



EXISTENCE AND COMPUTATION OF STATIONARY SOLUTIONS FOR CONGESTION-TYPE MEAN FIELD GAMES VIA BIFURCATION THEORY AND FORWARD-FORWARD PROBLEMS

JOSHUA SIN^{✉1}, JOHN W. BONNES^{✉2}, LUKE C. BROWN^{✉2,3}
AND DAVID M. AMBROSE^{✉*2}

¹Department of Physics, Drexel University, USA

²Department of Mathematics, Drexel University, USA

³Applied Physics Laboratory, Johns Hopkins University, USA

(Communicated by Diogo Gomes)

ABSTRACT. Time-dependent mean field games are a coupled system of a forward parabolic and backward parabolic partial differential equation. Stationary solutions are of interest, and then naturally the forward-backward structure in time becomes irrelevant. Forward-forward mean field games have been introduced with the rationale that they may be used to straightforwardly compute such stationary solutions. We perform some numerical simulations to find that typically stationary solutions of mean field games are unstable to the forward-forward evolution, i.e. frequently only trivial solutions can be found in this way. We then ask whether there are situations in which one would have reason to believe that the stationary solutions would be stable, and we use the exchange-of-stability phenomenon in bifurcation theory to give a class of examples for which the forward-forward solutions do converge to nontrivial stationary solutions as time increases.

1. Introduction. Ever since their development by Lasry and Lions in [20] and [21], and by Caines, Huang, and Malhalmé in [18] and [19], mean field games have become valuable tools for efficiently modeling the large-scale interaction of many agents. In a mean field game, a continuum of agents, assumed to be rational in the sense that their logic is interchangeable, engage in a continuous-in-time competition. An evolving utility function $u(t, x)$ reflects the satisfaction experienced by a representative agent found in state x at time t .

The agent with trajectory X_t uses the optimal control $c(t, x)$ to maximize $u(t, X_t)$ continuously in time, while responding to the relative state of the other agents,

2020 *Mathematics Subject Classification.* Primary: 91A16, 35B32; Secondary: 34C23, 35K40, 35Q89.

Key words and phrases. Stationary solutions, mean field games, forward-forward mean field games, bifurcation theory, spectral methods.

DMA gratefully acknowledges support from the National Science Foundation through grants DMS-1907684 and DMS-2307638.

*Corresponding author: David M. Ambrose.

subject to a Gaussian white noise representing environmental uncertainty. The resulting trajectory obeys the SDE

$$dX_t = c(t, X_t)dt + \sigma dW_t \quad (1)$$

where $\sigma > 0$. To model the interaction of N agents traditionally, a coupled system of N Hamilton-Jacobi equations would need to be solved simultaneously, yielding a utility function $u^i(t)$ reflecting the satisfaction of agent i at time $t \in [0, T]$ for each player. The innovation of mean field games is that one may sort the agents by state variable x (in our case, x is a spatial variable representing a position on $\mathbb{T}$) and, borrowing techniques from Statistical Mechanics, take the mean-field limit, reducing the model to a system of two PDE's: a Hamilton-Jacobi-Bellman (HJB) equation that governs $u(t, x)$, solved backwards in time from a terminal utility u^T , and a Fokker-Planck equation, solved forwards in time from an initial density m^0 , that governs the density $m(t, x)$ of agents whose trajectory is governed by (1). That is, by taking a statistical, rather than an atomic approach, one may use mean field games to simplify the modeling of large-scale player interactions. For a more general treatment of mean field game models, we direct the reader to [22]. The resulting general form of the mean field games system of partial differential equations is

$$\begin{cases} -u_t - \nu \Delta u + \mathcal{H}(t, x, m, \nabla u) = 0, \\ m_t - \nu \Delta m + \operatorname{div}(m \mathcal{H}_p(t, x, m, \nabla u)) = 0, \end{cases} \quad (2)$$

where the nonlinearity $\mathcal{H}$ is known as the Hamiltonian, and where $\mathcal{H}_p$ represents the derivative of $\mathcal{H}$ with respect to the p -variables when $\mathcal{H}$ is regarded as a function $\mathcal{H} = \mathcal{H}(t, x, q, p)$. This system is then augmented with initial data for m and terminal data, specified at some time $T > 0$, for u . The length of the time interval, T , is known as the time horizon.

In what follows, we examine one-dimensional mean field games with congestion, in which high-density regions are more costly for agents to traverse than low-density regions. We will pay the most attention to the following one-dimensional form of the mean field game system (2) given by Achdou and Porretta in [1], typical of a congestion model:

$$\begin{cases} -u_t - \nu \Delta u + \frac{1}{\beta} \frac{|\nabla u|^\beta}{(m+\mu)^\alpha} = V(t, x, m), \\ m_t - \nu \Delta m - \operatorname{div} \left\{ m \frac{|\nabla u|^{\beta-2}}{(m+\mu)^\alpha} \nabla u \right\} = 0. \end{cases} \quad (3)$$

The data for the system is then

$$u(T, x) = u^T(x), \quad m(0, x) = m^0(x).$$

Notice that the time-dependent system (3) has a forward-backward structure: the equation for m is a forward parabolic equation and is taken with initial data, while the equation for u is backward parabolic, and is taken instead with terminal data. Since m will be a probability distribution, we note that we are interested exclusively in solutions for which $m \geq 0$. Observe that in (3), $|\nabla u(t, x)|$ represents the expected marginal utility for the agent in state x at time t , and it is scaled by $(m + \mu)^{-\alpha}$ to reflect the fact that the agent is unsatisfied at a point where $m(t, x)$ is high relative to its surroundings. Observe also that if we let $\mu = 0$, one runs the risk of obtaining singularities when $m(t, x) = 0$. In [1], the authors prove the existence of weak solutions of (3) in the case that $\mu > 0$, along with the existence of weak subsolutions in the case that $\mu = 0$.

In addition to time-dependent solutions, stationary solutions of mean field games systems are frequently of interest; among other reasons, stationary solutions have in some cases been shown to give the limit of time-dependent solutions as the time horizon, T , goes to infinity [6], [7]. For congestion-type mean field games, stationary solutions have previously been proved to exist in the works [9], [10], [13], [14], [15]. Rather than being purely stationary, solutions are frequently taken to be relatively stationary, in the sense that while m is taken to satisfy $m_t = 0$, the equation for u becomes $u_t = \lambda$ for some fixed $\lambda \in \mathbb{R}$. The relatively stationary version of the system (3) is then

$$\begin{cases} -\nu\Delta u + \frac{1}{\beta} \frac{|\nabla u|^\beta}{(m+\mu)^\alpha} = V(t, x, m) + \lambda, \\ -\nu\Delta m - \operatorname{div} \left\{ m \frac{|\nabla u|^{\beta-2}}{(m+\mu)^\alpha} \nabla u \right\} = 0. \end{cases} \quad (4)$$

Numerical methods have been developed for computing stationary solutions of mean field games [3], [4]. As an alternative to these methods, forward-forward mean field games were introduced and studied in a series of papers [11], [12], [16], [17], with the stated purpose of being able to compute stationary solutions as steady-states of the forward-forward system. This has the advantage that computing solutions of parabolic initial value problems is in general a straightforward proposition, which can be accomplished with standard methods such as Runge-Kutta timestepping and spectral computation of spatial derivatives. The forward-forward system corresponding to (4) is

$$\begin{cases} u_t - \nu\Delta u + \frac{1}{\beta} \frac{|\nabla u|^\beta}{(m+\mu)^\alpha} = V(t, x, m), \\ m_t - \nu\Delta m - \operatorname{div} \left\{ m \frac{|\nabla u|^{\beta-2}}{(m+\mu)^\alpha} \nabla u \right\} = 0. \end{cases} \quad (5)$$

Then, a solution of (4) is a solution of (5) in which $u_t = -\lambda$ and $m_t = 0$.

Unfortunately, the forward-forward problem has proven to be difficult to use for the stated purpose of finding nontrivial stationary solutions of the mean field games system. In [16], it is shown that for certain Hamiltonians, the forward-forward system always converges to the trivial state. Subsequently, in [12], it is shown that traveling wave solutions exist for the forward-forward system with certain Hamiltonians. This implies that for these problems, one may choose initial data for which the problem does not converge at all to a stationary state. In general, it is entirely possible that nontrivial stationary solutions of the mean field games system may be unstable to the forward-forward evolution.

If one can expect the stability of the nontrivial stationary solutions, however, then we may expect the forward-forward evolution to be able to reach these. We take advantage of the “exchange-of-stability” phenomenon in bifurcation theory to find stationary solutions expected to be stable to the forward-forward evolution. Bifurcation theory has previously been used to prove existence of some stationary mean field games solutions in [8].

We will demonstrate the effectiveness of forward-forward mean field games for computing stationary solutions in the case that stationary solutions arise from a bifurcation by considering the following one-dimensional case of (3):

$$\begin{cases} -u_t - u_{xx} + \frac{1}{2} \frac{u_x^2}{(m+\mu)^2} = \gamma \left(m - \frac{1}{L} \right), \\ m_t - m_{xx} - \left(\frac{mu_x}{(m+\mu)^2} \right)_x = 0. \end{cases} \quad (6)$$

We consider the spatially periodic case, in which $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$, for fixed periodicity length L .

The paper is organized as follows: In Section 2, we will give an overview of the numerical method employed for the forward-forward system and give an overview of some computations unrelated to solutions arising from bifurcations. In Section 3, we will prove a bifurcation theorem for the relatively stationary version of (6). Guided by this theoretical result, we will then compute such relatively stationary solutions in Section 4.

2. Numerical method and equilibria of the Evangelista and Gomes model.

The numerical method we implement consists of a spectral method to compute the spatial derivatives Δ and ∂_x , together with a fourth-order Runge-Kutta time integration. Given a smooth function $f : \mathbb{T} \rightarrow \mathbb{R}$, one generates $\hat{f}$ using the Fast Fourier Transform (FFT) and computes

$$\Delta f = -\mathcal{F}^{-1} \left[\left(\frac{4\pi^2}{L^2} \right) |k|^2 \hat{f}(t, k) \right] (x), \quad f_x = \mathcal{F}^{-1} \left[\left(\frac{2\pi}{L} \right) ik \hat{f}(t, k) \right] (x)$$

using the Inverse Fast Fourier Transform (IFFT). We have implemented this scheme for the forward-forward mean field games system in Python as well as in Matlab.

This algorithm produces time-dependent solutions of the forward-forward system. As we have noted, if these solutions converge to equilibria, then the equilibria are also stationary solutions of the standard backward-forward system. We therefore generate solutions with the intention of numerically obtaining nontrivial equilibria of the original problem. We therefore also need to discuss the criterion we implement to determine whether a stationary solution has been reached. At the j th time-step we compute

$$K_j = \log \left(\|u(t_j, \cdot) - u(t_{j-1}, \cdot)\|_{\dot{L}^2(\mathbb{T})} \right)$$

where the L^2 norm is computed in the zero-mean subspace $\dot{L}^2$ (i.e., we subtract the mean of the function before computing the L^2 norm) to account for the mean-shift $u_t = \lambda$ in the solution of the time-reversed HJB equation. One may neglect to renormalize by Δt because our experiments all use the same step-size, $\Delta t = 10^{-4}$. In the numerical evidence we present, we shall include the convergence plots that show K_j over time, with the threshold $K_j = -10$ corresponding to the point at which we agree a limit has been obtained.

2.1. Sample preliminary calculations. Our investigation began with the following forward-forward system, based on the study of stationary congestion games in [10]. Let

$$\mathcal{H}(x, p) = (1 + p^2)^{\gamma/2}, \quad V(x, m) = \tan^{-1}(m)$$

for parameters $\alpha > 0$, $\gamma \in (0, 2)$. The forward-forward form of the system presented in [10] is given by

$$\begin{cases} u_t + u - u_{xx} + m^\alpha \mathcal{H}\left(x, \frac{u_x}{m^\alpha}\right) + V(x, m) = 0 \\ m_t + m - \partial_x \left\{ m \frac{\partial \mathcal{H}}{\partial p}\left(x, \frac{u_x}{m^\alpha}\right) \right\} = 1 \end{cases} \quad (7)$$

which exhibits a trivial equilibrium at

$$u_0^* = -1 - \frac{\pi}{4}, \quad m_0^* = 1.$$

(We take $L = 1$ for now; thus, this m_0^* is indeed a probability distribution.) We chose sample values of (α, γ) and implemented the forward-forward system using sinusoidal perturbations of (u_0^*, m_0^*) as initial data. Doing so for a variety of initial perturbations and parameter values overwhelmingly led to long-time behavior of convergence to the trivial equilibrium (u_0^*, m_0^*) .

However, in one interval of parameter values (specifically, for α between 2.3 and 2.6, with $\gamma = 1.55$), we did find nontrivial stationary solutions according to our criterion. In Figure 1, the terminal density is shown in pink and exhibits a minimum near $x = 0.2$ that sharpens into a cusp and approaches zero (a singular value for the model) as the congestion parameter α increases. In Figure (2), the terminal utility is shown in pink. While we know from the analytical results of [10] that the trivial state is not a unique equilibrium of (7), for most parameter ranges we conclude that the nontrivial stationary solutions are unstable to the forward-forward evolution. Rather than continuing to make a brute-force search through parameter space to look for small pockets of stable nontrivial stationary solutions, we will now change approaches.

This motivates our study of bifurcations in congestion models. In Section 3, we examine a different congestion model and demonstrate the exchange of stability from uniform to non-uniform equilibria. The non-uniform equilibria thus obtained are qualitatively similar to those obtained in this section.

FIGURE 1. Evolution to Nontrivial Stationary Densities

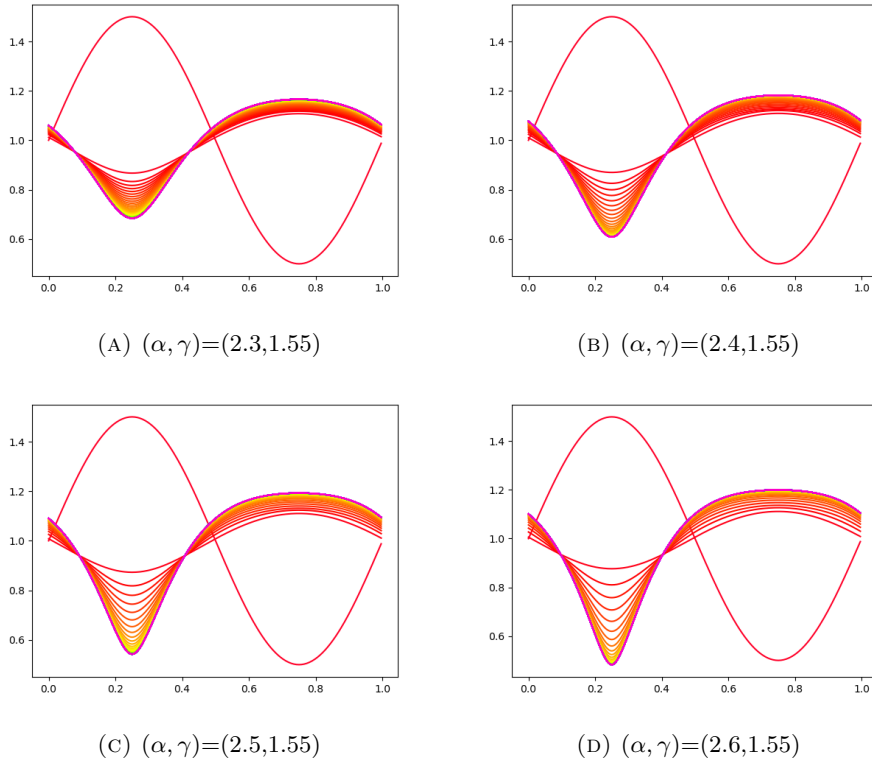
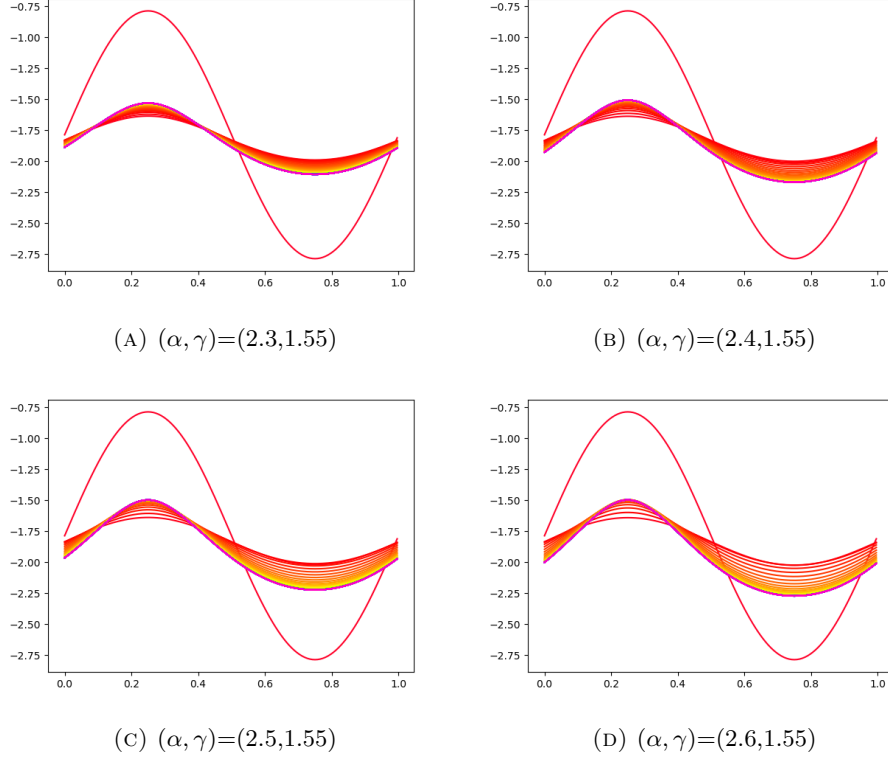


FIGURE 2. Evolution to Nontrivial Stationary Utilities



3. Existence of stationary solutions via bifurcation theory. In what follows, we demonstrate the existence of stationary solutions for congestion game (3) in the one-dimensional case with $\nu = 1$. That is, on the interval $[0, L]$ (with periodic boundary conditions) we will prove existence of solutions for the system

$$\begin{cases} -u_{xx} + \frac{1}{2} \frac{u_x^2}{(m+\mu)^2} = \gamma \left(m - \frac{1}{L} \right) + \lambda, \\ -m_{xx} - \left(\frac{mu_x}{(m+\mu)^2} \right)_x = 0. \end{cases}$$

We will be seeking solutions in which the mean of m is equal to $1/L$, so we may notice in the first equation that λ is equal to the mean of $\frac{1}{2} \frac{u_x^2}{(m+\mu)^2}$. We introduce the projection $\mathbb{P}$ which removes the mean of a periodic function. Then we rewrite the system as $F(u, m) = 0$, with F given by

$$F(u, m, \gamma) = \begin{pmatrix} -u_{xx} + \frac{1}{2} \mathbb{P} \frac{u_x^2}{(m+\mu)^2} - \gamma \left(m - \frac{1}{L} \right) \\ -m_{xx} - \left(\frac{mu_x}{(m+\mu)^2} \right)_x \end{pmatrix}.$$

For any γ , for any c , we have the trivial solution $F(c, \frac{1}{L}, \gamma) = 0$. We seek to bifurcate from this trivial family to nontrivial stationary solutions.

We use the following definition from the book of Zeidler [23].

Definition 3.1. Let X be a Banach space over the field $\mathbb{K}$ and consider the operator problem $G(u, p) = 0$ where $(u, p) \in X \times \mathbb{K}^n$. The point $(u_0, p_0) \in X \times \mathbb{K}^n$ is a bifurcation point of the problem if $G(u_0, p_0) = 0$ and there exist distinct sequences of solutions $\{(u_n, p_n) : n \in \mathbb{Z}^+\}$ and $\{(v_n, p_n) : n \in \mathbb{Z}^+\}$ that both converge to (u_0, p_0) .

Note that we will only take the field $\mathbb{K}$ to be $\mathbb{R}$.

We make use also of the following bifurcation theorem given in [23].

Theorem 3.2. *If $(u_0, p_0) \in X \times \mathbb{K}^n$ satisfies $G(u_0, p_0) = 0$ together with the following criteria, then (u_0, p_0) is a bifurcation point of the equation $G(u, p) = 0$.*

1. G is $\mathcal{C}^2$ in a neighborhood of (u_0, p_0) .
2. For all $p \in \mathbb{K}^n$ sufficiently close to p_0 , $G(u_0, p) = 0$.
3. The linearization $D_u G(u_0, p_0)$ is Fredholm and non-bijective.
4. For any nontrivial $b \in \text{Ker } D_u G(u_0, p_0)$ and basis $v_1, \dots, v_n$ of $\left(\text{Im } D_u G(u_0, p_0)\right)^\perp$, if $[T]_{jk} = \langle v_j, D_{p_k u}^2 G(u_0, p_0) b \rangle$ then $\det T \neq 0$.

We shall employ Theorem 3.2 to demonstrate the existence of a bifurcation point in the parameter γ .

If m is a probability distribution on $[0, L]$ then the stationary HJB equation

$$u_{xx} - \frac{1}{2} \frac{u_x^2}{(m + \mu)^2} = \gamma \left(m - \frac{1}{L} \right)$$

is only satisfied if $u_x = 0$, yielding the trivial equilibrium. Therefore a nontrivial stationary density $m(t, x) = \psi(x)$ requires that $u(t, x)$ approaches the long-time limit $u(t, x) = \phi(x) - \lambda t$ where $\phi(x)$ has a mean of zero and

$$\lambda = \frac{1}{2L} \int_{\mathbb{T}} \frac{\phi'^2}{(m + \mu)^2} dx.$$

Let $\mathbb{P}v = v - \frac{1}{\text{Vol}(\mathbb{T})} \int_{\mathbb{T}} v \, dx$ denote the projection onto the space of zero-mean functions. The HJB equation becomes

$$-\lambda - u_{xx} + \frac{1}{2} \frac{u_x^2}{(m + \mu)^2} - \gamma \left(m - \frac{1}{L} \right) = 0$$

yielding the stationary form of the forward-forward system

$$F(u, m, \gamma) = \begin{pmatrix} -u_{xx} + \frac{1}{2} \mathbb{P} \frac{u_x^2}{(m + \mu)^2} - \gamma \left(m - \frac{1}{L} \right) \\ -m_{xx} - \left(\frac{mu_x}{(m + \mu)^2} \right)_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Again, we take the periodic spatial domain $\mathbb{T} = \mathbb{R}/L\mathbb{Z}$. Choose base-point $(\bar{u}, \bar{m}) = (c, L^{-1})$ and let

$$w = u - c, \quad v = m - \frac{1}{L}.$$

The stationary form of the forward-forward system may be represented by the equivalent system

$$\tilde{F}(w, v, \gamma) = \begin{pmatrix} -w_{xx} + \frac{1}{2} \mathbb{P} \frac{w_x^2}{(v + \mu + L^{-1})^2} - \gamma v \\ -v_{xx} - \left(\frac{vw_x}{(v + \mu + L^{-1})^2} \right)_x - \left(\frac{w_x}{L(v + \mu + L^{-1})^2} \right)_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where we take solutions from the Sobolev space $H^2(\mathbb{T}) \times H^2(\mathbb{T})$.

Theorem 3.3. *Let $\gamma_0 = -4\pi^2(\mu + L^{-1})^2/L$. Then $(0, 0, \gamma_0)$ is a bifurcation point of the problem $\tilde{F} = 0$.*

Proof. We restrict $\tilde{F}$ to the space of odd functions (i.e. functions on the torus with a sine-series expansion) for both w and v . Let $\dot{H}_o^2(\mathbb{T})$ denote the space of all $\dot{H}^2$ functions on $\mathbb{T}$ that are odd. We linearize about the solution $(0, 0, \gamma_0)$ to obtain

$$A : \dot{H}_o^2(\mathbb{T}) \times \dot{H}_o^2(\mathbb{T}) \rightarrow \dot{L}^2(\mathbb{T}) \times \dot{L}^2(\mathbb{T}),$$

given by

$$A(v, w) = \begin{pmatrix} -w_{xx} - \gamma_0 v \\ -\frac{w_{xx}}{L(\mu + L^{-1})^2} - v_{xx} \end{pmatrix},$$

which one computes by taking the derivative in the sense of Frechet, i.e.,

$$\lim_{h \rightarrow 0} \frac{\tilde{F}(hw, hv, \gamma_0) - \tilde{F}(0, 0, \gamma_0)}{h}.$$

We see first that $\tilde{F}$ is $\mathcal{C}^2$ in a neighborhood of $(0, 0, \gamma_0)$ due to the presence of $\mu + L^{-1}$ in the denominators of the nonlinear terms. By direct calculation,

$$\begin{aligned} D\tilde{F}(w, v, \gamma)(w', v', \gamma') \\ = \begin{pmatrix} -w'_{xx} + \mathbb{P} \frac{w_x w'_x}{(v + \mu + L^{-1})^2} - \mathbb{P} \frac{w_x^2 v'}{(v + \mu + L^{-1})^3} - \gamma' v - \gamma v' \\ -v'_{xx} - \left(\frac{v' w_x + v w'_x}{(v + \mu + L^{-1})^2} - 2 \frac{v w_x v'}{(v + \mu + L^{-1})^3} + \frac{w'_x}{L(v + \mu + L^{-1})^2} - 2 \frac{w_x v'}{L(v + \mu + L^{-1})^3} \right)_x \end{pmatrix} \end{aligned}$$

and

$$D^2 \tilde{F}(w, v, \gamma) \left((w', v', \gamma'), (w'', v'', \gamma'') \right) = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix},$$

where

$$R_1 = -w'_{xx} + \mathbb{P} \frac{2w''_{xx} w'_x - 4w_x w'_x v''}{(v + \mu + L^{-1})^2} + \mathbb{P} \frac{3w_x^2 v' v''}{(v + \mu + L^{-1})^4} - \mathbb{P} \frac{2w_x w'_x v'}{(v + \mu + L^{-1})^3} - \gamma' v'' - \gamma'' v'$$

and

$$\begin{aligned} R_2 = & -v'_{xx} - \left(\frac{v' w''_{xx} + v'' w'_x}{(v + \mu + L^{-1})^2} \right)_x - 2 \left(\frac{2v' w_x v'' + v w''_x v' + v w'_x v''}{(v + \mu + L^{-1})^3} \right)_x \\ & + 6 \left(\frac{v w_x v' v''}{(v + \mu + L^{-1})^4} \right)_x - 2 \left(\frac{w'_x v'' + w''_x v'}{L(v + \mu + L^{-1})^3} \right)_x + 6 \left(\frac{w_x v' v''}{L(v + \mu + L^{-1})^4} \right)_x. \end{aligned}$$

Therefore if v is sufficiently close to 0 then there exist $c = c(w, v, \gamma, L, \mu) > 0$ and $c' = c'(w, v, \gamma, L, \mu) > 0$ such that

$$\left\| D\tilde{F}(w, v, \gamma)(w', v', \gamma) \right\|_{L^2 \times L^2} \leq c \left(\|w'\|_{H^2} + \|v'\|_{H^2} + |\gamma'| \right)$$

and

$$\begin{aligned} \left\| D^2 \tilde{F}(w, v, \gamma) \left((w', v', \gamma'), (w'', v'', \gamma'') \right) \right\|_{L^2 \times L^2} \\ \leq c' \left(\|w'\|_{H^2} + \|v'\|_{H^2} + |\gamma'| \right) \left(\|w''\|_{H^2} + \|v''\|_{H^2} + |\gamma''| \right). \end{aligned}$$

That is, if (w, v, γ) are sufficiently close to $(0, 0, \gamma_0)$ then F is $\mathcal{C}^2$. Clearly $\tilde{F}(0, 0, \gamma) = 0$ for all $\gamma \in \mathbb{R}$, so Conditions 1 and 2 of Theorem 3.2 are satisfied. In what follows,

we demonstrate that A is Fredholm and that $\tilde{F}$ satisfies the transversality condition at $(0, 0, \gamma_0)$.

If $A(w, v) = 0$ then

$$\begin{cases} -w_{xx} - \gamma_0 v = 0 \\ w_{xx} + L(\mu + L^{-1})^2 v_{xx} = 0 \end{cases}$$

so

$$v_{xx} = \frac{\gamma_0}{L(\mu + L^{-1})^2} v.$$

Note that it is crucial here that $\gamma_0 < 0$. If $\gamma_0 \geq 0$ then under periodic, zero-mean conditions, only $v = 0$ is possible, yielding a trivial kernel. Given

$$\gamma_0 = \frac{-4\pi^2}{L}(\mu + L^{-1})^2$$

we obtain that

$$v = c \sin\left(\frac{2\pi}{L}x\right)$$

for some $c \in \mathbb{R}$ and therefore

$$w_{xx} = -\gamma_0 c \sin\left(\frac{2\pi}{L}x\right)$$

implying

$$w = -cL(\mu + L^{-1})^2 \sin\left(\frac{2\pi}{L}x\right).$$

The linearization then has a one-dimensional kernel with a basis consisting only of

$$b = \left(-L(\mu + L^{-1}) \sin\left(\frac{2\pi}{L}x\right), \sin\left(\frac{2\pi}{L}x\right)\right).$$

We have already shown that $F \in \mathcal{C}^2$ in a neighborhood of $(0, 0, \gamma_0)$, and therefore A is continuous on its domain $\text{dom } A = H_o^2(\mathbb{T}) \times \dot{H}_o^2(\mathbb{T})$.

We now argue that $\text{Im}(A)$ is closed in $\dot{L}^2(\mathbb{T}) \times \dot{L}^2(\mathbb{T})$ which allows us to compute a basis of $\text{Coker } A$ by obtaining a basis of $\text{Ker } A^*$. We begin by decomposing the domain,

$$\dot{H}^2(\mathbb{T}) \times \dot{H}^2(\mathbb{T}) = \text{Ker}(A) \oplus W.$$

Since we have factored out the kernel, it is straightforward to see that $A \upharpoonright W$ has bounded inverse (see [5] for full details). Let $(f_n, g_n) = A(w_n, v_n)$ be a Cauchy sequence in $\text{Im}(A)$ with limit (f, g) . For all n , decompose (w_n, v_n) as

$$(w_n, v_n) = (r_n, s_n) + (y_n, z_n)$$

where $(r_n, s_n) \in \text{Ker}(A)$ and $(y_n, z_n) \in W$. Then clearly $A(y_n, z_n) = (f_n, g_n)$, and $(y_n, z_n) = (A \upharpoonright W)^{-1}(f_n, g_n)$. We may then bound a difference as

$$\begin{aligned} \|(y_n, z_n) - (y_m, z_m)\|_{H^2 \times H^2} &= \|(A \upharpoonright W)^{-1}((f_n, g_n) - (f_m, g_m))\|_{H^2 \times H^2} \\ &\leq C\|(f_n, g_n) - (f_m, g_m)\|_{L^2 \times L^2}. \end{aligned}$$

Since the sequence (f_n, g_n) is Cauchy, we conclude that the sequence (y_n, z_n) is also Cauchy, with limit (y, z) . Since A is continuous, we further conclude that $A(y, z) = (f, g)$. This proves that $\text{Im}(A)$ is closed, as desired. Therefore one may compute a basis of $\text{Coker } A$ by obtaining a basis of $\text{Ker } A^*$. Using Plancherel's

theorem, we may represent the action of A between $H^2 \times H^2$ and $L^2 \times L^2$ in Fourier space, and by taking the Hermitian adjoint,

$$\widehat{A}^* = \begin{pmatrix} k^2 & \frac{k^2}{L(\mu + L^{-1})^2} \\ -\gamma_0 & k^2 \end{pmatrix}.$$

If $(w, v) \in \text{Ker } A^*$ then $\widehat{A}(\widehat{w}, \widehat{v}) = 0$ and by the same calculation performed to compute a basis of $\text{Ker } A$, we obtain that the kernel of A^* is spanned by the vector

$$\widetilde{b} = \left(\sin\left(\frac{2\pi}{L}x\right), -L(\mu + L^{-1})^2 \sin\left(\frac{2\pi}{L}x\right) \right)$$

when we restrict to the subspace of odd functions. The linearization A is therefore Fredholm.

The parameter space of the operator problem is one-dimensional, so the validation of the transversality condition is simple:

$$D_\gamma D_{(w,v)} \widetilde{F}(0, 0, \gamma)(w, v) = (-v, 0),$$

so

$$\langle \widetilde{b}, D_\gamma D_{(w,v)} \widetilde{F}(0, 0, \gamma_0)(b) \rangle_{L^2 \times L^2} = - \int_{\mathbb{T}} \sin^2(x) \, dx < 0.$$

We have verified that $(0, 0, \gamma_0)$ satisfies the necessary and sufficient conditions to constitute a bifurcation point of the operator problem $\widetilde{F} = 0$. $\square$

Remark 3.4. Note that by taking $\gamma < 0$ in the forward-forward model, we obtain equilibria of the conventional backward-forward model, with a coupling that is increasing in m and therefore compliant with the assumptions typically placed on such a model.

Remark 3.5. Note also the dependence of γ_0 on L . In particular, note that the given congestion game on a short interval is more robust against the change of equilibrium than a longer interval (that is, $|\gamma|$ must be taken larger for $\gamma < 0$ to obtain a nontrivial distribution in the long-time limit).

4. Numerical demonstration of the exchange of stability. In what follows, we provide graphical results from our numerical experiments that demonstrate the fact that $\gamma_0 = -4\pi^2(\mu + L^{-1})^2/L$ is a bifurcation point in the congestion game.

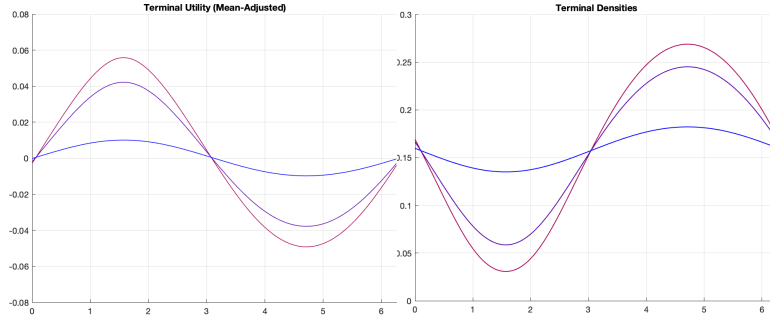
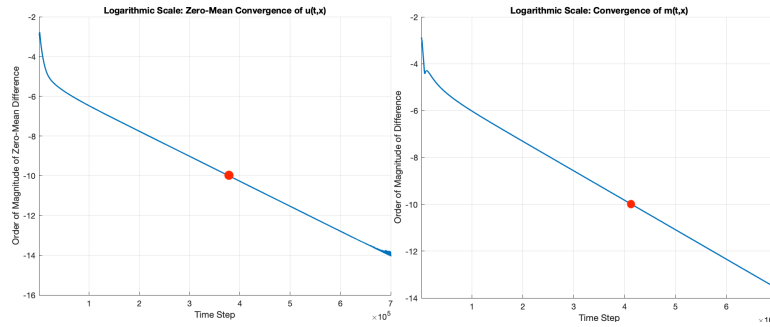
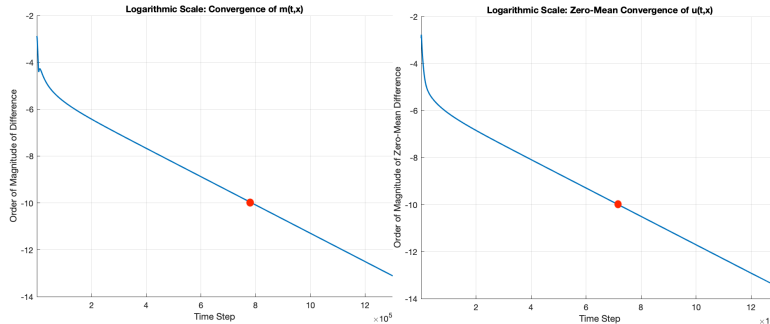
4.1. Nontrivial equilibria. The following experiments use

$$u^0(x) = \frac{1}{2} \sin\left(\frac{2\pi}{L}x\right), \quad m^0(x) = \frac{1}{L} + 5 * 10^{-5} \sin\left(\frac{2\pi}{L}x\right)$$

as initial data with $L = 2\pi$. Note that the initial distribution is taken quite close to the uniform distribution.

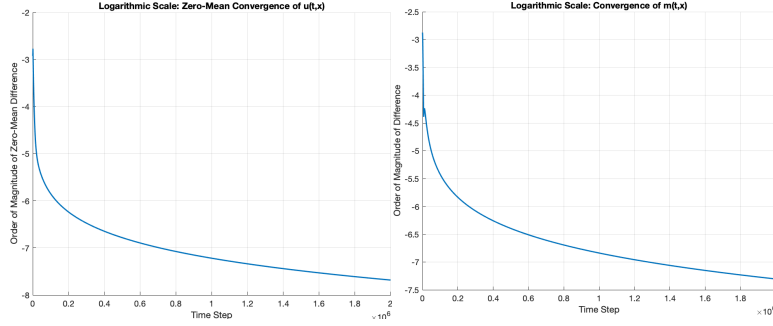
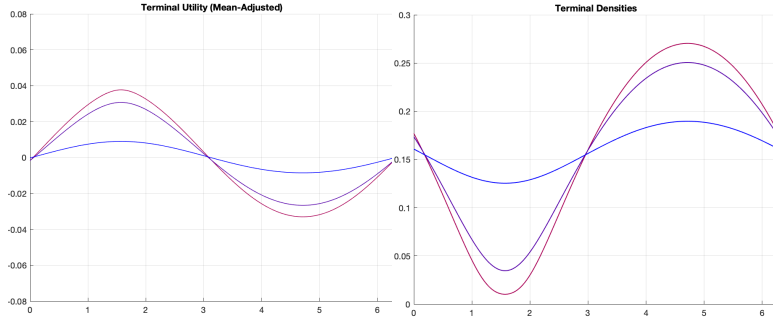
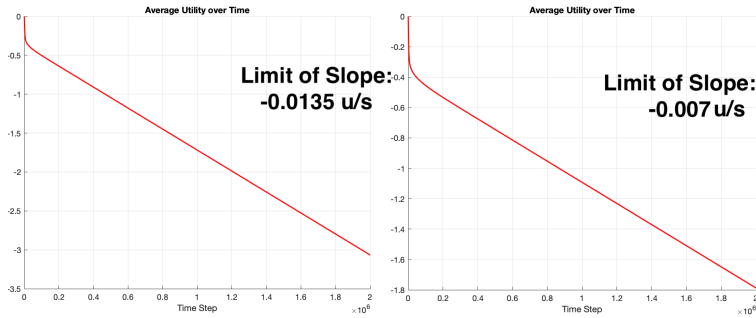
Figure 3 shows the mean-adjusted utility $\mathbb{P}u^T$ and density m^T at the end of the simulation, taking $\mu = 0.1$. The magenta (farthest from trivial), purple (middle), and blue (nearest to trivial) curves correspond to $b = -0.1$, $b = -0.0505$, and $b = -0.001$, respectively. The convergence of the solutions to equilibrium is demonstrated in Figures 4, 5, and 6, which show K_j as given in Section 2, with j ranging over the time steps. Note that in the case $b = -0.001$, convergence to the stationary solution is far slower than for the more extreme cases $b = -0.1$ and $b = -0.0505$.

The same experiment was repeated in the case $\mu = 0.05$, yielding $\mathbb{P}u^T$ and m^T as shown in Figure 7, with similar convergence rates to those shown in the $\mu = 0.1$ case.

FIGURE 3. Terminal Data, $\mu = 0.1$, $b < 0$ FIGURE 4. Convergence over Time; $b = -0.1$ K_j DecreasesFIGURE 5. Convergence over Time; $b = -0.0505$, K_j Decreases

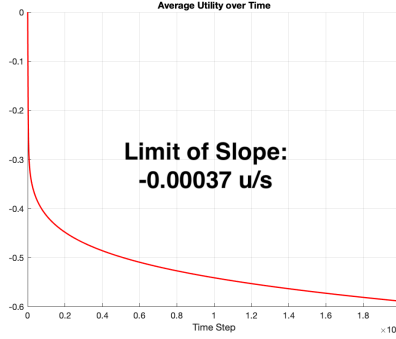
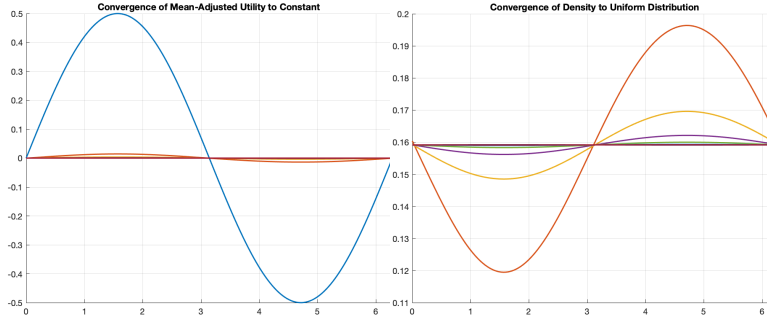
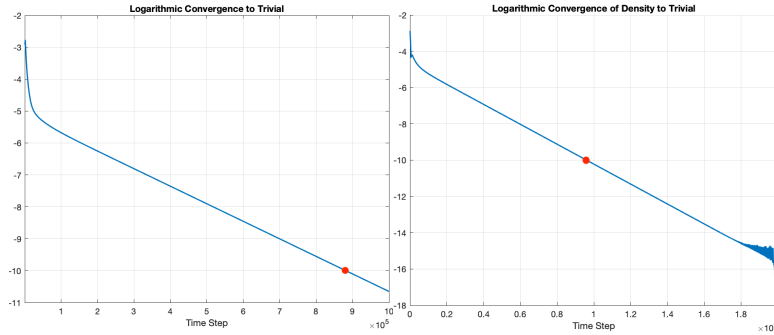
Note that the non-uniformity of the distribution of agents becomes more acute as μ grows smaller. Again, terminal data corresponding to $b = -0.1$ (Magenta, most extreme), $b = -0.0505$ (Purple, middle), and $b = -0.001$ (blue, nearest to trivial) are shown.

In Figures 8 and 9, the average value of $u(t, \cdot)$ over time is shown. As we have stated, the nontrivial long-time limits we obtain have the form $u(t, x) = \phi(x) - \lambda t$ and $m(t, x) = \psi(x)$. The approximate value of λ has been calculated and included in the plots. Note that λ is a translation rate in time (and therefore given in utils per second), while the horizontal axis is the time-step index, to coordinate with the convergence plots. In these experiments, we once again employed $\mu = 0.1$, with

FIGURE 6. Convergence over Time; $b = -0.001$, K_j DecreasesFIGURE 7. Terminal Data; $\mu = 0.05$, $b < 0$ FIGURE 8. Average Utility; $\mu = 0.1$, $b = -0.1$ and $b = -0.0505$

corresponding perturbations $b = -0.1$, $b = -0.0505$, and $b = -0.001$, respectively, from the bifurcation point.

4.2. Trivial equilibria. The same experiments performed for $b < 0$ were performed for various values of $b > 0$ and shown to yield trivial equilibria (although, as in the $b < 0$ case, convergence is extremely slow near the bifurcation point). As a representative example (Figure 10), a time-evolving plot shows the convergence of $\mathbb{P}u(t, x)$ and $m(t, x)$ to 0 and $1/L$, respectively, in the case of $b = 0.1$ with $\mu = 0.1$. Plots showing K_j over the time-steps j , corresponding to u and m , respectively, are shown in Figure 11.

FIGURE 9. Average Utility; $\mu = 0.1$, $b = -0.001$ FIGURE 10. Convergence to Trivial Equilibrium; $\mu = 0.1$, $b = 0.1$ FIGURE 11. Convergence over Time; $b = 0.1$, K_j decreases

4.3. Convergence rates near the bifurcation. We now examine the rate of convergence of the forward-forward system to equilibrium near the bifurcation point. Let $K_j^u(b)$ and $K_j^m(b)$ be as given in Section 2, measuring the zero-mean difference in u and m , respectively, after j time steps, taking the parameter value $\gamma = \gamma_0 + b$. With $Nt = 2 * 10^6$ and $\Delta t = 10^{-4}$, let

$$\Omega_u(b) = \frac{K_{Nt}^u(b) - K_{Nt-1}^u(b)}{\Delta t}$$

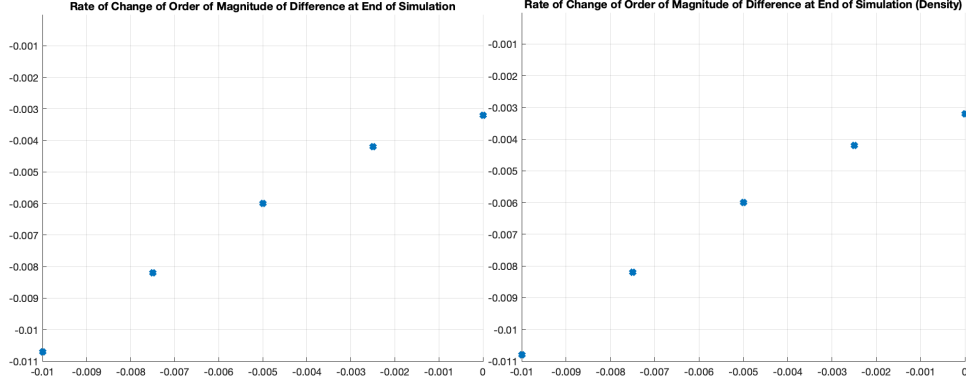


FIGURE 12. Rate of Convergence in Orders of Magnitude

and

$$\Omega_m(b) = \frac{K_{Nt}^m(b) - K_{Nt-1}^m(b)}{\Delta t}$$

represent the rate of change of the order of magnitude in the zero-mean L^2 -norm of the step-by-step difference between $u(t, \cdot)$ and $m(t, \cdot)$, respectively, at the end of the simulation. That is, $\Omega_u(b)$ represents the rate at which $u(t, \cdot)$ is converging to $\phi(x) - \lambda t$ in orders of magnitude per second, when $\gamma = \gamma_0 + b$ (and the same for $\Omega_m(b)$). In Figure 12, Ω_u and Ω_m are shown on the left and right, respectively, for $b = -0.01$, $b = -0.0075$, $b = -0.005$, $b = -0.0025$, and $b = 0$. Note that $\Omega_u, \Omega_m < 0$, showing that K_j is decreasing as the solution converges, but growing smaller in magnitude as the bifurcation point is approached.

5. Discussion. We have shown that the forward-forward form of a mean-field game system holds the potential to ease the detection of game equilibria by numerical experimentation. Our numerical experimentation focuses on a one-dimensional forward-forward mean-field game with congestion, and through numerical experimentation, we detect the emergence of nontrivial equilibria, motivating our search for the existence of a bifurcation point in the solutions parametrized by the coupling strength γ .

While several theoretical investigations into forward-forward mean-field games are known to the authors ([11], [12], [16], [17]), this is the first investigation, to our knowledge, that has used them to numerically detect nontrivial equilibria in the original backward-forward game. We therefore regard our results as a proof-of-concept for use of the forward-forward system to produce numerical results to complement the proof of theoretical ones. The same methodology may be employed to search for bifurcations in other mean-field game systems, or in generalizations of the same congestion model. One may perform a similar analysis of the general congestion game posed in [1], that is,

$$\begin{cases} -u_t - \nu \Delta u + \frac{1}{\beta} \frac{|\nabla u|^\beta}{(m+\mu)^\alpha} = V(t, x, m) \\ m_t - \nu \Delta m - \operatorname{div} \left\{ m \frac{|\nabla u|^{\beta-2}}{(m+\mu)^\alpha} \nabla u \right\} = 0 \\ u, m : [0, T] \times \mathbb{T}^d \rightarrow \mathbb{R}, \end{cases}$$

together with a parameter space consisting of (α, β) and parameters associated with a specific form of the coupling.

REFERENCES

- [1] Y. Achdou and A. Porretta, [Mean field games with congestion](#), *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **35** (2018), 443-480.
- [2] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd edition, Elsevier/Academic Press, Amsterdam, 2003.
- [3] N. Almulla, R. Ferreira and D. Gomes, [Two numerical approaches to stationary mean-field games](#), *Dyn. Games Appl.*, **7** (2017), 657-682.
- [4] L. M. Briceño-Arias, D. Kalise and F. J. Silva, [Proximal methods for stationary mean field games with local couplings](#), *SIAM J. Control Optim.*, **56** (2018), 801-836.
- [5] L. C. Brown, *Equilibria and Bifurcation Theory for Mean-Field Games*, Ph.D thesis, Drexel University, 2023.
- [6] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions and A. Porretta, [Long time average of mean field games](#), *Netw. Heterog. Media*, **7** (2012), 279-301.
- [7] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions and A. Porretta, [Long time average of mean field games with a nonlocal coupling](#), *SIAM J. Control Optim.*, **51** (2013), 3558-3591.
- [8] M. Cirant and G. Verzini, [Bifurcation and segregation in quadratic two-populations mean field games systems](#), *ESAIM Control Optim. Calc. Var.*, **23** (2017), 1145-1177.
- [9] D. Evangelista, R. Ferreira, D. A. Gomes, L. Nurbekyan and V. Voskanyan, [First-order, stationary mean-field games with congestion](#), *Nonlinear Anal.*, **173** (2018), 37-74.
- [10] D. Evangelista and D. A. Gomes, [On the existence of solutions for stationary mean-field games with congestion](#), *J. Dynam. Differential Equations*, **30** (2018), 1365-1388.
- [11] D. Gomes, L. Nurbekyan and M. Sedjro, [Conservation laws arising in the study of forward-forward mean-field games](#), *Theory, Numerics and Applications of Hyperbolic Problems, I*, Springer Proc. Math. Stat., Springer, Cham, **236** (2018), 643-649.
- [12] D. Gomes and M. Sedjro, [One-dimensional, forward-forward mean-field games with congestion](#), *Discrete Contin. Dyn. Syst. Ser. S*, **11** (2018), 901-914.
- [13] D. A. Gomes and H. Mitake, [Existence for stationary mean-field games with congestion and quadratic Hamiltonians](#), *NoDEA Nonlinear Differential Equations Appl.*, **22** (2015), 1897-1910.
- [14] D. A. Gomes, L. Nurbekyan and M. Prazeres, [Explicit solutions of one-dimensional, first-order, stationary mean-field games with congestion](#), *2016 IEEE 55th Conference on Decision and Control (CDC)*, Las Vegas, NV, USA, 2016, 4534-4539.
- [15] D. A. Gomes, L. Nurbekyan and M. Prazeres, [One-dimensional stationary mean-field games with local coupling](#), *Dyn. Games Appl.*, **8** (2018), 315-351.
- [16] D. A. Gomes, L. Nurbekyan and M. Sedjro, [One-dimensional forward-forward mean-field games](#), *Appl. Math. Optim.*, **74** (2016), 619-642.
- [17] D. A. Gomes and E. A. Pimentel, [Regularity for mean-field games systems with initial-initial boundary conditions: The subquadratic case](#), *Dynamics, Games and Science*, CIM Ser. Math. Sci., Springer, Cham, **1** (2015), 291-304.
- [18] M. Huang, P. E. Caines and R. P. Malhamé, [Large population stochastic dynamic games: Closed-Loop McKean-Vlasov systems and the Nash certainty equivalence principle](#), *Commun. Inf. Syst.*, **6** (2006), 221-252.
- [19] M. Huang, P. E. Caines and R. P. Malhamé, [Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized \$\varepsilon\$ -Nash equilibria](#), *IEEE T. Automat. Contr.*, **52** (2007), 1560-1571.
- [20] J.-M. Lasry and P.-L. Lions, [Jeux à champ moyen. I – Le cas stationnaire](#), *C. R. Math. Acad. Sci. Paris*, **343** (2006), 619-625.
- [21] J.-M. Lasry and P.-L. Lions, [Jeux à champ moyen. II – Horizon fini et controle optimal](#), *C. R. Math. Acad. Sci. Paris*, **343** (2006), 679-684.
- [22] J.-M. Lasry and P.-L. Lions, [Mean field games](#), *Jpn. J. Math.*, **2** (2007), 229-260.
- [23] E. Zeidler, *Applied Functional Analysis: Main Principles and their Applications*, Springer-Verlag, New York, 1995.

Received March 2023; revised September 2023; early access September 2023.