eta^* u_x - \eta^* u_x) - \eta^* u_x^*)\|_{L^2}.$$

Using the triangle inequality, this may be bounded as

$$\|z_2\|_{L^2} \leq c\,\|\eta u_x - \eta^* u_x\|_{L^2} + c\,\|\eta^* u_x - \eta^* u_x^*\|_{L^2}\,.$$

We may bring out u_x from the first term and η^* from the second term:

$$\|z_2\|_{L^2} \leq c\,\|\eta - \eta^*\|_{L^2}\,\|u_x\|_{L^\infty} + c\,\|\eta^*\|_{L^\infty}\,\|u - u^*\|_{H^1}\,.$$

By Sobolev embedding, since $s \geq 1$, we have $\|u_x\|_{L^\infty} \leq c\,\|u\|_{H^{s+1}}$ and $\|\eta^*\|_{L^\infty} \leq c\,\|\eta\|_{H^s}$. We therefore have

$$\|z_2\|_{L^2} \leq c(\|u\|_{H^{s+1}} + \|\eta\|_{H^s})Z(t)^{\frac{1}{2}}.$$

Using the uniform bound on the solutions, we then have

$$2\int_0^t (\eta - \eta^*)(z_2)dx \leq cZ(t).$$

Next, we will look at the third term. We begin by adding and subtracting in z_3,

$$z_3 = \eta_x^* u^* + (\eta_x^* u - \eta_x^* u) - \eta_x u.$$

We integrate by parts once, finding

$$2\int_0^t (\eta - \eta^*)(z_3)\mathrm{d}x = -2\int_0^t (\eta - \eta^*)(\eta^*)(u_x^* - u_x)\mathrm{d}x - 2\int_0^t \frac{1}{2}(\eta - \eta^*)^2(u_x)\ \mathrm{d}x.$$

This may then be bounded as

$$\begin{aligned} &2\int_0^t (\eta - \eta^*)(z_3)\mathrm{d}x \\ &\quad \leq c\,\|\eta - \eta^*\|_{L^2}\,\|\eta^*\|_{L^\infty}\,\|u^* - u\|_{H^1} + c\,\|\eta - \eta^*\|_{L^2}^2\,\|u_x\|_{L^\infty}\,. \end{aligned}$$

Again using Sobolev embedding and the uniform bound, we have

$$2\int_0^t (\eta - \eta^*)(z_3)\mathrm{d}x \leq cZ(t).$$

To complete the proof, it is sufficient to prove

$$\|z_k\|_{H^1} \leq cZ(t)^{\frac{1}{2}}, \quad \text{for} \quad k = 4, 5, 6, 7.$$

We begin with z_4. Recall A^{-1} is smoothing by two derivatives. We may then say

$$\|z_4\|_{H^1} \leq c\,\|\eta - \eta^*\|_{H^0} \leq cZ(t)^{\frac{1}{2}}.$$

To estimate $\|z_5\|_{H^1}$, we begin by adding and subtracting,

$$\|z_5\|_{H^1} = \left\|A^{-1}(uu_x + (u^* u_x - u^* u_x) - u^* u_x^*)\right\|_{H^1}.$$

We use the triangle inequality and $\|\cdot\|_{H^1} \leq \|\cdot\|_{H^2}$, so that we have

$$\|z_5\|_{H^1} \leq \left\|A^{-1}(uu_x - u^* u_x)\right\|_{H^2} + \left\|A^{-1}(u^* u_x - u^* u_x^*)\right\|_{H^2}.$$

Using the smoothing effect of A^{-1}, this becomes

$$\|z_5\|_{H^1} \leq c\,\|u - u^*\|_{H^0}\,\|u_x\|_{L^\infty} + c\,\|u^*\|_{L^\infty}\,\|u_x - u_x^*\|_{H^0}\,.$$

Using the definition of $Z(t)$, Sobolev embedding, and the uniform bound, we conclude

$$\|z_5\|_{H^1} \leq cZ(t)^{\frac{1}{2}}.$$

Proceeding similarly, we also have

$$\|z_6\|_{H^1} \leq cZ(t)^{\frac{1}{2}} \quad \text{and} \quad \|z_7\|_{H^1} \leq cZ(t)^{\frac{1}{2}}.$$

Hence, we may conclude that (2.19) is bounded as follows:

$$\frac{\mathrm{d}Z(t)}{\mathrm{d}t} \leq cZ(t).$$

By Gronwall's inequality, we therefore have that $Z(t) \leq e^{ct}Z(0) \leq e^{cT}Z(0)$. Together with Lemma 2.6, this implies (2.18). Uniqueness follows by taking $Z(0) = 0$, as this implies $Z(t) = 0$ for $t > 0$. □

3. Existence by the Cauchy–Kowalevski theorem

In this section, we prove an existence theorem for the system (1.1), (1.2), making use of an abstract Cauchy–Kowalevski theorem. This uses function spaces of analytic functions based on the Wiener algebra. We take this approach to proving existence because it is not clear that the system is well-posed in spaces of finite regularity, i.e., Sobolev spaces.

Consider the following simplified system, which is based upon the system (1.1), (1.2) with $r_{0x} \neq 0$, keeping only the terms with the most derivatives, and simplifying to the constant coefficient case:

$$\eta_t = -u_x, \quad u_t - u_{xxt} = -\eta_{xx}. \tag{3.1}$$

This system is ill-posed, as can be demonstrated with a calculation in Fourier space, which we now perform.

To begin, we take the second time derivative of η, finding

$$\eta_{tt} = \partial_x^3(1 - \partial_x^2)^{-1}\eta.$$

With the representation $\eta = \sum_k \eta_k e^{ikx}$, we have that the symbol of ∂_x is ik, and the coefficients η_k then satisfy

$$\eta_{k,tt} = -\frac{ik^3}{1+k^2}\eta_k.$$

If we set $k = N > 0$, then one solution for η_N is

$$\eta_N = \exp\left\{t\left(\frac{1-i}{\sqrt{2}}\right)\frac{N^{3/2}}{\sqrt{1+N^2}}\right\}. \tag{3.2}$$

One sees from (3.2) the source of the ill-posedness: similarly to solutions of the backward heat equation, the exponential growth rate is unbounded as $N \to \infty$. Linear combinations of solutions for the Fourier coefficients of the form (3.2) can be made, arriving at real-valued solutions for η, namely

$$\eta^N(x,t) = c\sin\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right)e^{tN^{3/2}/\sqrt{2+2N^2}}, \tag{3.3}$$

for some $c \in \mathbb{R}$. (Note that as an alternative to working through the Fourier series derivation of the formula (3.3), it is also straightforward to verify that this η^N, along with the corresponding function u^N, solves the system (3.1); see "Appendix A" for details.) We let $s \geq 0$, and we may then choose $c = c(N,s)$ so that $\|\eta^N(\cdot,0)\|_{H^s} = 1/N$. Then, at time $t > 0$, since $c\sin\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right)$ is simply a translate of $\eta^N(\cdot,0)$, the norm becomes

$$\|\eta^N(\cdot,t)\|_{H^s} = \frac{e^{tN^{3/2}/\sqrt{2+2N^2}}}{N}.$$

Letting a sequence of times t_N be defined as

$$t_N = \frac{\sqrt{2+2N^2}\ln(N)}{N^{3/2}},$$

we see that $t_N \to 0$ as $N \to \infty$, and also that at the times t_N we have

$$\|\eta^N(\cdot, t_N)\|_{H^s} = 1.$$

This is lack of continuous dependence on the initial data—we have demonstrated that η^N can initially be arbitrarily small, and also be of unit size arbitrarily fast, by taking N sufficiently large.

The above rigorously demonstrates ill-posedness of the toy linear model (3.1). We then view this as a heuristic argument suggesting ill-posedness of the system (1.1), (1.2) (with non-constant r_0) in Sobolev spaces. Furthermore, while the derivation of the model in [23] is based an asymptotic expansion of the velocity potential, Boussinesq models are frequently derived by instead making long-wave approximations; in such a long-wave model, one typically expects coefficients like $r_0(x)$ to be homogenized in the long-wave limit, see for instance [14] for an example of this phenomenon in the case of long-wave limits of polyatomic lattices. In this sense, while the system (1.1), (1.2) is more general, the system (1.3), (1.4) may be more fundamental.

3.1. The abstract Cauchy–Kowalevski theorem of Kano and Nishida

The following abstract Cauchy–Kowalevski theorem is proved by Kano and Nishida [16]; other, related abstract Cauchy–Kowalevski theorems may be found in [9,24,25].

Theorem 3.1. (Cauchy–Kowalevski Theorem). *Let $\{B_\rho\}_{\rho\ge 0}$ be a scale of Banach spaces, such that for any ρ, B_ρ is a linear subspace of B_0. Suppose that*

$$B_\rho \subset B_{\rho'}, \quad \|\cdot\|_{\rho'} \le \|\cdot\|_\rho \quad \textit{for} \quad \rho' \le \rho, \tag{3.4}$$

where $\|\cdot\|_\sigma$ denotes the norm of B_σ for any $\sigma \ge 0$. We assume the following conditions:
(H1) *There exist constants $R > R_0 > 0$, $T > 0$, and $\rho_0 > 0$, such that for any $0 \le \rho' < \rho < \rho_0$, $(z,t,s) \mapsto F(z,t,s)$ is a continuous operator of*

$$\{z \in B_\rho : \|z\|_\rho < R\} \times \{0 \le t \le T\} \times \{0 \le s \le T\} \quad \textit{into } B_{\rho'}.$$

(H2) *For every $\rho < \rho_0$, $F(0,t,s)$ is a continuous function of t with values in B_ρ and satisfies with a fixed constant K,*

$$\|F(0,t,s)\|_\rho \le \frac{K}{\rho_0 - \rho}.$$

(H3) *For any $0 \le \rho' < \rho < \rho_0$ and all $z, \tilde z \in B_\rho$, with $\|z\|_\rho < R$, $\|\tilde z\|_\rho < R$, F satisfies the following for all t, s in $[0,T]$,*

$$\|F(z,t,s) - F(\tilde z,t,s)\|_{\rho'} \le \frac{C\,\|z-\tilde z\|_\rho}{\rho - \rho'}$$

with a constant C independent of $\{t,s,z,\tilde z,\rho,\rho'\}$.
If **(H1)**–**(H3)** *hold, there exists a positive constant λ such that we have the unique continuous solution of*

$$z(t) = z_0(t) + \int_0^t F(z(s),t,s)\,\mathrm{d}s, \tag{3.5}$$

for all $0 < \rho < \rho_0$ and $|t| < \lambda(\rho_0 - \rho)$, with value in B_ρ.

Remark 3.2. We have stated the conclusion as in [16], but it can be rephrased in a way we will find more helpful. The existence of λ and the conditions $0 < \rho < \rho_0$ and $|t| < \lambda(\rho_0 - \rho)$ are equivalent to the existence of an upper bound on the time of existence for solutions. Namely, the solution can be continued to a time $t > 0$ as long as there exists a value $\rho \in (0, \rho_0)$ such that $t < \lambda(\rho_0 - \rho)$. Thus, the solution exists on the interval $[0, T)$, where $T = \lambda\rho_0$. In Theorem 3.6, rather than concluding the existence of λ, we will conclude the existence of T.

Remark 3.3. The form of Eq. (3.5) that Kano and Nishida consider is clearly intended to allow for semigroups from linear operators, such as would appear in a parabolic problem. In our intended application, we have no such semigroup present, so we do not need both variables s and t. In particular, adapting the initial value problem for the system (1.1), (1.2) to the form (3.5), we get $F = (F_1, F_2)$, where F_1 and F_2 are given by (suppressing dependence on the spatial variable)

$$F_1((\eta(s), u(s)), t, s) = -\frac{1}{2}(r_0 + \eta(s))u_x(s) - (r_0 + \eta(s))_x u(s), \tag{3.6}$$

$$\begin{aligned} F_2((\eta(s), u(s)), t, s) = A^{-1}\Bigg[&- (\bar\beta\eta(s))_x - u(s)u_x(s) - \frac{(3\bar\alpha + r_0)r_{0x}}{2}(\bar\beta\eta(s))_{xx} - \kappa u(s) \\ &+ \gamma\bar\beta\left(r_{0x}u(s) + \frac{r_0}{2}u_x(s)\right)\Bigg]. \end{aligned} \tag{3.7}$$

Here, the operator A is given by

$$Af = \left[1 - \bar\alpha r_{0xx} - \frac{(4\bar\alpha + r_0)r_0}{8}\partial_x^2\right] f.$$

We will use the exponentially weighted Wiener algebras as our spaces B_ρ; given $\rho \geq 0$, we say $f \in B_\rho$ if and only if

$$\|f\|_\rho = \sum_{k\in\mathbb{Z}} e^{\rho|k|}|\hat f_k| < \infty,$$

where $\{\hat f_k\}$ are the Fourier coefficients of f. These spaces satisfy (3.4). Furthermore, these spaces are Banach algebras, so that if $f \in B_\rho$ and $g \in B_\rho$, then $fg \in B_\rho$, with

$$\|fg\|_\rho \leq \|f\|_\rho \|g\|_\rho. \tag{3.8}$$

These spaces also have the Cauchy estimate; if $0 \leq \rho' < \rho$, then for all $f \in B_\rho$,

$$\|\partial_x f\|_{\rho'} \leq \frac{e}{\rho - \rho'}\|f\|_\rho. \tag{3.9}$$

Now, we move on to show that abstract Cauchy–Kowalevski theorem applies to our system (1.1), (1.2), by showing that F_1, F_2 given in (3.6), (3.7) satisfy the hypothesis **(H1)**–**(H3)**. Before verifying the hypotheses **(H1)**–**(H3)**, it is important to understand the action of the operator A^{-1} on the scale of spaces B_ρ; we consider this inverse operator next in Sect. 3.2, and then verify **(H1)**–**(H3)** in Sect. 3.3. In bounding the inverse operator, we necessarily make some assumptions on the function r_0. We will show that the assumptions on r_0 may be satisfied by furnishing a family of examples in Sect. 3.4.

3.2. The inverse operator

In the analysis of Sect. 2, we used the fact that $(1-\partial_x^2)^{-1}$ is a bounded linear operator from H^s to H^{s+2}; it of course is also a bounded linear operator from B_ρ to itself, for any ρ. The inverse operator we must deal with, however, is more complicated than $(1 - \partial_x^2)^{-1}$, as we now have to account for non-constant coefficients.

We begin by rewriting A to factor out the function multiplying ∂_x^2,

$$A = g_1 \left[g_2 - \partial_x^2\right],$$

where the functions g_1 and g_2 are given by

$$g_1 = \frac{(4\bar{\alpha} + r_0)r_0}{8}, \quad g_2 = \frac{8(1 - \bar{\alpha} r_{0xx})}{(4\bar{\alpha} + r_0)r_0}.$$

We make the following assumptions on g_1 and g_2 (of course, these are really assumptions about the function r_0).

(H4) We assume $g_1 > 0$ and there exists $\rho_0 > 0$ and $c_0 > 0$ such that $\partial_x^j g_1^{-1} \in B_{\rho_0}$ for $j \in \{0, 1, 2\}$, $g_2 \in B_{\rho_0}$, and

$$\left\| \frac{g_2 - c_0}{c_0} \right\|_{\rho_0} < 1. \tag{3.10}$$

We will focus now on inverting $g_2 - \partial_x^2$. We make the decomposition $g_2 - \partial_x^2 = A_1 + A_2$, where

$$A_1 = g_2 - c_0, \quad A_2 = c_0 - \partial_x^2.$$

As we are interested in $(A_1 + A_2)^{-1}$, we invert the identity $A_1 + A_2 = (1 + A_1 A_2^{-1})A_2$ to find the formula

$$(A_1 + A_2)^{-1} = A_2^{-1}(1 + A_1 A_2^{-1})^{-1}.$$

It is easily verified that, for any $0 \leq \rho \leq \rho_0$, A_2^{-1} is bounded from B_ρ to itself, with operator norm $1/c_0$. This implies that $A_1 A_2^{-1}$ is also bounded from B_ρ to itself, with

$$\|A_1 A_2^{-1}\|_{B_\rho \to B_\rho} \leq \left\| \frac{g_2 - c_0}{c_0} \right\|_\rho \leq \left\| \frac{g_2 - c_0}{c_0} \right\|_{\rho_0}.$$

Thus by (3.10), the operator norm of $A_1 A_2^{-1}$ is strictly less than 1. The operator $1 + A_1 A_2^{-1}$ can therefore be inverted by Neumann series. We conclude that A^{-1} is well-defined as a bounded linear operator mapping B_ρ to B_ρ. We have proved the following lemma.

Lemma 3.4. *Assume* **(H4)**. *For all ρ satisfying $0 \leq \rho \leq \rho_0$, A^{-1} is a well-defined bounded linear operator mapping from B_ρ to B_ρ.*

We also have the following corollary.

Corollary 3.5. *Assume* **(H4)**. *For any $0 \leq \rho' < \rho \leq \rho_0$, the operators $A^{-1}\partial_x$ and $A^{-1}\partial_x^2$ are bounded from B_ρ to $B_{\rho'}$, with the estimates*

$$\|A^{-1}\partial_x^j f\|_{\rho'} \leq \frac{c\|f\|_\rho}{\rho - \rho'}, \quad j \in \{1, 2\}.$$

Proof. We focus on the case $j = 2$, as the other case is simpler. We write $A^{-1} = (g_2 - \partial_x^2)^{-1} g_1^{-1}$, and for $f \in B_{\rho_0}$, we have

$$\begin{aligned} A^{-1}\partial_x^2 f &= (g_2 - \partial_x^2)^{-1} g_1^{-1} \partial_x^2 f \\ &= (g_2 - \partial_x^2)^{-1} \left[(\partial_x^2 (g_1^{-1} f) - 2(\partial_x g_1^{-1})(\partial_x f) - (\partial_x^2 g_1^{-1}) f\right]. \end{aligned}$$

We then add and subtract, finding

$$\begin{aligned} A^{-1}\partial_x^2 f = {} & (g_2 - \partial_x^2)^{-1} \left[\left(\partial_x^2 - g_2\right)\left(g_1^{-1} f\right)\right] + (g_2 - \partial_x^2)^{-1}(g_2 g_1^{-1} f) \\ & - 2(g_2 - \partial_x^2)^{-1}\left((\partial_x g_1^{-1})(\partial_x f)\right) - (g_2 - \partial_x^2)^{-1}\left((\partial_x^2 g_1^{-1}) f\right). \end{aligned}$$

The first term on the right-hand side simplifies, and this becomes

$$\begin{aligned} A^{-1}\partial_x^2 f = {} & -g_1^{-1} f + (g_2 - \partial_x^2)^{-1}(g_2 g_1^{-1} f) \\ & - 2(g_2 - \partial_x^2)^{-1}\left((\partial_x g_1^{-1})(\partial_x f)\right) - (g_2 - \partial_x^2)^{-1}\left((\partial_x^2 g_1^{-1}) f\right). \end{aligned}$$

There are four terms on the right-hand side, three of which involve zero derivatives of f and one of which involves $\partial_x f$. All operators applied here either to f or to $\partial_x f$ are bounded, and we may then use the inequalities (3.4) and (3.9) to reach the conclusion. $\square$

As we have said, we will discuss functions r_0 which satisfy **(H4)** in Sect. 3.4.

3.3. Verifying the hypotheses

We are now in a position to state our existence theorem for the initial value problem for the system (1.1), (1.2).

Theorem 3.6. *Assume that r_0 satisfies* **(H4)**. *Furthermore, assume r_0, r_{0x}, $\bar{\beta}$, and $\bar{\beta}_x$ are all in B_{ρ_0}. Let $\eta_0 \in B_{\rho_0}$ and $u_0 \in B_{\rho_0}$ be given. Then there exists $T > 0$ such that there exists a solution (η, u) of the initial value problem (1.1), (1.2) with initial conditions $\eta(\cdot, 0) = \eta_0$, $u(\cdot, 0) = u_0$, on the time interval $[0, T)$. At each time $t \in [0, T]$, each of $\eta(\cdot, t)$ and $u(\cdot, t)$ belong to the space B_ρ for all $0 \leq \rho < \rho_0 \left(1 - \frac{t}{T}\right)$.*

Proof. With the estimates we have established, it is immediate that $F = (F_1, F_2)$ maps B_ρ to $B_{\rho'}$, for $0 < \rho' < \rho < \rho_0$. This establishes **(H1)**. Next, clearly **(H2)** is automatically satisfied, as $F((0, 0), t, s) = 0$. What remains, then, is to establish **(H3)**.

To begin to verify **(H3)**, we consider $F_1((\eta, u), t, s) - F_1(\tilde{\eta}, \tilde{u}), t, s)$,

$$\left\| -\frac{r_0}{2}(u_x - \tilde{u}_x) - \frac{1}{2}(\eta u_x - \tilde{\eta}\tilde{u}_x) - (\eta_x u - \tilde{\eta}_x \tilde{u}) \right\|_{\rho'}$$
$$\leq \left\| -\frac{r_0}{2}(u_x - \tilde{u}_x) \right\|_{\rho'} + \left\| \frac{1}{2}(\eta u_x - \tilde{\eta}\tilde{u}_x) \right\|_{\rho'} + \left\| -r_{0x}(u - \tilde{u}) \right\|_{\rho'} + \left\| \eta_x u - \tilde{\eta}_x \tilde{u} \right\|_{\rho'}$$
$$\leq I + II + III + IV.$$

The first term, I, is readily bounded using the Cauchy estimate (3.9),

$$I \leq \frac{c \left\| u - \tilde{u} \right\|_\rho}{\rho - \rho'}.$$

For the second term, II, we add and subtract, and use the algebra property (3.8),

$$II \leq \frac{1}{2} \left\| \eta u_x - \tilde{\eta} u_x + \tilde{\eta} u_x - \tilde{\eta}\tilde{u}_x \right\|_{\rho'}$$
$$\leq \frac{1}{2} \left\| u_x \right\|_{\rho'} \left\| \eta - \tilde{\eta} \right\|_{\rho'} + \frac{1}{2} \left\| \tilde{\eta} \right\|_{\rho'} \left\| u_x - \tilde{u}_x \right\|_{\rho'}.$$

We may then apply the Cauchy estimate (3.9), finding

$$II \leq c \cdot \frac{\left\| \eta - \tilde{\eta} \right\|_\rho + \left\| u - \tilde{u} \right\|_\rho}{\rho - \rho'}.$$

The third and fourth terms, $III + IV$, may be estimated similarly, using the assumption on r_{0x} for the estimate for III,

$$III + IV \leq c \cdot \frac{\left\| \eta - \tilde{\eta} \right\|_\rho + \left\| u - \tilde{u} \right\|_\rho}{\rho - \rho'}.$$

We now consider F_2. Specifically, we estimate $F_2((\eta, u), t, s) - F_2((\tilde{\eta}, \tilde{u}), t, s):$

$$\left\| F_2((\eta, u), t, s) - F_2((\tilde{\eta}, \tilde{u}), t, s) \right\|_{\rho'} \leq V + VI + VII + VIII + IX + X,$$

where

$$V = \left\| A^{-1} \partial_x (\bar{\beta}(\eta - \tilde{\eta})) \right\|_{\rho'},$$

$$\begin{aligned} VI &= \|A^{-1}(uu_x - \tilde{u}\tilde{u}_x)\|_{\rho'}, \\ VII &= \left\| A^{-1}\left(\frac{(3\bar{\alpha}+r_0)r_{0x}}{2}\partial_x^2(\bar{\beta}(\eta-\tilde{\eta}))\right)\right\|_{\rho'}, \\ VIII &= \kappa\|A^{-1}(u-\tilde{u})\|_{\rho'}, \\ IX &= \gamma\|A^{-1}(\bar{\beta}r_{0x}u)\|_{\rho'}, \\ X &= \frac{\gamma}{2}\|A^{-1}(\bar{\beta}r_0\partial_x(u-\tilde{u}))\|_{\rho'}. \end{aligned}$$

We will omit some details, but each of V, VI, VII, $VIII$, IX, and X is bounded appropriately. We will demonstrate the estimate for a few terms, specifically for V, VI and X. The remaining terms are similar.

By Corollary 3.5 and (3.8), and by assumption on $\bar{\beta}$, we have

$$V \leq \frac{c\|\bar{\beta}\|_\rho\|\eta_x - \tilde{\eta}_x\|_\rho}{\rho-\rho'} \leq \frac{c\|\eta_x-\tilde{\eta}_x\|_\rho}{\rho-\rho'}.$$

For VI, we notice $uu_x = \frac{1}{2}\partial_x(u^2)$. Thus, we may write

$$VI = \frac{1}{2}\left\|A^{-1}\partial_x(u^2-\tilde{u}^2)\right\|_{\rho'}.$$

We then use Corollary 3.5, finding

$$VI \leq \frac{c\left\|u^2-\tilde{u}^2\right\|_\rho}{\rho-\rho'}.$$

Application of the algebra property (3.8) then yields

$$VI \leq \frac{c\|u-\tilde{u}\|_\rho}{\rho-\rho'}.$$

The final term for which we will provide details is X. We write

$$\bar{\beta}r_0(u_x-\tilde{u}_x) = \partial_x\left(\bar{\beta}r_0(u-\tilde{u})\right) - (\bar{\beta}r_0)_x(u-\tilde{u}),$$

and we bound X as

$$X \leq c\|A^{-1}\partial_x(\bar{\beta}r_0(u-\tilde{u}))\|_{\rho'} + c\|A^{-1}((\bar{\beta}r_0)_x(u-\tilde{u}))\|_{\rho'} = X_1 + X_2.$$

We use Corollary 3.5, the algebra property (3.8), and our assumptions on r_0 and $\bar{\beta}$ to bound X_1, finding

$$X_1 \leq \frac{c\left\|\bar{\beta}r_0(u-\tilde{u})\right\|_\rho}{\rho-\rho'} \leq \frac{c\|u-\tilde{u}\|_\rho}{\rho-\rho'}.$$

For X_2, we use Lemma 3.4, the algebra property (3.8), and our assumptions on r_0 and $\bar{\beta}$ to find

$$X_2 \leq c\|(\bar{\beta}r_0)_x(u-\tilde{u})\|_{\rho'} \leq c\|u-\tilde{u}\|_\rho \leq \frac{c\|u-\tilde{u}\|_\rho}{\rho-\rho'}.$$

The remaining terms are similar. This concludes the proof. □

3.4. A family of examples

Of course we wish to show that the set of functions which satisfy **(H4)** is nonempty; it is trivially nonempty since constant functions r_0 satisfy it. Going further, we wish to show that there are also non-constant functions which satisfy **(H4)**. To this end, we now demonstrate a simple family of functions r_0 which satisfy **(H4)**.

Let $R_0 > 0$; we consider $r_0 = R_0 + \varepsilon\sin(x)$, for sufficiently small ε. First, clearly, for sufficiently small ε, we have $g_1 > 0$, as required. Second, we see that for any $\rho \geq 0$, the function $8(1-\bar{\alpha})r_{0xx}$ is in B_ρ.

We next demonstrate that there exist values of $\rho > 0$ such that $\frac{1}{r_0} \in B_\rho$. We denote $\psi = \frac{1}{r_0}$, and let the Fourier coefficients of ψ be denoted as $\hat{\psi}_k$. We adapt this argument from the proof of Theorem IX.13 in [26].

Clearly, ψ has analytic extension to a strip of width $N > 0$ in the complex plane, for some $N > 0$ (we can even explicitly calculate this N if so desired); we call this extension $\tilde{\psi}$. For any given ρ such that $0 < \rho < N$, we denote by ψ_ρ the function such that $\psi_\rho(x) = \tilde{\psi}(x+i\rho)$, and we denote its Fourier coefficients as $\hat{\psi}_{\rho k}$. Since ψ_ρ is a bounded function on the torus, of course there exists $C > 0$ such $|\hat{\psi}_{\rho k}| \leq C$. By the Cauchy Integral Theorem, we have $\hat{\psi}_k = e^{-\rho k}\hat{\psi}_{\rho k}$. Thus, for $k \geq 0$, we have $|\hat{\psi}_k| \leq Ce^{-\rho|k|}$. Negative values of k can be treated similarly. This implies that for any $0 \leq \rho' < \rho$, we have $\psi \in B_{\rho'}$. Since ρ may be taken arbitrarily close to N, we conclude that for any $\rho \in (0, N)$, we have $\psi \in B_\rho$.

A similar argument naturally applies to the function $\frac{1}{(4\bar{\alpha}+r_0)}$, and to derivatives of $\frac{1}{r_0}$. Finally, by the algebra property for the B_ρ spaces, we conclude that there exists $\rho_0 > 0$ such that $\partial_x^j g_1^{-1}$ and g_2 are all in B_{ρ_0}.

Next we consider existence of the constant c_0 such that (3.10) holds. Denoting $K_0 = \frac{8}{(4\bar{\alpha}+R_0)R_0}$, we see that as $\varepsilon \to 0$, for any $c_0 \in (0, K_0)$, we have

$$\left\| \frac{g_2 - c_0}{c_0} \right\|_{\rho_0} \to \frac{K_0 - c_0}{c_0}.$$

As long as $c_0 \in (0, \frac{K_0}{2})$, for sufficiently small ε, we see that (3.10) holds.

We have therefore demonstrated that the set of functions r_0 satisfying **(H4)** is nontrivial. In Theorem 3.6, we also made the further assumption that r_0, r_{0x}, $\bar{\beta}$, and $\bar{\beta}_x$ are all in B_{ρ_0}. Clearly these properties hold as well (recall that $\bar{\beta}$ is proportional to ψ^2) for our family of examples.

4. Existence of periodic traveling waves

In this section, we establish the existence of periodic traveling waves for the system (1.3), (1.4). We will do this in the case $\kappa = \gamma = 0$. We prove existence by means of the following local bifurcation theorem [29]:

Theorem 4.1. (Bifurcation Theorem). *Let $\mathcal{H}'$ and $\mathcal{H}$ be Hilbert spaces, and let $(\eta_0, u_0) \in \mathcal{H}'$. Let U be an open neighborhood of (η_0, u_0) in $\mathcal{H}'$. Suppose*

(B1) *The map $\phi\colon U \times \mathbb{R} \to \mathcal{H}$ is C^2.*

(B2) *For all $c \in \mathbb{R}$, $\phi((\eta_0, u_0), c) = 0$.*

(B3) *For some c_0, $L(c_0) := \partial_{(\eta,u)}\phi((\eta_0, u_0), c)$ has a one-dimensional kernel and has zero Fredholm index.*

(B4) *If $h' \in \mathcal{H}'$ spans the kernel of $L(c_0)$ and $h^* \in \mathcal{H}$ spans the kernel of $L^*(c_0)$, then $\langle h^*, \partial_c L(c_0)h'\rangle_{\mathcal{H}} \neq 0$.*

If these four conditions hold, then there exists a sequence $\{(\eta_n, u_n), c_n\}_{n\in\mathbb{N}} \subset \mathcal{H}' \times \mathbb{R}$ with

a. $\lim_{n\to\infty} ((\eta_n, u_n), c_n) = ((\eta_0, u_0), c_0)$.
b. $(\eta_n, u_n) \neq (\eta_0, u_0)$ *for all $n \in \mathbb{N}$ and*
c. $\phi((\eta_n, u_n,)c_n) = 0$.

We will use Theorem 4.1 to prove the following theorem:

Theorem 4.2. *There exists a non-zero sequence $\{(\eta_n(x,t), u_n(x,t))\}_{n\in\mathbb{N}}$ such that for all n, for all t, $\eta_n \in H^1(\mathbb{T})$, $u_n \in H^3(\mathbb{T})$, and there exists a sequence of real numbers c_n such that for all n, the functions (η_n, u_n) constitute a nontrivial traveling wave solution of (1.3), (1.4) with speed c_n. There exists c_∞ such that as $n \to \infty$, $c_n \to c_\infty$ and $(\eta_n, u_n) \to (0, 0)$. Furthermore, at each time, each of η_n and u_n are even with zero mean.*

The rest of this section is the proof of Theorem 4.2. Specifically, we now demonstrate that the conditions **(B1)**, **(B2)**, **(B3)**, and **(B4)** hold for the traveling wave equations for (1.3), (1.4). This will be the content of Sects. 4.1, 4.2 and 4.3.

4.1. The mapping, (B1), and (B2)

Part of establishing that **(B1)** holds is specifying the function spaces and the mapping to be studied. We begin by defining the space $\mathcal{H}'$. We consider symmetric solutions, so we let

$$\mathcal{H}' := H^1_{e,0} \times H^3_{e,0}$$

where for any s,

$$\mathcal{H}^s_{e,0} := \left\{ f \in H^s : f \text{ is even and } \int_0^M f(x)dx = 0 \right\}.$$

We recall that H^s indicates the spatially periodic L^2-based Sobolev space of index s. As our choice of spaces shows, we will look for solutions η and u where both are even functions and have zero mean value.

We now give the traveling wave ansatz,

$$\eta = \eta(x - ct), \quad u = u(x - ct),$$

for some $c \in \mathbb{R}$. With this ansatz, and with parameter values $\kappa = 0$ and $\gamma = 0$, then the system (1.3), (1.4) becomes

$$-c\eta' + \frac{1}{2} r_0 u' + \frac{1}{2}\eta u' + \eta' u = 0, \tag{4.1}$$

$$-cu' + \overline{\beta}\eta' + uu' + \frac{c(4\overline{\alpha} + r_0)r_0}{8} u''' = 0. \tag{4.2}$$

The mapping $\phi(\eta, u)$ is given by the left-hand sides of (4.1), (4.2).

The space $\mathcal{H}'$ maps to odd functions under ϕ. Therefore, we take the codomain $\mathcal{H}$ to be

$$\mathcal{H} := L^2_{\text{odd}} \times L^2_{\text{odd}}.$$

With these definitions, **(B1)** and **(B2)** clearly hold, with the trivial solutions being $\eta = u = 0$ for any $c \in \mathbb{R}$.

4.2. The linearized operator and (B3)

We linearize the system about the equilibrium $\eta = 0$, $u = 0$. The linearization of the η equation is

$$\eta_1' = \frac{r_0}{2c} u_1', \tag{4.3}$$

which we may integrate, using the fact that our function spaces specify zero mean, finding

$$\eta_1 = \frac{r_0}{2c} u_1.$$

We turn to the u equation, which linearizes as

$$\frac{c(4\overline{\alpha} + r_0)r_0}{8} u_1''' - cu_1' + \overline{\beta}\eta_1' = 0. \tag{4.4}$$

The system (4.3), (4.4) is our linearized system, and the left-hand sides define our linearized operator, L. To investigate the dimension of the kernel and Fredholm properties, we rearrange the equations. Substituting η_1' from (4.3) in (4.4), and dividing by the leading coefficient, we arrive at

$$u_0''' + \left(\frac{8\overline{\beta}}{2c^2(4\overline{\alpha}+r_0)} - \frac{8}{(4\overline{\alpha}+r_0)r_0} \right) u_0' = 0. \tag{4.5}$$

The third-order differential Eq. (4.5) has characteristic polynomial

$$\xi^3 + \left(\frac{8\overline{\beta}}{2c^2(4\overline{\alpha}+r_0)} - \frac{8}{(4\overline{\alpha}+r_0)r_0} \right) \xi = 0. \tag{4.6}$$

We seek periodic solutions, for a fixed periodicity M, i.e., solutions satisfying

$$\eta(x+M,t) = \eta(x,t), \quad u(x+M,t) = u(x,t), \quad \forall x.$$

To have such periodic solutions, the cubic polynomial in (4.6) must have two pure imaginary roots, which we call $\pm iB$, and one real root, which we call D :

$$(r-iB)(r+iB)(r-D) = r^3 - Dr^2 + B^2 r - DB^2. \tag{4.7}$$

With such roots, there would be two independent spatially periodic solutions of (4.5),

$$v_1 = \cos(Bx), \quad v_2 = \sin(Bx).$$

Compatibility of our spatial period, M, and the wavelength require the existence of $k \in \mathbb{N}$ such that

$$\frac{2\pi k}{B} = M. \tag{4.8}$$

We now address the question of whether the roots of the cubic polynomial are in the desired form. We match coefficients in (4.7) to the coefficients of the cubic polynomial on the right-hand side of (4.6) so that we can find any restrictions on the parameters. Notice that (4.6) doesn't have an r^2 term, so $D = 0$. Continuing, we let

$$B^2 = \left(\frac{8\overline{\beta}}{2c^2(4\overline{\alpha}+r_0)} - \frac{8}{(4\overline{\alpha}+r_0)r_0} \right). \tag{4.9}$$

We can see that we may take B to be real (and positive) if

$$c^2 < \frac{\overline{\beta} r_0}{2}.$$

Let M be given; then, if c satisfies (4.9), to also satisfy (4.8), c must be given by

$$c^2 = \frac{8\overline{\beta} r_0 M^2}{16M^2 + 8\pi^2 k^2 (4\overline{\alpha}+r_0) r_0}. \tag{4.10}$$

That is, given a choice of M, there are infinitely many values c (one corresponding to each $k \in \mathbb{N}$), which give a nontrivial periodic kernel for L. Henceforth, we will take $M = 2\pi$.

Moreover, here we can define the linear operator $\partial_{(\eta,u)}\phi((\eta_0, u_0), c)$:

$$L(c)(\eta_0, u_0) = \begin{pmatrix} \partial_x & -\frac{r_0}{2c}\partial_x \\ 0 & \partial_x^3 + B^2 \partial_x \end{pmatrix} \begin{pmatrix} \eta_0 \\ u_0 \end{pmatrix}$$

Given any $k_0 \in \mathbb{N}$, with $k_0 \neq 0$, with the choice $M = 2\pi$, the formula (4.10) for c becomes

$$c_0 = \left(\frac{4\overline{\beta} r_0}{k_0^2 r_0 (4\overline{\alpha}+r_0) + 8} \right)^{\frac{1}{2}}.$$

Then, the possible kernel functions of $L(c_0)$ are:

$$z_1 = \cos(k_0 x) + i\sin(k_0 x) \quad \text{and} \quad z_2 = \cos(k_0 x) - i\sin(k_0 x).$$

Recall that we are considering the domain $\mathcal{H}' : H^1_{e,0} \times H^3_{e,0}$; therefore, we may eliminate odd functions from the kernel. As a result, we have a one-dimensional kernel of $L(c_0)$:

$$\ker L(c_0) = \left\{ \mathrm{span}\{h'\} | \; h' = \begin{pmatrix} \frac{r_0}{2c_0} \cos(k_0 x) \\ \cos(k_0 x) \end{pmatrix} \in \mathcal{H}' \right\}.$$

Now, we need to show that the Fredholm index of the linear operator $L(c_0)$ is zero. For $L(c_0)$ to be Fredholm, the following must hold:

- $\ker(L(c_0))$ is finite dimensional,
- $\mathrm{coker}(L(c_0))$ is finite dimensional, and
- $\mathrm{Range}(L(c_0))$ is closed.

If $L(c_0)$ is Fredholm, the index of $L(c_0)$ is

$$\mathrm{Ind}(L(c_0)) = \dim(\ker L(c_0)) - \dim(\mathrm{coker} L(c_0)).$$

Notice we have already shown the first condition: the dimension of the kernel of $L(c_0)$ is one-dimensional. We will show the dimension of the kernel of $L(c_0)$ is same as the dimension of the cokernel of $L(c_0)$, so $\mathrm{Ind}(L(c_0))$ becomes zero.

We begin by demonstrating that the kernel of the adjoint of $L(c_0)$ is one-dimensional. The adjoint of $L(c_0)$ is given by

$$L^*(c_0)(\eta_1, u_1) = \begin{pmatrix} -\eta_1' \\ \frac{r_0}{2c_0}\eta_1' - u_1''' - {k_0}^2 u_1' \end{pmatrix}.$$

Now, we will find kernel of L^*, as a subset of $\mathcal{H} : L^2_{odd} \times L^2_{odd}$. Notice when $\eta_1' = 0$, we have $\eta_1 = 0$. Then the equation $\frac{r_0}{2c_0} - u_1''' - {k_0}^2 u_1' = 0$ becomes $-u_1''' - {k_0}^2 u_1' = 0$. This is a third-order linear differential equation with characteristic polynomial $-r^3 - {k_0}^2 r = 0$, and the roots of this are $r = 0$ and $r = \pm i k_0$. On our domain $\mathcal{H}$, then, the kernel becomes

$$\ker L^*(c_0) = \left\{ \mathrm{span}\{h^*\} \mid h^* = \begin{pmatrix} 0 \\ \sin(k_0 x) \end{pmatrix} \in \mathcal{H} \right\}.$$

Hence, $\ker(L^*(c_0))$ is one-dimensional. It remains to establish that this is the same as the dimension of the cokernel.

Now, we move on to prove that the range of $L(c_0)$ is closed. Recall that $L(c_0)$ maps $\mathcal{H}'$ to $\mathcal{H}$. We introduce the decompositions $\mathcal{H}' = X_{k_0} \oplus \tilde{X}$ and $\mathcal{H} = Y_{k_0} \oplus \tilde{Y}$, where X_{k_0} and Y_{k_0} are given by

$$X_{k_0} = \mathrm{span}\left\{ \begin{pmatrix} \cos(k_0 x) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \cos(k_0 x) \end{pmatrix} \right\},$$
$$Y_{k_0} = \mathrm{span}\left\{ \begin{pmatrix} \sin(k_0 x) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sin(k_0 x) \end{pmatrix} \right\},$$

and $\tilde{X}$ and $\tilde{Y}$ are the complementary subspaces. The operator $L(c_0)$ maps X_{k_0} to Y_{k_0} and maps $\tilde{X}$ to $\tilde{Y}$. We will now show $L(c_0) \restriction \tilde{X}$ is bijective.

Since $\ker(L(c_0)) \subseteq X_{k_0}$, we see that $L(c_0) \restriction \tilde{X}$ has only trivial kernel and is thus injective. We need to show it is surjective as well. That is, we will show

$$\forall y \in \tilde{Y}, \quad \exists x \in \tilde{X} \quad \text{such that} \quad L(c_0)x = y. \tag{4.11}$$

To do so, we consider the inverse of $\hat{L}(c_0)$,

$$\hat{\Gamma}(k) = \frac{1}{k^4 - k^2 {k_0}^2} \begin{pmatrix} -i\,|k|^3 + i\,|k|\,{k_0}^2 & \frac{r_0}{2c_0} \cdot i\,|k| \\ 0 & i\,|k| \end{pmatrix}.$$

Notice the denominator is nonzero when $k \neq 0$ and $k \neq k_0$; the remaining wavenumbers correspond to $\tilde{X}$. To show (4.11), we will demonstrate

$$\forall y \in \tilde{Y},\ \exists\, x \in \tilde{X} \text{ s.t } \Gamma(y) = x.$$

That is, we will demonstrate that $\Gamma[\tilde{Y}] = \tilde{X}$.

We consider the four components of the operator as $\hat{\Gamma}_{ij}$, where $i, j \in \{1, 2\}$,

$$\begin{aligned}
\hat{\Gamma}_{1,1}(k) &= \frac{1}{i\,|k|},\\
\hat{\Gamma}_{1,2}(k) &= \frac{-i\frac{r_0}{2c_0}}{|k|^3 - |k|\,k_0^2},\\
\hat{\Gamma}_{2,1}(k) &= 0,\\
\hat{\Gamma}_{2,2}(k) &= \frac{-i}{|k|^3 - |k|\,k_0^2}.
\end{aligned}$$

Notice $\hat{\Gamma}_{1,1}$ is smoothing by one derivative since its symbol behaves like k^{-1} when $k \to \infty$, while $\hat{\Gamma}_{1,2}$ and $\hat{\Gamma}_{2,2}$ are smoothing by three derivatives since their symbols behave like k^{-3} as $k \to \infty$. Of course, $\hat{\Gamma}_{2,1}$ is simply the zero operator. Furthermore, all four symbols are pure imaginary, and thus map odd functions to even functions. Thus, we have that the $\hat{\Gamma}_{ij}$ are bounded linear operators between the following spaces:

$$\begin{aligned}
\hat{\Gamma}_{1,1} &: L^2_{\text{odd}} \to \mathcal{H}^1_{e,0},\\
\hat{\Gamma}_{1,2} &: L^2_{\text{odd}} \to \mathcal{H}^1_{e,0},\\
\hat{\Gamma}_{2,1} &: L^2_{\text{odd}} \to \mathcal{H}^3_{e,0},\\
\hat{\Gamma}_{2,2} &: L^2_{\text{odd}} \to \mathcal{H}^3_{e,0}.
\end{aligned}$$

This implies that Γ is a bounded mapping from $\tilde{Y} \to \tilde{X}$. The existence of this inverse implies $L(c_0)[\tilde{X}] = \tilde{Y}$.

The range of $L(c_0)$ is therefore equal to $L(c_0)[X_{k_0}] \oplus \tilde{Y}$. Since $\tilde{Y}$ is closed and X_{k_0} is finite-dimensional, we conclude that the range of $L(c_0)$ is closed. This also implies that the dimension of the cokernel of $L(c_0)$ is equal to the dimension of $\ker(L^*(c_0))$. Therefore, we have demonstrated that $L(c_0)$ is Fredholm, with Fredholm index zero. We have now established **(B3)**.

4.3. Establishing (B4)

Next, we will show that $\langle h^*, \partial_c L(c_0) h' \rangle_{\mathcal{H}}$ is nonzero. Taking the derivative of L with respect to c, and evaluating at c_0, we have:

$$\partial_c L(c_0) = \begin{pmatrix} 0 & \frac{r_0}{2c_0^2}\partial_x \\ 0 & \frac{-8\beta}{c_0^3(4\overline{\alpha}+r_0)}\partial_x \end{pmatrix}.$$

Then we apply this to h', computing

$$\partial_c L(c_0) h' = \begin{pmatrix} 0 & \frac{r_0}{2c_0^2}\partial_x \\ 0 & \frac{-8\beta}{c_0^3(4\overline{\alpha}+r_0)}\partial_x \end{pmatrix} \begin{pmatrix} \frac{r_0}{2c_0}\cos(k_0 x) \\ \cos(k_0 x) \end{pmatrix} = \begin{pmatrix} -\frac{r_0 k_0}{2c_0^2}\sin(k_0 x) \\ \frac{8\beta k_0}{(4\overline{\alpha}+r_0)c_0^3}\sin(k_0 x) \end{pmatrix}.$$

Then, taking the inner product with h^*, we find

$$\langle h^*, \partial_c L(c_0)h' \rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} 0 \\ \sin(k_0 x) \end{pmatrix}, \begin{pmatrix} -\frac{r_0 k_0}{2c_0^2} \sin(k_0 x) \\ \frac{8\overline{\beta} k_0}{(4\overline{\alpha}+r_0)c_0^3} \sin(k_0 x) \end{pmatrix} \right\rangle_{\mathcal{H}}$$

$$= \frac{8\overline{\beta}k_0}{(4\overline{\alpha}+r_0)c_0^3} \int_0^M \sin^2(k_0 x)dx \neq 0,$$

with this being nonzero since $\overline{\beta}$ and k_0 are nonzero. Hence, we have proved **(B4)** and we may apply Theorem 4.1, completing the proof of Theorem 4.2.

5. Discussion

We mention here a few future directions for this line of analysis. First, while we have given an argument that the system (1.1), (1.2) has an ill-posed initial value problem when r_0 is non-constant, we have not rigorously demonstrated ill-posedness. Generally speaking, ill-posedness can be more challenging to prove than well-posedness, and most often this is approached by demonstrating lack of continuous dependence on the initial data. Now that we have demonstrated a family of solutions for the general problem in Sect. 3, it is possible that a detailed analysis of these solutions could yield insight into a lack of continuous dependence on the data.

There are of course also future directions regarding traveling waves. We have proved the existence of periodic traveling waves in the case $\kappa = \gamma = 0$. This restriction on the parameters leads to a one-dimensional kernel of the linearized operator, which is a hypothesis of Theorem 4.1. In the general case, the kernel is two-dimensional. We have considered applying one-dimensional bifurcation theorems with two-dimensional kernels such as [17,18], but have found that the conditions of these theorems are not satisfied by our system. Considering a genuinely two-dimensional bifurcation will be the subject of future work.

Additionally, there are other models of fluid flow in viscoelastic vessels, such as the work of [8]. Analysis of further models, and comparison of features of solutions across different models, is another direction for future work. Asymptotic models such as the system (1.3), (1.4) should also be validated, in the sense that it should be proved that solutions of the model equation and solutions of the full equations (i.e., the Navier–Stokes equations) remain close, if they begin with the appropriately scaled, equivalent initial data. There is a long history of validation results for model equations in free-surface fluid dynamics [19], and extending such results to the present setting will be valuable to understand the sense in which these models are indeed a good approximation to the phenomena under consideration.

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Appendix A: Explicit solution of the toy linear model

In this appendix, we demonstrate an explicit solution of the toy linear model (3.1). We have already stated the formula for $\eta = \eta^N$ in (3.3). The corresponding formula for u is

$$u = \frac{cN^{1/2}}{\sqrt{2+2N^2}} \sin\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}} + \frac{cN^{1/2}}{\sqrt{2+2N^2}} \cos\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}}.$$

We then immediately have

$$\eta_t = -\frac{cN^{3/2}}{\sqrt{2+2N^2}} \cos\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}} + \frac{cN^{3/2}}{\sqrt{2+2N^2}} \sin\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}} = -u_x.$$

Therefore, the first equation in (3.1) is satisfied.

We next calculate the time derivative of u :

$$\begin{aligned} u_t = &-\frac{cN^2}{2+2N^2} \cos\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}} \\ &+ \frac{cN^2}{2+2N^2} \sin\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}} \\ &+ \frac{cN^2}{2+2N^2} \sin\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}} \\ &+ \frac{cN^2}{2+2N^2} \cos\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}}. \end{aligned}$$

This simplifies considerably, as the first and fourth terms on the right-hand side cancel and the second and third terms combine, yielding simply

$$u_t = \frac{cN^2}{1+N^2} \sin\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}}.$$

We then take two derivatives of this with respect to x, finding

$$u_{xxt} = -\frac{cN^4}{1+N^2} \sin\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}}.$$

Finally, we compute

$$\begin{aligned} u_t - u_{xxt} &= \frac{cN^2 + cN^4}{1+N^2} \sin\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}} \\ &= cN^2 \sin\left(Nx - t\frac{N^{3/2}}{\sqrt{2+2N^2}}\right) e^{tN^{3/2}/\sqrt{2+2N^2}} = -\eta_{xx}. \end{aligned}$$

This is the second equation in (3.1), so our calculation is complete.

We have mentioned before that $\|\eta^N(\cdot, 0)\|_{H^s} \to 0$ as $N \to \infty$. We finally note that it is straightforward to check that also $\|u^N(\cdot, 0)\|_{H^s} \to 0$ as $N \to \infty$, using our given definition of c, the definition of $u = u^N$, the fact that sin and cos functions are translates of each other, and that Sobolev norms are invariant under translation.

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