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Well-posedness of a two-dimensional coordinate-free model for the motion of flame fronts[☆]

Shunlian Liu [a], David M. Ambrose [b,*]

[a] School of Science, Hunan University of Technology, Zhuzhou, Hunan 412007, China
[b] Department of Mathematics, Drexel University, 3141 Chestnut St., Philadelphia, PA 19104, USA

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ABSTRACT

We study a two-dimensional coordinate-free model for the motion of flame fronts. The model specifies the normal velocity of the interface in terms of geometric information, such as the mean curvature and the Gaussian curvature of the front. As the tangential velocities do not determine the position of the interface, we choose them to maintain a favorable parameterization. We choose this to be an isothermal parameterization. After appropriately reformulating the equations of motion, we use the energy method to prove short-time well-posedness in Sobolev spaces.

1. Introduction

The Kuramoto–Sivashinsky equation is a weakly nonlinear model equation for the motion of flame fronts [1,2]. It has been widely studied in one spatial dimension [3–7]. In two spatial dimensions, much work is either in the case of thin domains so that one-dimensional dynamics dominate [8–11], or is computational [12]. The second author and Mazzucato have established some existence results for the two-dimensional Kuramoto–Sivashinsky equation without an assumption of thinness of the domain [13,14].

While the Kuramoto–Sivashinsky equation has attracted much interest, as we have said, it is a weakly nonlinear model and with this comes some limitations. The biggest such limitation is that the flame front must be a graph with respect to the horizontal coordinates. In both one and two spatial dimensions, Frankel and Sivashinsky introduced more general, coordinate-free models of the motion of flame fronts [15,16]. These coordinate-free models are fully specified by giving a formula for the normal velocity of the flame front in terms of the front's intrinsic geometric information (such as its curvature). In addition to developing the coordinate-free models, Frankel and Sivashinsky demonstrate how the one-dimensional and two-dimensional Kuramoto–Sivashinsky equations may be derived from them.

In the decades since these coordinate-free models were introduced, there has been some limited mathematical theory developed for them. Temperature effects were incorporated into the one-dimensional model in [17]. A number of approximations to this model were then made, including quasi-steady approximations and weakly nonlinear approximations, in a series of papers [18–21]. None of these papers developed rigorous analytical theory for the full coordinate-free model of [17]. More recently, the first rigorous theory for the one-dimensional coordinate-free model of [15] was developed in [22] by the second author, Hadadifard, and Wright; there, it is demonstrated that the one-dimensional coordinate-free model is well-posed for small data, and that solutions of the coordinate-free model and solutions of the one-dimensional Kuramoto–Sivashinsky remain close if their initial conditions are close (thus this is a validation theorem for Kuramoto–Sivashinsky as a weakly nonlinear model). In the present work, we give the first analytical theory for the two-dimensional coordinate-free model of [16], proving a short-time well-posedness result for data of arbitrary size in Sobolev spaces.

We believe that there are two reasons for the dearth of rigorous theory for the full coordinate-free models of [15–17]. First, the models are not stated in evolutionary form, and instead are given as formulas for the normal velocity of the flame front. Second, even when one makes the effort to then restate the model in evolutionary form, the equations of motion for the

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* Corresponding author.
E-mail addresses: 06yslsl@163.com (S. Liu), dma68@math.drexel.edu (D.M. Ambrose).

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front involve high derivatives of curvature; this means that if one were to attempt to evolve the Cartesian coordinates of the flame front, the leading-order terms in the evolution equations would be highly nonlinear. We deal with both of these difficulties by adapting ideas originating in the numerical work of Hou, Lowengrub, and Shelley for the motion of one-dimensional vortex sheets with surface tension [23,24]. In this work, Hou, Lowengrub, and Shelley observed that only the normal velocity of the fluid interface was needed to provide for the motion of the interface, as the tangential velocity could be artificially chosen so as to enforce a preferred parameterization. They also chose to evolve geometric dependent variables such as tangent angle and arclength of the interface, as curvature is essentially linear in terms of these variables, and curvature enters the problem through the Laplace–Young jump condition for the pressure. Thus, in the one-dimensional case, the Hou, Lowengrub, Shelley work demonstrates how one might work with a normal velocity related to the curvature of an interface. The second author and Akers have adapted the numerical method of [23,24] to the one-dimensional coordinate-free model of Frankel and Sivashinsky [15] in [25]. The second author and Masmoudi used the ideas of [23,24] to prove well-posedness of the vortex sheet with surface tension and related problems [26–29]. The second author and Masmoudi then generalized these ideas for analysis of two-dimensional fluid interface problems [30–32]. We may view the present work as the adaptation of the analysis of the second author and Masmoudi from these papers to prove well-posedness of the two-dimensional coordinate-free model of [16].

The method by which we prove well-posedness of the initial value problem for the two-dimensional coordinate-free model is to first specify tangential velocities for the flame front; recall that the normal velocity is the content of the model of [16]. We choose tangential velocities so as to maintain a favorable parameterization, and as in [30–33], we choose an isothermal parameterization. Having fully specified the velocity of the flame front, we are able to write the evolution equations for the front, and to write evolution equations for related quantities. In particular, we need the evolution of the mean curvature of the front. This is because (again, as in the papers [30–33] for two-dimensional fluid interface problems) we are able to make energy estimates for the mean curvature, and we can use these estimates to establish the regularity of the front itself. The energy estimates we make are not for the mean curvature of the actual front, but instead are performed in the context of an iterative scheme. We set up an iterative approximation of the equations of motion for the flame front, prove existence of solutions for the iterated equations, demonstrate bounds on the solutions (by means of the energy estimates for mean curvature) which are uniform with respect to the iteration parameter, and then pass to the limit as the iteration parameter goes to infinity, finding solutions of the original problem.

The plan of the paper is as follows. In Section 2 we specify the model of [16] and we choose the tangential velocities for the flame front. We also state our main theorem at the end of Section 2. We explore the consequences of the equations of motion of the surface for the evolution of geometric quantities in Section 3. In Section 4, we then give some useful estimates related to commutators and to geometric quantities. In Section 5, we prove our main theorem by introducing our iterative scheme, carrying out the energy estimates for the iterates, and passing to the limit.

2. The equations of motion

We consider a two-dimensional flame front moving in three-dimensional space, with Cartesian coordinates

$$\mathbf{X}(\alpha,\beta,t)=(x(\alpha,\beta,t),y(\alpha,\beta,t),z(\alpha,\beta,t)).$$

Here, naturally, the two parameters along the surface are α and β, while t is time. We define a frame of normal and tangential vectors,

$$\hat{\mathbf{t}}^1=\frac{\mathbf{X}_\alpha}{|\mathbf{X}_\alpha|},\qquad \hat{\mathbf{t}}^2=\frac{\mathbf{X}_\beta}{|\mathbf{X}_\beta|},\qquad \hat{\mathbf{n}}=\frac{\mathbf{X}_\alpha\times\mathbf{X}_\beta}{|\mathbf{X}_\alpha\times\mathbf{X}_\beta|}. \tag{2.1}$$

The surface $\mathbf{X}$ moves according to normal velocity U and tangential velocities V_1 and V_2,

$$\mathbf{X}_t=U\hat{\mathbf{n}}+V_1\hat{\mathbf{t}}^1+V_2\hat{\mathbf{t}}^2. \tag{2.2}$$

The normal velocity will be specified in Section 2.1, and the tangential velocities will be specified in Section 2.2. We take an initial condition for the surface $\mathbf{X}$, namely

$$\mathbf{X}(\alpha,\beta,0)=\mathbf{X}_0(\alpha,\beta). \tag{2.3}$$

The geometry we consider is horizontally doubly periodic. The surface $\mathbf{X}$ at all times, including at the initial time, is such that

$$\mathbf{X}(\alpha+2\pi,\beta,t)=(2\pi,0,0)+\mathbf{X}(\alpha,\beta,t),$$
$$\mathbf{X}(\alpha,\beta+2\pi,t)=(0,2\pi,0)+\mathbf{X}(\alpha,\beta,t),$$

for all α and β.

We will now describe the surface and its curvature in terms of the first and second fundamental forms; the interested reader might refer to [34] for more background on these quantities. We define the coefficients of the first fundamental form for the surface $\mathbf{X}$ as

$$E=\mathbf{X}_\alpha\cdot\mathbf{X}_\alpha,\qquad F=\mathbf{X}_\alpha\cdot\mathbf{X}_\beta,\qquad G=\mathbf{X}_\beta\cdot\mathbf{X}_\beta.$$

The coefficients of the second fundamental form for the surface $\mathbf{X}$ are

$$L=-\mathbf{X}_\alpha\cdot\hat{\mathbf{n}}_\alpha,\qquad M=-\mathbf{X}_\alpha\cdot\hat{\mathbf{n}}_\beta=-\mathbf{X}_\beta\cdot\hat{\mathbf{n}}_\alpha,\qquad N=-\mathbf{X}_\beta\cdot\hat{\mathbf{n}}_\beta. \tag{2.4}$$

In terms of the first and second fundamental forms, the mean curvature is then

$$\kappa=\frac{EN+GL-2FM}{2(EG-F^2)}.$$

The Gaussian curvature is given by

$$q=\frac{LN-M^2}{EG-F^2}.$$

We will be choosing an isothermal parameterization for the flame front, meaning

$$E=G,\qquad F=0, \tag{2.5}$$

for all (α,β) and for all t. To enforce this parameterization, we will assume the initial surface $\mathbf{X}_0$ is parameterized accordingly, and then the tangential velocities V_1 and V_2 will be chosen so as to maintain the parameterization at positive times. The authors and Masmoudi have used this parameterization to good effect in a number of problems in interfacial fluid dynamics [30–33]. The implications of this choice for V_1 and V_2 are detailed in Section 2.2. Using an isothermal parameterization, the mean curvature and Gaussian curvature simplify to

$$\kappa=\frac{L+N}{2E},\qquad q=\frac{LN-M^2}{E^2}. \tag{2.6}$$

Also, it will be useful to note that with an isothermal parameterization, the surface $\mathbf{X}$ satisfies

$$\Delta\mathbf{X}=2\kappa\mathbf{X}_\alpha\times\mathbf{X}_\beta. \tag{2.7}$$

We mention that with the geometry under consideration, namely that the flame front is doubly periodic, a global isothermal parameterization may be found [35]. Thus we are not making a restrictive assumption on the class of initial data.

2.1. The normal velocity

The normal velocity for the flame front is developed in [16] as

$$U = -1 + (1-\sigma)\kappa - \left(1+\frac{\sigma^2}{2}\right)\kappa^2 + \left(\frac{\sigma^3}{3} - 5\sigma^2 - 2\sigma\right)\kappa^3 + 2(\sigma^2+1)q + (20\sigma^2+8\sigma-4)\kappa q - \sigma^2(\sigma+3)\Delta_S\kappa. \quad (2.8)$$

The parameter σ satisfies $\sigma > 1$; this allows the term $(1-\sigma)\kappa$ to destabilize the front at low frequencies, leading to nontrivial dynamics (as in the Kuramoto–Sivashinsky equation [12]). The operator Δ_S indicates the Laplace–Beltrami operator of the front. We use the formula for the Laplace–Beltrami operator found in the appendix of [36],

$$\Delta_S u = \frac{1}{\sqrt{EG-F^2}}\left(\frac{Eu_\beta - Fu_\alpha}{\sqrt{EG-F^2}}\right)_\beta + \frac{1}{\sqrt{EG-F^2}}\left(\frac{Gu_\alpha - Fu_\beta}{\sqrt{EG-F^2}}\right)_\alpha .$$

If we have an isothermal parameterization with $E = G$ and $F = 0$, then the Laplace–Beltrami operator simplifies to

$$\Delta_S u = \frac{u_{\alpha\alpha} + u_{\beta\beta}}{E}.$$

We define $W(\kappa, q)$ and τ as

$$W(\kappa, q) = (1-\sigma)\kappa - \left(1+\frac{\sigma^2}{2}\right)\kappa^2 + \left(\frac{\sigma^3}{3} - 5\sigma^2 - 2\sigma\right)\kappa^3 + 2(\sigma^2+1)q + (20\sigma^2+8\sigma-4)\kappa q, \quad (2.9)$$

$$\tau = \sigma^2(\sigma+3) > 0.$$

With these definitions, we may rewrite the normal velocity as

$$U = -\tau\Delta\kappa/E + W(\kappa, q) - 1. \quad (2.10)$$

As we have said in the introduction, the model of Frankel and Sivashinsky developed in [16] consists of the specification of the normal velocity of the flame front in terms of its intrinsic geometric information. That is, (2.8) (or equivalently (2.10)) is the model under consideration. One contribution of the present work is to rewrite this model as a system of evolution equations for the position of the flame front. This requires setting a parameterization of the front, which itself consists of two steps: setting the initial parameterization, and defining tangential velocities to maintain the chosen parameterization. The definition of the tangential velocities is the subject of the next subsection.

2.2. The tangential velocities and choice of parameterization

As we have said, while the normal velocity comes from the physical problem, the tangential velocities may be freely chosen so as to enforce a favored parameterization. That is, moving the surface tangent to itself does not change the location of the surface.

The tangential velocities may be determined by using (2.2) together with the time derivative of (2.5), $E_t = G_t$ and $F_t = 0$. This is the same choice made for the motion of a vortex sheet in three-dimensional fluids by the second author and Masmoudi, and the calculation of the tangential velocities may be found in [31]. The result is that the tangential velocities V_1, V_2 satisfy

$$\left(\frac{V_1}{\sqrt{E}}\right)_\alpha - \left(\frac{V_2}{\sqrt{E}}\right)_\beta = \frac{U(L-N)}{E}, \quad (2.11)$$

$$\left(\frac{V_1}{\sqrt{E}}\right)_\beta + \left(\frac{V_2}{\sqrt{E}}\right)_\alpha = \frac{2UM}{E}. \quad (2.12)$$

Then, if V_1 and V_2 satisfy (2.11) and (2.12), and if the initial surface $\mathbf{X}_0$ satisfies (2.5), then at positive times the surface will satisfy (2.5). We will prove well-posedness of the initial value problem (2.2), (2.3), with V_1 and V_2 enforcing the isothermal parameterization and U given by (2.8).

We will discuss solvability of this elliptic system in Section 4.

2.3. The main result

We take $s \in \mathbb{Z}$, with $s \geq 6$. In the calculations which follow in the next several sections, we will sometimes state that our reasoning is valid because s is "sufficiently large;" this simply refers to this fact that $s \geq 6$. Let c_0 be a positive constant. We define an open subset $\mathcal{O}_{c_0} \subseteq H^{s+2}$, such that for every $\mathbf{X} \in \mathcal{O}_{c_0}$, the following condition holds:

$$E(\alpha, \beta) > c_0.$$

Theorem 2.1. *We assume that the surface $\mathbf{X}_0 \in \mathcal{O}_{c_0}$ is globally parameterized by isothermal coordinates (namely (2.5) holds). Then, there exist a time $T > 0$ and a unique solution $\mathbf{X} \in C\left([0,T), \overline{\mathcal{O}_{\frac{c_0}{2}}}\right)$ of the Cauchy problem*

$$\begin{cases} \mathbf{X}_t = U\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2, \\ U = -\tau\Delta\kappa/E + W(\kappa, q) - 1, \\ \mathbf{X}(t=0) = \mathbf{X}_0. \end{cases} \quad (2.13)$$

Remark 1. When we say $\mathbf{X} \in H^s$, this means that $\mathbf{X}(\alpha, \beta) - (\alpha, \beta, 0)$ is actually in H^s, since the surface $\mathbf{X}$ is doubly periodic.

3. Geometric identities and evolution of geometric quantities

In this section, we first give some useful geometric identities. We then study the regularity of E and $\mathbf{X}$, and find evolution equations for E and κ. Versions of these equations and further discussion may be found in [31].

3.1. Geometric identities

We will frequently need to differentiate the normal and tangential vectors to the front, so formulas for these derivatives (in the context of our isothermal parameterization) will be helpful. The derivatives of the normal and tangential vectors satisfy the following:

$$\hat{\mathbf{n}}_\alpha = -\frac{L}{E^{1/2}}\hat{\mathbf{t}}^1 - \frac{M}{E^{1/2}}\hat{\mathbf{t}}^2,$$
$$\hat{\mathbf{n}}_\beta = -\frac{M}{E^{1/2}}\hat{\mathbf{t}}^1 - \frac{N}{E^{1/2}}\hat{\mathbf{t}}^2,$$
$$\hat{\mathbf{t}}^1_\alpha = -\frac{E_\beta}{2E}\hat{\mathbf{t}}^2 + \frac{L}{E^{1/2}}\hat{\mathbf{n}},$$
$$\hat{\mathbf{t}}^1_\beta = \frac{E_\alpha}{2E}\hat{\mathbf{t}}^2 + \frac{M}{E^{1/2}}\hat{\mathbf{n}},$$
$$\hat{\mathbf{t}}^2_\alpha = \frac{E_\beta}{2E}\hat{\mathbf{t}}^1 + \frac{M}{E^{1/2}}\hat{\mathbf{n}},$$
$$\hat{\mathbf{t}}^2_\beta = -\frac{E_\alpha}{2E}\hat{\mathbf{t}}^1 + \frac{N}{E^{1/2}}\hat{\mathbf{n}}.$$

These formulas are all directly verifiable by using (2.1), (2.5), and the definitions of the first and second fundamental forms. For instance, to compute $\hat{\mathbf{t}}^1_\alpha$, we first note that $\hat{\mathbf{t}}^1_\alpha \cdot \hat{\mathbf{t}}^1 = 0$, so that

$$\hat{\mathbf{t}}^1_\alpha = (\hat{\mathbf{t}}^1_\alpha \cdot \hat{\mathbf{t}}^2)\hat{\mathbf{t}}^2 + (\hat{\mathbf{t}}^1_\alpha \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}.$$

We then substitute $\hat{\mathbf{t}}^1 = \mathbf{X}_\alpha/E^{1/2}$, and arrive at

$$\hat{\mathbf{t}}^1_\alpha = \frac{(\mathbf{X}_{\alpha\alpha}\cdot\hat{\mathbf{t}}^2)}{E^{1/2}}\hat{\mathbf{t}}^2 + \frac{(\mathbf{X}_{\alpha\alpha}\cdot\hat{\mathbf{n}})}{E^{1/2}}\hat{\mathbf{n}} = \frac{(\mathbf{X}_{\alpha\alpha}\cdot\mathbf{X}_\beta)}{E}\hat{\mathbf{t}}^2 + \frac{L}{E^{1/2}}\hat{\mathbf{n}} = -\frac{(\mathbf{X}_\alpha\cdot\mathbf{X}_\alpha)_\beta}{2E}\hat{\mathbf{t}}^2 + \frac{L}{E^{1/2}}\hat{\mathbf{n}} = -\frac{E_\beta}{2E}\hat{\mathbf{t}}^2 + \frac{L}{E^{1/2}}\hat{\mathbf{n}}.$$

The remaining formulas are similar, and we omit the details.

3.2. Gain of regularity for E and $\mathbf{X}$

In the estimates we will be making, we will be using the mean curvature, κ, as our primary dependent variable. We will make estimates for κ in the Sobolev space H^s. We then need to infer regularity for $\mathbf{X}$ and E. We will be able to conclude that $\mathbf{X}$ is $(s+2)$-times differentiable (specifically, we will say $\mathbf{X} \in H^{s+2}$ which will mean that $\mathbf{X}-(\alpha,\beta,0)$ is actually in the space H^{s+2}). It may appear at first glance, then, that $E \in H^{s+1}$, but we may infer higher regularity and find in fact that $E \in H^{s+2}$ as well. This gain is a consequence of the isothermal parameterization.

The gain of one derivative for E may be seen by calculating ΔE. Recalling that $E = \mathbf{X}_\alpha \cdot \mathbf{X}_\alpha = \mathbf{X}_\beta \cdot \mathbf{X}_\beta$ and $\mathbf{X}_\alpha \cdot \mathbf{X}_\beta = 0$, we have

$$\Delta E = 2(\mathbf{X}_{\alpha\beta} \cdot \mathbf{X}_{\alpha\beta}) - 2(\mathbf{X}_{\alpha\alpha} \cdot \mathbf{X}_{\beta\beta}). \tag{3.1}$$

So, if $\mathbf{X}$ is in H^{s+2}, then the right-hand side of (3.1) is in H^s. We conclude that E is also in H^{s+2}. We have proved the following lemma.

Lemma 3.1. *If $(\mathbf{X}-(\alpha,\beta,0)) \in H^{s+2}$ then E is in H^{s+2}.*

(We remark that this gain of regularity is related to Gauss's Theorema egregium, and we also remark that there is a similar gain of regularity for E_t.) Finally, we mention that regularity of $\mathbf{X}$ may be inferred from the regularity of κ through the formula (2.7).

3.3. Evolution of E and κ

As we have said, we will perform energy estimates for the mean curvature, κ; as such, we must develop the evolution equation satisfied by κ. The evolution equation for κ can be inferred from (2.2), using the formula for κ in (2.6) with the definitions of the first and second fundamental coefficients. For the moment, a convenient way to write the evolution equation for the curvature is

$$(\sqrt{E}\kappa)_t = \frac{\Delta U}{2\sqrt{E}} + \frac{V_1}{\sqrt{E}}(\sqrt{E}\kappa)_\alpha + \frac{V_2}{\sqrt{E}}(\sqrt{E}\kappa)_\beta + \frac{UM^2}{\sqrt{E}} + \frac{L}{2\sqrt{E}}\left(\frac{V_1}{\sqrt{E}}\right)_\alpha + \frac{N}{2\sqrt{E}}\left(\frac{V_2}{\sqrt{E}}\right)_\beta. \tag{3.2}$$

Further details of the derivation of (3.2) may be found in [31]. Of course, to fully specify κ_t, we also must have an evolution equation for E. Such an evolution equation for E may be inferred from (2.2), using the definition $E = \mathbf{X}_\alpha \cdot \mathbf{X}_\alpha$, or alternatively $E = \mathbf{X}_\beta \cdot \mathbf{X}_\beta$. We therefore have the evolution equation

$$E_t = 2\sqrt{E}\left(V_{1,\alpha} - \frac{UL}{\sqrt{E}} + \frac{V_2 E_\beta}{2E}\right) = 2\sqrt{E}\left(V_{2,\beta} - \frac{UN}{\sqrt{E}} + \frac{V_1 E_\alpha}{2E}\right). \tag{3.3}$$

Since $\kappa_t = (\sqrt{E}\kappa)_t/\sqrt{E} - E_t\kappa/2E$, using (3.2) and (3.3), we conclude that the evolution of κ is given by the following:

$$\kappa_t = \frac{\Delta U}{2E} + \frac{V_1}{E}(\sqrt{E}\kappa)_\alpha + \frac{V_2}{E}(\sqrt{E}\kappa)_\beta + \frac{UM^2}{E} + \frac{L}{2E}\left(\frac{V_1}{\sqrt{E}}\right)_\alpha + \frac{N}{2E}\left(\frac{V_2}{\sqrt{E}}\right)_\beta - \frac{\kappa}{\sqrt{E}}\left(V_{1,\alpha} - \frac{UL}{\sqrt{E}} + \frac{V_2E_\beta}{2E}\right).$$

Using (2.10) to substitute for U, we have the evolution of κ as being

$$\kappa_t = -\frac{\tau}{2E}\Delta\left(\frac{\Delta\kappa}{E}\right) + \frac{\Delta W(\kappa,q)}{2E} + \frac{V_1}{E}(\sqrt{E}\kappa)_\alpha + \frac{V_2}{E}(\sqrt{E}\kappa)_\beta + \frac{UM^2}{E} + \frac{L}{2E}\left(\frac{V_1}{\sqrt{E}}\right)_\alpha + \frac{N}{2E}\left(\frac{V_2}{\sqrt{E}}\right)_\beta - \frac{\kappa}{\sqrt{E}}\left(V_{1,\alpha} - \frac{UL}{\sqrt{E}} + \frac{V_2E_\beta}{2E}\right).$$

4. Preliminary estimates and useful formulas

In this section we record some basic estimates which will be useful a number of times. Before doing so, we define the operator Λ to be the operator with symbol $\hat{\Lambda}(k) = |k|$. This can also be represented using the Riesz transforms H_1 and H_2, as we also have the formula $\Lambda = H_1\partial_\alpha + H_2\partial_\beta$. The Riesz transforms may either be defined in terms of their symbols or as singular integrals [37]. For any $\ell > 0$, we define the Sobolev space H^ℓ to be the space of functions for which the norm

$$\|f\|_\ell = \left(\|f\|_{L^2}^2 + \|\Lambda^\ell f\|_{L^2}^2\right)^{1/2}$$

is finite. Notice that $\Lambda^2 = -\Delta$.

The first lemma concerns commutators; as usual, the commutator notation means $[A,B]f = ABf - BAf$.

Lemma 4.1. *Let $s > 1$. If $f \in H^{s+2}$ and $g \in H^{s+1}$, then $[\Lambda^2, f]g$ is in H^s, with the estimate*

$$\|[\Lambda^2, f]g\|_s \le c\|f\|_{s+2}\|g\|_{s+1}.$$

Proof. Notice that

$$\left[\Lambda^2, f\right]g = -\Delta(fg) + f\Delta g = -g\Delta f - 2\nabla f \cdot \nabla g.$$

The Sobolev space $H^s(\mathbb{T}^d)$ is an algebra for $s > d/2$; since we are dealing with $d = 2$ and since $s > 1$, we do indeed have that H^s is an algebra. We therefore have

$$\|-g\Delta f\|_s \le c\|g\|_s\|\Delta f\|_s \le c\|f\|_{s+2}\|g\|_s$$

and

$$\|-2\nabla f \cdot \nabla g\|_s \le c\|\nabla f\|_s\|\nabla g\|_s \le c\|f\|_{s+1}\|g\|_{s+1}.$$

This completes the proof of the lemma. □

We next have another commutator estimate, which is a version of a standard Sobolev product estimate [38–40]; the authors previously used it in [33] as well.

Lemma 4.2. *For $s > 0$, then*

$$\|[\Lambda^s, f]g\| \le C\left(\|\nabla f\|_{L^\infty}\|g\|_{s-1} + \|f\|_s\|g\|_{L^\infty}\right).$$

Next we give a standard elementary interpolation estimate.

Lemma 4.3. *For $0 < m < s$, and $f \in H^s$, then*

$$\|\Lambda^m f\| \le \|\Lambda^s f\|^{m/s}\|f\|^{1-m/s}$$

The proof of Lemma 4.3 can be found in many places, such as [26]; see also [41].

For the final result of this section, we comment on the regularity of the velocities U, V_1, and V_2.

Lemma 4.4. *If $(\mathbf{X}-(\alpha,\beta,0)) \in H^{s+2}$, $E \in H^{s+2}$ and $\kappa \in H^s$, then $U \in H^{s-2}$ and $V_i \in H^{s-1}$.*

Proof. Recall that

$$U + 1 = -\tau\Delta\kappa/E + W(\kappa, q).$$

We immediately get $U \in H^{s-2}$ when κ and q are in H^s; this uses the definition of W in (2.9) and the fact that H^s is an algebra for $s > 1$.

Turning to V_1 and V_2, we apply ∂_α to (2.11) and we apply ∂_β to (2.12), and then add the results. It follows that V_1 satisfies

$$\Delta\left(\frac{V_1}{\sqrt{E}}\right)=\left(\frac{U(L-N)}{E}\right)_\alpha+\left(\frac{2UM}{E}\right)_\beta.$$

That this Poisson equation is solvable is immediate since our spatial domain is $\mathbb{T}^2$ and the right-hand side has zero mean (this is evident since the right-hand side is a sum of derivatives); the solvability may be seen in Fourier space. With the assumed regularities of $\mathbf{X}$ and E, and the demonstrated regularity for U, we have $V_1\in H^{s-1}$. Similarly, we have $V_2\in H^{s-1}$ since

$$\Delta\left(\frac{V_2}{\sqrt{E}}\right)=-\left(\frac{U(L-N)}{E}\right)_\beta+\left(\frac{2UM}{E}\right)_\alpha.$$

The interested reader could find more on the solution of Poisson equations in doubly periodic geometry (using a potential theory approach) in [42]. □

5. Well-posedness

In this section we prove Theorem 2.1, showing that the two-dimensional coordinate-free model for the motion of flame fronts is well-posed. We will use an iteration method. In Section 5.1, we set up an iterated system of evolution equations. We then set up a general linear Cauchy problem in Section 5.2, and demonstrate energy estimates for this problem in Section 5.3. In Section 5.4 we then use the results of Sections 5.2 and 5.3 to show that the iterates in our iteration scheme obey bounds uniform with respect to the iteration parameter. We use the uniform bounds to take the limit of the sequence of iterates, and demonstrate that this limit solves the original system, in Section 5.5. We then discuss uniqueness and continuous dependence in 2.1 in Section 5.6.

5.1. The iterated system

We now set up iterated evolution equations for $\mathbf{X}$ and κ. We take initial data $\mathbf{X}_0\in H^{s+2}$, such that the surface has a global isothermal parameterization with $E_0=\mathbf{X}_{0\alpha}\cdot\mathbf{X}_{0\alpha}=\mathbf{X}_{0\beta}\cdot\mathbf{X}_{0\beta}>c_0$, for some constant $c_0>0$. Of course, in taking a global isothermal parameterization for $\mathbf{X}_0$, we also have $\mathbf{X}_{0\alpha}\cdot\mathbf{X}_{0\beta}=0$.

We initialize the iterative scheme with the initial iterate being the flat surface $\mathbf{X}^0=(\alpha,\beta,0)$. Then $E^0=1$, and $\kappa^0=0$. Then clearly the initial iterates are C^∞. Similarly we also have the other quantities corresponding to $\mathbf{X}^0$, namely L^0, M^0, N^0, and q^0, with these all being C^∞. Recall that these quantities are defined through (2.4) and (2.6), using $\mathbf{X}^0$ as the surface. We then also may define U^0 through (2.10), using κ^0, E^0, and q^0. Then (V_1^0,V_2^0) is determined by solving the following system,

$$\left(\frac{V_1^0}{\sqrt{E^0}}\right)_\alpha-\left(\frac{V_2^0}{\sqrt{E^0}}\right)_\beta=\frac{U^0(L^0-N^0)}{E^0}=0,$$
$$\left(\frac{V_1^0}{\sqrt{E^0}}\right)_\beta+\left(\frac{V_2^0}{\sqrt{E^0}}\right)_\alpha=\frac{2U^0M^0}{E^0}=0.$$

That this elliptic system is solvable follows as in the proof of Lemma 4.4. Specifically, we can apply ∂_α to the first equation and ∂_β to the second equation, and add and subtract the two equations to find Poisson equations. The Poisson equations are solvable since the spatial domain is $\mathbb{T}^2$ and the right-hand sides have zero mean. Of course, this solution is $V_1^0=V_2^0=0$.

Assume that for some $l\geq 0$ we have already constructed $(\mathbf{X}^l,\kappa^l,E^l)$, and these are all C^∞ functions. We then compute the related quantities

$$\hat{\mathbf{t}}^{1,l}=\frac{\mathbf{X}^l_\alpha}{|\mathbf{X}^l_\alpha|},\qquad \hat{\mathbf{t}}^{2,l}=\frac{\mathbf{X}^l_\beta}{|\mathbf{X}^l_\beta|},\qquad \hat{\mathbf{n}}^l=\frac{\mathbf{X}^l_\alpha\times\mathbf{X}^l_\beta}{|\mathbf{X}^l_\alpha\times\mathbf{X}^l_\beta|},$$

$$L^l=\mathbf{X}^l_{\alpha\alpha}\cdot\hat{\mathbf{n}}^l,\qquad N^l=\mathbf{X}^l_{\beta\beta}\cdot\hat{\mathbf{n}}^l,\qquad M^l=\mathbf{X}^l_{\alpha\beta}\cdot\hat{\mathbf{n}}^l,$$
$$q^l=\frac{L^lN^l-(M^l)^2}{(\mathbf{X}^l_\alpha\cdot\mathbf{X}^l_\alpha)(\mathbf{X}^l_\beta\cdot\mathbf{X}^l_\beta)-(\mathbf{X}^l_\alpha\cdot\mathbf{X}^l_\beta)^2}.$$

To begin finding the next iterates, we construct κ^{l+1} to solve the linear Cauchy problem

$$\kappa^{l+1}_t=-\frac{\tau}{2E^l}\Delta\left(\frac{\Delta\kappa^{l+1}}{E^l}\right)+Q_1^l,\tag{5.1}$$

where Q_1^l is given by

$$\begin{aligned}Q_1^l=&\frac{\Delta W(\kappa^l,q^l)}{2E^l}+\frac{V_1^l}{E^l}\left(\sqrt{E^l}\kappa^l\right)_\alpha+\frac{V_2^l}{E^l}\left(\sqrt{E^l}\kappa^l\right)_\beta\\&+\frac{U^l(M^l)^2}{E^l}+\frac{L^l}{2E^l}\left(\frac{V_1^l}{\sqrt{E^l}}\right)_\alpha+\frac{N^l}{2E^l}\left(\frac{V_2^l}{\sqrt{E^l}}\right)_\beta\\&-\frac{\kappa^l}{\sqrt{E^l}}\left(V^l_{1,\alpha}-\frac{U^lL^l}{\sqrt{E^l}}+\frac{V_2^lE^l_\beta}{2E^l}\right),\end{aligned}\tag{5.2}$$

and with initial condition $\kappa^{l+1}|_{t=0}=\chi_l\kappa_0$. Here, χ_l is a smoothing operator, which projects onto Fourier modes with wavenumbers at most $|l|$. We see that the data for κ^{l+1} is therefore infinitely smooth for all l; as long as the previous iterates are infinitely smooth, we will have κ^{l+1}, being the solution of a linear parabolic equation, is infinitely smooth as well. After we demonstrate that this problem is solvable, we can then define U^{l+1} as

$$U^{l+1}=-\tau\Delta\kappa^{l+1}/E^l+W(\kappa^l,q^l)-1.\tag{5.3}$$

We next determine (V_1^{l+1},V_2^{l+1}) by solving the system

$$\left(\frac{V_1^{l+1}}{\sqrt{E^l}}\right)_\alpha-\left(\frac{V_2^{l+1}}{\sqrt{E^l}}\right)_\beta=\frac{U^{l+1}(L^l-N^l)}{E^l}=-\tau\frac{\Delta\kappa^{l+1}(L^l-N^l)}{(E^l)^2}+f_1^l,\tag{5.4}$$

$$\left(\frac{V_1^{l+1}}{\sqrt{E^l}}\right)_\beta+\left(\frac{V_2^{l+1}}{\sqrt{E^l}}\right)_\alpha=\frac{2U^{l+1}M^l}{E^l}=-\tau\frac{2\Delta\kappa^{l+1}M^l}{(E^l)^2}+f_2^l.\tag{5.5}$$

Here, the functions f_1^l,f_2^l are defined by

$$f_1^l=\frac{(W(\kappa^l,q^l)-1)(L^l-N^l)}{E^l},\tag{5.6}$$
$$f_2^l=\frac{2(W(\kappa^l,q^l)-1)M^l}{E^l}.\tag{5.7}$$

These equations can be combined to more clearly have solvable Poisson equations,

$$\begin{aligned}\Delta\left(\frac{V_1^{l+1}}{\sqrt{E^l}}\right)=&-\left(\frac{\tau\Delta\kappa^{l+1}(L^l-N^l)}{(E^l)^2}\right)_\alpha-\left(\frac{2\tau\Delta\kappa^{l+1}M^l}{(E^l)^2}\right)_\beta\\&+(f_1^l)_\alpha+(f_2^l)_\beta,\end{aligned}\tag{5.8}$$

$$\begin{aligned}\Delta\left(\frac{V_2^{l+1}}{\sqrt{E^l}}\right)=&\left(\frac{\tau\Delta\kappa^{l+1}(L^l-N^l)}{(E^l)^2}\right)_\beta-\left(\frac{2\tau\Delta\kappa^{l+1}M^l}{(E^l)^2}\right)_\alpha\\&-(f_1^l)_\beta+(f_2^l)_\alpha.\end{aligned}\tag{5.9}$$

As discussed in the proof of Lemma 4.4, these Poisson equations are solvable since the spatial domain is $\mathbb{T}^2$ and the right-hand sides have zero mean. Specifically, the solvability is clear in Fourier space. Furthermore, we mention that since κ^{l+1} and the previous iterates are all C^∞, all of these quantities such as U^{l+1} and V_i^{l+1} are as well.

We define the next iterate of the surface, $\mathbf{X}^{l+1}$, in a few stages. To begin we let $\mathbf{Z}_1^{l+1}$ be the solution of the initial value problem

$$(\mathbf{Z}_1)^{l+1}_t=U^{l+1}\hat{\mathbf{n}}^l+V_1^{l+1}\hat{\mathbf{t}}^{1,l}+V_2^{l+1}\hat{\mathbf{t}}^{2,l},\quad (\mathbf{Z}_1)^{l+1}|_{t=0}=\chi_l\mathbf{X}_0.\tag{5.10}$$

Here, we understand that χ_l applied to $\mathbf{X}_0$ truncates the Fourier series of $\mathbf{X}_0 - (\alpha, \beta, 0)$. We therefore again have that $\mathbf{X}^l$ is in C^∞ for all l. Substituting for U^{l+1} using Eq. (5.3), we can write this as

$$(\mathbf{Z}_1)_t^{l+1} = (-\tau \Delta\kappa^{l+1}/E^l)\hat{\mathbf{n}}^l + V_1^{l+1}\hat{\mathbf{t}}^{1,l} + V_2^{l+1}\hat{\mathbf{t}}^{2,l} + Q_2^l, \tag{5.11}$$

where Q_2^l is given by

$$Q_2^l = (W(\kappa^l, q^l) - 1)\hat{\mathbf{n}}^l. \tag{5.12}$$

We will have one more intermediate variable $\mathbf{Z}_2^{l+1}$, which is given by solving the elliptic equation

$$\Delta\mathbf{Z}_2^{l+1} - \mathbf{Z}_2^{l+1} = 2\kappa^l(\mathbf{Z}_1^{l+1})_\alpha \times (\mathbf{Z}_1^{l+1})_\beta - \mathbf{Z}_1^{l+1}. \tag{5.13}$$

Note that this formula is based upon Eq. (2.7). Now we are ready to construct $\mathbf{X}^{l+1}$ by solving the following elliptic equation (again influenced by (2.7)),

$$\Delta\mathbf{X}^{l+1} - \mathbf{X}^{l+1} = 2\kappa^l(\mathbf{Z}_2^{l+1})_\alpha \times (\mathbf{Z}_2^{l+1})_\beta - \mathbf{Z}_1^{l+1}. \tag{5.14}$$

Finally, we will define E^{l+1} also by solving an elliptic equation,

$$\Delta E^{l+1} - E^{l+1} = 2(\mathbf{X}_{\alpha\beta}^{l+1}\cdot\mathbf{X}_{\alpha\beta}^{l+1} - \mathbf{X}_{\alpha\alpha}^{l+1}\cdot\mathbf{X}_{\beta\beta}^{l+1}) - \frac{1}{2}(\mathbf{X}_\alpha^{l+1}\cdot\mathbf{X}_\alpha^{l+1} + \mathbf{X}_\beta^{l+1}\cdot\mathbf{X}_\beta^{l+1}). \tag{5.15}$$

Note that this equation is based upon (3.1) as well as the fact that in an isothermal parameterization, we have $E = \frac{1}{2}\mathbf{X}_\alpha \cdot \mathbf{X}_\alpha + \frac{1}{2}\mathbf{X}_\beta \cdot \mathbf{X}_\beta$. Again, all of the quantities we have defined as the $(l+1)$-st iterates are C^∞.

Remark 2. Note that for any of the versions of the iterated surfaces, they are not expected to be parameterized isothermally. The isothermal parameterization of the solution will be recovered after taking the limit as $l \to \infty$.

5.2. A linear Cauchy problem

To deal with the iterated system, we study the well-posedness of the linearized Cauchy problem for κ^{l+1} and $\mathbf{Z}_1^{l+1}$. More precisely, we will consider the linear Cauchy problems for η and $\mathbf{Y}$, where η satisfies the evolution equation

$$\eta_t = -\frac{\tau}{2E}\Delta\left(\frac{\Delta\eta}{E}\right) + Q_1, \tag{5.16}$$

with initial condition $\eta|_{t=0} = \eta_0$, and $\mathbf{Y}$ satisfies evolution equation

$$\mathbf{Y}_t = (-\tau\Delta\eta/E)\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2 + Q_2, \tag{5.17}$$

with initial condition $\mathbf{Y}|_{t=0} = \mathbf{Y}_0$. The tangential velocities V_1 and V_2 solve

$$\left(\frac{V_1}{\sqrt{E}}\right)_\alpha - \left(\frac{V_2}{\sqrt{E}}\right)_\beta = -\tau\frac{\Delta\eta(L-N)}{(E)^2} + f_1, \tag{5.18}$$

$$\left(\frac{V_1}{\sqrt{E}}\right)_\beta + \left(\frac{V_2}{\sqrt{E}}\right)_\alpha = -\tau\frac{2\Delta\eta M}{(E)^2} + f_2. \tag{5.19}$$

Here, Q_1, Q_2, f_1, and f_2 are given nonhomogeneous terms. We are assuming here that $\tau > 0$ and that there is a given surface $\mathbf{X}$ and function E. The functions L, M, and N are the second fundamental coefficients of $\mathbf{X}$, and the vectors $\hat{\mathbf{t}}^i$, $\hat{\mathbf{n}}$ are the unit tangent and normal vectors to $\mathbf{X}$. As we have discussed several times, the tangential velocities may be solved for by applying ∂_α to (5.18) and ∂_β to (5.19), and adding and subtracting to arrive at Poisson equations. The Poisson equations may then be solved, as discussed in the proof of Lemma 4.4.

5.3. The *a priori* estimate

The iterative scheme which we set up involves the solution of a sequence of linear equations, all of the form (5.16)–(5.19). Bounds for the solutions of these equations are therefore fundamental to our existence proof. We now establish the needed estimates.

Theorem 5.1. *Suppose that there exists $T > 0$ such that for every $j \in \mathbb{N}$,*

$$E \in C([0,T], H^j)\bigcap C^1([0,T], H^j),$$

$$\mathbf{X} \in C([0,T], H^j)\bigcap C^1([0,T], H^j),$$

and

$$Q_1 \in L^2([0,T], H^j), Q_2 \in L^2([0,T], H^j), f_i \in L^2([0,T], H^j).$$

Assume there exists $C_0 > 0$ such that for all $t \in [0,T]$, we have $E(\cdot,t) \geq C_0 > 0$. Let initial data $(\eta_0, \mathbf{Y}_0)$ be given such that $\eta_0 \in H^j$ and $\mathbf{Y}_0 \in H^j$ for every $j \in \mathbb{N}$. Then there exist a constant $m > 0$ and a unique solution to the Cauchy problem $\eta \in C([0,T], H^j) \cap L^2([0,T], H^{j+2})$ and $\mathbf{Y} \in C([0,T], H^j)$ for every j, such that the bounds

$$\|\eta\|_s^2 + \int_0^t \frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{4}\,dt' \leq e^{Ct}\left(\|\eta_0\|_s^2 + \int_0^t \frac{4}{m\tau}\|Q_1\|_{s-2}^2\,dt'\right), \tag{5.20}$$

$$\|\mathbf{Y}\|_s^2 \leq e^{Ct}\left(\|\mathbf{Y}_0\|_s^2 + C\|\eta_0\|_s^2 + C\int_0^t \|Q_1\|_{s-2}^2 + \|Q_2\|_s^2 + \|f_1\|_{s-1}^2 + \|f_2\|_{s-1}^2\,dt'\right), \tag{5.21}$$

are satisfied for all $t \in [0,T]$. The constant C depends on the functions E, $\mathbf{X}$, Q_1, Q_2, f_1, and f_2 only through the norms

$$E \in C([0,T], H^{s+2})\bigcap C^1([0,T], H^{s-2}),$$

$$\mathbf{X} \in C([0,T], H^{s+2})\bigcap C^1([0,T], H^{s-2}),$$

and

$$Q_1 \in L^2([0,T], H^{s-2}), Q_2 \in L^2([0,T], H^s), f_i \in L^2([0,T], H^{s-1}).$$

Remark 3. We note that we have assumed that the functions E, $\mathbf{X}$, and so on are C^∞, which will ensure that all integrals below converge. However we will only rely on the regularity specified at the end of the theorem for estimates, i.e. while our E may be C^∞, the size of $\|\eta\|_{H^s}$ will only depend on its norm $\|E\|_{H^{s+2}}$. The corresponding statement is true for the rest of the given quantities $\mathbf{X}$, Q_i, f_i. That is, all constants which are denoted by C in what follows may be expressed in terms of these norms of $\mathbf{X}$ and E, and the given functions Q_1, Q_2, f_1, and f_2, as well as the parameters of the problem such as τ.

Proof. The proof of well-posedness for the Cauchy problem for (5.16) and (5.17) follows classical steps, namely approximation, existence for the approximate problems, establishing uniform estimates, passage to the limit, and establishing further estimates for uniqueness and continuous dependence. By far the most important and most interesting of these steps is the uniform bound, and this will be the focus of our presentation here. We will give the relevant energy estimate, and omit the other details.

We will first establish the energy estimate for η. We define the energy

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1,$$

with $\mathcal{E}_0 = \frac{1}{2}\|\eta\|_0^2$ and $\mathcal{E}_1 = \frac{1}{2}\|\Lambda^s\eta\|_0^2$. In what follows, we will be taking the time derivative of E_0 and E_1. We may exchange the time derivative and the integral because all of the iterates are, as we have mentioned several times, C^∞. The relevance of this is that the integrals of the time derivatives are absolutely convergent, and this justifies the exchange.

To begin, we take the time derivative of $\mathcal{E}_0$:

$$\frac{d\mathcal{E}_0}{dt} = \iint \eta\eta_t \, d\alpha d\beta.$$

The evolution equation for η, (5.16), involves up to fourth derivatives of η, second derivatives of E, and zero derivatives of Q_1. Since s is sufficiently large (recall that we have assumed $s \geq 6$), E and its first two derivatives are bounded by assumption. With s sufficiently large, up to the fourth derivatives of η are bounded by $\mathcal{E}$, and Q_1 is bounded in H^{s-2}. We therefore may immediately conclude

$$\frac{d\mathcal{E}_0}{dt} \leq C(\mathcal{E} + \|Q_1\|_{s-2}^2).$$

We next take the time derivative of $\mathcal{E}_1$,

$$\frac{d\mathcal{E}_1}{dt} = -\tau \iint (\Lambda^s\eta)\Lambda^s\left(\frac{\Delta}{2E}\left(\frac{\Delta\eta}{E}\right)\right) d\alpha d\beta + \iint (\Lambda^{s+2}\eta)(\Lambda^{s-2}Q_1)\, d\alpha d\beta. \quad (5.22)$$

For the second term on the right-hand side of (5.22), we have used the fact that Λ is a self-adjoint operator (which can be seen from its symbol in Fourier space).

We first deal with the first term on the right-hand side of (5.22). Recalling that $\Delta = -\Lambda^2$, and pulling $\frac{1}{E}$ through Λ^2 (incurring a commutator), we have

$$\begin{aligned}\iint (\Lambda^s\eta)\Lambda^s\left(\frac{\Delta}{2E}\left(\frac{\Delta\eta}{E}\right)\right) d\alpha d\beta &= \iint (\Lambda^s\eta)\Lambda^s\left(\frac{\Lambda^2}{2E}\left(\frac{\Lambda^2\eta}{E}\right)\right) d\alpha d\beta \\ &= \iint (\Lambda^{s+2}\eta)\Lambda^{s-2}\left(\frac{\Lambda^4\eta}{2E^2}\right) d\alpha d\beta \\ &\quad + \iint (\Lambda^{s+2}\eta)\Lambda^{s-2}\left(\frac{1}{2E}\left[\Lambda^2, \frac{1}{E}\right]\Lambda^2\eta\right) d\alpha d\beta.\end{aligned}$$

We then incur another commutator, this time pulling $\frac{1}{2E^2}$ through Λ^{s-2}; this yields

$$\begin{aligned}\iint (\Lambda^s\eta)\Lambda^s\left(\frac{\Delta}{2E}\left(\frac{\Delta\eta}{E}\right)\right) d\alpha d\beta &= \iint \frac{1}{2E^2}(\Lambda^{s+2}\eta)^2 \, d\alpha d\beta \\ &\quad + \iint (\Lambda^{s+2}\eta)\left[\Lambda^{s-2}, \frac{1}{2E^2}\right](\Lambda^4\eta)\, d\alpha d\beta \\ &\quad + \iint (\Lambda^{s+2}\eta)\Lambda^{s-2}\left(\frac{1}{2E}\left[\Lambda^2, \frac{1}{E}\right]\Lambda^2\eta\right) d\alpha d\beta.\end{aligned}$$

By Lemma 4.2, for sufficiently large s, we have

$$\begin{aligned}\left\|\left[\Lambda^{s-2}, \frac{1}{2E^2}\right]\Lambda^4\eta\right\|_0 &\leq c\left(\left\|\nabla\left(\frac{1}{2E^2}\right)\right\|_{L^\infty}\|\Lambda^4\eta\|_{s-3} + \left\|\frac{1}{2E^2}\right\|_{s-2}\|\Lambda^4\eta\|_{L^\infty}\right) \\ &\leq c\|\eta\|_{s+1} \leq c(\|\eta\|_0 + \|\Lambda^{s+1}\eta\|_0).\end{aligned}$$

And, by Lemma 4.1, we have

$$\left\|\Lambda^{s-2}\left(\frac{1}{2E}\left[\Lambda^2, \frac{1}{E}\right]\Lambda^2\eta\right)\right\|_0 \leq c\|\Lambda^2\eta\|_{s-1} \leq c(\|\eta\|_0 + \|\Lambda^{s+1}\eta\|_0).$$

By Lemma 4.3, $\|\Lambda^{s+1}\eta\|_0 \leq c\|\Lambda^{s+2}\eta\|_0^{\frac{s+1}{s+2}}\|\eta\|_0^{\frac{1}{s+2}}$. By first applying Hölder's inequality and then Young's inequality (with parameter $n > 0$ to be chosen), we have

$$\begin{aligned}&\left|\iint (\Lambda^{s+2}\eta)\left[\Lambda^{s-2}, \frac{1}{2E^2}\right](\Lambda^4\eta)\, d\alpha d\beta\right| \\ &\leq c\|\eta\|_0^{\frac{1}{s+2}}\|\Lambda^{s+2}\eta\|_0^{\frac{2s+3}{s+2}} + c\|\eta\|_0\|\Lambda^{s+2}\eta\|_0 \leq \frac{\|\Lambda^{s+2}\eta\|_0^2}{n} + C\mathcal{E}.\end{aligned}$$

In just the same way, we have

$$\left|\iint (\Lambda^{s+2}\eta)\Lambda^{s-2}\left(\frac{1}{2E}\left[\Lambda^2, \frac{1}{E}\right]\Lambda^2\eta\right) d\alpha d\beta\right| \leq \frac{\|\Lambda^{s+2}\eta\|_0^2}{n} + C\mathcal{E}.$$

We make our first conclusion that

$$\begin{aligned}&-\tau\iint (\Lambda^s\eta)\Lambda^s\left(\frac{\Delta}{2E}\left(\frac{\Delta\eta}{E}\right)\right) d\alpha d\beta \\ &\leq -\tau\iint \frac{1}{2E^2}(\Lambda^{s+2}\eta)^2\, d\alpha d\beta + \frac{2\tau\|\Lambda^{s+2}\eta\|_0^2}{n} + C\tau\mathcal{E}. \quad (5.23)\end{aligned}$$

For the second term on the right-hand side of (5.22), by Young's inequality, we have

$$\left|\iint (\Lambda^{s+2}\eta)\,\Lambda^{s-2}Q_1 d\alpha d\beta\right| \leq \frac{\tau\|\Lambda^{s+2}\eta\|_0^2}{n} + \frac{n\|Q_1\|_{s-2}^2}{\tau}.$$

Now we make the conclusion that

$$\frac{d\mathcal{E}}{dt} \leq -\tau\iint \frac{1}{2E^2}(\Lambda^{s+2}\eta)^2\, d\alpha d\beta + \frac{3\tau\|\Lambda^{s+2}\eta\|_0^2}{n} + C\mathcal{E} + n\|Q_1\|_{s-2}^2/\tau.$$

We know $E > 0$, and $E \in L^\infty$ when $s > 1$. Then there exists $m > 0$ such that $-\frac{1}{2E^2} \leq -m$. Now we take $n = 4/m$. Then

$$\frac{d\mathcal{E}}{dt} + \frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{4} \leq C\mathcal{E} + \frac{4}{m\tau}\|Q_1\|_{s-2}^2.$$

By Grönwall's inequality, it follows that

$$\mathcal{E}(t) + \int_0^t e^{C(t-t')}\frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{4}\, dt' \leq e^{Ct}\left(\mathcal{E}(0) + \int_0^t \frac{4}{m\tau}e^{-Ct'}\|Q_1\|_{s-2}^2\, dt'\right).$$

Moreover, since $e^{C(t-t')} \geq 1$ and $e^{-Ct'} \leq 1$, we have

$$\mathcal{E}(t) + \int_0^t \frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{4}\, dt' \leq e^{Ct}\left(\mathcal{E}(0) + \int_0^t \frac{4}{m\tau}\|Q_1\|_{s-2}^2\, dt'\right).$$

Now we will do the energy estimate for $\mathbf{Y}$, where $\mathbf{Y}$ evolves according to (5.17). We define energy

$$\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_0 + \widetilde{\mathcal{E}}_1,$$

with $\widetilde{\mathcal{E}}_0 = \frac{1}{2}\|\mathbf{Y}\|_0^2$ and $\widetilde{\mathcal{E}}_1 = \frac{1}{2}\|\Lambda^s\mathbf{Y}\|_0^2$. To begin, we take the time derivative of $\widetilde{\mathcal{E}}_0$:

$$\frac{d\widetilde{\mathcal{E}}_0}{dt} = \iint \mathbf{Y}\cdot\mathbf{Y}_t\, d\alpha d\beta.$$

The evolution equation for $\mathbf{Y}$ involves up to second derivatives of η, zero derivatives of E, first derivatives of $\mathbf{X}$ (through the tangent and normal vectors), zero derivatives of V_1 and V_2, and zero derivatives of Q_2. Recall that the energy bounds s derivatives of η, and that the constants in our bound are allowed to depend upon up to $s+2$ derivatives of E and $\mathbf{X}$. Also recalling that $s \geq 6$, using the evolution of $\mathbf{Y}$ and the estimates of Section 4, we may immediately conclude

$$\frac{d\widetilde{\mathcal{E}}_0}{dt} \leq C(\widetilde{\mathcal{E}} + \|\eta\|_s^2 + \|Q_2\|_s^2 + \|f_1\|_{s-1}^2 + \|f_2\|_{s-1}^2).$$

Next, we take the time derivative of $\widetilde{\mathcal{E}}_1$, and make a use of Young's inequality:

$$\begin{aligned}\frac{d\widetilde{\mathcal{E}}_1}{dt} &= \iint \Lambda^s\mathbf{Y}\cdot\Lambda^s\mathbf{Y}_t\, d\alpha d\beta \\ &= \iint \Lambda^s\mathbf{Y}\cdot\Lambda^s((-\tau\Delta\eta/E)\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2 + Q_2)\, d\alpha d\beta \\ &\le \frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{8} + \iint \Lambda^s\mathbf{Y}\cdot\Lambda^s(V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2)\, d\alpha d\beta \\ &\quad + \|Q_2\|_s^2 + C\widetilde{\mathcal{E}} + C\|\eta\|_s^2.\end{aligned}$$

Since V_1 and V_2 solve the elliptic system (5.18), (5.19), for sufficiently large s we have

$$\begin{aligned}\|\Lambda^s(V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2)\|_0 &\le c\tau\|\eta\|_{s+1} + c(\|f_1\|_{s-1} + \|f_2\|_{s-1}) \\ &\le c\tau(\|\eta\|_0 + \|\Lambda^{s+2}\eta\|_0) + c(\|f_1\|_{s-1} + \|f_2\|_{s-1}).\end{aligned}$$

By first applying Hölder's inequality and then Young's inequality, we have

$$\begin{aligned}&\left|\iint \Lambda^s\mathbf{Y}\cdot\Lambda^s(V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2)\, d\alpha d\beta\right| \\ &\le \|\Lambda^s\mathbf{Y}\|_0\left(c(\|\eta\|_0 + \|\Lambda^{s+2}\eta\|_0) + c(\|f_1\|_{s-1} + \|f_2\|_{s-1})\right) \\ &\le \frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{8} + C\widetilde{\mathcal{E}} + C(\|f_1\|_{s-1}^2 + \|f_2\|_{s-1}^2 + \|\eta\|_s^2).\end{aligned}$$

We may now make the conclusion that

$$\frac{d\widetilde{\mathcal{E}}}{dt} \le C\widetilde{\mathcal{E}} + \frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{4} + C(\|Q_2\|_s^2 + \|f_1\|_{s-1}^2 + \|f_2\|_{s-1}^2 + \|\eta\|_s^2).$$

By Grönwall's inequality, it follows

$$\widetilde{\mathcal{E}}(t) \le e^{Ct}\left(\widetilde{\mathcal{E}}(0) + \int_0^t \frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{4}\, dt' + \int_0^t C\|Q_2\|_s^2 + \|f_1\|_{s-1}^2 + \|f_2\|_{s-1}^2 + \|\eta\|_s^2\, dt'\right).$$

Using our estimate for η, (5.20), we have

$$\begin{aligned}\widetilde{\mathcal{E}}(t) &\le e^{Ct}\left(\widetilde{\mathcal{E}}(0) + C\int_0^t \|Q_2\|_s^2 + \|f_1\|_{s-1}^2 + \|f_2\|_{s-1}^2\, dt'\right) \\ &\quad + e^{Ct}\left(Ce^{Ct}\left(\|\eta_0\|_s^2 + \int_0^t \frac{4}{m\tau}\|Q_1\|_{s-2}^2\, dt'\right)\right) \\ &\le e^{Ct}\left(\widetilde{\mathcal{E}}(0) + \|\eta_0\|_s^2 + C\int_0^t \|Q_2\|_s^2 + \|f_1\|_{s-1}^2 + \|f_2\|_{s-1}^2 + \|Q_1\|_{s-2}^2\, dt'\right).\end{aligned}$$

This concludes the proof of the theorem. □

5.4. Estimates for the iteration scheme

To prove that the sequences of iterations converge, we will need the following lemma.

Lemma 5.2. *The family of iterates $(\mathbf{X}^l, E^l, \kappa^l)$ are defined for all l and there exist $T > 0$ and positive constants C_0, C_1, C_2, C_3 and C_4 such that for all l,*

$$E^l \ge C_0 > 0, \quad |\mathbf{X}^l_\alpha \times \mathbf{X}^l_\beta| \ge C_0 > 0, \tag{5.24}$$

$$\|\kappa^l\|_{C^0([0,T];H^s)} \le C_1, \tag{5.25}$$

$$\|\mathbf{X}^l\|_{C^0([0,T];H^{s+2})} + \|E^l\|_{C^0([0,T];H^{s+2})} \le C_2, \tag{5.26}$$

$$\|\partial_t\kappa^l\|_{C^0([0,T];H^{s-4})} \le C_3, \tag{5.27}$$

$$\|\partial_t\mathbf{X}^l\|_{C^0([0,T];H^{s-2})} + \|\partial_t E^l\|_{C^0([0,T];H^{s-2})} \le C_4. \tag{5.28}$$

Proof. We proceed by induction. We take $C_0 = c_0/2$. We will determine appropriate values for C_1, C_2, C_3, and C_4 as we go. Given the definition of our initial iterates, the needed bounds are satisfied for $(\mathbf{X}^0, E^0, \kappa^0)$. Assume that $(\mathbf{X}^l, E^l, \kappa^l)$ satisfies (5.24), (5.25) (5.26), (5.27) and (5.28). By the definition of Q_1^l in (5.2), applying $s-2$ derivatives to Q_1^l involves at most s-derivatives of κ^l and at most $(s+2)$-derivatives of $\mathbf{X}^l$, and at most s-derivatives of E^l. Thus $\|Q_1^l\|_{C([0,T];H^{s-2})} \le C(C_0, C_1, C_2)$, with this constant being independent of l. (In fact, in what follows, all constants of the form $C(C_0, C_1, C_2)$, or $C(C_0, C_1, C_2, C_3)$ are independent of l.) By the result of our energy estimates, (5.20), κ^{l+1} satisfies

$$\begin{aligned}&\|\kappa^{l+1}(t)\|_s^2 + \int_0^t \frac{\tau m\|\Lambda^{s+2}\kappa^{l+1}(s)\|_0^2}{4}\, dt' \\ &\le e^{C(C_0,C_1,C_2)t}\|\kappa_0\|_s^2 + C(C_0, C_1, C_2)te^{C(C_0,C_1,C_2)t}.\end{aligned}$$

Hence taking $C_1 = 2\|\kappa_0\|_s$, we may take T small enough so that

$$\|\kappa^{l+1}(t)\|_{C([0,T];H^s)}^2 + \int_0^t \frac{\tau m\|\Lambda^{s+2}\kappa^{l+1}(s)\|_0^2}{4}\, dt' \le C_1^2. \tag{5.29}$$

Inspecting the definitions of f_1^l, f_2^l, and Q_2^l in (5.6), (5.7), and (5.12), we see that we may bound these as $\|Q_2^l\|_{C([0,t];H^s)} \le C(C_0, C_1, C_2)$ and $\|f_i^l\|_{C([0,t];H^{s-1})} \le C(C_0, C_1, C_2)$. Using (5.21), the estimate of $\mathbf{Z}_1^{l+1}$ is then

$$\|\mathbf{Z}_1^{l+1}(t)\|_s^2 \le e^{C(C_0,C_1,C_2)t}(\|\kappa_0\|_s^2 + \|\mathbf{X}_0\|_s^2) + C(C_0, C_1, C_2)te^{C(C_0,C_1,C_2)t}.$$

Hence we may again choose T sufficiently small so that $\|\mathbf{Z}_1^{l+1}\|_{C([0,T];H^s)} \le 2(\|\mathbf{X}_0\|_s + \|\kappa_0\|_s)$. Solving the elliptic Eqs. (5.13) and (5.14), we then have $\mathbf{X}^{l+1} \in C^0([0,T]; H^{s+2})$ and

$$\begin{aligned}&\|\mathbf{Z}_2^{l+1}\|_{C([0,T];H^{s+1})} \le C(\|\mathbf{X}_0\|_s, \|\kappa_0\|_s), \\ &\|\mathbf{X}^{l+1}\|_{C([0,T];H^{s+2})} \le C(\|\mathbf{X}_0\|_s, \|\kappa_0\|_s).\end{aligned}$$

Then, solving the elliptic Eq. (5.15), we have $E^{l+1} \in C^0([0,T]; H^{s+2})$ with estimate

$$\|E^{l+1}\|_{C([0,T];H^{s+2})} \le C(\|\mathbf{X}_0\|_s, \|\kappa_0\|_s)$$

We may then take $C_2 = 2C(\|\mathbf{X}_0\|_s, \|\kappa_0\|_s)$, and we see that the estimate (5.26) holds.

Using the inductive hypotheses and the bound (5.29), we may then bound the right-hand side of (5.1), finding

$$\begin{aligned}\|\partial_t\kappa^{l+1}(t)\|_{s-4} &\le C(C_0, C_1, C_2)(1 + \|\kappa^{l+1}(t)\|_s) \\ &\le C(C_0, C_1, C_2)(1 + C_1).\end{aligned}$$

We may similarly bound the right-hand side of (5.10), finding

$$\|\partial_t\mathbf{Z}_1^{l+1}\|_{C([0,T];H^{s-2})} \le C(C_0, C_1, C_2).$$

Taking C_3 such that $C_3 \ge C(C_0, C_1, C_2)(1 + C_1)$, we have the estimate (5.27).

Taking the time derivative of (5.13) and (5.14), we may conclude that

$$\|\partial_t\mathbf{X}^{l+1}\|_{C([0,T];H^{s-2})} \le C(C_0, C_1, C_2, C_3).$$

Similarly, taking the time derivative of (5.15), we may conclude the bound

$$\|\partial_t E^{l+1}\|_{C([0,T];H^{s-2})} \le C(C_0, C_1, C_2, C_3).$$

We then may take C_4 to be such that $C_4 \ge C(C_0, C_1, C_2, C_3)$, getting the estimate (5.28).

Notice that when s is sufficiently large so that $H^{s-2} \subseteq L^\infty$,

$$|E^{l+1}(t)| \ge E_0 - \left|\int_0^t \partial_t E^{l+1}(t')\, dt'\right| \ge c_0 - tC_4$$

and

$$|\mathbf{X}_\alpha^{l+1}(t) \times \mathbf{X}_\beta^{l+1}(t)|$$
$$\geq |\mathbf{X}_\alpha^{l+1}(0) \times \mathbf{X}_\beta^{l+1}(0)| - \int_0^t \partial_\tau(\mathbf{X}_\alpha^{l+1}(s) \times \mathbf{X}_\beta^{l+1})(t')\,dt' \geq c_0 - tC_4C_2.$$

So, we can take T small enough such that $c_0 - TC_4 \geq C_0$ and $c_0 - TC_4C_2 \geq C_0$. This completes the proof of the lemma. □

5.5. The limit of the iterated system

In this section we take the limit of (a subsequence of) our iterates. Since the spatial domain, $\mathbb{T}^2$, is compact, we may do so by the Arzela–Ascoli theorem.

Since the iterates κ^l, $\mathbf{Z}_1^l$ are uniformly bounded in H^s and H^{s+2}, respectively, and since the time derivatives κ_t^l and $\mathbf{X}_t^l$ involve (as the most singular terms) four derivatives of κ^l, we see that κ_t^l and $(\mathbf{Z}_1^l)_t$ are bounded, uniformly with respect to (α,β,t) and also l. Furthermore κ_α^l, κ_β^l, $(\mathbf{Z}_1^l)_\alpha$ and $(\mathbf{Z}_1^l)_\beta$ are similarly uniformly bounded with respect to (α,β,t) and l. We conclude that κ^l and $\mathbf{Z}_1^l$ are bounded equicontinuous families. By the Arzela–Ascoli theorem, there is a subsequence (which we do not relabel) and limits κ and $\mathbf{Z}_1$ such that κ^l converges uniformly to κ and $\mathbf{Z}_1^l$ converges uniformly to $\mathbf{Z}_1$. With this uniform convergence, we immediately conclude that the convergence also holds in $C([0,T];L^2(\mathbb{T}^2))$.

We next use the interpolation lemma, Lemma 4.3, to find convergence in more regular spaces. Specifically, for any $s' \in (0,s)$, convergence in $C([0,T];L^2)$ and boundedness in $C([0,T];H^s)$ implies that κ^l converges to κ in $C([0,T];H^{s'})$ and $\mathbf{Z}_1^l$ converges to $\mathbf{Z}_1$ in $C([0,T];H^{s'})$. Then, from (5.13) we have

$$\mathbf{Z}_2^{l+1} = -(1-\Delta)^{-1}\left(2\kappa^l(\mathbf{Z}_1^{l+1})_\alpha \times (\mathbf{Z}_1^{l+1})_\beta - \mathbf{Z}_1^{l+1}\right),$$

and we can pass to the limit on the right-hand side in $H^{s'-1}$ finding that $\mathbf{Z}_2^l$ converges to $\mathbf{Z}_2$ in $C([0,T];H^{s'+1})$, where

$$\Delta\mathbf{Z}_2 - \mathbf{Z}_2 = 2\kappa(\mathbf{Z}_1)_\alpha \times (\mathbf{Z}_1)_\beta - \mathbf{Z}_1.$$

Similarly, then, by (5.14), we find that $\mathbf{X}^l$ converges to $\mathbf{X}$ in $C([0,T];H^{s'+2})$, where

$$\Delta\mathbf{X} - \mathbf{X} = 2\kappa(\mathbf{Z}_2)_\alpha \times (\mathbf{Z}_2)_\beta - \mathbf{Z}_1.$$

We then are able to similarly pass to the limit in (5.15), finding that E^l converges to E in $C([0,T];H^{s'+2})$, where E satisfies

$$\Delta E - E = 2(\mathbf{X}_{\alpha\beta}\cdot\mathbf{X}_{\alpha\beta} - \mathbf{X}_{\alpha\alpha}\cdot\mathbf{X}_{\beta\beta}) - (\mathbf{X}_\alpha\cdot\mathbf{X}_\alpha + \mathbf{X}_\beta\cdot\mathbf{X}_\beta).$$

We now conclude that κ and $\mathbf{Z}_1$ satisfy the appropriate evolution equations. Integrating (5.1) with respect to time, we have

$$\kappa^{l+1}(\cdot,t) = \kappa^{l+1}\Big|_{t=0} + \int_0^t \left(-\frac{\tau}{2E^l(\cdot,t')}\Delta\left(\frac{\Delta\kappa^{l+1}(\cdot,t')}{E^l(\cdot,t')}\right) + Q_1^l(\cdot,t')\right)\,dt'.$$

We have established enough regularity to pass to the limit on both the left-hand side and the right-hand side of this equation. To pass to the limit on the right-hand side, we need uniform convergence to exchange the limit as l goes to infinity with the time integral. Fortunately, the convergences we have established, such as $\kappa^l \to \kappa$ in $C([0,T];H^{s'})$ for any $s' \in (0,s)$ with s fixed (recall we have taken $s \geq 6$) is sufficient to be able to conclude this needed uniform convergence. We conclude

$$\kappa(\cdot,t) = \kappa_0 + \int_0^t \left(-\frac{\tau}{2E(\cdot,t')}\Delta\left(\frac{\Delta\kappa(\cdot,t')}{E(\cdot,t')}\right) + Q_1(\cdot,t')\right)\,dt'.$$

This immediately implies $\kappa(\cdot,0) = \kappa_0$, and differentiating with respect to time, we have

$$\kappa_t = -\frac{\tau}{2E}\Delta\left(\frac{\Delta\kappa}{E}\right) + Q_1.$$

In just the same way, we may pass to the limit in the evolution equation for $\mathbf{Z}_1^{l+1}$, (i.e. we integrate (5.10) with respect to time, pass to the limit, and then differentiate with respect to time) finding

$$(\mathbf{Z}_1)_t = (-\tau\Delta\kappa/E)\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2 + Q_2.$$

We must also prove that the following relations hold:

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{X}, \quad E = \mathbf{X}_\alpha\cdot\mathbf{X}_\alpha = \mathbf{X}_\beta\cdot\mathbf{X}_\beta, \quad \mathbf{X}_\alpha\cdot\mathbf{X}_\beta = 0, \quad \kappa = \frac{L+N}{2E}. \tag{5.30}$$

We omit the details of the proof here as all of the details demonstrating (5.30) are included in Section 5.4 of [43]. (Like the present work, [43] also uses the isothermal parameterization and iterative scheme of [31]. Therefore, after taking the limit of the iterates, they demonstrate that exactly the relationships given in (5.30) hold.)

The higher regularity of the solution $\mathbf{X}$ must still be established. We have already shown that the solutions are continuous in time in a low norm, and the boundedness in a high norm together with the interpolation result Lemma 4.3 implies that $\mathbf{X} \in C([0,T];H^{s'+2})$ for any $s' < s$. All that remains to show is that $\mathbf{X} \in C^0([0,T];H^{s+2})$. We do not include the remaining details, but this can be done by adapting the corresponding argument for regularity of solutions for the Navier–Stokes equations in Chapter 3 of [44].

5.6. Uniqueness and continuous dependence

In this section we sketch the proof uniqueness of solutions and continuous dependence of solutions upon the initial data. The proof relies on energy estimates very similar to those in the proof of Theorem 5.1.

We assume that there exists a time $T > 0$ such that both $\mathbf{X}$ and $\mathbf{X}'$ are elements of $C^0([0,T),\overline{\mathcal{O}})$ which solve the Cauchy problem (2.13). We denote by $(\delta\mathbf{X},\delta\kappa)$ the difference $(\mathbf{X}-\mathbf{X}',\kappa-\kappa')$, and define an energy functional

$$D = \frac{1}{2}\|\delta\kappa\|_0^2 + \frac{1}{2}\|\Lambda^2\delta\kappa\|_0^2 + \frac{1}{2}\|\delta\mathbf{X}\|_0^2 + \frac{1}{2}\|\Lambda^2\delta\mathbf{X}\|_0^2. \tag{5.31}$$

We also denote $\delta E = E - E'$, $\delta U = U - U'$, $\delta V_i = V_i - V_i'$, and so on.

We can write evolution equations for $\delta\mathbf{X}$ and $\delta\kappa$ as follows:

$$\delta\mathbf{X}_t = \delta U\hat{\mathbf{n}} + (\delta V_1)\hat{\mathbf{t}}^1 + (\delta V_2)\hat{\mathbf{t}}^2 + R_1, \tag{5.32}$$

where the remainder R_1 is defined as

$$R_1 = V_1(\hat{\mathbf{t}}^1 - \hat{\mathbf{t}}'^1) + V_2(\hat{\mathbf{t}}^2 - \hat{\mathbf{t}}'^2) + U(\hat{\mathbf{n}} - \hat{\mathbf{n}}'), \tag{5.33}$$

and

$$\delta\kappa_t = -\frac{\tau}{2E}\Delta\left(\frac{\Delta\delta\kappa}{E}\right) + R_2, \tag{5.34}$$

where the remainder R_2 is defined as

$$R_2 = -\frac{\tau}{2E}\Delta\left(\frac{\Delta\kappa}{E}\right) + \frac{\tau}{2E'}\Delta\left(\frac{\Delta\kappa}{E'}\right) + Q_1 - Q_1'. \tag{5.35}$$

We have the formula

$$Q_1 = \frac{\Delta W(\kappa,q)}{2E} + \frac{V_1}{E}(\sqrt{E}\kappa)_\alpha + \frac{V_2}{E}(\sqrt{E}\kappa)_\beta + \frac{U(M)^2}{E} + \frac{L}{2E}\left(\frac{V_1}{\sqrt{E}}\right)_\alpha + \frac{N}{2E}\left(\frac{V_2}{\sqrt{E}}\right)_\beta + \frac{\kappa}{\sqrt{E}}\left(V_{1,\alpha} - \frac{UL}{\sqrt{E}} + \frac{V_2E_\beta}{2E}\right), \tag{5.36}$$

and naturally Q_1' is defined accordingly.

To begin, we estimate $\delta\mathbf{X}$ in H^4 using (2.7). Specifically, we may take the difference of (2.7), finding

$$\Delta\delta\mathbf{X} = 2\delta\kappa(\mathbf{X}_\alpha \times \mathbf{X}_\beta) + 2\kappa'(\delta\mathbf{X}_\alpha \times \mathbf{X}_\beta) + 2\kappa'(\mathbf{X}'_\alpha \times \delta\mathbf{X}_\beta). \quad (5.37)$$

This then implies

$$\|\delta\mathbf{X}\|_3 \le C(\|\delta\kappa\|_1 + \|\delta\mathbf{X}\|_2). \quad (5.38)$$

Then using (5.37) again, but then substituting the result of (5.38), we find

$$\|\delta\mathbf{X}\|_4 \le C(\|\delta\kappa\|_2 + \|\delta\mathbf{X}\|_3) \le C(\|\delta\kappa\|_2 + \|\delta\mathbf{X}\|_2).$$

We similarly get an estimate for δE by considering differences in (3.1); the result is

$$\|\delta E\|_4 \le C\|\delta\mathbf{X}\|_4 \le C(\|\delta\kappa\|_2 + \|\delta\mathbf{X}\|_2).$$

It is immediate that we may bound R_1 in H^0 by the energy, $\|R_1\|_0^2 \le CD$, since R_1 includes only first derivatives of $\delta\mathbf{X}$. We furthermore may bound R_2 in H^0 in terms of the energy as $\|R_2\|_0^2 \le CD$; this requires a number of routine estimates for differences, For instance, to estimate $W(\kappa, q) - W(\kappa', q')$ requires writing

$$W(\kappa, q) - W(\kappa, q') = (1-\sigma)\delta\kappa - \left(1 + \frac{\sigma^2}{2}\right)\delta\kappa(\kappa + \kappa')$$
$$+ \left(\frac{\sigma^3}{3} - 5\sigma^2 - 2\sigma\right)\delta\kappa(\kappa^2 + \kappa\kappa' + \kappa'^2) + 2(\sigma^2+1)\delta q$$
$$+ (20\sigma_8^2\sigma - 4)(q\delta\kappa + \kappa'\delta q).$$

We may then estimate this as

$$\|W(\kappa, q) - W(\kappa', q')\|_0 \le CD.$$

The corresponding estimate for δU then follows, as does the estimate for δV_i, and so on.

Now we are in a position to take the time derivative of D, finding

$$\frac{dD}{dt} = \iint \delta\kappa\delta\kappa_t + (\Lambda^2\delta\kappa)\Lambda^2\delta\kappa_t d\alpha d\beta$$
$$+ \iint \delta\mathbf{X}\delta\mathbf{X}_t + (\Lambda^2\delta\mathbf{X})(\Lambda^2\delta\mathbf{X}_t)\, d\alpha d\beta.$$

We then substitute from the evolution equations for $\delta\kappa$ and $\delta\mathbf{X}$, and we use the fact that Λ is self-adjoint, and the estimates we have already established. These considerations lead us to the bound

$$\frac{dD}{dt} \le \iint (\Lambda^2\delta\kappa)\Lambda^2\left(-\frac{\tau}{2E}\Delta\left(\frac{\Delta\delta\kappa}{E}\right)\right)\, d\alpha d\beta$$
$$+ \iint (\Lambda^4\delta\kappa)R_2\, d\alpha d\beta$$
$$+ \iint (\Lambda^4\delta\mathbf{X})\left(\delta U\hat{\mathbf{n}} + (\delta V_1)\hat{\mathbf{t}}^1 + (\delta V_2)\hat{\mathbf{t}}^2 + R_1\right)\, d\alpha d\beta + CD.$$

Then, as in the proof of Theorem 5.1, the first term on the right-hand side may be written as a negative term which controls $\Lambda^4\delta\kappa$ (the remainder is a commutator which may be bounded, as in (5.23)). For the second term on the right-hand side we may use Young's inequality, bounding the resulting term $(\Lambda^4\delta\kappa)^2$ by the previous term (making use of the negative sign in the previous term). The remaining term from Young's inequality is zero derivatives of R_2^2, which we have already argued is bounded by the energy. Similarly for the third term on the right-hand side, we have already argued that it is bounded in terms of the energy. We are able to conclude the bound

$$\frac{dD}{dt} \le CD, \quad (5.39)$$

which immediately implies via Grönwall's inequality that

$$D(t) \le D(0)e^{Ct}. \quad (5.40)$$

With $D(0) = 0$, then this implies the solution of Cauchy problem (2.13) is unique.

The bound (5.40) also implies continuous dependence on the initial data, in a low norm (i.e., with $\mathbf{X}$ measured in H^4). Using Lemma 4.3, since the solutions we have proved to exist are bounded with $\mathbf{X}$ in H^{s+2}, then for any $s' \in (0, s)$, this also implies continuous dependence in $H^{s'+2}$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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