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Well-Posedness of a Model Equation for Water Waves in Fluids with Odd Viscosity

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Abstract
We study an asymptotic model for the motion of capillary-gravity waves in a fluid with non-Newtonian viscosity (known as *odd viscosity*). This model was one of three which were introduced recently by Granero-Belinchón and Ortega; they showed that two of their models were well-posed in Sobolev spaces and one was well-posed in analytic function spaces. For the model previously shown to have analytic solutions, we improve the theory to establish well-posedness in Sobolev spaces. This is accomplished through careful use of commutator estimates. We discuss related applications of our approach using these commutator estimates.

Keywords Odd viscosity · Well-posedness · Model equation · Commutator estimates · Sobolev spaces · Non-Newtonian fluid

1 Introduction

We consider the free surface dynamics of a viscous two-dimensional incompressible fluid bounded above by a free surface, subject to the effects of gravity and surface tension; notably, the viscous effect we consider here is non-Newtonian, and is known as *odd viscosity*. Studies of fluids that exhibit odd viscosity (or Hall viscosity) have been made since the work of Avron, Seiler, and Zongraf [11]. They showed that the viscosity of quantum fluids with an energy gap at zero temperature is non-dissipative. Other studies of systems with odd viscosity

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effects include the motion of two-dimensional quantum Hall fluids [5, 36], vortex dynamics [17], [38], polyatomic gases [25], and chiral active matter or chiral active fluids [12, 33].

While the free-surface problem for an incompressible fluid subject to the usual, Newtonian viscosity (even viscosity) has been well-studied [7, 13, 14, 16, 21–24, 26, 30, 32, 35] the corresponding problem with odd viscosity has only recently been analyzed in certain cases. In particular, the dynamics of surface waves with odd viscosity were studied in [2, 18], [19]. We concern ourselves here with extending the results of [19], in which one-dimensional model equations were developed for the odd viscosity capillary-gravity free surface problem, and the initial value problem for these systems were shown to be well-posed in certain settings. We now describe their models and results, and our extension.

Following the approach of [10, 17, 19], we consider a two-dimensional fluid occupying a domain $\Omega(t)$ at each time, t. The Navier–Stokes equations with odd viscosity are

$$\rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) = -\nabla p + \nu_0\Delta u^{\perp}, \quad \text{in } \Omega(t)\times[0,T],$$
$$\nabla\cdot u = 0, \quad \text{in } \Omega(t)\times[0,T],$$
$$\frac{\partial\rho}{\partial t} + \nabla\cdot(u\rho) = 0, \quad \text{in } \Omega(t)\times[0,T],$$

where as usual u and p denote the velocity and pressure of the fluid, respectively. The parameters ν_0 and ρ are the odd viscosity coefficient and the fluid density, respectively. The region $\Omega(t)$ is bounded above by a free surface, and there is vacuum above (at zero pressure). The Laplace-Young condition for the pressure, then, is that at the free surface, the pressure is proportional to the curvature of the surface, with the constant of proportionality being the positive, constant coefficient of surface tension.

We denote the domain as

$$\Omega(t) = \{(x,y)\in\mathbb{R}\times\mathbb{R} : y < \eta(t,x)\}$$

with free boundary of the form

$$\Gamma(t) = \{(x,y)\in\mathbb{R}\times\mathbb{R} : y = \eta(t,x)\}$$

so that the function $\eta(t,x)$ denotes the location of the free surface. Following the work of Zakharov [39], there are two classical kinematic and dynamic boundary conditions on the free surface $\Gamma(t)$, and a harmonic function $\phi(t,x,y)$ such that $u = \nabla\phi$. Using the trace of velocity potential,

$$\varphi(t,x,y) = \phi(t,x,\eta(t,x))$$

the main dynamics of surface waves under the effect of gravity, capillary forces, and odd viscosity, which is analogous to water waves with surface tension, can be written as

$$\begin{cases} \Delta\phi = 0, & \text{in } \Omega(t)\times[0,T] \\ \rho(\phi_t - \frac{1}{2}\phi_x^2 - \frac{1}{2}\phi_y^2 + g\eta) = \kappa\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} + \frac{2\nu_0}{(1+\eta_x^2)^{1/2}}\left(\frac{\eta_t}{(1+\eta_x^2)^{1/2}}\right)_x, & \text{on } \Gamma(t)\times[0,T] \\ \eta_t = -\eta_x\phi_x + \phi_y, & \text{on } \Gamma(t)\times[0,T] \end{cases}$$

where g and κ denote the gravitational acceleration and the coefficient of surface tension. Starting from this point, further reformulations may be made, and asymptotic expansions developed. Making a multiscale expansion in the steepness of the wave, the authors of [19] develop the quadratic model

$$u_{tt} = -\Lambda u - \beta\Lambda^3 u + \alpha_0 \Lambda u_{tx} + \varepsilon(-H((Hu_t)^2) + [H, u]\Lambda u)_x \\ +\varepsilon(-\alpha_0[H, u]\Lambda u_{tx} + \beta[H, u]\Lambda^3 u)_x, \tag{1.1}$$

where $\Lambda = H\partial_x$, H is the Hilbert transform, and α_0 and ε are positive constants. This is one of three models developed in [19], and the modeling work there extends that in the prior works [1, 9, 15, 18, 20] and [21]. The results in [19] include that the initial value problem for (1.1) with analytic initial data is locally well-posed. In the present work, we improve this to demonstrate well-posedness of the initial value problem for (1.1) with initial data in Sobolev spaces. Clearly the commutator structure of the right-hand side of (1.1) is important, and we achieve this improved result through application of a delicate commutator estimate. We now state our main result:

Theorem 1.1 *Let $\theta > 0$ be given. Let $(u_0, u_1) \in H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R})$ be given. There exists $T > 0$ and a unique solution u of the initial value problem for (1.1), with $u(0, x) = u_0$ and $u_t(0, x) = u_1$ such that*

$$(u, u_t) \in C^0([0, T], H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R})).$$

The plan of the paper is as follows: we provide some preliminary results, including our commutator estimates, in Sect. 2. We prove existence of solutions in Sect. 3, and prove uniqueness and continuous dependence in Sect. 4. We make some concluding remarks in Sect. 5.

2 Commutator Estimates and Preliminaries

In this section we present lemmas, especially commutator estimates, which will be useful to us throughout the sequel. We first define the L^2-based Sobolev spaces, H^s.

Definition 2.1 For $s \in \mathbb{R}$, $H^s(\mathbb{R})$ is the space of tempered distributions u such that their Fourier transform, $\hat{u}$, is locally integrable and

$$\|u\|^2_{H^s(\mathbb{R})} := \frac{1}{(2\pi)} \int \langle\xi\rangle^{2s} \left|\hat{u}(\xi)\right|^2 \, d\xi < \infty,$$

where $\langle\xi\rangle = \sqrt{1+\xi^2}$. We also define the Fourier multiplier operator J through its symbol, $\widehat{Jf}(\xi) = \langle\xi\rangle \hat{f}(\xi)$.

We will need an elementary interpolation lemma for Sobolev spaces, which we now state. The spatially periodic version appears in [6] among other places, and there is no essential difference in the version on the real line.

Lemma 2.1 *Let $s' \geq 0$ and $s \geq s'$ be given. There exists $c > 0$ such that for every $f \in H^s$, the following inequality holds:*

$$\|f\|_{H^{s'}} \leq c\|f\|^{1-\frac{s'}{s}}_{H^0} \|f\|^{\frac{s'}{s}}_{H^s}.$$

The Hilbert transform on $\mathbb{R}$ may be defined via its symbol as follows:

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi).$$

Of course, there is also the representation of the Hilbert transform as a singular integral,

$$Hf(x) = \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{f(x')}{x-x'}\,dx'.$$

Details on the Hilbert transform and the equivalence of these definitions may be found many places, such as [34].

Throughout the present work, many commutators of the form $[H,\phi]f = H(\phi f) - \phi H(f)$ will need to be estimated. The next two lemmas contain the estimates which we will use to bound such commutators.

Lemma 2.2 *Let $s \geq 0$ be given. Let $\sigma > 1/2$ be given. For any $\phi \in H^s(\mathbb{R})$ and $f \in H^\sigma$, the commutator $[H,\phi]f$ is in H^s, with the estimate*

$$\|[H,\phi]f\|_{H^s} \leq C\|\phi\|_{H^s}\|f\|_{H^\sigma}.$$

We next give a somewhat generalized version of Lemma 2.2 in Lemma 2.3.

Lemma 2.3 *Let $s \geq 0$ be given, and let $\gamma \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ satisfy $\gamma \geq 0$ and $\sigma > 1/2$. For any $\phi \in H^{s+\gamma}$ and $f \in H^{\sigma-\gamma}$, we have*

$$\|[H,\phi]f\|_{H^s} \leq C\|\phi\|_{H^{s+\gamma}}\|f\|_{H^{\sigma-\gamma}}.$$

We also need a lemma about the commutator of powers of J and multiplication by a smooth function.

Lemma 2.4 *Let $s > 3/2$ be given. For any $\phi \in H^s(\mathbb{R})$ and $f \in H^{s-1}$, the commutator $[J^s,\phi]f$ is in L^2, with the estimate*

$$\|[J^s,\phi]f\|_{L^2} \leq C\|\phi\|_{H^s}\|f\|_{H^{s-1}}.$$

To prove the above three Lemmas, we first need introduce the following useful lemma.

Lemma 2.5 *Assume that $F(\xi,\eta)$ is piecewise continuous function, and let*

$$T_F(f,g)(\xi) = \int F(\xi,\eta)f(\eta)g(\xi-\eta)\,d\eta, \quad f,g \in C_0.$$

If there exists $M > 0$ such that either

$$\int |F(\xi,\eta)|^2\,d\eta \leq M^2 \quad \text{for all } \xi \tag{2.1}$$

or

$$\int |F(\xi,\eta)|^2\,d\xi \leq M^2 \quad \text{for all } \eta \tag{2.2}$$

holds, then $T_F : L^2 \times L^2 \to L^2$, with the estimate

$$\|T_F(f,g)\|_{L^2} \leq M\|f\|_{L^2}\|g\|_{L^2}.$$

The proof of Lemma 2.5 may be found in [28], where this appears as Lemma 2.1.

Since Lemma 2.2 follows immediately from Lemma 2.3, we only give the proof of Lemma 2.3.

Proof *(The proof of Lemma 2.3)* To compute the H^s-norm of the commutator, we first write the formula for $\langle\xi\rangle^s \widehat{[H,\phi]f}(\xi)$:

$$\langle\xi\rangle^s \widehat{[H,\phi]f}(\xi) = \int \left((2\pi)^{-1}\langle\xi\rangle^s (i\,\mathrm{sgn}(\xi) - i\,\mathrm{sgn}(\eta))\widehat{\phi}(\xi-\eta)\widehat{f}(\eta)\right)\,d\eta.$$

We multiply and divide:

$$\langle\xi\rangle^s \widehat{[H,\phi]f}(\xi) = \int \Big((2\pi)^{-1}\langle\xi\rangle^s (i\,\mathrm{sgn}(\xi) - i\,\mathrm{sgn}(\eta))\langle\xi-\eta\rangle^{-s-\gamma}\langle\eta\rangle^{-\sigma+\gamma} \\ \langle\xi-\eta\rangle^{s+\gamma}\langle\eta\rangle^{\sigma-\gamma}\widehat{\phi}(\xi-\eta)\widehat{f}(\eta)\Big)\,d\eta.$$

Define $F(\xi,\eta) = (2\pi)^{-1}\langle\xi\rangle^s(i\,\mathrm{sgn}(\xi) - i\,\mathrm{sgn}(\eta))\langle\xi-\eta\rangle^{-s-\gamma}\langle\eta\rangle^{-\sigma+\gamma}$, and furthermore, make the auxiliary definitions $\phi_1(\xi) = \langle\xi\rangle^{s+\gamma}\widehat{\phi}(\xi)$ and $f_1(\xi) = \langle\xi\rangle^{\sigma-\gamma}\widehat{f}(\xi)$.

We wish to apply Lemma 2.5, so we begin to verify that condition (2.1) is satisfied:

$$\int |F(\xi,\eta)|^2 d\eta = \int \left|(2\pi)^{-1}\langle\xi\rangle^s(i\,\mathrm{sgn}(\xi) - i\,\mathrm{sgn}(\eta))\langle\xi-\eta\rangle^{-s-\gamma}\langle\eta\rangle^{-\sigma+\gamma}\right|^2\,d\eta.$$

Notice that $\mathrm{sgn}(\xi) - \mathrm{sgn}(\eta)$ is nonzero only when ξ and η have opposite signs. Then, we have $|\xi-\eta| = |\xi| + |\eta| \geq |\xi|$ and also $|\xi-\eta| = |\xi| + |\eta| \geq |\eta|$. With this in mind, we continue:

$$\begin{aligned}\int |F(\xi,\eta)|^2 d\eta &\leq C\int \left|\langle\xi\rangle^s\langle|\xi|+|\eta|\rangle^{-s-\gamma}\langle\eta\rangle^{\gamma}\langle\eta\rangle^{-\sigma}\right|^2\,d\eta \\ &\leq C\int \langle\eta\rangle^{-2\sigma}\,d\eta \leq M^2.\end{aligned}$$

Here, we have used the inequality $\langle\xi\rangle^s\langle|\xi|+|\eta|\rangle^{-s-\gamma}\langle\eta\rangle^{\gamma} \leq 1$ for $s \geq 0$ and $\gamma \geq 0$, and the fact that the final integral converges for $\sigma > \frac{1}{2}$.

Therefore, we apply Lemma 2.5, finding

$$\left\|\langle\xi\rangle^s\widehat{[H,\phi]f}(\xi)\right\|_{L^2} \leq M\|\phi_1\|_{L^2}\|f_1\|_{L^2} \leq M\|\phi\|_{H^{s+\gamma}}\|f\|_{H^{\sigma-\gamma}}.$$

This completes the proof of the Lemma 2.3. □

Proof of Lemma 2.4 We first expand the Fourier transform of the commutator $[J^s,\phi]f$:

$$\widehat{[J^s,\phi]f}(\xi) = \int \left((2\pi)^{-1}(\langle\xi\rangle^s - \langle\eta\rangle^s)\widehat{\phi}(\xi-\eta)\widehat{f}(\eta)\right)\,d\eta.$$

Define the function $g(w) = \langle w\rangle^s$, and let real numbers a and b be given. Then for some $\theta \in (0,1)$, we have $g(b) - g(a) = g'(a+\theta(b-a))(b-a)$. This implies the following:

$$\begin{aligned}|g(b)-g(a)| &= |g'(a+\theta(b-a))(b-a)| = |s\langle a+\theta(b-a)\rangle^{s-2}(a+\theta(b-a))(b-a)| \\ &\leq s|\langle a+\theta(b-a)\rangle^{s-1}(b-a)| \leq s\langle 2\max\{a,\theta(b-a)\}\rangle^{s-1}|b-a| \\ &\leq s2^s\langle\max\{a,\theta(b-a)\}\rangle^{s-1}|b-a| \leq s2^s(\langle a\rangle^{s-1} + \langle b-a\rangle^{s-1})|b-a|.\end{aligned}$$

Thus, from $|\langle\xi\rangle^s - \langle\eta\rangle^s| \leq C|\xi-\eta|(\langle\xi-\eta\rangle^{s-1} + \langle\eta\rangle^{s-1}) \leq C\langle\xi-\eta\rangle^s + C\langle\eta\rangle^{s-1}\langle\xi-\eta\rangle$, we have

$$\begin{aligned}&\widehat{[J^s,\phi]f}(\xi) \\ &\leq C\int\langle\xi-\eta\rangle^s\widehat{\phi}(\xi-\eta)\widehat{f}(\eta)\,d\eta + C\int\langle\eta\rangle^{s-1}\langle\xi-\eta\rangle\widehat{\phi}(\xi-\eta)\widehat{f}(\eta)\,d\eta. \quad (2.3)\end{aligned}$$

Let $\sigma > 1/2$ be given. We multiply and divide the first integrand on the right-hand side of (2.3) by $\langle\eta\rangle^{\sigma}$, and we multiply and divide the second integrand by $\langle\xi-\eta\rangle^{\sigma}$. To apply Lemma 2.5 to the first integral on the right-hand side of (2.3), we let F be given by $F(\xi,\eta)=\langle\eta\rangle^{-\sigma}$, and find that (2.1) is satisfied:

$$\int |F(\xi,\eta)|^2 d\eta = \int \langle\eta\rangle^{-2\sigma} d\eta \leq C.$$

For the second integral on the right-hand side of (2.3), we let $F(\xi,\eta)=\langle\xi-\eta\rangle^{\sigma}$, and we see that (2.2) is satisfied:

$$\int |F(\xi,\eta)|^2 d\eta = \int \langle\xi-\eta\rangle^{-2\sigma} d\xi \leq C.$$

Therefore, we have

$$\widehat{[J^s,\phi]f}(\xi) \leq C(\|\phi\|_{H^s}\|f\|_{H^\sigma} + \|\phi\|_{H^{\sigma+1}}\|f\|_{H^{s-1}})$$

To complete the proof of the Lemma 2.4, we only need to set $\sigma = s-1 > 1/2$ since $s > 3/2$. □

3 Existence

In this section we prove existence of solutions for the initial value problem for the model Eq. (1.1). We first prove existence of solutions for a mollified system, making use of the Picard theorem. We then establish an energy estimate and use this to show that all the solutions (for different values of the mollification parameter) exist uniformly in time. After this, we are able to pass to the limit as the regularization vanishes, arriving at a solution of the non-mollified system.

3.1 The Approximate System

The scalar Eq. (1.1) will be viewed as a system

$$\begin{cases} u_t = v, \\ v_t = -\Lambda u - \beta\Lambda^3 u + \alpha_0\Lambda v_x + \varepsilon(-H((Hv)^2) + [H,u]\Lambda u)_x \\ \quad +\varepsilon(-\alpha_0[H,u]\Lambda v_x + \beta[H,u]\Lambda^3 u)_x. \end{cases} \tag{3.1}$$

We now introduce the mollifier $\chi_\rho$with the parameter ρ: this is a classical mollifier, regularizing through convolution. As such, χ_ρ is a real multiplier in Fourier space, and therefore is self-adjoint and commutes with derivatives and Hilbert transforms. A detailed discussion of mollifiers and their properties may be found in Chapter 3 of [29]. We introduce our approximate problem:

$$\begin{cases} u_t = \chi_\rho^2 v, \\ v_t = -\chi_\rho\Lambda\chi_\rho u - \beta\chi_\rho\Lambda^3\chi_\rho u + \varepsilon\chi_\rho(-H((\chi_\rho Hv)^2) + [H,\chi_\rho u]\Lambda u)_x \\ \quad +\alpha_0\chi_\rho\Lambda\chi_\rho\, v_x + \varepsilon\chi_\rho(-\alpha_0[H,\chi_\rho u]\Lambda v_x + \beta[H,\chi_\rho u]\Lambda^3 u)_x, \end{cases} \tag{3.2}$$

with the initial data $(u_0,u_1)\in H^{3.5+\theta}(\mathbb{R})\times H^{2+\theta}(\mathbb{R})$, for some $\theta>0$. For the approximate Cauchy problem, we have local existence and uniqueness by the Picard Theorem. Without loss of generality, from now we fix $\alpha_0=1$, $\varepsilon=1$.

Proposition 3.1 *Let $\beta > 0$ be a constant, and for any $\theta > 0$, $(u_0, u_1) \in H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R})$ be initial data for Eq. (3.2) with $u(0,x) = u_0$, $v(0,x) = u_1$. Then, there exists a time $T_\rho > 0$ and a unique solution*

$$(u^\rho, v^\rho) \in C^1([0, T_\rho]; H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R})).$$

Proof Since χ_ρ is a smoothing opertator, it is easy to check that the right hand side of system (3.2) is locally Lipschtiz from $H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R}))$ to itself. By the Picard theorem, for any $\rho > 0$, there is $T_\rho > 0$, such that the system (3.2) has a unique solution $(u^\rho, v^\rho) \in C^1([0, T_\rho]; H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R}))$. □

3.2 Uniform Estimates and Existence

We have demonstrated the existence of solutions to the mollified system. We would like to pass to the limit as $\rho \to 0^+$. However, we cannot do this yet, as the size of the time interval guaranteed to exist in Proposition 3.1 could go to zero as ρ vanishes. Our next step is to prove an energy estimate, uniformly in ρ, for the solutions (u^ρ, v^ρ).

Proposition 3.2 *Let $\beta > 0$ be given. Let $\theta > 0$ and $\rho > 0$ be given. Let $(u^\rho, v^\rho) \in C^1([0, T_\rho]; H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R}))$ be a solution of (3.1), with initial conditions (u_0, u_1). Then there exist $T_0 > 0$, independent of ρ, such that for all $t \in (0, T_0)$*

$$\|(u^\rho, v^\rho)\|_{H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R})} \leq 2\|(u_0, u_1)\|_{H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R})}.$$

Proof Recall the evolution for (u, v),

$$\begin{cases} u_t = \chi_\rho^2 v, \\ v_t = -\chi_\rho \Lambda \chi_\rho u - \beta \chi_\rho \Lambda^3 \chi_\rho u + \chi_\rho(-H((\chi_\rho H v)^2) + [H, \chi_\rho u]\Lambda u)_x \\ \quad +\chi_\rho \Lambda \chi_\rho\, v_x + \chi_\rho(-[H, \chi_\rho u]\Lambda v_x + \beta[H, \chi_\rho u]\Lambda^3 u)_x. \end{cases}$$

We define the energy $E = E_0 + E_1$, with

$$\begin{aligned} E_0(t) &= \frac{1}{2}\|u\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}} u\Lambda u + \beta(\Lambda u)\Lambda^2 u\, dx \\ &\quad + \int (J^{2+\theta} u) J^{2+\theta} \Lambda u + \beta(J^{2+\theta}\Lambda u) J^{2+\theta}\Lambda^2 u\, dx, \\ E_1(t) &= \frac{1}{2}\|v\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}} (J^{2+\theta} v)^2\, dx. \end{aligned}$$

First, we take the time derivative of $\frac{1}{2}\|u\|_{L^2}$, finding $\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} u^2\, dx \leq E$. Second, we take the time derivative of $\|v\|_{L^2}^2$:

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} v^2\, dx &= \int_{\mathbb{R}} v v_t\, dx = \int_{\mathbb{R}} v\left(-\chi_\rho \Lambda \chi_\rho u - \beta \chi_\rho \Lambda^3 \chi_\rho u\right)\, dx + \int_{\mathbb{R}} v(\chi_\rho \Lambda \chi_\rho v_x)\, dx \\ &+ \int_{\mathbb{R}} v\chi_\rho \left((-H((\chi_\rho H v)^2) + [H, \chi_\rho u]\Lambda u)_x + (-[H, \chi_\rho u]\Lambda v_x + \beta[H, \chi_\rho u]\Lambda^3 u)_x\right)\, dx. \end{aligned}$$

Using the fact that χ_ρ is self-adjoint and commutes with Λ, we find

$$\begin{aligned} \int_{\mathbb{R}} v\chi_\rho(-\Lambda\chi_\rho u - \beta\Lambda^3\chi_\rho u)\, dx &= \int_{\mathbb{R}} u_t(-\Lambda u - \beta\Lambda^3 u)\, dx \\ &= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u\Lambda u + \beta(\Lambda u)\Lambda^2 u\, dx. \end{aligned}$$

Taking similar steps, we may also calculate the following:

$$\int_{\mathbb{R}} v(\chi_\rho \Lambda \chi_\rho v_x)\,dx = \int_{\mathbb{R}} (\Lambda^{1/2}\chi_\rho v)\partial_x(\Lambda^{1/2}\chi_\rho v)\,dx = 0,$$
$$\int_{\mathbb{R}} v\chi_\rho(-H((\chi_\rho Hv)^2)_x\,dx = -\int_{\mathbb{R}} (H\chi_\rho v)_x(H\chi_\rho v)^2\,dx = -\frac{1}{3}\int_{\mathbb{R}} (H\chi_\rho v)^3_x\,dx = 0.$$

We then must estimate three commutator terms. We usually estimate commutators using the smoothing properties given by the lemmas of Sect. 2. However, for the first of these, we do not need the smoothing properties, and it may be estimated directly as

$$\left\|([H,\chi_\rho u]\Lambda u)_x\right\|_{L^2(\mathbb{R})} \le C\|u\|^2_{H^2(\mathbb{R})}.$$

For the other two commutators, however, we must use Lemma 2.3. Taking $\sigma=\gamma=1$, and $s=1$, we apply this lemma to find

$$\|-([H,\chi_\rho u]\Lambda v_x)_x\|_{L^2(\mathbb{R})} \le C\|\chi_\rho u\|_{H^2(\mathbb{R})}\|\Lambda v_x\|_{L^2(\mathbb{R})} \le C\|u\|_{H^2}\|v\|_{H^2(\mathbb{R})}.$$

We again use Lemma 2.3 for the third commutator, again with $\sigma=\gamma=1$ and $s=1$, finding

$$\|([H,\chi_\rho u]\Lambda^3 u)_x\|_{L^2(\mathbb{R})} \le C\|\chi_\rho u\|_{H^2(\mathbb{R})}\|\Lambda^3 u\|_{L^2(\mathbb{R})} \le C\|u\|^2_{H^3(\mathbb{R})}.$$

Consolidating these calculations, we arrive at the bound

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} v^2 + u\Lambda u + \beta(\Lambda u)\Lambda^2 u\,dx \le CE^{3/2}$$

Third, we take the time derivative of $\frac{1}{2}\int_{\mathbb{R}}(J^{2+\theta}v)^2dx$:

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}}(J^{2+\theta}v)^2\,dx &= \int_{\mathbb{R}}(J^{2+\theta}v)J^{2+\theta}\chi_\rho(-\Lambda\chi_\rho u - \beta\Lambda^3\chi_\rho u)\,dx\\ &+\int_{\mathbb{R}}(J^{2+\theta}v)J^{2+\theta}\chi_\rho\Lambda\chi_\rho v_x\,dx - \int_{\mathbb{R}}(J^{2+\theta}v)J^{2+\theta}\chi_\rho H((\chi_\rho Hv)^2)_x\,dx\\ &+\int_{\mathbb{R}}(J^{2+\theta}v)J^{2+\theta}\chi_\rho\left([H,\chi_\rho u]\Lambda u - [H,\chi_\rho u]\Lambda v_x + \beta[H,\chi_\rho u]\Lambda^3 u\right)_x\,dx. \end{aligned}\tag{3.3}$$

We will now estimate the various terms on the right-hand side of (3.3).

The first term of the right-hand side of (3.3) may be rewritten by recognizing an exact time derivative, namely

$$\begin{aligned}&\int_{\mathbb{R}}(J^{2+\theta}v)J^{2+\theta}\chi_\rho(-\Lambda\chi_\rho u - \beta\Lambda^3\chi_\rho u)\,dx\\ &=\int_{\mathbb{R}}(J^{2+\theta}u_t)J^{2+\theta}(-\Lambda u-\beta\Lambda^3 u)\,dx = -\frac{d}{dt}\frac{1}{2}\\ &\int_{\mathbb{R}}(J^{2+\theta}u)J^{2+\theta}\Lambda u + \beta(J^{2+\theta}\Lambda u)J^{2+\theta}\Lambda^2 u\,dx.\end{aligned}$$

The second term of the right-side of (3.3) equals zero, since

$$\int_{\mathbb{R}}(J^{2+\theta}v)J^{2+\theta}\chi_\rho\Lambda\chi_\rho v_x\,dx = \int_{\mathbb{R}}(J^{2+\theta}\chi_\rho\Lambda^{1/2}v)\partial_x(J^{2+\theta}\chi_\rho\Lambda^{1/2}v)\,dx = 0.$$

The third term of the right-side of the Eq. (3.3) may be rewritten to bring out a commutator:

$$
\begin{aligned}
&-\int_{\mathbb{R}}(J^{2+\theta}v)J^{2+\theta}\chi_\rho H((H\chi_\rho v)^2)_x\,dx\\
&=\int_{\mathbb{R}}(J^{2+\theta}H\chi_\rho v)J^{2+\theta}((H\chi_\rho v)^2)_x\,dx=\int_{\mathbb{R}}(J^{2+\theta}H\chi_\rho v)J^{2+\theta}(2H\chi_\rho vH\chi_\rho v_x)\,dx\\
&=\int_{\mathbb{R}}2H\chi_\rho v(J^{2+\theta}H\chi_\rho v)(J^{2+\theta}H\chi_\rho v)_x+2J^{2+\theta}H\chi_\rho v[J^{2+\theta},H\chi_\rho v]H\chi_\rho v_x\,dx\\
&=-\int_{\mathbb{R}}H\chi_\rho v_x(J^{2+\theta}H\chi_\rho v)^2\,dx+2\int_{\mathbb{R}}J^{2+\theta}H\chi_\rho v[J^{2+\theta},H\chi_\rho v]H\chi_\rho v_x\,dx.
\end{aligned}
$$

The first term on the right-hand side may be immediately bounded using Sobolev embedding: $\int_{\mathbb{R}}H\chi_\rho v_x(J^{2+\theta}H\chi_\rho v)^2dx\le CE^{3/2}$. For the second term on the right-hand side, by Lemma 2.4 we have

$$
\begin{aligned}
2\int_{\mathbb{R}}(J^{2+\theta}H\chi_\rho v)[J^{2+\theta},H\chi_\rho v]H\chi_\rho v_x\,dx&\le C\|v\|_{H^{2+\theta}}\|[J^{2+\theta},H\chi_\rho v]H\chi_\rho v_x\|_{L^2}\\
&\le C\|v\|^2_{H^{2+\theta}}\|\chi_\rho v_x\|_{H^{1+\theta}}\le CE^{3/2}.
\end{aligned}
$$

Thus the third term of the right-side of (3.3) may be bounded as

$$
-\int_{\mathbb{R}}(J^{2+\theta}v)J^{2+\theta}\chi_\rho H((H\chi_\rho v)^2)_x\,dx\le CE^{3/2}.
$$

The fourth term on the right-side of (3.3) may be bounded as

$$
\begin{aligned}
&\int_{\mathbb{R}}(J^{2+\theta}\chi_\rho v)J^{2+\theta}\chi_\rho\left([H,\chi_\rho u]\Lambda u-[H,\chi_\rho u]\Lambda v_x+\beta[H,\chi_\rho u]\Lambda^3u\right)_x\,dx\\
&\le C\|v\|_{H^{2+\theta}}\left(\|[H,\chi_\rho u]\Lambda u\|_{H^{3+\theta}}+\|[H,\chi_\rho u]\Lambda v_x\|_{H^{3+\theta}}+\beta\|[H,\chi_\rho u]\Lambda^3u\|_{H^{3+\theta}}\right).
\end{aligned}
$$

By Lemma 2.2, we have $\|[H,\chi_\rho u]\Lambda u\|_{H^{3+\theta}}\le C\|u\|_{H^{3+\theta}}\|\Lambda u\|_\sigma$ for any $\sigma>1/2$; we may take $\sigma=1$. We use Lemma 2.3 to bound the other two commutators; taking $\sigma=\frac{1}{2}+\theta$, $\gamma=\frac{1}{2}$, and $s=3+\theta$, we have

$$
\begin{aligned}
\|[H,\chi_\rho u]\Lambda v_x\|_{H^{3+\theta}}&\le C\|u\|_{H^{3.5+\theta}}\|\Lambda v_x\|_{H^\theta}\le C\|u\|_{H^{3.5+\theta}}\|v\|_{H^{2+\theta}},\\
\|[H,\chi_\rho u]\Lambda^3u\|_{H^{3+\theta}}&\le C\|u\|_{H^{3.5+\theta}}\|\Lambda^3u\|_{H^\theta}\le C\|u\|^2_{H^{3.5+\theta}}.
\end{aligned}
$$

Now, we can conclude that

$$
\frac{dE}{dt}\le C(E^{3/2}+E).
$$

The conclusion of the proposition now follows. □

Proposition 3.3 *There exists* $0\le T_1\le T$ *such that* $\{(u^\rho,v^\rho)\}$ *is a Cauchy sequence in* $C^0([0,T_1];H^{1.5}(\mathbb{R})\times L^2(\mathbb{R}))$.

We omit the proof of Proposition 3.3 because the proof is entirely similar to the proof of uniqueness, which we provide in the subsequent section. Both results require estimating the norm of a difference; for Proposition 3.3, we estimate the norm of two solutions with different values of the regularization parameter. For uniqueness, by contrast, we estimate the norm of two solutions with possibly different initial data. The only difference is that the proof of Proposition 3.3 requires dealing with terms involving $\chi_{\rho_1}-\chi_{\rho_2}$. The following fact allows these to be estimated in a straightforward way.

Lemma 3.1 *For any* $0 < \rho_1 < \rho_2$ *and* $m > 0$, *we have*

$$\|\chi_{\rho_1} - \chi_{\rho_2}\|_{H^\mu \to H^{\mu-m}} \leq C\rho_2^m.$$

3.3 Remaining Details

In the previous subsection, we have proved the existence of a limit (u, v) in $C^0([0, T]; H^{1.5}(\mathbb{R}) \times L^2(\mathbb{R}))$. It remains to show that this limit (u, v) satisfies the non-mollified system, and that it has the same regularity as the initial data. We begin by establishing almost the highest regularity.

The sequence (u^ρ, v^ρ) is uniformly bounded with respect to both ρ and t in $H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R})$. The unit ball of the Hilbet space is weakly compact, so there exists a weak limit along a subsequence, and this weak limit must clearly be (u, v). This implies $(u, v) \in L^\infty([0, T]; H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R}))$. By the basic interpolation result (Lemma 2.1), for any $s < 2$, it is true that $(u, v) \in C^0([0, T]; H^{s+1.5+\theta}(\mathbb{R}) \times H^{s+\theta}(\mathbb{R}))$.

This is enough regularity to conclude that the limit satisfies the non-mollified system. To see this, we may represent the mollified solutions as

$$\begin{aligned} u^\rho &= u_0 + \int_0^t \chi_\rho^2 v^\rho \, ds, \\ v^\rho &= u_1 + \int_0^t -\chi_\rho \Lambda \chi_\rho u^\rho - \beta \chi_\rho \Lambda^3 \chi_\rho u^\rho \, ds \\ &\quad + \int_0^t \chi_\rho \left(-H((\chi_\rho H v^\rho)^2) + [H, \chi_\rho u^\rho]\Lambda u^\rho\right)_x + \chi_\rho \Lambda \chi_\rho v_x^\rho \, ds \\ &\quad + \int_0^t \chi_\rho \left(-[H, \chi_\rho u^\rho]\Lambda v_x^\rho + \beta [H, \chi_\rho u^\rho]\Lambda^3 u^\rho\right)_x \, ds. \end{aligned}$$

Since we have established that (u^ρ, v^ρ) converges to (u, v) in $C^0([0, T]; H^{3.5+\tilde{\theta}}(\mathbb{R}) \times H^{2+\tilde{\theta}}(\mathbb{R}))$ for some $\tilde{\theta} > 0$, we have uniform convergence of the integrands. We therefore may pass to the limit under the integral, finding

$$\begin{aligned} u &= u_0 + \int_0^t v \, ds, \\ v &= u_1 + \int_0^t -\Lambda u - \beta \Lambda^3 u + \left(-H((Hv)^2) + [H, u]\Lambda u\right)_x + \Lambda v_x \, ds \\ &\quad + \int_0^t \left(-[H, u]\Lambda v_x + \beta [H, u]\Lambda^3 u\right)_x \, ds. \end{aligned}$$

Setting $t = 0$ we see that the initial conditions are satisfied, and differentiating with respect to t, we see that (u, v) satisfies the non-mollified system.

All that remains is the highest regularity in time. That is, we now prove that the solution (u, v) is continuous in time in $H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R})$. To show continuity in time in, we must show that for any $t_* \in [0, T]$, we have

$$\lim_{t \to t_*} \|u(t, \cdot) - u(t_*, \cdot)\|_{H^{3.5+\theta}} + \|v(t, \cdot) - v(t_*, \cdot)\|_{H^{2+\theta}} = 0.$$

Note that the above limit is a one-sided limit when $t_* = 0$ or $t_* = T$. We know that the space $H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R})$ is a Hilbert space. To establish convergence in a Hilbert space, it is sufficient to establish weak convergence, plus convergence of the norm. We omit the details of this, as the argument is identical to the corresponding argument in [27] for hydroelastic waves, or the corresponding argument in Chapter 3 of [29] for the incompressible Euler equations.

The proof of the existence portion of Theorem 1.1 is now complete.

4 Uniqueness and Continuous Dependence

We assume that we have two solutions (u, u_t) and (u', u_t') of (1.1) in the space $H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R})$, and we estimate the difference in a lower regularity space, $H^{1.5}(\mathbb{R}) \times L^2(\mathbb{R})$. This space is chosen to be high enough so that the estimates have positive powers of derivatives, but low enough so that the terms can be bounded by $\|u\|_{H^{3.5+\theta}}$ and $\|u_t\|_{H^{2+\theta}}$. The estimates for $(u - u', u_t - u_t')$ in $H^{1.5}(\mathbb{R}) \times L^2(\mathbb{R})$, we will be very similar to the energy estimate above in the existence proof.

Theorem 4.1 *Let $\theta > 0$ and $T_0 > 0$ be given. Let both (u, u_t) and (u', u_t') be in $C^0([0, T_0]; H^{3.5+\theta}(\mathbb{R}) \times H^{2+\theta}(\mathbb{R}))$ and satisfy (1.1), with initial data (u_0, u_1) and (u_0', u_1'), respectively. Then the following estimate is satisfied:*

$$\begin{aligned}&\|(u, u_t) - (u', u_t')\|_{L^\infty([0,T_0]; H^{1.5}(\mathbb{R}) \times L^2(\mathbb{R}))}\\ &\quad \le C\|(u_0, u_1) - (u_0', u_1')\|_{H^{1.5}(\mathbb{R}) \times L^2(\mathbb{R})}.\end{aligned}$$

Proof We define the difference $\delta u = u - u'$, and then we also have $\delta u_t = u_t - u_t'$. We recall that we have fixed $\alpha_0 = 1$ and $\varepsilon = 1$. The equation satisfied by $(\delta u, \delta u_t)$ is

$$\begin{aligned}\delta u_{tt} &= -\Lambda\delta u - \beta\Lambda^3\delta u + \Lambda\delta u_{tx}\\ &\quad +(-H((Hu_t + Hu_t')(H\delta u_t)) + [H, \delta u]\Lambda u + [H, u']\Lambda\delta u)_x\\ &\quad +(-[H, \delta u]\Lambda u_{tx} - [H, u']\Lambda\delta u_{tx} + \beta[H, \delta u]\Lambda^3 u + \beta[H, u']\Lambda^3\delta u)_x.\end{aligned}$$

We define the energy for the difference, E_d:

$$E_d(t) = \frac{1}{2}\|\delta u_t\|_{L^2}^2 + \frac{1}{2}\|\delta u\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}} (\delta u)\Lambda\delta u + \beta(\Lambda\delta u)\Lambda^2\delta u\, dx.$$

We will estimate the growth of this energy, leading to a Grönwall argument.

First, we take the time derivative of $\|\delta u_t\|_{L^2}^2$:

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} (\delta u_t)^2 dx = \int_{\mathbb{R}} (\delta u_t)(\delta u_{tt})\, dx\\ &\quad = \int_{\mathbb{R}} (\delta u_t)(-\Lambda\delta u - \beta\Lambda^3\delta u)\, dx + \int_{\mathbb{R}} (\delta u_t)(\Lambda\delta u_{tx})\, dx\\ &\qquad + \int_{\mathbb{R}} \delta u_t \left(-H((Hu_t + Hu_t')(H\delta u_t)) + [H, \delta u]\Lambda u + [H, u']\Lambda\delta u\right)_x\, dx\\ &\qquad + \int_{\mathbb{R}} \delta u_t \left(-[H, \delta u]\Lambda u_{tx} - [H, u']\Lambda\delta u_{tx} + \beta[H, \delta u]\Lambda^3 u + \beta[H, u']\Lambda^3\delta u\right)_x\, dx.\end{aligned}$$

We first recognize an exact time derivative,

$$\int_{\mathbb{R}} \delta u_t(-\Lambda\delta u - \beta\Lambda^3\delta u)dx = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (\delta u)(\Lambda\delta u) + \beta(\Lambda\delta u)(\Lambda^2\delta u)\, dx.$$

Then we notice an exact spatial derivative, which integrates to zero:

$$\int_{\mathbb{R}} (\delta u_t)(\Lambda \delta u_{tx})\, dx = \frac{1}{2}\int_{\mathbb{R}} \partial_x((\Lambda^{1/2}\delta u_t)^2)\, dx = 0.$$

Now we estimate the remaining terms. We next have the following:

$$\begin{aligned}
&\int_{\mathbb{R}} \delta u_t(-H((Hu_t + Hu_t')(H\delta u_t))_x dx \\
&= -\int_{\mathbb{R}} (H\delta u_t)_x (Hu_t + Hu_t')(H\delta u_t) dx = \frac{1}{2}\int_{\mathbb{R}} (H\delta u_t)^2 (Hu_t + Hu_t')_x dx \\
&\le C\|(Hu_t + Hu_t')_x\|_{L^\infty(\mathbb{R})}\|H\delta u_t\|^2_{L^2(\mathbb{R})} \le C\|\delta u_t\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Here, we have used the fact that H and ∂_x are both skew-adjoint, and we have recognized a perfect spatial derivative and then integrated by parts; we then used Sobolev embedding. It remains to estimate the commutator terms. We let $\tilde{\theta}$ be given such that $\tilde{\theta} \in (0, \min\{\frac{1}{2}, \theta\})$. Applying Lemma 2.3 with $\sigma = \tilde{\theta} + \frac{1}{2} > 1/2$, $\gamma = 1/2$ and $s = 1$, we have the following estimates:

$$\begin{aligned}
&\|([H, \delta u]\Lambda u)_x\|_{L^2(\mathbb{R})} \le C\|\delta u\|_{H^{1.5}(\mathbb{R})}\|\Lambda u\|_{H^{\tilde{\theta}}(\mathbb{R})} \le C\|\delta u\|_{H^{1.5}(\mathbb{R})}, \\
&\|([H, u']\Lambda \delta u)_x\|_{L^2(\mathbb{R})} \le C\|u'\|_{H^{1.5}(\mathbb{R})}\|\Lambda \delta u\|_{H^{\tilde{\theta}}(\mathbb{R})} \le C\|\delta u\|_{H^{1.5}(\mathbb{R})}, \\
&\|(-[H, \delta u]\Lambda u_{tx})_x\|_{L^2(\mathbb{R})} \le C\|\delta u\|_{H^{1.5}(\mathbb{R})}\|\Lambda u_{tx}\|_{H^{\tilde{\theta}}(\mathbb{R})} \le C\|\delta u\|_{H^{1.5}(\mathbb{R})}, \\
&\|([H, \delta u]\Lambda^3 u)_x\|_{L^2(\mathbb{R})} \le C\|\delta u\|_{H^{1.5}(\mathbb{R})}\|\Lambda^3 u\|_{H^{\tilde{\theta}}(\mathbb{R})} \le C\|\delta u\|_{H^{1.5}(\mathbb{R})}.
\end{aligned}$$

Then, again applying Lemma 2.3 with $\sigma = \tilde{\theta} + \frac{1}{2} > 1/2$ but now with $\gamma = \frac{5}{2} + \tilde{\theta}$,$s = 1$, we have our remaining estimates:

$$\begin{aligned}
&\|(-[H, u']\Lambda \delta u_{tx})_x\|_{L^2(\mathbb{R})} \le C\|u'\|_{H^{3.5+\tilde{\theta}}(\mathbb{R})}\|\Lambda \delta u_{tx}\|_{H^{-2}(\mathbb{R})} \le C\|\delta u_t\|_{L^2(\mathbb{R})}, \\
&\|([H, u']\Lambda^3 \delta u)_x\|_{L^2(\mathbb{R})} \le C\|u'\|_{H^{3.5+\tilde{\theta}}(\mathbb{R})}\|\Lambda^3 \delta u\|_{H^{-2}(\mathbb{R})} \le C\|\delta u\|_{H^{1.5}(\mathbb{R})}.
\end{aligned}$$

All that remains is to take the time derivative of $\frac{1}{2}\|\delta u\|^2_{L^2}$:

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} \delta u^2 dx = \int_{\mathbb{R}} \delta u \delta u_t dx \le \|\delta u\|_{L^2}\|\delta u_t\|_{L^2}.$$

Consolidating all of the above calculations, we have found the estimate

$$\frac{dE_d}{dt} \le CE_d.$$

Grönwall's inequality now implies the conclusion. □

5 Discussion

We have proved that the initial value problem for the model (1.1) is well-posed for initial data in Sobolev spaces, improving the result of [19], in which it was demonstrated to have solutions with analytic data. Two other models were considered in [19].

In the model (1.1), if one neglects terms of size $\mathcal{O}(\varepsilon\alpha_0)$ and $\mathcal{O}(\varepsilon\beta)$, then one arrives at the model

$$u_{tt} = -\Lambda u - \beta\Lambda^3 u + \alpha_0 \Lambda u_{tx} + \varepsilon[-H((Hu_t)^2) + [H, u]\Lambda u]_x. \quad (5.1)$$

In [19], it was shown that the initial value problem for (5.1) is well-posed in Sobolev spaces, specifically with $(u, u_t)|_{t=0} \in H^{4.5} \times H^3$. While we do not provide the details, we are able to provide a uniform bound for approximate solutions of this initial value problem in the space $H^{1.5} \times H^0$. At this level of regularity one would not find a classical solution, but still a suitable notion of weak solution, or even a classical solution at lower regularity than that demonstrated in [19], could be proved to exist.

Another model developed in [19] is for unidirectional waves, and is given by

$$\begin{aligned} 2u_t + \alpha_0 \Lambda u_t = & \frac{1}{\varepsilon}(u_x + Hu + (\alpha_0 - \beta) H u_{xx}) \\ & + H((\Lambda u)^2) - [H, u]\Lambda u + (\alpha_0 - \beta)[H, u]\Lambda^3. \end{aligned} \tag{5.2}$$

The initial value problem for (5.2) was shown in [19] to be well-posed with initial data in H^3. The method of the present work is able to lower this threshold to $H^{2.5+\theta}$ for any $\theta > 0$.

Another possible direction to extend the present results is to consider forces other than surface tension at the wave surface, such as elastic forces. The authors and collaborators have previously studied hydroelastic waves in Newtonian fluids in a series of papers [3, 4, 8, 27]. Such models have been also studied by a number of other researchers, and stem from model equations developed by Plotnikov and Toland [31, 37]; including the hydroelastic force can model situations such as an ice sheet on the fluid surface. If one were interested in allowing the hydroelastic force on the surface of the non-Newtonian fluid, the resulting model would use the pressure at the free surface being given by

$$P = \tau \frac{\partial_x}{\sqrt{1+\eta_x^2}} \frac{\partial_x}{\sqrt{1+\eta_x^2}} \left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \right).$$

Developing the models along the lines of (1.1), (5.1), and (5.2) using this formula for the pressure instead of the Laplace-Young condition (in which the jump in pressure is proportional to the curvature), one arrives at models for which the initial value problems can be shown to be well-posed in Sobolev spaces, arguing along the lines of our main theorem.

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