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INVERSE PROBLEMS FOR NONLINEAR MAGNETIC SCHRÖDINGER EQUATIONS ON CONFORMALLY TRANSVERSALLY ANISOTROPIC MANIFOLDS

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We study the inverse boundary problem for a nonlinear magnetic Schrödinger operator on a conformally transversally anisotropic Riemannian manifold of dimension $n \geq 3$. Under suitable assumptions on the nonlinearity, we show that the knowledge of the Dirichlet-to-Neumann map on the boundary of the manifold determines the nonlinear magnetic and electric potentials uniquely. No assumptions on the transversal manifold are made in this result, whereas the corresponding inverse boundary problem for the linear magnetic Schrödinger operator is still open in this generality.

1. Introduction and statement of results

Let (M, g) be a smooth compact oriented Riemannian manifold of dimension $n \geq 3$ with smooth boundary. Let $A \in C^\infty(M, T^*M)$ be a 1-form with complex-valued C^∞ coefficients, and let

$$d_A = d + iA : C^\infty(M) \to C^\infty(M, T^*M),$$

where $d : C^\infty(M) \to C^\infty(M, T^*M)$ is the de Rham differential. We define the formal L^2-adjoint of d_A, $d_A^* : C^\infty(M, T^*M) \to C^\infty(M)$, as

$$(d_A u, v)_{L^2(M,T^*M)} = (u, d_A^* v)_{L^2(M)}, \quad u \in C_0^\infty(M^{\text{int}}), \quad v \in C_0^\infty(M^{\text{int}}, T^*M^{\text{int}}),$$

where $M^{\text{int}} = M \setminus \partial M$ stands for the interior of M. Here and in what follows, when $u, v \in C^\infty(M)$, we write

$$(u, v)_{L^2(M)} = \int_M u\bar{v}\, dV_g$$

for the natural L^2-scalar product, where dV_g is the Riemannian volume element on M. Similarly, when $\alpha, \beta \in C^\infty(M, T^*M)$ are 1-forms, we define the L^2-scalar product

$$(\alpha, \beta)_{L^2(M,T^*M)} = \int_M \langle \alpha, \bar{\beta} \rangle_g \, dV_g(x),$$

where $\langle \cdot, \cdot \rangle_g$ is the pointwise scalar product in the space of 1-forms induced by the Riemannian metric g. In the local coordinates $(x_1, \ldots, x_n)$, in which $\alpha = \sum_{j=1}^n \alpha_j \, dx_j$, $\beta = \sum_{j=1}^n \beta_j \, dx_j$, and (g^{jk}) is the

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matrix inverse of (g_{jk}) with $g = \sum_{j,k=1}^n g_{jk}\, dx_j\, dx_k$, we have

$$\langle \alpha, \beta \rangle_g = \sum_{j,k=1}^n g^{jk} \alpha_j \beta_k.$$

We also have

$$d_A^* = d^* - i\langle \bar{A}, \cdot \rangle_g.$$

In local coordinates, we see that

$$d^* v = -\sum_{j,k=1}^n |g|^{-1/2} \partial_{x_j} (|g|^{1/2} g^{jk} v_k), \tag{1-1}$$

where $|g| = \det(g_{jk})$ and $v = \sum_{j=1}^n v_j\, dx_j$.

In this paper we shall consider 1-forms and scalar functions depending holomorphically on a parameter $z \in \mathbb{C}$. Specifically, let $A : M \times \mathbb{C} \to T^*M$ and $V : M \times \mathbb{C} \mapsto \mathbb{C}$ satisfy the following conditions:

- (A_i) The map $\mathbb{C} \ni z \mapsto A(\cdot, z)$ is holomorphic with values in $C^{1,1}(M, T^*M)$, the space of 1-forms with complex-valued $C^{1,1}(M)$ coefficients.
- (V_i) The map $\mathbb{C} \ni z \mapsto V(\cdot, z)$ is holomorphic with values in $C^{1,1}(M)$.
- (V_{ii}) $V(x, 0) = 0$, for all $x \in M$.

Here $C^{1,1}(M)$ is the space of C^1 functions on M with a Lipschitz gradient.

It follows from (A_i), (V_i), and (V_{ii}) that A and V can be expanded into the power series

$$A(x, z) = \sum_{k=0}^\infty A_k(x) \frac{z^k}{k!}, \quad A_k(x) := \partial_z^k A(x, 0) \in C^{1,1}(M, T^*M), \tag{1-2}$$

converging in the $C^{1,1}(M, T^*M)$ topology, and

$$V(x, z) = \sum_{k=1}^\infty V_k(x) \frac{z^k}{k!}, \quad V_k(x) := \partial_z^k V(x, 0) \in C^{1,1}(M), \tag{1-3}$$

converging in the $C^{1,1}(M)$ topology.

Let us introduce the nonlinear magnetic Schrödinger operator defined by

$$\begin{aligned} L_{A,V} u &= d^*_{\overline{A(\cdot,u)}}\, d_{A(\cdot,u)} u + V(\cdot, u) \\ &= -\Delta_g u + d^*(i A(\cdot, u) u) - i\langle A(\cdot, u), du \rangle_g + \langle A(\cdot, u), A(\cdot, u) \rangle_g u + V(\cdot, u), \end{aligned} \tag{1-4}$$

for $u \in C^\infty(M)$. Notice that the first-order linearization of $L_{A,V}$ is the standard linear magnetic Schrödinger operator $d^*_{\bar{A}_0} d_{A_0} + V_1$. Furthermore, we also assume that $A_0 \in C^\infty(M, T^*M)$, $V_1 \in C^\infty(M)$, and that

(i) 0 is not a Dirichlet eigenvalue of the operator $d^*_{\bar{A}_0} d_{A_0} + V_1$.

Consider the Dirichlet problem for the nonlinear magnetic Schrödinger operator

$$\begin{cases} L_{A,V} u = 0 \quad \text{in } M^{\text{int}}, \\ u|_{\partial M} = f. \end{cases} \tag{1-5}$$

It is shown in Theorem B.1 that under the above assumptions, there exist $\delta > 0$ and $C > 0$ such that when $f \in B_\delta(\partial M) := \{f \in C^{2,\alpha}(\partial M) : \|f\|_{C^{2,\alpha}(\partial M)} < \delta\}$, $0 < \alpha < 1$, the problem (1-5) has a unique solution $u = u_f \in C^{2,\alpha}(M)$ satisfying $\|u\|_{C^{2,\alpha}(M)} < C\delta$. Here $C^{2,\alpha}(M)$ stands for the standard Hölder space of functions on M. Associated to the problem (1-5), we define the Dirichlet-to-Neumann map

$$\Lambda_{A,V} f = \partial_\nu u_f|_{\partial M}, \tag{1-6}$$

where $f \in B_\delta(\partial M)$ and ν is the unit outer normal to the boundary.

The inverse problem that we are interested in is whether the knowledge of the Dirichlet-to-Neumann map $\Lambda_{A,V}$ determines the nonlinear magnetic and electric potentials, A and V, respectively.

When $A = 0$ and $V(x, z) = V_1(x)z$, the inverse problem for the linear Schrödinger operator $-\Delta_g + V_1$ is related to the Calderón problem, which has been the object of intense study but remains open in the case of a general smooth Riemannian manifold (M, g) of dimension $n \geq 3$ with smooth boundary. Let us mention that the unique determination of the potential V_1 from the knowledge of the Dirichlet-to-Neumann map Λ_{0,V_1} was established in [Sylvester and Uhlmann 1987] in the Euclidean setting, in [Isozaki 2004] for hyperbolic manifolds, and in [Kohn and Vogelius 1984; Lassas and Uhlmann 2001; Lee and Uhlmann 1989] in the analytic case. The uniqueness in the inverse boundary problem for the linear magnetic Schrödinger operator $d^*_{\bar{A}_0} d_{A_0} + V_1$ up to a suitable gauge transformation was obtained in [Nakamura et al. 1995] in the Euclidean setting; see also [Krupchyk and Uhlmann 2014]. Going beyond these settings, the most general uniqueness results were obtained in the case when the manifold (M, g) is conformally transversally anisotropic and the transversal manifold satisfies some additional assumptions. Following [Dos Santos Ferreira et al. 2009; 2016], let us recall the definition of a conformally transversally anisotropic manifold.

Definition 1.1. A compact smooth oriented Riemannian manifold (M, g) of dimension $n \geq 3$ with smooth boundary is said to be conformally transversally anisotropic if there exists an $(n-1)$-dimensional smooth compact Riemannian manifold (M_0, g_0) with smooth boundary such that $M \Subset \mathbb{R} \times M_0$ and $g = c(e \oplus g_0)$, where e is the Euclidean metric on $\mathbb{R}$ and c is a positive smooth function on M.

In the case when (M, g) is conformally transversally anisotropic, assuming that the transversal manifold (M_0, g_0) is simple in the sense that the boundary ∂M_0 is strictly convex and, for any point $p \in M_0$, the exponential map $\exp_p$ with its maximal domain of definition in $T_p M_0$ is a diffeomorphism onto M_0, the global uniqueness for the inverse boundary problem for the linear magnetic Schrödinger equation up to a gauge was proven in [Dos Santos Ferreira et al. 2009]; see also [Krupchyk and Uhlmann 2018]. Note that the geodesic ray transform on functions and 1-forms is invertible on simple manifolds; see [Anikonov 1978; Muhometov 1977].

These uniqueness results were strengthened in [Dos Santos Ferreira et al. 2016], where the global uniqueness in the inverse boundary problem for the linear Schrödinger equation was established under the assumption that the geodesic ray transform on the transversal manifold is injective. Similar results for the inverse boundary problem for the linear magnetic Schrödinger equation were obtained in [Cekić 2017; Krupchyk and Uhlmann 2018]. The injectivity of the geodesic ray transform is open in general, and has only been established under certain geometric assumptions. In particular, the injectivity of the

geodesic ray transform is proven in [Stefanov et al. 2018; Uhlmann and Vasy 2016] when M_0 has strictly convex boundary and is foliated by strictly convex hypersurfaces, and in [Guillarmou 2017; Guillarmou et al. 2021] when M_0 has a hyperbolic trapped set and no conjugate points. As an example of the latter, one can consider a negatively curved manifold M_0. We refer to [Dos Santos Ferreira et al. 2020] where the linearized anisotropic Calderón problem was studied on a transversally anisotropic manifold under certain mild conditions on the transversal manifold related to the geometry of pairs of intersecting geodesics.

Turning the attention to inverse problems for nonlinear PDEs, it was discovered in [Kurylev et al. 2018] that nonlinearity can be helpful in solving inverse problems for hyperbolic equations; see also [Feizmohammadi et al. 2021; Lassas et al. 2018]. Similar phenomena for inverse problems for semilinear elliptic PDEs have been revealed in [Feizmohammadi and Oksanen 2020; Lassas et al. 2021a]; see also [Krupchyk and Uhlmann 2020a; 2020b; Lai and Zhou 2020; Lassas et al. 2021b]. A common feature of all of the aforementioned works is that the presence of a nonlinearity allows one to solve inverse problems for nonlinear equations in cases where the corresponding inverse problem in the linear setting is open.

In particular, the inverse boundary problem for the nonlinear Schrödinger equation

$$L_{0,V}u = -\Delta_g u + V(\,\cdot\,, u) = 0$$

on a conformally transversally anisotropic manifold (M, g) of dimension $n \geq 3$ was studied in [Feizmohammadi and Oksanen 2020; Lassas et al. 2021a], and the following result was obtained: if V satisfies the assumptions (V_i), (V_{ii}), and

(V_{iii}) $\partial_z V(x, 0) = \partial_z^2 V(x, 0) = 0$, for all $x \in M$,

then the knowledge of the Dirichlet-to-Neumann map $\Lambda_{0,V}$ determines V in $M \times \mathbb{C}$ uniquely. Notice that remarkably there are no assumptions on the transversal manifold in this result while the inverse problem for the linear Schrödinger equation is still open in this generality. The proof of this result relies on higher-order linearizations of the Dirichlet-to-Neumann map, which allow one to reduce the inverse problem to the following density result; see [Lassas et al. 2021a].

Proposition 1.2. *Let (M, g) be a conformally transversally anisotropic manifold of dimension $n \geq 3$, and let $q \in C^{1,1}(M)$. If*

$$\int_M q u_1 u_2 u_3 u_4 \, dV_g = 0, \tag{1-7}$$

for all harmonic functions $u_j \in C^\infty(M)$, $j = 1, 2, 3, 4$, then $q \equiv 0$.

The purpose of this paper is to extend the aforementioned result of [Feizmohammadi and Oksanen 2020; Lassas et al. 2021a] to the nonlinear magnetic Schrödinger equation $L_{A,V}u = 0$ given by (1-4). To state our result, similarly to the assumption (V_{iii}) on the potential V, we shall also suppose that the nonlinear magnetic potential A satisfies

(A_{ii}) $A(x, 0) = \partial_z A(x, 0) = 0$, for all $x \in M$.

Our main result is as follows.

Theorem 1.3. *Let (M, g) be a conformally transversally anisotropic manifold of dimension $n \geq 3$. Let $A^{(1)}, A^{(2)} : M \times \mathbb{C} \to T^*M$ and $V^{(1)}, V^{(2)} : M \times \mathbb{C} \mapsto \mathbb{C}$ satisfy the assumptions (A_i), (A_{ii}), and (V_i), (V_{ii}), (V_{iii}), respectively. If $\Lambda_{A^{(1)},V^{(1)}} = \Lambda_{A^{(2)},V^{(2)}}$ then $A^{(1)} = A^{(2)}$ and $V^{(1)} = V^{(2)}$ in $M \times \mathbb{C}$.*

Remark 1.4. Let us point out that there are no assumptions on the transversal manifold in Theorem 1.3, whereas the corresponding inverse boundary problem for the linear magnetic Schrödinger operator is still open in this generality.

Remark 1.5. Notice that as opposed to the inverse boundary problem for the linear magnetic Schrödinger equation, where one can determine the magnetic potential up to a gauge transformation only, in our nonlinear setting the unique determination of both potentials is possible, due to the assumptions (A_i), (A_{ii}), and (V_i), (V_{ii}), (V_{iii}), which imply that the first-order linearization of the nonlinear equation is given by $-\Delta_g u = 0$, rather than by the linear magnetic Schrödinger equation.

Similarly to [Feizmohammadi and Oksanen 2020; Lassas et al. 2021a], the proof of Theorem 1.3 relies on higher-order linearizations of the Dirichlet-to-Neumann map $\Lambda_{A,V}$, as well as a suitable consequence of the following density result, which may be of some independent interest.

Proposition 1.6. *Let (M, g) be a conformally transversally anisotropic manifold of dimension $n \geq 3$, and let $A \in C^{1,1}(M, T^*M)$ be a 1-form. If*

$$\int_M \langle A, d(u_1u_2u_3)\rangle_g u_4 \, dV_g = 0, \tag{1-8}$$

for all harmonic functions $u_j \in C^\infty(M)$, $j = 1, 2, 3, 4$, then $A \equiv 0$.

The starting point in the proof of Proposition 1.6 consists of showing that the boundary traces of the 1-form A, as well as of its normal derivative, vanish, as a consequence of the integral identity (1-8). This allows us to extend A by zero to $\mathbb{R} \times M_0 \setminus M$, while preserving its regularity. The proof of Proposition 1.6 then follows the strategy of the proof of Proposition 1.2 established in [Lassas et al. 2021a]. Specifically, we construct harmonic functions to be used in (1-8), based on suitable Gaussian beams quasimodes associated to two nontangential intersecting geodesics on the transversal manifold M_0. Using the freedom of working with four harmonic functions, we construct a pair of harmonic functions based on a Gaussian beam quasimode v and its complex conjugate $\bar{v}$, concentrated near one geodesic, and another pair of harmonic functions based on a Gaussian beam quasimode w and its complex conjugate $\overline{w}$, concentrated near the other geodesic. The product $d(v\bar{v}w)\overline{w}$ is supported near the finitely many points of intersections of these geodesics, and the product does not have high oscillations. This makes it possible to conclude that $A = 0$, using both nonstationary as well as stationary phase arguments (the Laplace method).

Remark 1.7. Our regularity assumption on A in Proposition 1.6 is motivated by the fact that the continuity of the zero extension of A to $\mathbb{R} \times M_0 \setminus M$ is needed for a rough stationary phase argument and the Lipschitz continuity of the gradient of the zero extension of A is needed for a nonstationary phase argument in the proof of Proposition 1.6.

Returning to the proof of Theorem 1.3, let us mention that due to the assumptions (A_{ii}) and (V_{ii}), (V_{iii}), only the linearizations of the Dirichlet-to-Neumann map of order ≥ 3 become useful when recovering the

nonlinear potentials $A(x,z)$ and $V(x,z)$. Considering the m-th order linearization, $m \geq 3$, leads to the integral identity

$$\int_M ((m+1)i\langle A, d(u_1\cdots u_m)\rangle_g u_{m+1} - (mid^*(A)+V)u_1\cdots u_{m+1})\,dV_g = 0, \tag{1-9}$$

where $A = A^{(1)}_{m-1} - A^{(2)}_{m-1}$ and $V = V^{(1)}_m - V^{(2)}_m$, which is valid for any harmonic function $u_l \in C^{2,\alpha}(M)$ with $l = 1,\dots,m+1$. Setting $u_1 = \cdots = u_{m-3} = 1$ in (1-9) gives the identity

$$(m+1)i\int_M \langle A,\ d(u_{m-2}u_{m-1}u_m)\rangle_g u_{m+1}\,dV_g = \int_M (mid^*(A)+V)u_{m-2}u_{m-1}u_m u_{m+1})\,dV_g. \tag{1-10}$$

To proceed, we first show that (1-10) implies that $A|_{\partial M} = 0$ and $\partial_\nu A|_{\partial M} = 0$, and then use a consequence of Proposition 1.6 to obtain that $A \equiv 0$; see Corollary 4.1 below. To recover V, we substitute $A = 0$ in (1-10) and rely on Proposition 1.2.

Remark 1.8. The assumptions (A_i), (A_{ii}), (V_i), (V_{ii}), and (V_{iii}) in Theorem 1.3 are made precisely so that the higher-order linearizations of the Dirichlet-to-Neumann map $\Lambda_{A,V}$ lead to the integral identities (1-9) involving at least four harmonic functions, and the freedom of working with four harmonic functions allows one to solve the inverse boundary problem without any assumption on the transversal manifold; see also [Lassas et al. 2021a].

Let us point out that inverse boundary problems for the nonlinear magnetic Schrödinger equation in the Euclidean space, both in the case of full and partial data, have been studied in [Lai and Zhou 2020]. The density of certain products of gradients of harmonic functions in the Euclidean space has been recently established in [Cârstea and Feizmohammadi 2021], when solving an inverse boundary problem for certain anisotropic quasilinear elliptic equations.

Finally, let us remark that inverse boundary problems for nonlinear elliptic PDEs have been studied extensively in the literature. We refer to [Cârstea and Feizmohammadi 2021; Cârstea et al. 2019; Feizmohammadi and Oksanen 2020; Hervas and Sun 2002; Isakov and Nachman 1995; Isakov and Sylvester 1994; Kang and Nakamura 2002; Krupchyk and Uhlmann 2020a; 2020b; Lai and Zhou 2020; Lassas et al. 2021a; 2021b; Sun 1996; 2004; 2010, Sun and Uhlmann 1997].

The paper is organized as follows. In Section 2 we recall the construction of harmonic functions on a conformally transversally anisotropic manifold based on Gaussian beams quasimodes constructed on $\mathbb{R}\times M_0$ and localized near nontangential geodesics on the transversal manifold M_0. For the convenience of the reader, in Section 3 we provide a proof of Proposition 1.6 in a simplified setting. Section 4 is devoted to the proof of Proposition 1.6 in the general case. The proof of Theorem 1.3 occupies Section 5. Appendix A discusses a standard rough version of stationary phase needed in the proof of Proposition 1.6. In Appendix B, we show the well-posedness of the Dirichlet problem for the nonlinear magnetic Schrödinger equation, in the case of small boundary data. The determination of the first-order boundary traces of a scalar function and a 1-form, via suitable orthogonality relations involving harmonic functions on the manifold M, is presented in Appendix C. Finally, Appendix D discusses some basic properties of geodesics which are used in the body of the paper.

2. Gaussian beams quasimodes and construction of harmonic functions

Let (M, g) be a conformally transversally anisotropic manifold so that $(M, g) \Subset (\mathbb{R} \times M_0, c(e \oplus g_0))$. Let us write $x = (x_1, x')$ for local coordinates in $\mathbb{R} \times M_0$. Note that $\phi(x) = \pm\alpha x_1,\ \alpha > 0$, is a limiting Carleman weight for $-h^2\Delta_g$; see [Dos Santos Ferreira et al. 2009].

Letting $\tilde{g} = e \oplus g_0$, we have

$$c^{(n+2)/4} \circ (-\Delta_g) \circ c^{-(n-2)/4} = -\Delta_{\tilde{g}} + q, \tag{2-1}$$

where

$$q = -c^{(n+2)/4}\Delta_g(c^{-(n-2)/4});$$

see [Dos Santos Ferreira et al. 2016]. Here $q \in C^\infty(\mathbb{R} \times M_0; \mathbb{R})$. It follows from (2-1) that in order to construct harmonic functions on (M, g) based on Gaussian beams quasimodes, we shall need to have Gaussian beams quasimodes for the Schrödinger operator $-\Delta_{\tilde{g}} + q$, conjugated by an exponential weight corresponding to the limiting Carleman weight ϕ. Our quasimodes will be constructed on the manifold $\mathbb{R} \times M_0$ and will be localized to nontangential geodesics on the transversal manifold M_0. A unit speed geodesic $\gamma : [-S_1, S_2] \to M_0,\ 0 < S_1, S_2 < \infty$, is called nontangential if $\gamma(-S_1), \gamma(S_2) \in \partial M_0$, $\dot\gamma(-S_1), \dot\gamma(S_2)$ are nontangential vectors to ∂M_0, and $\gamma(t) \in M_0^{\text{int}}$ for all $-S_1 < t < S_2$; see [Dos Santos Ferreira et al. 2016]. As in [Lassas et al. 2021a], it will be convenient to normalize our quasimodes in $L^4(M_0)$, as later we shall have to deal with products of four such quasimodes. We shall need the following essentially well-known result, see [Feizmohammadi and Oksanen 2020, Section 4.1]; see also [Dos Santos Ferreira et al. 2016; Lassas et al. 2021a].

Proposition 2.1. *Let $\alpha > 0$, and let $\tau = s + i\lambda,\ s \geq 1$, with $\lambda \in \mathbb{R}$ fixed. Then for any $k \in \mathbb{N}$ and $R \geq 1$, there exist $N \in \mathbb{N}$ and families of Gaussian beam quasimodes $v_1(\cdot\,; s), v_2(\cdot\,; s) \in C^\infty(\mathbb{R} \times M_0)$ such that*

$$\begin{aligned} \|e^{-\alpha\tau x_1}(-\Delta_{\tilde{g}} + q)e^{\alpha\tau x_1}v_1(\cdot\,; s)\|_{H^k((I\times M_0)^{\text{int}})} &= \mathcal{O}(s^{-R}), \\ \|e^{\alpha\tau x_1}(-\Delta_{\tilde{g}} + q)e^{-\alpha\tau x_1}v_2(\cdot\,; s)\|_{H^k((I\times M_0)^{\text{int}})} &= \mathcal{O}(s^{-R}), \end{aligned} \tag{2-2}$$

and

$$\|v_j(\cdot\,; s)\|_{L^4(I\times M_0)} = \mathcal{O}(1), \quad \|v_j(\cdot\,; s)\|_{L^\infty(I\times M_0)} = \mathcal{O}(1)s^{(n-2)/8}, \quad j = 1, 2, \tag{2-3}$$

as $s \to \infty$. Here $I \subset \mathbb{R}$ is an arbitrary bounded interval. The local structure of the quasimodes is as follows: Let $p \in \gamma([-S_1, S_2])$ and let $t_1 < \cdots < t_P$ be the times in $[-S_1, S_2]$ when $\gamma(t_l) = p$. In a sufficiently small neighborhood U of p, the quasimode v_j is a finite sum

$$v_j|_U = v_j^{(1)} + \cdots + v_j^{(P)}.$$

Each $v_j^{(l)}$ has the form

$$v_1^{(l)} = s^{(n-2)/8}e^{i\alpha\tau\varphi^{(l)}}a^{(l)}, \quad v_2^{(l)} = s^{(n-2)/8}e^{i\alpha\tau\varphi^{(l)}}b^{(l)}, \quad l = 1, \ldots, P,$$

where $\varphi = \varphi^{(l)} \in C^\infty(\overline{U}; \mathbb{C})$ satisfies, for t close to t_l,

$$\varphi(\gamma(t)) = t, \quad \nabla\varphi(\gamma(t)) = \dot\gamma(t), \quad \operatorname{Im}(\nabla^2\varphi(\gamma(t))) \geq 0, \quad \operatorname{Im}(\nabla^2\varphi)|_{\dot\gamma(t)^\perp} > 0,$$

and $a^{(l)}, b^{(l)} \in C^\infty(\mathbb{R} \times \overline{U})$ *are of the form*

$$a^{(l)}(x_1, t, y; s) = \left(\sum_{j=0}^{N} \tau^{-j} a_j^{(l)}\right) \chi\left(\frac{y}{\delta'}\right), \quad b^{(l)}(x_1, t, y; s) = \left(\sum_{j=0}^{N} \tau^{-j} b_j^{(l)}\right) \chi\left(\frac{y}{\delta'}\right),$$

where $a_0^{(l)} = b_0^{(l)}$ *is independent of* x_1 *and the potential* q,

$$a_0^{(l)}(t, y) = a_{00}^{(l)}(t) + \mathcal{O}(|y|), \qquad a_{00}^{(l)}(t) \neq 0, \quad \text{for all } t,$$
$$a_1^{(l)}(x_1, t, y) = a_{10}^{(l)}(x_1, t) + \mathcal{O}(|y|), \quad b_1^{(l)}(x_1, t, y) = b_{10}^{(l)}(x_1, t) + \mathcal{O}(|y|).$$

Here $a_{10}^{(l)}(x_1, t) = e^{f^{(l)}(t)} \tilde{a}_{10}^{(l)}(x_1, t)$ *and* $b_{10}^{(l)}(x_1, t) = e^{f^{(l)}(t)} \tilde{b}_{10}^{(l)}(x_1, t)$, *where* $f^{(l)}$ *is independent of the potential* q, *and further we have that* $\tilde{a}_{10}^{(l)}$ *and* $\tilde{b}_{10}^{(l)}$ *satisfy the equations*

$$(\partial_{x_1} + i\partial_t)\tilde{a}_{10}^{(l)} = \frac{1}{\alpha}\Big(-\frac{1}{2} e^{-f^{(l)}} (\Delta_{\tilde{g}} a_0^{(l)})|_{y=0} + C_0^{(l)} q(x_1, t, 0)\Big),$$
$$(\partial_{x_1} - i\partial_t)\tilde{b}_{10}^{(l)} = \frac{1}{\alpha}\Big(\frac{1}{2} e^{-f^{(l)}} (\Delta_{\tilde{g}} a_0^{(l)})|_{y=0} - C_0^{(l)} q(x_1, t, 0)\Big),$$

where $C_0^{(l)} \neq 0$ *is a constant, independent of the potential* q. *Here* (t, y) *are the Fermi coordinates for* γ *for* t *close to* t_l, $\chi \in C_0^\infty(\mathbb{R}^{n-2})$ *is such that* $0 \leq \chi \leq 1$, $\chi = 1$ *for* $|y| \leq \frac{1}{4}$ *and* $\chi = 0$ *for* $|y| \geq \frac{1}{2}$, *and* $\delta' > 0$ *is a fixed number that can be taken arbitrarily small.*

Remark 2.2. In the special case when the conformal factor c is equal to 1, we have $q = 0$, $g = \tilde{g}$, and

$$e^{\mp \alpha\tau x_1} \circ (-\Delta_g) \circ e^{\pm \alpha\tau x_1} = -\Delta_g \mp 2\alpha\partial_{x_1} - (\alpha\tau)^2.$$

Thus, we can take the Gaussian beams quasimodes in (2-2) to be equal, $v_1 = v_2$, and independent of x_1.

Next we shall construct harmonic functions on (M, g) based on the Gaussian beams quasimodes of Proposition 2.1. To that end, we shall use the approach of [Dos Santos Ferreira et al. 2009], based on Carleman estimates with limiting Carleman weights. The construction is standard, see [Dos Santos Ferreira et al. 2016; Lassas et al. 2021a], and is presented here for the convenience of the reader only.

Assume, as we may, that (M, g) is embedded in a compact smooth manifold (N, g) without boundary of the same dimension. Our starting point is the following Carleman estimates for the Schrödinger operator, which is established in [Dos Santos Ferreira et al. 2009, Lemma 4.3].

Proposition 2.3. *Let* $q \in C^\infty(M)$. *Then given any* $t \in \mathbb{R}$, *we have for all* $h > 0$ *small enough and all* $u \in C_0^\infty(M^{\text{int}})$ *that*

$$h\|u\|_{H^t_{\text{scl}}(N)} \leq C\|e^{\phi/h}(-h^2\Delta + h^2 q)e^{-\phi/h} u\|_{H^t_{\text{scl}}(N)}, \quad C > 0. \tag{2-4}$$

Here $H^t(N)$, $t \in \mathbb{R}$, is the standard Sobolev space, equipped with the natural semiclassical norm

$$\|u\|_{H^t_{\text{scl}}(N)} = \|(1 - h^2\Delta_g)^{t/2} u\|_{L^2(N)}.$$

Using a standard argument, see [Dos Santos Ferreira et al. 2009], we convert the Carleman estimate (2-4) into the following solvability result.

Proposition 2.4. *Let $t \in \mathbb{R}$. If $h > 0$ is small enough, then for any $v \in H^t(M^{\text{int}})$, there is a solution $u \in H^t(M^{\text{int}})$ of the equation*

$$e^{\phi/h}(-h^2\Delta + h^2 q)e^{-\phi/h}u = v \quad \text{in } M^{\text{int}}$$

which satisfies

$$\|u\|_{H^t_{\text{scl}}(M^{\text{int}})} \leq \frac{C}{h}\|v\|_{H^t_{\text{scl}}(M^{\text{int}})}.$$

Here

$$H^t(M^{\text{int}}) = \{V|_{M^{\text{int}}} : V \in H^t(N)\}, \quad t \in \mathbb{R},$$

with the norm

$$\|v\|_{H^t_{\text{scl}}(M^0)} = \inf_{V \in H^t_{\text{scl}}(N), v = V|_{M^{\text{int}}}} \|V\|_{H^t_{\text{scl}}(N)}.$$

Let $\alpha > 0$, and let

$$\tau = s + i\lambda \quad \text{with } 1 \leq s = \frac{1}{h}, \quad \lambda \in \mathbb{R}, \quad \lambda \text{ fixed.}$$

In view of (2-1), to construct suitable harmonic functions on (M, g), we shall find complex geometric optics solution to the equation

$$(-\Delta_{\tilde{g}} + q)\tilde{u} = 0 \quad \text{in } M^{\text{int}} \tag{2-5}$$

having the form

$$\tilde{u}_1 = e^{\alpha\tau x_1}(v_1 + r_1) \quad \text{and} \quad \tilde{u}_2 = e^{-\alpha\tau x_1}(v_2 + r_2),$$

where v_1 and v_2 are the Gaussian beam quasimodes given in Proposition 2.1, and r_1 and r_2 are the remainder terms. Thus, $\tilde{u}_1$ is a solution of (2-5) provided that

$$e^{-\alpha x_1/h}(-h^2\Delta_{\tilde{g}} + h^2 q)e^{\alpha x_1/h}(e^{i\alpha\lambda x_1}r_1) = -e^{i\alpha\lambda x_1}e^{-\alpha\tau x_1}(-h^2\Delta_{\tilde{g}} + h^2 q)e^{\alpha\tau x_1}v_1. \tag{2-6}$$

For any $k \in \mathbb{N}$ and $R \geq 1$ arbitrarily large, Propositions 2.4 and 2.1 imply that there is $r_1 \in H^k(M^{\text{int}})$ such that

$$\|r_1\|_{H^k_{\text{scl}}(M^{\text{int}})} \leq \mathcal{O}(h^{-1})\|e^{-\alpha\tau x_1}(-h^2\Delta_{\tilde{g}} + h^2 q)e^{\alpha\tau x_1}v_1\|_{H^k_{\text{scl}}(M^{\text{int}})} = \mathcal{O}(h^{R-1}),$$

and therefore, for any K, there is R large enough so that

$$\|r_1\|_{H^k(M^{\text{int}})} \leq h^{-k}\|r_1\|_{H^k_{\text{scl}}(M^{\text{int}})} = \mathcal{O}(h^K).$$

Similarly, one can construct r_2. This together with (2-1) gives the following result concerning the construction of harmonic functions on (M, g) based on Gaussian beams quasimodes.

Proposition 2.5. *Let $\alpha > 0$, and let $\tau = s + i\lambda$, $s = 1/h$, with $\lambda \in \mathbb{R}$ being fixed. For all k, K, and $h > 0$ small enough, there are $u_1, u_2 \in H^k(M^{\text{int}})$ solutions of $-\Delta_g u_j = 0$ in M^{int} having the form*

$$u_1 = e^{\alpha\tau x_1}c^{-(n-2)/4}(v_1 + r_1) \quad \text{and} \quad u_2 = e^{-\alpha\tau x_1}c^{-(n-2)/4}(v_2 + r_2),$$

where $v_1 = v_1(\,\cdot\,; s)$, $v_2 = v_2(\,\cdot\,; s) \in C^\infty(\mathbb{R} \times M_0)$ are the Gaussian beam quasimodes from Proposition 2.1, and $r_1, r_2 \in H^k(M^{\text{int}})$ are such that $\|r_j\|_{H^k(M^{\text{int}})} = \mathcal{O}(h^K)$ as $h \to 0$.

Remark 2.6. Taking $k > \frac{1}{2}n + 3$ and using the Sobolev embedding $H^k(M^{\text{int}}) \subset C^3(M)$, we see that $u_j \in C^3(M)$ with

$$\|r_j\|_{C^3(M)} = \mathcal{O}(h^K),$$

as $h \to 0,\ j = 1, 2$.

3. Proof of Proposition 1.6 in a simplified setting

The proof of Proposition 1.6 will follow along the lines of the proof of [Lassas et al. 2021a, Proposition 5.1]. Before we prove Proposition 1.6 in the general case, let us explain the main ideas in a simplified setting.

Let us assume that each point $p \in M_0^{\text{int}}$ is the unique intersection point of two distinct nontangential non-self-intersecting geodesics γ and η. Assume furthermore that the conformal factor c equals 1. As we shall see below, in this simplified setting the continuity of A suffices, and therefore to extend A by 0 to the continuous form on $\mathbb{R} \times M_0 \setminus M$, we only need to show $A|_{\partial M} = 0$. This follows by taking $u_2 = u_3 = 1$ in (1-8) and applying Proposition C.3.

In view of Proposition C.5, we see that (1-8) also holds for all harmonic functions $u_j \in C^{2,\alpha}(M)$, $0 < \alpha < 1,\ j = 1, \dots, 4$.

Let $s = 1/h$, and let $\lambda \in \mathbb{R}$ be fixed. Our choice of the harmonic functions below will be similar to [Lassas et al. 2021a]. Specifically, using Proposition 2.5 and Remark 2.6, we see that there exist harmonic functions $u_j \in C^3(M),\ j = 1, \dots, 4$, on (M, g) of the form

$$\begin{aligned} u_1 &= e^{-(s+i\lambda)x_1}(v + r_1), \quad & u_2 &= \overline{e^{(s+i\lambda)x_1}(v + r_2)}, \\ u_3 &= e^{-sx_1}(w + r_3), & u_4 &= \overline{e^{sx_1}(w + r_4)}, \end{aligned} \tag{3-1}$$

where

$$\|r_j\|_{C^1(M)} = \mathcal{O}(s^{-K}), \tag{3-2}$$

as $s \to \infty,\ K \gg 1$, and $v = v(\,\cdot\,; s)$, $w = w(\,\cdot\,; s) \in C^\infty(M_0)$ are Gaussian beams quasimodes concentrating near the geodesics η and γ, respectively, constructed in Proposition 2.1; see also Remark 2.2. We have

$$v(x'; s) = s^{(n-2)/8} e^{i(s+i\lambda)\varphi(x')} a(x'; s) \quad \text{and} \quad w(x'; s) = s^{(n-2)/8} e^{is\psi(x')} b(x'; s), \tag{3-3}$$

where

$$\begin{aligned} &\varphi(\eta(t)) = t, \quad \nabla\varphi(\eta(t)) = \dot\eta(t), \quad \operatorname{Im}(\nabla^2\varphi(\eta(t))) \geq 0, \quad \operatorname{Im}(\nabla^2\varphi)|_{\dot\eta(t)^\perp} > 0, \\ &\psi(\gamma(\tau)) = \tau, \quad \nabla\psi(\gamma(\tau)) = \dot\gamma(\tau), \quad \operatorname{Im}(\nabla^2\psi(\gamma(\tau))) \geq 0, \quad \operatorname{Im}(\nabla^2\psi)|_{\dot\gamma(\tau)^\perp} > 0, \end{aligned} \tag{3-4}$$

and

$$a(t, y; s) = \Big(\sum_{j=0}^{N} \tau^{-j} a_j\Big)\chi\Big(\frac{y}{\delta'}\Big), \quad b(\tau, z; s) = \Big(\sum_{j=0}^{N} \tau^{-j} b_j\Big)\chi\Big(\frac{z}{\delta'}\Big), \tag{3-5}$$

where

$$\begin{aligned} a_0(t, y) &= a_{00}^{(l)}(t) + \mathcal{O}(|y|), \quad a_{00}(t) \neq 0, \quad \text{for all } t, \\ b_0(\tau, z) &= a_{00}(\tau) + \mathcal{O}(|z|), \quad b_{00}(\tau) \neq 0, \quad \text{for all } \tau. \end{aligned} \tag{3-6}$$

Here (t, y) and (τ, z) are the Fermi coordinates for the geodesics η and γ, $\chi \in C_0^\infty(\mathbb{R}^{n-2})$ is such that $0 \le \chi \le 1$, $\chi = 1$ for $|y| \le \frac{1}{4}$ and $\chi = 0$ for $|y| \ge \frac{1}{2}$, and $\delta' > 0$ is a fixed number that can be taken arbitrarily small. We also have

$$\|v\|_{L^4(M_0)} = \|w\|_{L^4(M_0)} = \mathcal{O}(1), \quad \|v\|_{L^\infty(M_0)} = \|w\|_{L^\infty(M_0)} = \mathcal{O}(s^{(n-2)/8}), \tag{3-7}$$

as $s \to \infty$. Similarly, we find that

$$\begin{gathered} \|s^{(n-2)/8} e^{i(s+i\lambda)\varphi} \nabla a\|_{L^4(M_0)} = \|s^{(n-2)/8} e^{is\psi} \nabla b\|_{L^4(M_0)} = \mathcal{O}(1), \\ \|\nabla v\|_{L^4(M_0)} = \mathcal{O}(s), \qquad \|\nabla w\|_{L^4(M_0)} = \mathcal{O}(s), \\ \|\nabla v\|_{L^\infty(M_0)} = \mathcal{O}(s^{(n+6)/8}), \quad \|\nabla w\|_{L^\infty(M_0)} = \mathcal{O}(s^{(n+6)/8}), \end{gathered} \tag{3-8}$$

as $s \to \infty$.

Now it follows from (3-1) that

$$(u_1 u_2 u_3)(x) = e^{-2i\lambda x_1 - s x_1}(|v(x')|^2 w(x') + R(x)),$$

where

$$R = |v|^2 r_3 + (w + r_3)(v\bar{r}_2 + \bar{v} r_1 + r_1 \bar{r}_2).$$

Using (3-2), (3-7), and (3-8), we see that

$$\|R\|_{C^1(M)} = \mathcal{O}(s^{-L}), \tag{3-9}$$

where L is large depending on K. Hence, we have

$$\partial_{x_1}(u_1 u_2 u_3) = e^{-2i\lambda x_1 - s x_1}[(-2i\lambda - s)(|v|^2 w + R) + \partial_{x_1} R],$$

and therefore, using (3-9), (3-2), and (3-7), we get

$$\partial_{x_1}(u_1 u_2 u_3) u_4 = -s e^{-2i\lambda x_1} |v|^2 |w|^2 + \mathcal{O}_{L^1(M)}(1), \tag{3-10}$$

as $s \to \infty$. We also get

$$\partial_{x_k}(u_1 u_2 u_3) = e^{-2i\lambda x_1 - s x_1}(\partial_{x_k}(|v|^2 w) + \partial_{x_k}(R))$$

for $k = 2, \ldots, n$, and therefore, (3-9), (3-2), (3-7), and (3-8) yield

$$\partial_{x_k}(u_1 u_2 u_3) u_4 = e^{-2i\lambda x_1} \partial_{x_k}(|v|^2 w)\overline{w} + \mathcal{O}_{L^1(M)}(1), \tag{3-11}$$

as $s \to \infty$. Writing $A = (A_1, A')$ and using (3-10) and (3-11), we conclude that

$$\langle A, d(u_1 u_2 u_3)\rangle_g u_4 = e^{-2i\lambda x_1}(-s A_1 |v|^2 |w|^2 + \langle A', d_{x'}(|v|^2 w)\overline{w}\rangle_{g_0}) + \mathcal{O}_{L^1(M)}(1), \tag{3-12}$$

as $s \to \infty$. It follows from (1-8) with the help of (3-12) that

$$\int_M e^{-2i\lambda x_1}(-s A_1 |v|^2 |w|^2 + \langle A', d_{x'}(|v|^2 w)\overline{w}\rangle_{g_0})\, dV_g = \mathcal{O}(1), \tag{3-13}$$

as $s \to \infty$.

Extending A by zero to $\mathbb{R}\times M_0\setminus M$, and denoting the extension again by A, we now see that $A\in C(\mathbb{R}\times M_0, T^*(\mathbb{R}\times M_0))$ as $A|_{\partial M}=0$. Denoting the partial Fourier transform of A in the x_1 variable by $\hat{A}(\lambda, x')$, we get from (3-13) that

$$\int_{M_0}(-s\hat{A}_1(2\lambda,\cdot)|v|^2|w|^2+\langle\hat{A}'(2\lambda,\cdot), d_{x'}(|v|^2w)\overline{w}\rangle_{g_0})\,dV_{g_0}=\mathcal{O}(1), \tag{3-14}$$

as $s\to\infty$. Since v and w can be chosen to be supported in arbitrarily small but fixed neighborhoods of η and γ, respectively, and since η and γ only intersect at p, the products $|v|^2|w|^2$ and $d_{x'}(|v|^2w)\overline{w}$ concentrate in a small neighborhood U of p. Using (3-3) and (3-5), we see that in U,

$$\begin{aligned}|v|^2|w|^2&=s^{(n-2)/2}e^{-2s(\operatorname{Im}\varphi+\operatorname{Im}\psi)}e^{-2\lambda\operatorname{Re}\varphi}(|a_0|^2|b_0|^2+\mathcal{O}_{L^\infty(M_0)}(1/s))\\&=s^{(n-2)/2}e^{-2s(\operatorname{Im}\varphi+\operatorname{Im}\psi)}e^{-2\lambda\operatorname{Re}\varphi}|a_0|^2|b_0|^2+\mathcal{O}_{L^1(M_0)}(1/s),\end{aligned} \tag{3-15}$$

and

$$\begin{aligned}d_{x'}(|v|^2w)\overline{w}&=s^{(n-2)/2}e^{-2s(\operatorname{Im}\varphi+\operatorname{Im}\psi)}e^{-2\lambda\operatorname{Re}\varphi}\big[(is(2i\,d\operatorname{Im}\varphi+d\psi)(|a_0|^2|b_0|^2+\mathcal{O}_{L^\infty(M_0)}(1/s))\\&\qquad-2\lambda(d\operatorname{Re}\varphi)|a|^2|b|^2+d_{x'}(|a|^2b)\bar{b}\big]\\&=s^{(n-2)/2}e^{-2s(\operatorname{Im}\varphi+\operatorname{Im}\psi)}e^{-2\lambda\operatorname{Re}\varphi}is(2i\,d\operatorname{Im}\varphi+d\psi)|a_0|^2|b_0|^2+\mathcal{O}_{L^1(M_0)}(1),\end{aligned} \tag{3-16}$$

as $s\to\infty$. Substituting (3-15) and (3-16) into (3-14) and dividing by $s^{1/2}$, we obtain

$$s^{(n-1)/2}\int_U(-\hat{A}_1(2\lambda,\cdot)+i\langle\hat{A}'(2\lambda,\cdot), 2i\,d\operatorname{Im}\varphi+d\psi\rangle_{g_0})e^{-2\lambda\operatorname{Re}\varphi}|a_0|^2|b_0|^2e^{-s\Psi}\,dV_{g_0}=\mathcal{O}(s^{-1/2}), \tag{3-17}$$

as $s\to\infty$, where

$$\Psi=2(\operatorname{Im}\varphi+\operatorname{Im}\psi).$$

It follows from (3-4) that

$$\Psi(p)=0,\quad d\Psi(p)=0,\quad \nabla^2\Psi(p)>0,$$

where the later inequality is a consequence of the fact that the Hessians of $\operatorname{Im}\varphi$ and $\operatorname{Im}\psi$ at p are positive definite in the directions orthogonal to η and γ, respectively.

Let us now denote by $z=(z_1,\dots,z_{n-1})$ the geodesic normal coordinates in (M_0, g_0) with the origin at p. Then

$$g_0(z)=1+\mathcal{O}(|z|^2), \tag{3-18}$$

see [Petersen 2006, Chapter 2, Section 8, p. 56], and $dV_{g_0}=|g_0(z)|^{1/2}\,dz$. Passing to the limit as $s\to\infty$ in (3-17) and using the rough version of the stationary phase Lemma A.1, as well as (3-18), we obtain

$$(-\hat{A}_1(2\lambda,\,p)+i\hat{A}'(2\lambda,\,p)(\dot{\gamma}(t_0)))e^{-2\lambda\operatorname{Re}\varphi(p)}|a_{00}(p)|^2|b_{00}(p)|^2=0,$$

where $p=\gamma(t_0)$, for all $\lambda\in\mathbb{R}$. As $a_{00}(p)\neq 0,\ b_{00}(p)\neq 0$, and λ is arbitrary, we see that

$$-A_1(x_1,\,p)+iA'(x_1,\,p)(\dot{\gamma}(t_0))=0,$$

which is equivalent to

$$(iA_1,\,A')(x_1,\,p)(1,\,\dot{\gamma}(t_0))=0. \tag{3-19}$$

Here we may replace $\dot\gamma(t_0)$ by $-\dot\gamma(t_0)$. Thus, (3-19) gives that $A_1(x_1, p) = 0$, and since the point (x_1, p) is an arbitrary point in $\mathbb{R} \times M_0$, we get $A_1 \equiv 0$. Hence, we only need to show that the 1-form $A'(x_1, \cdot)$ vanishes on M_0, knowing that

$$A'(x_1, p)(\dot\gamma(t_0)) = 0. \tag{3-20}$$

To that end, we assume without loss of generality that $v_1 = \dot\gamma(t_0) = (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$, and consider the small perturbations of v_1 given by

$$v_2 = \frac{1}{\sqrt{1+\varepsilon^2}}(1, \varepsilon, 0, \dots, 0), \quad \dots, \quad v_{n-1} = \frac{1}{\sqrt{1+\varepsilon^2}}(1, 0, \dots, 0, \varepsilon), \tag{3-21}$$

for $\varepsilon > 0$ small. The unit vectors $v_1, \dots, v_{n-1}$ are linearly independent, and thus, they span the tangent space T_pM_0. By Proposition D.2, for $\varepsilon > 0$ sufficiently small, the unit speed geodesic γ_{p,v_j} through (p, v_j), $j = 2, \dots, n-1$, is nontangential between boundary points, does not have self-intersections, and intersects η at the point p only. Applying the discussion above with $\gamma = \gamma_{p,v_j}$, we obtain that $A'(x_1, p)(v_j) = 0$, $j = 2, \dots, n-1$. This together with (3-20) gives that $A'(x_1, p) = 0$. The proof of Proposition 1.6 in the simplified case is complete.

4. Proof of Proposition 1.6 in the general setting

In the case of a general transversal manifold M_0, the nontangential geodesics γ and η might have self-intersections and may intersect more than in one point, which complicates the proof. To proceed we shall follow [Lassas et al. 2021a] and introduce additional parameters in the construction of harmonic functions. Furthermore, we shall implement the presence of the conformal factor c which is assumed to be equal to 1 in [Lassas et al. 2021a].

Let us proceed to discuss the choice of two nontangential geodesics to be used when constructing Gaussian beams quasimodes. When doing so let us first observe that arguing as in the proof of Theorem 1.2 of [Salo 2017], we may assume that (M_0, g_0) has a strictly convex boundary. An application of [Salo 2017, Lemma 3.1] gives therefore that there exists a null set E in (M_0, g_0) such that all points in $M_0 \setminus E$ lie on some nontangential geodesic joining boundary points. Fix a point $y_0 \in M_0^{\text{int}} \setminus E$ and let $\gamma : [-S_1, S_2] \to M_0$, $0 < S_1, S_2 < \infty$, be a unit speed nontangential geodesic such that $\gamma(0) = y_0$. Then by Proposition D.1, moving the point y_0 along γ a little and reparametrizing the geodesic, if necessary, there exists a small neighborhood $W \subset S_{y_0}M_0$ of $w_0 = \dot\gamma(0)$ such that for every $w \in W,\ w \neq w_0$, the unit speed geodesic $\eta : [-T_1, T_2] \to M_0,\ 0 < T_1, T_2 < \infty$, such that $\eta(0) = y_0$ and $\dot\eta(0) = w$ is also nontangential, and γ and η do not intersect each other at the boundary of M_0. Notice that γ and η are distinct and are not reverses of each other. As we shall see below, the fact that γ and η do not intersect each other at the boundary of M_0 allows us to avoid the use of stationary and nonstationary phase on the boundary of M_0.

By [Lassas et al. 2021a], we know that γ and η can intersect only finitely many times. Let us denote by $p_1, \dots, p_N \in M_0^{\text{int}}$ the distinct intersection points of γ and η. For each $r,\ r = 1, \dots, N$, let $t_1^{(r)} < \cdots < t_{P_r}^{(r)}$ be the times in $[-T_1, T_2]$ when $\eta(t_j^r) = p_r$, and let $\tau_1^{(r)} < \cdots < \tau_{Q_r}^{(r)}$ be the times in $[-S_1, S_2]$ when $\gamma(\tau_j^{(r)}) = p_r$. Let U_r be a small neighborhood of $p_r,\ r = 1, \dots, N$.

Choosing harmonic functions. First it follows from Proposition C.5 that (1-8) continues to hold for all harmonic functions $u_j \in C^{2,\alpha}(M),\ 0<\alpha<1,\ j=1,\dots,4$.

Let $s \geq 1$ and let $L>0,\ \lambda, \mu \in \mathbb{R}$ be fixed. By Proposition 2.5 and Remark 2.6, there are harmonic functions $u_j \in C^3(M)$ of the form

$$\begin{aligned} u_1 &= e^{(s+i\mu)x_1} c^{-(n-2)/4}(v_1+r_1), \qquad u_2 = \overline{e^{-(s+i\mu)x_1} c^{-(n-2)/4}(v_2+r_2)}, \\ u_3 &= e^{-L(s+i\lambda)x_1} c^{-(n-2)/4}(w_1+r_3), \quad u_4 = \overline{e^{L(s+i\lambda)x_1} c^{-(n-2)/4}(w_2+r_4)}, \end{aligned} \tag{4-1}$$

where

$$\|r_j\|_{C^1(M)} = \mathcal{O}(s^{-K}), \tag{4-2}$$

as $s \to \infty,\ K \gg 1$, and $v_j \in C^\infty(\mathbb{R}\times M_0),\ j=1,2$, and $w_j \in C^\infty(\mathbb{R}\times M_0),\ j=1,2$, are the Gaussian beam quasimodes constructed in Proposition 2.1 and associated to the nontangential geodesics η and γ, respectively, such that

$$\operatorname{supp}(v_j(\,\cdot\,;s)) \subset \mathbb{R}\times \text{small neigh}(\eta) \quad \text{and} \quad \operatorname{supp}(w_j(\,\cdot\,;s)) \subset \mathbb{R}\times \text{small neigh}(\gamma). \tag{4-3}$$

Notice that here we follow [Lassas et al. 2021a], and the minor differences are as follows: in order to incorporate the presence of the conformal factor our Gaussian beams quasimodes are constructed on all of $\mathbb{R}\times M_0$ rather than on M_0 as in that work, and the parameters μ and λ are real.

Let us now recall a local description of the quasimodes v_j and w_j near the intersection points p_r of γ and η. In doing so, let us fix p to be one of the intersection points p_r and let us set $U=U_r$. In the open set U, the quasimodes v_j are of the form

$$v_j|_U = \sum_{k=1}^{P} v_j^{(k)}, \quad j=1,2, \tag{4-4}$$

where $t_1<\cdots<t_P$ are the times in $[-T_1,T_2]$ when $\eta(t_k)=p$. Each $v_1^{(k)}$ and $v_2^{(k)}$ in (4-4) has the form

$$v_1^{(k)} = s^{(n-2)/8} e^{i(s+i\mu)\varphi^{(k)}} a^{(k)}, \quad v_2^{(k)} = s^{(n-2)/8} e^{i(s+i\mu)\varphi^{(k)}} b^{(k)}, \quad k=1,\dots,P, \tag{4-5}$$

where $\varphi=\varphi^{(k)} \in C^\infty(\overline{U};\mathbb{C})$ satisfies, for t close to t_k,

$$\varphi(\eta(t))=t, \quad \nabla\varphi(\eta(t))=\dot\eta(t), \quad \operatorname{Im}(\nabla^2\varphi(\eta(t))) \geq 0, \quad \operatorname{Im}(\nabla^2\varphi)|_{\dot\eta(t)^\perp} > 0, \tag{4-6}$$

and each $a^{(k)}, b^{(k)} \in C^\infty(\mathbb{R}\times\overline{U})$ is of the form

$$a^{(k)}(x_1,t,y;s) = \Big(\sum_{j=0}^{N} \tau^{-j} a_j^{(k)}\Big)\chi\Big(\frac{y}{\delta'}\Big), \quad b^{(k)}(x_1,t,y;s) = \Big(\sum_{j=0}^{N} \tau^{-j} b_j^{(k)}\Big)\chi\Big(\frac{y}{\delta'}\Big), \tag{4-7}$$

where $a_0^{(k)} = b_0^{(k)}$ is independent of x_1 and

$$a_0^{(k)}(t,y) = a_{00}^{(k)}(t) + \mathcal{O}(|y|), \quad a_{00}^{(k)}(t) \neq 0, \quad \text{for all } t. \tag{4-8}$$

Here (t,y) are the Fermi coordinates for η for t close to t_k, $\chi \in C_0^\infty(\mathbb{R}^{n-2})$ is such that $0\leq\chi\leq 1,\ \chi=1$ for $|y|\leq\frac14$ and $\chi=0$ for $|y|\geq\frac12$, and $\delta'>0$ is a fixed number that can be taken arbitrarily small.

Furthermore, in U, the quasimodes w_1 and w_2 are finite sums

$$w_j|_U = \sum_{k=1}^{Q} w_j^{(k)}, \quad j = 1, 2, \tag{4-9}$$

where $\tau_1 < \cdots < \tau_Q$ are the times in $[-S_1, S_2]$ when $\gamma(\tau_k) = p$. Each $w_1^{(k)}$ and $w_2^{(k)}$ in (4-9) has the form

$$w_1^{(k)} = s^{(n-2)/8} e^{Li(s+i\lambda)\psi^{(k)}} c^{(k)}, \quad w_2^{(k)} = s^{(n-2)/8} e^{Li(s+i\lambda)\psi^{(k)}} d^{(k)}, \quad k = 1, \ldots, Q, \tag{4-10}$$

where each $\psi = \psi^{(k)} \in C^\infty(\overline{U}; \mathbb{C})$ satisfies, for τ close to τ_k,

$$\psi(\gamma(\tau)) = \tau, \quad \nabla\psi(\gamma(\tau)) = \dot\gamma(\tau), \quad \operatorname{Im}(\nabla^2\psi(\gamma(\tau))) \geq 0, \quad \operatorname{Im}(\nabla^2\psi)|_{\dot\gamma(\tau)^\perp} > 0, \tag{4-11}$$

and each $c^{(k)}, d^{(k)} \in C^\infty(\mathbb{R} \times \overline{U})$ is of the form

$$c^{(k)}(x_1, \tau, z; s) = \left(\sum_{j=0}^{N} \tau^{-j} c_j^{(k)}\right) \chi\left(\frac{z}{\delta'}\right), \quad d^{(k)}(x_1, \tau, z; s) = \left(\sum_{j=0}^{N} \tau^{-j} d_j^{(k)}\right) \chi\left(\frac{z}{\delta'}\right), \tag{4-12}$$

where $c_0^{(k)} = d_0^{(k)}$ is independent of x_1 and

$$c_0^{(k)}(\tau, z) = c_{00}^{(k)}(\tau) + \mathcal{O}(|z|), \quad c_{00}^{(k)}(\tau) \neq 0, \quad \text{for all } \tau. \tag{4-13}$$

Here (τ, z) are the Fermi coordinates for γ for t close to t_k.

We also have

$$\begin{aligned} \|v_j\|_{L^4(M)} &= \mathcal{O}(1), & \|\nabla v_j\|_{L^4(M)} &= \mathcal{O}(s), \\ \|w_j\|_{L^4(M)} &= \mathcal{O}(1), & \|\nabla w_j\|_{L^4(M)} &= \mathcal{O}(s), \\ \|v_j\|_{L^\infty(M)} &= \mathcal{O}(s^{(n-2)/8}), & \|\nabla v_j\|_{L^\infty(M)} &= \mathcal{O}(s^{(n+6)/8}), \\ \|w_j\|_{L^\infty(M)} &= \mathcal{O}(s^{(n-2)/8}), & \|\nabla w_j\|_{L^\infty(M)} &= \mathcal{O}(s^{(n+6)/8}), \end{aligned} \tag{4-14}$$

as $s \to \infty$, $j = 1, 2$.

Now it follows from (4-1) that

$$u_1 u_2 u_3 = e^{(-Ls+2i\mu-Li\lambda)x_1} c^{-3(n-2)/4} (v_1 \bar v_2 w_1 + \tilde R), \tag{4-15}$$

where

$$\tilde R = r_3 v_1 \bar v_2 + (w_1 + r_3)(v_1 \bar r_2 + \bar v_2 r_1 + r_1 \bar r_2).$$

Using (4-2) and (4-14), we get

$$\|\tilde R\|_{C^1(M)} = \mathcal{O}(s^{-L}), \tag{4-16}$$

where L is large. Hence, we have

$$\begin{aligned} \partial_{x_1}(u_1 u_2 u_3) = e^{(-Ls+2i\mu-Li\lambda)x_1} \big[&(-Ls + 2i\mu - Li\lambda) c^{-3(n-2)/4} (v_1 \bar v_2 w_1 + \tilde R) \\ &+ \partial_{x_1}(c^{-3(n-2)/4})(v_1 \bar v_2 w_1 + \tilde R) + c^{-3(n-2)/4}(\partial_{x_1}(v_1 \bar v_2 w_1) + \partial_{x_1}\tilde R)\big], \end{aligned}$$

and therefore, in view of (4-16), (4-2), and (4-14), we get

$$\partial_{x_1}(u_1 u_2 u_3) u_4 = e^{2i(\mu - L\lambda)x_1} c^{-(n-2)} [-Ls v_1 \bar v_2 w_1 \overline{w}_2 + \partial_{x_1}(v_1 \bar v_2 w_1)\overline{w}_2] + \mathcal{O}_{L^1(M)}(1), \tag{4-17}$$

as $s \to \infty$. We also have from (4-15) that

$$\partial_{x_k}(u_1u_2u_3) = e^{(-Ls+2i\mu-Li\lambda)x_1}[c^{-3(n-2)/4}(\partial_{x_k}(v_1\bar{v}_2w_1) + \partial_{x_k}\tilde{R}) + \partial_{x_k}(c^{-3(n-2)/4})(v_1\bar{v}_2w_1 + \tilde{R})],$$

for $k = 2, \dots, n$, and therefore, in view of (4-16), (4-2), and (4-14), we get

$$\partial_{x_k}(u_1u_2u_3)u_4 = e^{2i(\mu-L\lambda)x_1}c^{-(n-2)}\partial_{x_k}(v_1\bar{v}_2w_1)\overline{w}_2 + \mathcal{O}_{L^1(M)}(1), \tag{4-18}$$

as $s \to \infty$.

For future reference, we also note that

$$u_1u_2u_3u_4 = e^{2i(\mu-L\lambda)x_1}c^{-(n-2)}(v_1\bar{v}_2w_1\overline{w}_2 + \tilde{R}w_2 + (v_1\bar{v}_2w_1 + \tilde{R})\bar{r}_2) = \mathcal{O}_{L^1(M)}(1), \tag{4-19}$$

as $s \to \infty$.

Using (4-17) and (4-18), we obtain

$$\begin{aligned}\langle A, d(u_1u_2u_3)\rangle_g u_4 = e^{2i(\mu-L\lambda)x_1}c^{1-n}\big(&A_1(-Lsv_1\bar{v}_2w_1\overline{w}_2 + \partial_{x_1}(v_1\bar{v}_2w_1)\overline{w}_2) \\ &+ \langle A', d_{x'}(v_1\bar{v}_2w_1)\rangle_{g_0}\overline{w}_2\big) + \mathcal{O}_{L^1(M)}(1),\end{aligned} \tag{4-20}$$

as $s \to \infty$.

It follows from (1-8) in view of (4-20) that

$$\int_M (A_1(-Lsv_1\bar{v}_2w_1\overline{w}_2 + \partial_{x_1}(v_1\bar{v}_2w_1)\overline{w}_2) + \langle A', d_{x'}(v_1\bar{v}_2w_1)\rangle_{g_0}\overline{w}_2)e^{2i(\mu-L\lambda)x_1}c^{1-n}\,dV_g = \mathcal{O}(1), \tag{4-21}$$

as $s \to 0$.

Now taking $u_2 = u_3 = 1$ in (1-8) and applying Proposition C.3, we obtain that $A|_{\partial M} = 0$ and $\partial_\nu A|_{\partial M} = 0$. Let us extend A by zero to $(\mathbb{R} \times M_0) \setminus M$ and denote this extension by A again. Since $A \in C^{1,1}(M, T^*M)$ and $A|_{\partial M} = 0$, $\partial_\nu A|_{\partial M} = 0$, we see that $A \in C^{1,1}(\mathbb{R} \times M_0, T^*(\mathbb{R} \times M_0))$. Now (4-21) implies that

$$\begin{aligned}\int_{\mathbb{R}\times M_0} (A_1(-Lsv_1\bar{v}_2w_1\overline{w}_2 + \partial_{x_1}(v_1\bar{v}_2w_1)\overline{w}_2) + \langle A', d_{x'}(v_1\bar{v}_2w_1)\rangle_{g_0}\overline{w}_2) \\ \times e^{2i(\mu-L\lambda)x_1}c^{1-n}\,dV_g = \mathcal{O}(1),\end{aligned} \tag{4-22}$$

as $s \to 0$. In view of (4-3), (4-22) gives

$$\begin{aligned}\sum_{r=1}^{N}\int_{\mathbb{R}\times U_r} (A_1(-Lsv_1\bar{v}_2w_1\overline{w}_2 + \partial_{x_1}(v_1\bar{v}_2w_1)\overline{w}_2) + \langle A', d_{x'}(v_1\bar{v}_2w_1)\rangle_{g_0}\overline{w}_2) \\ \times e^{2i(\mu-L\lambda)x_1}c^{1-n}\,dV_g = \mathcal{O}(1),\end{aligned} \tag{4-23}$$

as $s \to 0$, where the U_r are sufficiently small neighborhoods of the points p_r of the intersections of γ and η. Using (4-4), (4-5), (4-7), (4-10), and (4-12), we obtain that in U_r,

$$v_1\bar{v}_2w_1\overline{w}_2 = s^{(n-2)/2}\sum_{1\le k,l\le P_r}\sum_{1\le m,j\le Q_r} e^{is\Psi^r_{klmj}}e^{\Phi^r_{klmj}}a_0^{(k),r}\overline{a_0^{(l),r}}c_0^{(m),r}\overline{c_0^{(j),r}} + \mathcal{O}_{L^1(I\times M_0)}(1/s), \tag{4-24}$$

where

$$\Psi^r_{klmj} = \varphi^{(k),r} - \overline{\varphi^{(l),r}} + L\psi^{(m),r} - L\overline{\psi^{(j),r}}, \tag{4-25}$$

$$\Phi^r_{klmj} = -\mu\varphi^{(k),r} - \mu\overline{\varphi^{(l),r}} - L\lambda\psi^{(m),r} - L\lambda\overline{\psi^{(j),r}}, \tag{4-26}$$

and $I \subset \mathbb{R}$ is a bounded interval. Recall that all $a_0^{(k),r}$ and $c^{(m),r}$ are independent of x_1. This fact also implies that

$$\partial_{x_1}(v_1\bar{v}_2 w_1)\overline{w}_2 = \mathcal{O}_{L^1(I\times M_0)}(1/s). \tag{4-27}$$

Using (4-4), (4-5), (4-7), (4-10), and (4-12), we also get that in U_r,

$$d_{x'}(v_1\bar{v}_2 w_1)\overline{w}_2 = s^{(n-2)/2}\sum_{1\le k,l\le P_r}\sum_{1\le m,j\le Q_r} is(d\varphi^{(k),r} - d\overline{\varphi^{(l),r}} + L\,d\psi^{(m),r}) \times e^{is\Psi^r_{klmj}} e^{\Phi^r_{klmj}} a_0^{(k),r}\overline{a_0^{(l),r}} c_0^{(m),r}\overline{c_0^{(j),r}} + \mathcal{O}_{L^1(I\times M_0)}(1). \tag{4-28}$$

Substituting (4-24), (4-27), and (4-28) into (4-23), using that $dV_g = c^{n/2}\,dx_1\,dV_{g_0}$, and dividing (4-23) by $s^{1/2}$, we obtain

$$s^{(n-1)/2}\sum_{r=1}^{N}\sum_{1\le k,l\le P_r}\sum_{1\le m,j\le Q_r}\int_{U_r} B^r_{klmj} e^{is\Psi^r_{klmj}}\,dV_{g_0} = \mathcal{O}(s^{-1/2}), \tag{4-29}$$

where

$$B^r_{klmj} = [-L\widehat{A_1 c^{1-n/2}}(2(\mu - L\lambda),\cdot) + i\langle\widehat{A' c^{1-n/2}}(2(\mu-L\lambda),\cdot)\,d\varphi^{(k),r} - d\overline{\varphi^{(l),r}} + L\,d\psi^{(m),r}\rangle_{g_0}] \times e^{\Phi^r_{klmj}} a_0^{(k),r}\overline{a_0^{(l),r}} c_0^{(m),r}\overline{c_0^{(j),r}}. \tag{4-30}$$

Notice that the occurrence of the factor $s^{(n-1)/2}$ is natural here, in view of a subsequent application of the stationary phase method, in its rough version, to the integral in the left-hand side of (4-29).

Choosing L. The argument below follows [Lassas et al. 2021a] closely and is presented here for completeness and the convenience of the reader only. We claim that $L > 0$ can be chosen sufficiently large but fixed so that $d\Psi^r_{klmj}(p_r) = 0$ for all points $p_r,\ 1 \le r \le N$, if and only if $k = l$ and $m = j$. Indeed, it follows from (4-25) that

$$\begin{aligned}\nabla\Psi^r_{klmij}(p_r) &= (\nabla\varphi^{(k),r} - \nabla\overline{\varphi^{(l),r}} + L\nabla\psi^{(m),r} - L\nabla\overline{\psi^{(j),r}})(p_r)\\ &= \dot\eta(t_k^r) - \dot\eta(t_l^r) + L\dot\gamma(\tau_m^r) - L\dot\gamma(\tau_j^r).\end{aligned} \tag{4-31}$$

If $k = l$ and $m = j$, (4-31) implies that $\nabla\Psi^r_{klmij}(p_r) = 0$ for all $1 \le r \le N$. Now since the geodesic γ is nontangential, and therefore not closed, we have $\dot\gamma(\tau_m^r) - \dot\gamma(\tau_j^r) \ne 0$, for all $m \ne j$, for all $r,\ 1 \le r \le N$. Let

$$\alpha = \min\{|\dot\gamma(\tau_m^r) - \dot\gamma(\tau_j^r)| : m \ne j,\ 1 \le m, j \le Q_r,\ 1 \le r \le N\} > 0.$$

Then in view of the fact that η is a unit speed geodesic, it follows from (4-31) that for all $r,\ 1 \le r \le N$, and for all $m \ne j$,

$$|\nabla\Psi^r_{klmj}(p_r)| \ge L\alpha - 2 \ge 1, \tag{4-32}$$

provided that $L \ge 3/\alpha$. Hence, if $d\Psi^r_{klmj}(p_r) = 0$ then $m = j$, and therefore, (4-31) implies that

$$\nabla\Psi^r_{klmk}(p_r) = \dot\eta(t_k^r) - \dot\eta(t_l^r). \tag{4-33}$$

This completes the proof of the claim since $\dot\eta(t_k^r) - \dot\eta(t_l^r) \ne 0$ for all $k \ne l$ and all $r,\ 1 \le r \le N$.

In what follows we choose $L \geq 3/\alpha$. Furthermore, it follows from (4-32) and (4-33) that for such L, there exists $\beta > 0$ such that

$$|\nabla \Psi^r_{klmj}(p_r)| \geq \beta > 0, \tag{4-34}$$

for $(k, l, m, j) \in \{(k, l, m, j) : 1 \leq k, l \leq P_r,\ 1 \leq m, j \leq Q_r\} \setminus \{(k, l, m, j) : k = l,\ m = j\},\ 1 \leq r \leq N$.

Returning to (4-29), we write the integral there as

$$I = s^{(n-1)/2} \sum_{r=1}^{N} \sum_{1\leq k,l\leq P_r} \sum_{1\leq m,j\leq Q_r} \int_{U_r} B^r_{klmj} e^{is\Psi^r_{klmj}}\, dV_{g_0} = \sum_{r=1}^{N} (I^r_1 + I^r_2), \tag{4-35}$$

where

$$\begin{aligned} I^r_1 &= s^{(n-1)/2} \sum_{1\leq k\leq P_r} \sum_{1\leq m\leq Q_r} \int_{U_r} B^r_{kkmm} e^{is\Psi^r_{kkmm}}\, dV_{g_0}, \\ I^r_2 &= s^{(n-1)/2} \sum_{1\leq k\neq l\leq P_r} \sum_{1\leq m\neq j\leq Q_r} \int_{U_r} B^r_{klmj} e^{is\Psi^r_{klmj}}\, dV_{g_0}. \end{aligned} \tag{4-36}$$

Rough stationary phase calculation. Here the analysis is concerned with the integrals I^r_1. It follows from (4-25) that

$$\Psi^r_{kkmm} = 2i(\operatorname{Im}\varphi^{(k),r} + L \operatorname{Im}\psi^{(m),r}),$$

and therefore, $d\Psi^r_{kkmm}(p_r) = 0$, $\Psi^r_{kkmm}(p_r) = 0$, and $\operatorname{Im}\nabla^2\Psi^r_{kkmm}(p_r) > 0$, where $p_r \in M_0^{\text{int}}$ is the point of intersection of γ and η. Note that $U_r \subset M_0^{\text{int}}$, and hence, there will be no contributions from the boundary.

Let us denote by $z = (z_1, \ldots, z_{n-1})$ the geodesic normal coordinates in (M_0, g_0) with origin at p_r. Writing $dV_{g_0} = |g_0(z)|^{1/2}\, dz$, applying Lemma A.1, and using (4-30) and (4-26), we obtain that

$$\begin{aligned} &\lim_{s\to\infty} s^{(n-1)/2} \int_{U_r} B^r_{kkmm} e^{is\Psi^r_{kkmm}}\, dV_{g_0} \\ &\quad = \lim_{s\to\infty} s^{(n-1)/2} \int_{\text{neigh}(0,\mathbb{R}^{n-1})} B^r_{kkmm}(z)|g_0(z)|^{1/2} e^{is\Psi^r_{kkmm}(z)}\, dz = C^r_{kkmm} B^r_{kkmm}(p_r) \\ &\quad = C^r_{kkmm}[-L\widehat{A_1 c^{1-n/2}}(2(\mu - L\lambda), p_r) + iL\widehat{A' c^{1-n/2}}(2(\mu - L\lambda), p_r)(\dot\gamma(\tau^r_m))] \\ &\qquad\qquad \times e^{-2\mu t^r_k - 2L\lambda\tau^r_m} |a^{(k),r}_{00}(p_r)|^2 |c^{(m),r}_{00}(p_r)|^2, \end{aligned} \tag{4-37}$$

where

$$C^r_{kkmm} = \frac{(2\pi)^{(n-1)/2}}{(\det \operatorname{Im}\nabla^2\Psi^r_{kkmm}(p_r))^{1/2}} > 0.$$

Here we also used that

$$\varphi^{(k),r}(p_r) = t^r_k \quad \text{and} \quad \psi^{(m),r}(p_r) = \tau^r_m.$$

Thus, we see from (4-36) and (4-37) that

$$\begin{aligned} \lim_{s\to\infty} I^r_1 = \sum_{1\leq k\leq P_r} \sum_{1\leq m\leq Q_r} & C^r_{kkmm}[-L\widehat{A_1 c^{1-n/2}}(2(\mu - L\lambda), p_r) + iL\widehat{A' c^{1-n/2}}(2(\mu - L\lambda), p_r)(\dot\gamma(\tau^r_m))] \\ & \times e^{-2\mu t^r_k - 2L\lambda\tau^r_m} |a^{(k),r}_{00}(p_r)|^2 |c^{(m),r}_{00}(p_r)|^2. \end{aligned} \tag{4-38}$$

Nonstationary phase calculation. Here the analysis is concerned with I_2^r in (4-36). It follows from (4-25) that

$$\Psi_{klmj}^r = \tilde{\Psi}_{klmj}^r + i\,\mathrm{Im}\,\varphi^{(k),r} + i\,\mathrm{Im}\,\varphi^{(l),r} + Li\,\mathrm{Im}\,\psi^{(m),r} + Li\,\mathrm{Im}\,\psi^{(j),r}, \tag{4-39}$$

where

$$\tilde{\Psi}_{klmj}^r = \mathrm{Re}\,\varphi^{(k),r} - \mathrm{Re}\,\varphi^{(l),r} + L\,\mathrm{Re}\,\psi^{(m),r} - L\,\mathrm{Re}\,\psi^{(j),r} \in C^\infty \tag{4-40}$$

is real such that $|\nabla\tilde{\Psi}_{klmj}^r(p_r)| = |\nabla\Psi_{klmj}^r(p_r)| \geq \beta > 0$ provided $L > 3/\alpha$ in view of (4-34).

Let us denote by $z = (z_1, \ldots, z_{n-1})$ the geodesic normal coordinates in (M_0, g_0) with origin at p. Motivated by (4-30) and (4-39), we set

$$f(z) = [-L\widehat{A_1 c^{1-n/2}}(2(\mu - L\lambda), z) + i\langle\widehat{A' c^{1-n/2}}(2(\mu - L\lambda), z)d\varphi^{(k),r} - d\overline{\varphi^{(l),r}} + Ld\psi^{(m),r}\rangle_{g_0}] \times e^{\Phi_{klmj}^r}|g_0(z)|^{1/2} \in C_0^{1,1}(M_0),$$

and

$$\hat{a}_0^{(k),r} = s^{(n-2)/8}e^{-s\,\mathrm{Im}\,\varphi^{(k),r}}a_0^{(k),r}, \quad \hat{c}_0^{(m),r} = s^{(n-2)/8}e^{-s\,\mathrm{Im}\,\psi^{(m),r}}c_0^{(m),r}. \tag{4-41}$$

Thus,

$$I_{2,klmj}^r := s^{(n-1)/2}\int_{U_r} B_{klmj}^r e^{is\Psi_{klmj}^r}\,dV_g = s^{1/2}\int_{\mathrm{neigh}(0,\mathbb{R}^{n-1})} f(z)\hat{a}_0^{(k),r}\overline{\hat{a}_0^{(l),r}}\hat{c}_0^{(m),r}\overline{\hat{c}_0^{(j),r}}e^{is\tilde{\Psi}_{klmj}^r(z)}\,dz. \tag{4-42}$$

Note that f is independent of s, and

$$\|\hat{a}_0^{(k),r}\|_{L^4(M_0)} = \mathcal{O}(1), \quad \|\hat{c}_0^{(m),r}\|_{L^4(M_0)} = \mathcal{O}(1), \tag{4-43}$$

as $s \to \infty$. We next claim that

$$\|\nabla\hat{a}_0^{(k),r}\|_{L^4(M_0)} = \mathcal{O}(s^{1/2}), \quad \|\nabla\hat{c}_0^{(m),r}\|_{L^4(M_0)} = \mathcal{O}(s^{1/2}), \tag{4-44}$$

as $s \to \infty$; see [Lassas et al. 2021a]. Let us recall the argument briefly. It is enough to show the first bound in (4-44). To that end, we have from (4-41) that

$$\nabla\hat{a}_0^{(k),r} = s^{(n-2)/8}e^{-s\,\mathrm{Im}\,\varphi^{(k),r}}(-s(\nabla\,\mathrm{Im}\,\varphi^{(k),r})a_0^{(k),r} + \nabla a_0^{(k),r}). \tag{4-45}$$

It suffices to control the first term in the right-hand side of (4-45), and to this end we note that in the Fermi coordinates (t, y), associated with the geodesic η, we have

$$|\nabla\,\mathrm{Im}\,\varphi^{(k),r}(t, y)| = \mathcal{O}(|y|) \tag{4-46}$$

and

$$\mathrm{Im}\,\varphi^{(k),r}(t, y) \geq c|y|^2, \tag{4-47}$$

for some $c > 0$; see (4-6). Thus, using (4-46) and (4-47), we get

$$\|s^{(n-2)/8}e^{-s\,\mathrm{Im}\,\varphi^{(k),r}}s(\nabla\,\mathrm{Im}\,\varphi^{(k),r})a_0^{(k),r}\|_{L^4(M_0)} = \mathcal{O}(s^{(n-2)/8}s)\left(\int_{|y|\leq 1/2} e^{-4s\,\mathrm{Im}\,\varphi^{(k),r}}|y|^4\,dy\right)^{1/4} = \mathcal{O}(s^{1/2}).$$

This bound together with (4-45) shows the first bound in (4-44). Similarly to (4-44), we also have

$$\|\partial^\alpha\hat{a}_0^{(k),r}\|_{L^4(M_0)} = \mathcal{O}(s^{|\alpha|/2}), \quad \|\partial^\alpha\hat{c}_0^{(m),r}\|_{L^4(M_0)} = \mathcal{O}(s^{|\alpha|/2}), \quad \text{for all } \alpha, \tag{4-48}$$

as $s \to \infty$. Furthermore, as $\delta' > 0$ can be chosen as small as we wish, we see that $\hat{a}_0^{(k),r}$ and $\hat{c}_0^{(k),r}$ have compact support in U_r.

Letting

$$L = \frac{\nabla \tilde{\Psi}^r_{klmij} \cdot \nabla}{i|\nabla \tilde{\Psi}^r_{klmij}|^2},$$

we have $L(e^{is\tilde{\Psi}^r_{klmij}}) = se^{is\tilde{\Psi}^r_{klmij}}$. Integrating by parts in (4-42), we get

$$I^r_{2,klmj} = s^{-1/2} \int_{\mathrm{neigh}(0,\mathbb{R}^{n-1})} e^{is\tilde{\Psi}^r_{klmj}(z)} L^t(f(z)\hat{a}_0^{(k),r}\overline{\hat{a}_0^{(l),r}}\hat{c}_0^{(m),r}\overline{\hat{c}_0^{(j),r}})\, dz,$$

where $L^t = -L - \operatorname{div} L$. Now in view of (4-40) and (4-43), we see that

$$s^{-1/2}\left| \int_{\mathrm{neigh}(0,\mathbb{R}^{n-1})} e^{is\tilde{\Psi}^r_{klmj}(z)} (\operatorname{div} L)(f(z)\hat{a}_0^{(k),r}\overline{\hat{a}_0^{(l),r}}\hat{c}_0^{(m),r}\overline{\hat{c}_0^{(j),r}})\, dz \right| = \mathcal{O}(s^{-1/2}),$$

and in view of (4-44),

$$s^{-1/2}\left| \int_{\mathrm{neigh}(0,\mathbb{R}^{n-1})} e^{is\tilde{\Psi}^r_{klmj}(z)} f(z)\nabla(\hat{a}_0^{(k),r}\overline{\hat{a}_0^{(l),r}}\hat{c}_0^{(m),r}\overline{\hat{c}_0^{(j),r}})\, dz \right| = \mathcal{O}(1),$$

as $s \to \infty$. As f is independent of s, we see, after one integration by parts in (4-42), that $I^r_{2,klmj} = \mathcal{O}(1)$. Since ∇f is Lipschitz, we can integrate by parts a second time, and using (4-48), we get

$$I^r_{2,klmj} = \mathcal{O}(s^{-1/2}), \tag{4-49}$$

as $s \to \infty$. Notice that it is precisely here that we need the assumption that our 1-form A is an element of $C_0^{1,1}(\mathbb{R} \times M_0, T^*(\mathbb{R} \times M_0))$.

We get, in view of (4-36) and (4-49),

$$I^r_2 = \mathcal{O}(s^{-1/2}), \tag{4-50}$$

as $s \to \infty$.

Completion of the proof. Passing to the limit $s \to \infty$ in (4-29) and using (4-35), (4-36), (4-38), and (4-50), we obtain

$$\sum_{r=1}^{N}\sum_{k=1}^{P_r}\sum_{m=1}^{Q_r} C^r_{kkmm}[-L\widehat{A_1 c^{1-n/2}}(2(\mu - L\lambda),\, p_r) + iL\widehat{A' c^{1-n/2}}(2(\mu - L\lambda),\, p_r)(\dot{\gamma}(\tau^r_m))] \times e^{-2\mu t^r_k - 2L\lambda\tau^r_m}|a_{00}^{(k),r}(p_r)|^2|c_{00}^{(m),r}(p_r)|^2 = 0. \tag{4-51}$$

Next we would like to determine each term in the sum in (4-51) separately. To do this, we shall follow [Lassas et al. 2021a]. First choosing $\mu = (1 - L)\lambda$, we get

$$\sum_{r=1}^{N}\sum_{k=1}^{P_r}\sum_{m=1}^{Q_r}[-L\widehat{A_1 c^{1-n/2}}(2\lambda(1 - 2L),\, p_r) + iL\widehat{A' c^{1-n/2}}(2\lambda(1 - 2L),\, p_r)(\dot{\gamma}(\tau^r_m))] \times C^r_{kkmm} e^{-2\lambda[L(\tau^r_m - t^r_k) + t^r_k]}|a_{00}^{(k),r}(p_r)|^2|c_{00}^{(m),r}(p_r)|^2 = 0. \tag{4-52}$$

It is shown in [Lassas et al. 2021a] that for all $L \geq 1$ sufficiently large,

$$L(\tau_{m_1}^{r_1} - t_{k_1}^{r_1}) + t_{k_1}^{r_1} \neq L(\tau_{m_2}^{r_2} - t_{k_2}^{r_2}) + t_{k_2}^{r_2} \tag{4-53}$$

when $(r_1, k_1, m_1) \neq (r_2, k_2, m_2)$, and fixing $L \geq 3/\alpha$ large enough, we may assume in what follows that (4-53) holds. We shall next need Lemma 5.2 from [Lassas et al. 2021a] which can be stated as follows: let $f_1, \ldots, f_N \in \mathcal{E}'(\mathbb{R})$ be such that for some distinct real numbers $a_1, \ldots, a_N$, one has

$$\sum_{j=1}^{N} \hat{f}_j(\lambda) e^{a_j \lambda} = 0, \quad \lambda \in \mathbb{R},$$

then $f_1 = \cdots = f_N = 0$. Applying this result, we get for all r, k, m, λ,

$$(-\widehat{A_1 c^{1-n/2}}(2\lambda(1-2L), p_r) + i\widehat{A' c^{1-n/2}}(2\lambda(1-2L), p_r)(\dot\gamma(\tau_m^r)))C_{kkmm}^r |a_{00}^{(k),r}(p_r)|^2 |c_{00}^{(m),r}(p_r)|^2 = 0,$$

and as $C_{kkmm}^r \neq 0$, $a_{00}^{(k),r}(p_r) \neq 0$, and $c_{00}^{(m),r}(p_r) \neq 0$, we get, taking the inverse Fourier transform in x_1,

$$-A_1(x_1, p_r) + i A'(x_1, p_r)(\dot\gamma(\tau_m^r)) = 0,$$

for all $x_1 \in \mathbb{R}$, p_r, and τ_m^r. Since y_0 was one of the points p_r, and $\gamma(\tau_m^r) = y_0$, we know

$$(i A_1, A')(x_1, y_0)(1, \dot\gamma(\tau_m^r)) = 0. \tag{4-54}$$

Here we may replace $\dot\gamma(\tau_m^r)$ by $-\dot\gamma(\tau_m^r)$, and thus, (4-54) implies that $A_1(x_1, y_0) = 0$, for all $x_1 \in \mathbb{R}$ and almost all $y_0 \in M_0$, and therefore, by continuity, $A_1 \equiv 0$. Hence, we are left with proving that the 1-form $A'(x_1, \cdot)$ vanishes on M_0 from the fact that

$$A'(x_1, y_0)(\dot\gamma(\tau_m^r)) = 0. \tag{4-55}$$

To proceed we assume without loss of generality that $v_1 := \dot\gamma(\tau_m^r) = (1, 0, \ldots, 0) \in \mathbb{R}^{n-1}$ and consider its small perturbations $v_2, \ldots, v_{n-1}$ given by (3-21). The unit vectors $v_1, \ldots, v_{n-1}$ are linearly independent, and therefore, they span the tangent space $T_{y_0} M_0$. By Proposition D.1, for $\varepsilon > 0$ sufficiently small, the unit speed geodesic γ_{y_0, v_j}, $j = 2, \ldots, n-1$, through (y_0, v_j) is nontangential between boundary points, and γ and γ_{y_0, v_j} do not intersect each other at the boundary of M_0. Applying the discussion above with $\eta = \gamma$ and $\gamma = \gamma_{y_0, v_j}$, we get

$$A'(x_1, y_0)(v_j) = 0, \quad j = 2, \ldots, n-1. \tag{4-56}$$

It follows from (4-55) and (4-56) that the 1-form $A'(x_1, y_0)$ equals 0, and therefore, $A' \equiv 0$. This completes the proof of Proposition 1.6 in the general setting.

In the course of the proof of Proposition 1.6, we also proved the following result.

Corollary 4.1. *Let (M, g) be a conformally transversally anisotropic manifold of dimension $n \geq 3$. Let $A \in C^{1,1}(M, T^*M)$ be a 1-form such that $A|_{\partial M} = 0$ and $\partial_\nu A|_{\partial M} = 0$. If*

$$\int_M \langle A, d(u_1 u_2 u_3) \rangle_g u_4 \, dV_g = \mathcal{O}(1),$$

as $s \to \infty$, for all harmonic functions $u_l \in C^3(M)$, $l = 1, \ldots, 4$, of the form (4-1), *then $A \equiv 0$.*

5. Proof of Theorem 1.3

Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{C}^m,\ m \geq 3$, and consider the Dirichlet problem (1-5) with

$$f = \sum_{k=1}^{m} \varepsilon_k f_k, \quad f_k \in C^{2,\alpha}(\partial M), \quad k = 1, \dots, m.$$

Then for all $|\varepsilon|$ sufficiently small, the problem (1-5) has a unique small solution $u(\,\cdot\,, \varepsilon) \in C^{2,\alpha}(M)$, which depends holomorphically on $\varepsilon \in \text{neigh}(0, \mathbb{C}^m)$; see Theorem B.1.

We shall use an induction argument on $m \geq 3$ to show that all the coefficients A_m and V_m in (1-2) and (1-3), see also (1-5), can be determined from the Dirichlet-to-Neumann map $\Lambda_{A,V}$ given in (1-6).

First, let $m = 3$, and let us proceed to carry out a third-order linearization of the Dirichlet-to-Neumann map. Let $u_j = u_j(x, \varepsilon)$ be the unique small solution of the Dirichlet problem

$$\begin{cases} -\Delta_g u_j + id^*\big(\sum_{k=2}^{\infty} A_k^{(j)}(x)(u_j^k/k!)u_j\big) - i\big\langle \sum_{k=2}^{\infty} A_k^{(j)}(x)(u_j^k/k!), du_j\big\rangle_g \\ \quad + \big\langle \sum_{k=2}^{\infty} A_k^{(j)}(x)(u_j^k/k!), \sum_{k=2}^{\infty} A_k^{(j)}(x)(u_j^k/k!)\big\rangle_g u_j + \sum_{k=3}^{\infty} V_k^{(j)}(x)(u_j^k/k!) = 0 & \text{in } M, \\ u_j = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \varepsilon_3 f_3 & \text{on } \partial M, \end{cases} \tag{5-1}$$

for $j = 1, 2$. Differentiating (5-1) with respect to $\varepsilon_l,\ l = 1, 2, 3$, and using that $u_j(x, 0) = 0$, we get

$$\begin{cases} -\Delta_g v_j^{(l)} = 0 & \text{in } M, \\ v_j^{(l)} = f_l & \text{on } \partial M, \end{cases} \tag{5-2}$$

where $v_j^{(l)} = \partial_{\varepsilon_l} u_j|_{\varepsilon=0}$. By the uniqueness and the elliptic regularity for the Dirichlet problem (5-2), we have that $v^{(l)} := v_1^{(l)} = v_2^{(l)} \in C^{2,\alpha}(M),\ l = 1, 2, 3$; see [Gilbarg and Trudinger 1983, Theorem 6.15].

Applying $\partial_{\varepsilon_k}\partial_{\varepsilon_l}|_{\varepsilon=0},\ k, l = 1, 2, 3$, to (5-1), we next get

$$\begin{cases} -\Delta_g w_j^{(k,l)} = 0 & \text{in } M, \\ w_j^{(k,l)} = 0 & \text{on } \partial M, \end{cases} \tag{5-3}$$

where $w_j^{(k,l)} = \partial_{\varepsilon_k}\partial_{\varepsilon_l} u_j|_{\varepsilon=0}$, and therefore, $w_j^{(k,l)} = 0$ for all $j = 1, 2$ and $k, l = 1, 2, 3$. Finally, applying $\partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}|_{\varepsilon=0}$ to (5-1), we obtain the third-order linearization

$$\begin{cases} -\Delta_g w_j + 3id^*(A_2^{(j)} v^{(1)} v^2 v^{(3)}) - i\langle A_2^{(j)}, d(v^{(1)} v^{(2)} v^{(3)})\rangle_g + V_3^{(j)} v^{(1)} v^{(2)} v^{(3)} = 0 & \text{in } M, \\ w_j = 0 & \text{on } \partial M, \end{cases} \tag{5-4}$$

where $w_j = \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_j|_{\varepsilon=0}$. Using that

$$d^*(Av) = (d^*A)v - \langle A, dv\rangle_g, \tag{5-5}$$

for any 1-form A and a function v, we can rewrite (5-4) as

$$\begin{cases} -\Delta_g w_j - 4i\langle A_2^{(j)}, d(v^{(1)} v^{(2)} v^{(3)})\rangle_g + (3id^*(A_2^{(j)}) + V_3^{(j)}) v^{(1)} v^{(2)} v^{(3)} = 0 & \text{in } M, \\ w_j = 0 & \text{on } \partial M. \end{cases} \tag{5-6}$$

The fact that

$$\Lambda_{A^{(1)},V^{(1)}}(\varepsilon_1 f_1+\varepsilon_2 f_2+\varepsilon_3 f_3)=\Lambda_{A^{(2)},V^{(2)}}(\varepsilon_1 f_1+\varepsilon_2 f_2+\varepsilon_3 f_3)$$

for all small ε and all $f_j\in C^{2,\alpha}(\partial M)$ implies that $\partial_\nu u_1|_{\partial M}=\partial_\nu u_2|_{\partial M}$. Therefore, an application of $\partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}|_{\varepsilon=0}$ yields $\partial_\nu w_1|_{\partial M}=\partial_\nu w_2|_{\partial M}$. Multiplying (5-6) by $v^{(4)}\in C^{2,\alpha}(M)$ harmonic in (M,g) and applying Green's formula, we get

$$\int_M (4i\langle A, d(v^{(1)}v^{(2)}v^{(3)})\rangle_g v^{(4)}-(3id^*(A)+V)v^{(1)}v^{(2)}v^{(3)}v^{(4)})\,dV_g=0, \tag{5-7}$$

for all $v^{(l)}\in C^{2,\alpha}(M)$ harmonic in (M,g), $l=1,\dots,4$. Here $A=A_2^{(1)}-A_2^{(2)}$ and $V=V_3^{(1)}-V_3^{(2)}$. An application of Proposition C.4 implies that $A|_{\partial M}=0$ and $\partial_\nu A|_{\partial M}=0$.

Choosing $v^{(l)}=u_l\in C^3(M)$, $l=1,\dots,4$, to be harmonic functions of the form (4-1), and using (4-19), we first observe that (5-7) implies that

$$\int_M \langle A, d(u_1u_2u_3)\rangle_g u_4\,dV_g=\mathcal{O}(1),$$

as $s\to\infty$. By Corollary 4.1, we get $A\equiv 0$, and therefore, $A_2^{(1)}=A_2^{(2)}$. Substituting $A=0$ into (5-7), we get

$$\int_M V v^{(1)}v^{(2)}v^{(3)}v^{(4)}\,dV_g=0,$$

for all harmonic functions $v^{(l)}\in C^{2,\alpha}(M)$, $l=1,\dots,4$. Using Proposition 1.2, we obtain that $V=0$, and thus, $V_3^{(1)}=V_3^{(2)}$.

Let $m\geq 4$ and assume that

$$A_k=A_k^{(1)}=A_k^{(2)},\quad \text{for } k=2,\dots,m-2,\qquad V_k=V_k^{(1)}=V_k^{(2)},\quad \text{for } k=3,\dots,m-1. \tag{5-8}$$

To show that $A_{m-1}^{(1)}=A_{m-1}^{(2)}$ and $V_m^{(1)}=V_m^{(2)}$, we shall perform the m-th order linearization of the Dirichlet-to-Neumann map. To that end, let $u_j=u_j(x,\varepsilon)$ be the unique small solution of the Dirichlet problem

$$\begin{cases} -\Delta_g u_j+id^*\big(\sum_{k=2}^\infty A_k^{(j)}(x)(u_j^k/k!)u_j\big)-i\big\langle\sum_{k=2}^\infty A_k^{(j)}(x)(u_j^k/k!), du_j\big\rangle_g \\ \quad+\big\langle\sum_{k=2}^\infty A_k^{(j)}(x)(u_j^k/k!), \sum_{k=2}^\infty A_k^{(j)}(x)(u_j^k/k!)\big\rangle_g u_j+\sum_{k=3}^\infty V_k^{(j)}(x)(u_j^k/k!)=0 & \text{in } M, \\ u_j=\varepsilon_1 f_1+\cdots+\varepsilon_m f_m & \text{on } \partial M, \end{cases} \tag{5-9}$$

for $j=1,2$. We would like to apply $\partial_{\varepsilon_1}\cdots\partial_{\varepsilon_m}|_{\varepsilon=0}$ to (5-9). First we observe that

$$\partial_{\varepsilon_1}\cdots\partial_{\varepsilon_m}\left(id^*\left(\sum_{k=m}^\infty A_k^{(j)}(x)\frac{u_j^k}{k!}u_j\right)-i\left\langle\sum_{k=m}^\infty A_k^{(j)}(x)\frac{u_j^k}{k!}, du_j\right\rangle_g+\sum_{k=m+1}^\infty V_k^{(j)}(x)\frac{u_j^k}{k!}\right)$$

is a sum of terms, each of them containing positive powers of u_j which vanish when $\varepsilon=0$. The only term in the expression for $\partial_{\varepsilon_1}\cdots\partial_{\varepsilon_m}(V_m^{(j)}(x)u_j^m/m!)$ which does not contain a positive power of u_j

is $V_m^{(j)}(x)\partial_{\varepsilon_1}u_j\cdots\partial_{\varepsilon_m}u_j$. Furthermore, the only term in the expression for

$$\partial_{\varepsilon_1}\cdots\partial_{\varepsilon_m}\left(id^*\left(A_{m-1}^{(j)}\frac{u_j^m}{(m-1)!}\right)\right)$$

which does not contain a positive power of u_j is $mid^*(A_{m-1}^{(j)}\partial_{\varepsilon_1}u_j\cdots\partial_{\varepsilon_m}u_j)$. The only terms in

$$\partial_{\varepsilon_1}\cdots\partial_{\varepsilon_m}\left\langle A_{m-1}^{(j)}\frac{u_j^{m-1}}{(m-1)!},du_j\right\rangle_g$$

which do not contain a positive power of u_j can be written as $\langle A_{m-1}^{(j)},d(\partial_{\varepsilon_1}u_j\cdots\partial_{\varepsilon_m}u_j)\rangle_g$. The expression

$$\partial_{\varepsilon_1}\cdots\partial_{\varepsilon_m}\left(id^*\left(\sum_{k=2}^{m-2}A_k^{(j)}(x)\frac{u_j^k}{k!}u_j\right)-i\left\langle\sum_{k=2}^{m-2}A_k^{(j)}(x)\frac{u_j^k}{k!},du_j\right\rangle_g+\sum_{k=3}^{m-1}V_k^{(j)}(x)\frac{u_j^k}{k!}\right)$$

is independent of $j=1,2$, in view of (5-8) and the fact that it contains only derivatives of u_j of the form $\partial^k_{\varepsilon_{l_1},\dots,\varepsilon_{l_k}}u_j|_{\varepsilon=0}$ with $k=1,\dots,m-2$ and $\varepsilon_{l_1},\dots,\varepsilon_{l_k}\in\{\varepsilon_1,\dots,\varepsilon_m\}$. Here we use the fact that

$$\partial^k_{\varepsilon_{l_1},\dots,\varepsilon_{l_k}}u_1|_{\varepsilon=0}=\partial^k_{\varepsilon_{l_1},\dots,\varepsilon_{l_k}}u_2|_{\varepsilon=0}$$

for $k=1,\dots,m-1$ and $\varepsilon_{l_1},\dots,\varepsilon_{l_k}\in\{\varepsilon_1,\dots,\varepsilon_m\}$. This follows by applying the operators $\partial^k_{\varepsilon_{l_1},\dots,\varepsilon_{l_k}}|_{\varepsilon=0}$ to (5-9), using (5-8) and the unique solvability of the Dirichlet problem for the Laplacian.

The terms in the expression for

$$\partial_{\varepsilon_1}\cdots\partial_{\varepsilon_m}\left(\left\langle\sum_{k=2}^{\infty}A_k^{(j)}(x)\frac{u_j^k}{k!},\sum_{k=2}^{\infty}A_k^{(j)}(x)\frac{u_j^k}{k!}\right\rangle_g u_j\right)$$

which do not contain a positive power of u_j, only contain $A_2^{(j)},\dots,A_{m-3}^{(j)}$, and only derivatives of u_j of the form $\partial^k_{\varepsilon_{l_1},\dots,\varepsilon_{l_k}}u_j|_{\varepsilon=0}$ with $k=1,\dots,m-4$ and $\varepsilon_{l_1},\dots,\varepsilon_{l_k}\in\{\varepsilon_1,\dots,\varepsilon_m\}$, which are independent of $j=1,2$.

Hence, the m-th order linearization has the form

$$\begin{cases}-\Delta_g w_j+mid^*(A_{m-1}^{(j)}v^{(1)}\cdots v^{(m)})-i\langle A_{m-1}^{(j)},d(v^{(1)}\cdots v^{(m)})\rangle_g+V_m^{(j)}v^{(1)}\cdots v^{(m)}=H_m & \text{in } M,\\ w_j=0 & \text{on } \partial M,\end{cases}$$

where $w_j=\partial_{\varepsilon_1}\cdots\partial_{\varepsilon_m}u_j|_{\varepsilon=0}$ and H_m is known and independent of $j=1,2$. Using (5-5), the previous system can be written as

$$\begin{cases}-\Delta_g w_j-(m+1)i\langle A_{m-1}^{(j)},d(v^{(1)}\cdots v^{(m)})\rangle_g+(mid^*(A_{m-1}^{(j)})+V_m^{(j)})v^{(1)}\cdots v^{(m)}=H_m & \text{in } M,\\ w_j=0 & \text{on } \partial M.\end{cases}$$

Proceeding as in the case $m=3$, we see that

$$\int_M((m+1)i\langle A,d(v^{(1)}\cdots v^{(m)})\rangle_g v^{(m+1)}-(mid^*(A)+V)v^{(1)}\cdots v^{(m+1)})\,dV_g=0,$$

for any $v^{(l)}\in C^{2,\alpha}(M)$ harmonic, $l=1,\dots,m+1$. Here $A=A_{m-1}^{(1)}-A_{m-1}^{(2)}$ and $V=V_m^{(1)}-V_m^{(2)}$. Setting $v^{(1)}=\cdots=v^{(m-3)}=1$ and arguing as in the case $m=3$, we complete the proof of Theorem 1.3.

Appendix A: A rough stationary phase argument

We need the following rough version of the stationary phase; see [Lassas et al. 2021a].

Lemma A.1. *Let $\Psi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ be such that*

$$\Psi(0) = 0, \quad \Psi'(0) = 0, \quad \textit{and} \quad \Psi''(0) > 0. \tag{A-1}$$

Let $V \subset \mathbb{R}^n$ be a sufficiently small neighborhood of zero, and let $a \in C(\overline{V})$. Then

$$\lim_{s\to\infty} s^{n/2} \int_V e^{-s\Psi(z)} a(z)\, dz = \frac{(2\pi)^{n/2}}{(\det \Psi''(0))^{1/2}} a(0). \tag{A-2}$$

Proof. Taylor expanding the phase function Ψ and using (A-1), we get

$$\Psi(z) = \tfrac{1}{2}\Psi''(0)z \cdot z + \mathcal{O}(|z|^3),$$

and therefore,

$$\Psi(z) \geq c|z|^2, \tag{A-3}$$

with some $c > 0$, for all $z \in V$, a sufficiently small neighborhood of zero. Making the change of variables $z \mapsto s^{1/2}z$ in the integral in (A-2) and using the dominated convergence theorem, we obtain that

$$\begin{aligned}\lim_{s\to\infty} s^{n/2} \int_V e^{-s\Psi(z)} a(z)\, dz &= \lim_{s\to\infty} \int_{s^{1/2}V} e^{-s\Psi(z/s^{1/2})} a(z/s^{1/2})\, dz \\ &= \left(\int_{\mathbb{R}^n} e^{-\Psi''(0)z\cdot z/2}\, dz\right) a(0) = \frac{(2\pi)^{n/2}}{(\det \Psi''(0))^{1/2}} a(0).\end{aligned}$$

Here we use the following consequence of (A-3),

$$|\chi_{s^{1/2}V} e^{-s\Psi(z/s^{1/2})} a(z/s^{1/2})| \leq \mathcal{O}(1) e^{-c|z|^2} \in L^1(\mathbb{R}^n),$$

where $\chi_{s^{1/2}V}$ is the characteristic function of the set $s^{1/2}V$. Thus, (A-2) follows. □

Appendix B: Well-posedness of the Dirichlet problem for a nonlinear magnetic Schrödinger equation

The purpose of this appendix is to show the well-posedness of the Dirichlet problem for a nonlinear magnetic Schrödinger equation with small boundary data. The argument is standard, see [Krupchyk and Uhlmann 2020a; Lassas et al. 2021a], and is given here for completeness and the convenience of the reader.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary. Let $C^{k,\alpha}(M)$ stand for the Hölder space on M, where $k \in \mathbb{N} \cup \{0\}$ and $0 < \alpha < 1$; see [Hörmander 1976, Appendix A]. Let us note that $C^{k,\alpha}(M)$ is an algebra under pointwise multiplication, and

$$\|uv\|_{C^{k,\alpha}(M)} \leq C(\|u\|_{C^{k,\alpha}(M)}\|v\|_{L^\infty(M)} + \|u\|_{L^\infty(M)}\|v\|_{C^{k,\alpha}M)}), \quad u, v \in C^{k,\alpha}(M); \tag{B-1}$$

see [Hörmander 1976, Theorem A.7].

Consider the Dirichlet problem for the nonlinear magnetic Schrödinger operator

$$\begin{cases} L_{A,V}u = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases} \tag{B-2}$$

where $L_{A,V}$ is given in (1-4). Here the 1-form A mapping $M \times \mathbb{C}$ to T^*M and the function V mapping $M \times \mathbb{C}$ to $\mathbb{C}$ satisfy the following conditions:

- (A) The map $\mathbb{C} \ni z \mapsto A(\cdot, z)$ is holomorphic with values in $C^{1,\alpha}(M, T^*M)$, the space of 1-forms with complex-valued $C^{1,\alpha}(M)$ coefficients.
- (V_i) The map $\mathbb{C} \ni z \mapsto V(\cdot, z)$ is holomorphic with values in $C^{\alpha}(M)$.
- (V_{ii}) $V(x, 0) = 0$, for all $x \in M$.

The condition (V_{ii}) guarantees that $u = 0$ is a solution to (B-2) when $f = 0$. It follows from (A), (V_i), and (V_{ii}) that A and V can be expanded into the power series

$$A(x, z) = \sum_{k=0}^{\infty} A_k(x)\frac{z^k}{k!}, \quad A_k(x) := \partial_z^k A(x, 0) \in C^{1,\alpha}(M, T^*M), \tag{B-3}$$

converging in the $C^{1,\alpha}(M, T^*M)$ topology, and

$$V(x, z) = \sum_{k=1}^{\infty} V_k(x)\frac{z^k}{k!}, \quad V_k(x) := \partial_z^k V(x, 0) \in C^{\alpha}(M), \tag{B-4}$$

converging in the $C^{\alpha}(M)$ topology. We also assume that $A_0 \in C^{\infty}(M, T^*M)$ and $V_1 \in C^{\infty}(M)$. Let us assume furthermore that

- (i) 0 is not a Dirichlet eigenvalue of the operator $d^*_{A_0} d_{A_0} + V_1$.

Under all of the assumptions above, we have the following result.

Theorem B.1. *There exist $\delta > 0$ and $C > 0$ such that for any*

$$f \in B_\delta(\partial M) := \{f \in C^{2,\alpha}(\partial M) : \|f\|_{C^{2,\alpha}(\partial M)} < \delta\},$$

the problem (B-2) *has a solution $u = u_f \in C^{2,\alpha}(M)$ which satisfies*

$$\|u\|_{C^{2,\alpha}(M)} \leq C\|f\|_{C^{2,\alpha}(\partial M)}.$$

The solution u is unique within the class $\{u \in C^{2,\alpha}(M) : \|u\|_{C^{2,\alpha}(M)} < C\delta\}$ and it depends holomorphically on $f \in B_\delta(\partial M)$. Furthermore, the map

$$B_\delta(\partial M) \to C^{1,\alpha}(M), \quad f \mapsto \partial_\nu u_f|_{\partial M}$$

is holomorphic.

Proof. We shall follow [Lassas et al. 2021a], see also [Krupchyk and Uhlmann 2020a], and in order to prove this result we shall rely on the implicit function theorem for holomorphic maps between complex Banach spaces; see [Pöschel and Trubowitz 1987, p. 144]. To that end, we let

$$B_1 = C^{2,\alpha}(\partial M), \quad B_2 = C^{2,\alpha}(M), \quad \text{and} \quad B_3 = C^{\alpha}(M) \times C^{2,\alpha}(\partial M),$$

and introduce the map

$$F : B_1 \times B_2 \to B_3, \quad F(f, u) = (L_{A,V}u, u|_{\partial M} - f). \tag{B-5}$$

Let us verify that the map F indeed enjoys the mapping properties given in (B-5). To that end, let $u \in C^{2,\alpha}(M)$ and note first that $-\Delta_g u \in C^\alpha(M)$. Let us check that $A(\cdot, u(\cdot)) \in C^{1,\alpha}(M, T^*M)$. By Cauchy's estimates, the coefficients A_k in (B-3) satisfy

$$\|A_k\|_{C^{1,\alpha}(M,T^*M)} \leq \frac{k!}{R^k} \sup_{|z|=R} \|A(\cdot, z)\|_{C^{1,\alpha}(M,T^*M)}, \quad R > 0, \tag{B-6}$$

for all $k = 0, 1, \ldots$. Using (B-1) and (B-6), we obtain

$$\left\| \frac{A_k}{k!} u^k \right\|_{C^{1,\alpha}(M,T^*M)} \leq \frac{C^k}{R^k} \|u\|^k_{C^{1,\alpha}(M)} \sup_{|z|=R} \|A(\cdot, z)\|_{C^{1,\alpha}(M,T^*M)}, \tag{B-7}$$

for all $k = 0, 1, \ldots$. Choosing $R = 2C\|u\|_{C^{1,\alpha}(M)}$, it follows from (B-7) that the series $\sum_{k=0}^\infty A_k(x)u^k/k!$ converges in $C^{1,\alpha}(M, T^*M)$, and thus, $A(\cdot, u(\cdot)) \in C^{1,\alpha}(M, T^*M)$. Similarly, $V(\cdot, u(\cdot)) \in C^\alpha(M)$; see also [Krupchyk and Uhlmann 2020a]. Hence, using (1-4), we see that $L_{A,V}u \in C^\alpha(M)$.

We next claim that the map F in (B-5) is holomorphic. To this end, we first note that F is locally bounded as F is continuous in (f, u). Thus, it suffices to show that F is weakly holomorphic; see [Pöschel and Trubowitz 1987, p. 133]. In doing so, let $(f_0, u_0), (f_1, u_1) \in B_1 \times B_2$, and let us prove that the map

$$\lambda \mapsto F((f_0, u_0) + \lambda(f_1, u_1))$$

is holomorphic in $\mathbb{C}$ with values in B_3. It suffices to check that the map $\lambda \mapsto A(x, u_0(x) + \lambda u_1(x))$ is holomorphic in $\mathbb{C}$ with values in $C^{1,\alpha}(M, T^*M)$, as the fact that the map $\lambda \mapsto V(x, u_0(x) + \lambda u_1(x))$ is holomorphic in $\mathbb{C}$ with values in $C^\alpha(M)$ can be proved similarly; see [Krupchyk and Uhlmann 2020a]. The holomorphy of $\lambda \mapsto A(x, u_0(x) + \lambda u_1(x))$ follows from the fact that in view of (B-7), the series

$$\sum_{k=0}^\infty \frac{A_k}{k!}(u_0 + \lambda u_1)^k$$

converges in $C^{1,\alpha}(M, T^*M)$, locally uniformly in $\lambda \in \mathbb{C}$.

We have $F(0, 0) = 0$, and the partial differential $\partial_u F(0, 0) : B_2 \to B_3$ is given by

$$\partial_u F(0, 0)v = (d^*_{\bar{A}_0} d_{A_0} v + V_1 v, v|_{\partial M}).$$

By the assumption (i), we have that the map $\partial_u F(0, 0) : B_2 \to B_3$ is a linear isomorphism; see [Gilbarg and Trudinger 1983, Theorem 6.15].

An application of the implicit function theorem, see [Pöschel and Trubowitz 1987, p. 144], allows us to conclude that there exists $\delta > 0$ and a unique holomorphic map $S : B_\delta(\partial M) \to C^{2,\alpha}(M)$ such that $S(0) = 0$ and $F(f, S(f)) = 0$ for all $f \in B_\delta(\partial M)$. Setting $u = S(f)$ and noting that S is Lipschitz continuous with $S(0) = 0$, we see that

$$\|u\|_{C^{2,\alpha}(M)} \leq C\|f\|_{C^{2,\alpha}(\partial M)}. \qquad \square$$

Appendix C: First-order boundary determination of potentials

When proving Theorem 1.3 and Proposition 1.6, an important step consists in determining the boundary values, as well as the normal derivatives, of a scalar function and a 1-form, via suitable orthogonality relations involving harmonic functions on the manifold. The purpose of this section is to carry out this step. In doing so, we shall rely on the methods developed in [Brown 2001; Brown and Salo 2006], with suitable modifications in [Guillarmou and Tzou 2011, Appendix], where the boundary values of a scalar potential and a vector field are recovered. The main contribution of this section is that we push the methods a little further, in order to recover the first-order normal derivatives of the potential and the 1-form under limited regularity assumptions; see also [Alessandrini et al. 2018]. We would like to mention the works [Brown and Salo 2006; García and Zhang 2016, Appendix], where the gradient of a C^1-conductivity at the boundary of a Euclidean domain is recovered; see also [Alessandrini 1990; Caro and Garcia 2017; Caro and Meroño 2020]. We refer to [Kohn and Vogelius 1984; Lee and Uhlmann 1989; Nakamura et al. 1995; Sylvester and Uhlmann 1988], where the entire Taylor series at the boundary of C^∞-coefficients are recovered.

To proceed, we shall need the following density result for the space of L^2-harmonic functions; see also [Choe et al. 2004, Corollary 2.14] for a different approach in the Euclidean setting.

Proposition C.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary. The set of harmonic functions on M^{int} that are smooth up to the boundary is dense in the space of L^2-harmonic functions in the L^2 topology.*

Proof. Let $u \in L^2(M)$ be harmonic, i.e., $-\Delta_g u = 0$ in M^{int}. Then by the partial hypoellipticity of the Laplacian, see [Eskin 2011, Theorem 26.1], we have $f = u|_{\partial M} \in H^{-1/2}(\partial M)$. There exists therefore a sequence $f_j \in C^\infty(\partial M)$, $j = 1, 2, \ldots$, such that $\|f_j - f\|_{H^{-1/2}(\partial M)} \to 0$, as $j \to \infty$. The Dirichlet problem

$$\begin{cases} -\Delta_g u_j = 0 & \text{in } M^{\text{int}}, \\ u_j|_{\partial M} = f_j, \end{cases}$$

has a unique solution $u_j \in H^1(M)$, and by the boundary elliptic regularity, $u_j \in C^\infty(M)$. By [Eskin 2011, Theorem 26.3], we get

$$\|u_j - u\|_{L^2(M)} \leq C\|f_j - f\|_{H^{-1/2}(\partial M)} \to 0,$$

as $j \to \infty$, establishing the proposition. □

Our first boundary determination result follows. While this result is not used in this work, the construction of a family of harmonic functions given in the proof is needed for the proof of Proposition C.3 below. Furthermore, we state this result and provide the proof for completeness and the convenience of the reader.

Proposition C.2. *Let (M, g) be a conformally transversally anisotropic manifold of dimension $n \geq 3$, and let $V \in C^{1,1}(M)$. If*

$$\int_M V u_1 u_2 \, dV_g = 0, \tag{C-1}$$

for all harmonic functions $u_1, u_2 \in C^\infty(M)$, then $V|_{\partial M} = 0$ and $\partial_\nu V|_{\partial M} = 0$.

Proof. By Proposition C.1, we see that (C-1) continues to hold for all harmonic functions $u_1, u_2 \in L^2(M)$. To proceed, we shall follow [Brown 2001; Brown and Salo 2006], constructing a family of functions, whose boundary values have a highly oscillatory behavior while becoming increasingly concentrated near a given point on the boundary of M. To convert such functions to harmonic functions, we follow the idea of [Guillarmou and Tzou 2011, Appendix] and rely on a Carleman estimate for the conjugated Laplacian with a gain of two derivatives, established in [Salo and Tzou 2009, Lemma 2.1] in the Euclidean case and in [Krupchyk and Uhlmann 2018, Proposition 2.2] in the conformally transversally anisotropic case.

Let $x_0 \in \partial M$ and let $(x_1, \dots, x_n)$ be the boundary normal coordinates centered at x_0 so that in these coordinates, $x_0 = 0$, the boundary ∂M is given by $\{x_n = 0\}$, and M^{int} is given by $\{x_n > 0\}$. We have, see [Lee and Uhlmann 1989],

$$g(x', x_n) = \sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta}(x)\, dx_\alpha\, dx_\beta + (dx_n)^2, \tag{C-2}$$

and we may also assume that the coordinates $x' = (x_1, \dots, x_{n-1})$ are chosen so that

$$g^{\alpha\beta}(x', 0) = \delta^{\alpha\beta} + \mathcal{O}(|x'|^2), \quad 1 \le \alpha, \ \ \beta \le n-1; \tag{C-3}$$

see [Petersen 2006, Chapter 2, Section 8, p. 56]. Therefore,

$$g^{\alpha\beta}(x', x_n) = g^{\alpha\beta}(x', 0) + \mathcal{O}(x_n) = \delta^{\alpha\beta} + \mathcal{O}(|x'|^2) + \mathcal{O}(x_n). \tag{C-4}$$

In view of (C-3), we have

$$-\Delta_g = D_{x_n}^2 + \sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(x) D_{x_\alpha} D_{x_\beta} + f(x) D_{x_n} + R(x, D_{x'}), \tag{C-5}$$

where f is a smooth function and R is a differential operator of order 1 in x' with smooth coefficients; see [Lee and Uhlmann 1989]. Notice that in the local coordinates, $T_{x_0}\partial M = \mathbb{R}^{n-1}$, equipped with the Euclidean metric. The unit tangent vector τ is then given by $\tau = (\tau', 0)$, where $\tau' \in \mathbb{R}^{n-1}$, $|\tau'| = 1$. Associated to the tangent vector τ' is the covector $\xi'_\alpha = \sum_{\beta=1}^{n-1} g_{\alpha\beta}(0)\tau'_\beta = \tau'_\alpha \in T^*_{x_0}\partial M$.

Let $\eta \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ be such that $\operatorname{supp}(\eta)$ is in a small neighborhood of 0, and

$$\int_{\mathbb{R}^{n-1}} \eta(x', 0)^2\, dx' = 1. \tag{C-6}$$

Let $\frac{1}{3} \le \alpha \le \frac{1}{2}$. Following [Brown and Salo 2006], in the boundary normal coordinates, we set

$$v_0(x) = \eta\left(\frac{x}{\lambda^\alpha}\right) e^{i(\tau'\cdot x' + i x_n)/\lambda}, \quad 0 < \lambda \ll 1, \tag{C-7}$$

so that $v_0 \in C^\infty(M)$, with $\operatorname{supp}(v_0)$ in an $\mathcal{O}(\lambda^\alpha)$ neighborhood of $x_0 = 0$. Here τ' is viewed as a covector.

A direct computation

$$\|v_0\|^2_{L^2(M)} = \mathcal{O}(1) \int_{|x| \le c\lambda^\alpha,\, x_n \ge 0} e^{-2x_n/\lambda}\, dx'\, dx_n = \mathcal{O}(\lambda^{\alpha(n-1)}) \int_0^\infty e^{-2t}\lambda\, dt = \mathcal{O}(\lambda^{\alpha(n-1)+1}), \tag{C-8}$$

as $\lambda \to 0$, shows that

$$\|v_0\|_{L^2(M)} = \mathcal{O}(\lambda^{\alpha(n-1)/2+1/2}). \tag{C-9}$$

Following [Guillarmou and Tzou 2011, Appendix], we shall construct a harmonic function $u \in L^2(M)$ of the form

$$u = v_0 + r,$$

and therefore, we need to find $r \in L^2(M)$ satisfying

$$\Delta_g r = -\Delta_g v_0 \quad \text{in } M^{\text{int}}. \tag{C-10}$$

To that end, we shall rely on the following Carleman estimate for the conjugated Laplacian with a gain of two derivatives established in [Salo and Tzou 2009, Lemma 2.1; Krupchyk and Uhlmann 2018, Proposition 2.2]: for all $0 < h \ll 1$ and all $v \in C_0^\infty(M^{\text{int}})$, we have

$$\|v\|_{H^2_{\text{scl}}(M^{\text{int}})} \leq \frac{C}{h} \|e^{\varphi/h} \circ (-h^2\Delta_g) \circ e^{-\varphi/h} v\|_{L^2(M)}. \tag{C-11}$$

Here the limiting Carleman weight $\varphi(x)$ equals x_1. Using a standard argument, one can convert the Carleman estimate (C-11) into a solvability result. Applying this solvability result with $h > 0$ small but fixed, we see that there exists a solution $r \in L^2(M)$ of (C-10) such that

$$\|r\|_{L^2(M)} \leq C \|\Delta_g v_0\|_{H^{-2}(M^{\text{int}})}. \tag{C-12}$$

Next we claim that

$$\|\Delta_g v_0\|_{H^{-2}(M^{\text{int}})} = \mathcal{O}(\lambda^{\alpha(n-3)/2+3/2}), \quad \tfrac{1}{3} \leq \alpha \leq \tfrac{1}{2}, \tag{C-13}$$

as $\lambda \to 0$. In order to prove (C-13), we first compute the Euclidean Laplacian acting on v_0:

$$\begin{aligned} \Delta v_0 &= e^{i(\tau'\cdot x' + ix_n)/\lambda} \left[\lambda^{-2\alpha} (\Delta\eta)\left(\frac{x}{\lambda^\alpha}\right) + 2i\lambda^{-\alpha-1} (\nabla\eta)\left(\frac{x}{\lambda^\alpha}\right) \cdot (\tau', i) - \lambda^{-2} (\tau', i)\cdot(\tau', i)\eta\left(\frac{x}{\lambda^\alpha}\right)\right] \\ &= e^{i(\tau'\cdot x' + ix_n)/\lambda} \left[\lambda^{-2\alpha} (\Delta\eta)\left(\frac{x}{\lambda^\alpha}\right) + 2i\lambda^{-\alpha-1} (\nabla\eta)\left(\frac{x}{\lambda^\alpha}\right) \cdot (\tau', i)\right], \end{aligned} \tag{C-14}$$

where we have used that $(\tau', i)\cdot(\tau', i) = 0$. The second term in the right-hand side of (C-14) has the worst growth as $\alpha \to 0$ and we will analyze it. The first term in the right-hand side of (C-14) can be treated in a similar fashion. To that end, we note that the second term in the right-hand side of (C-14) has the form

$$\lambda^{-\alpha-1} \chi\left(\frac{x}{\lambda^\alpha}\right) e^{i(\tau'\cdot x' + ix_n)/\lambda},$$

where $\chi \in C^\infty(\mathbb{R}^n)$ is supported in a small neighborhood of 0, and we can proceed similarly to [Guillarmou and Tzou 2011, Appendix]. Setting

$$L = \frac{\nabla\bar\phi\cdot\nabla}{i|\nabla\phi|^2} = \frac{1}{2i}\nabla\bar\phi\cdot\nabla, \quad \phi = \tau'\cdot x' + ix_n,$$

we get $Le^{i(\tau'\cdot x'+ix_n)/\lambda} = \lambda^{-1}e^{i(\tau'\cdot x'+ix_n)/\lambda}$. Letting $\psi \in C_0^\infty(M^{\text{int}})$ and integrating by parts twice using the operator L, we obtain

$$\lambda^{-\alpha-1}\int_M \chi\left(\frac{x}{\lambda^\alpha}\right)\psi(x)e^{i(\tau'\cdot x'+ix_n)/\lambda}\,dV_g = \lambda^{-\alpha-1}\lambda^2\int_M (L)^2\left(\chi\left(\frac{x}{\lambda^\alpha}\right)\psi(x)|g(x)|^{1/2}\right)e^{i(\tau'\cdot x'+ix_n)/\lambda}\,dx, \quad \text{(C-15)}$$

since the transpose L^t equals $-L$. The term in the right-hand side of (C-15), where the bound cannot be improved integrating by parts further, will occur when the operator $(L)^2$ falls on ψ, and in this case, using the Cauchy–Schwarz inequality and a computation similar to (C-8), we get

$$\left|\lambda^{-\alpha+1}\int_M \chi\left(\frac{x}{\lambda^\alpha}\right)e^{i(\tau'\cdot x'+ix_n)/\lambda}(L)^2(\psi(x))\,dV_g\right| \le \lambda^{-\alpha+1}\left\|\chi\left(\frac{x}{\lambda^\alpha}\right)e^{i(\tau'\cdot x'+ix_n)/\lambda}\right\|_{L^2(M)}\|\psi\|_{H^2(M^{\text{int}})} \le \mathcal{O}(\lambda^{\alpha(n-3)/2+3/2})\|\psi\|_{H^2(M^{\text{int}})}. \quad \text{(C-16)}$$

Proceeding similarly, integrating by parts using the operator L, if needed, we can bound all the other terms in (C-15) with the same bound as in (C-16). Therefore, it follows from (C-14) and (C-16) that for $0 < \alpha \le \frac{1}{2}$, we have

$$\|\Delta v_0\|_{H^{-2}(M^{\text{int}})} = \mathcal{O}(\lambda^{\alpha(n-3)/2+3/2}), \quad \text{(C-17)}$$

as $\lambda \to 0$. To get the bound (C-13) for the Laplace–Beltrami operator, we notice that in view of (C-3), (C-5), and (C-17), we have to bound

$$\sum_{\alpha,\beta=1}^{n-1}(g^{\alpha\beta}(x)-\delta^{\alpha\beta})D_{x_\alpha}D_{x_\beta}v_0 + f(x)D_{x_n}v_0 + R(x, D_{x'})v_0 \quad \text{(C-18)}$$

in $H^{-2}(M^{\text{int}})$. Let us proceed to bound the first term. To that end, we compute

$$D_{x_\alpha}D_{x_\beta}v_0 = e^{i(\tau'\cdot x'+ix_n)/\lambda}\left[\lambda^{-2\alpha}(D_{x_\alpha}D_{x_\beta}\eta)\left(\frac{x}{\lambda^\alpha}\right) + \lambda^{-1-\alpha}(D_{x_\alpha}\eta)\left(\frac{x}{\lambda^\alpha}\right)\tau_\beta + \lambda^{-1-\alpha}(D_{x_\beta}\eta)\left(\frac{x}{\lambda^\alpha}\right)\tau_\alpha + \lambda^{-2}\tau_\alpha\tau_\beta\eta\left(\frac{x}{\lambda^\alpha}\right)\right]. \quad \text{(C-19)}$$

The worst growth as $\lambda \to 0$ is in the fourth term in (C-19), and therefore, in view of (C-18), we proceed to bound

$$\lambda^{-2}(g^{\alpha\beta}-\delta^{\alpha\beta})\chi\left(\frac{x}{\lambda^\alpha}\right)e^{i(\tau'\cdot x'+ix_n)/\lambda}, \quad \chi(x) = \tau_\alpha\tau_\beta\eta(x),$$

in $H^{-2}(M^{\text{int}})$. The other terms in the first term in (C-18) can be bounded similarly. As before, integrating by parts twice using the operator L, we get

$$\lambda^{-2}\int_M (g^{\alpha\beta}-\delta^{\alpha\beta})\chi\left(\frac{x}{\lambda^\alpha}\right)e^{i(\tau'\cdot x'+ix_n)/\lambda}\psi\,dV_g = \int_M (L)^2\left((g^{\alpha\beta}-\delta^{\alpha\beta})\chi\left(\frac{x}{\lambda^\alpha}\right)\psi|g(x)|^{1/2}\right)e^{i(\tau'\cdot x'+ix_n)/\lambda}\,dx. \quad \text{(C-20)}$$

The term in the right-hand side of (C-20) where the bound cannot be improved occurs when the operator $(L)^2$ falls on ψ, and in this case, using the Cauchy–Schwarz inequality, (C-4), and a computation similar to (C-8), we get

$$\begin{aligned}\left|\int_M (g^{\alpha\beta}-\delta^{\alpha\beta})\chi\left(\frac{x}{\lambda^\alpha}\right)e^{i(\tau'\cdot x'+ix_n)/\lambda}(L)^2\psi\,dV_g\right| &\\ \leq \left(\int_M (\mathcal{O}(|x'|^4)+\mathcal{O}(x_n^2))\chi^2\left(\frac{x}{\lambda^\alpha}\right)e^{-2x_n/\lambda}\,dV_g\right)^{1/2}\|\psi\|_{H^2(M^{\mathrm{int}})} &\\ \leq \left(\mathcal{O}(\lambda^{2\alpha}\lambda^{\alpha(n-1)/2+1/2})+\mathcal{O}(\lambda^{\alpha(n-1)/2})\left(\int_0^\infty x_n^2e^{-2x_n/\lambda}\,dx_n\right)^{1/2}\right)\|\psi\|_{H^2(M^{\mathrm{int}})} &\\ = (\mathcal{O}(\lambda^{\alpha(n+3)/2+1/2})+\mathcal{O}(\lambda^{\alpha(n-1)/2+3/2}))\|\psi\|_{H^2(M^{\mathrm{int}})}. &\end{aligned} \tag{C-21}$$

The growth in λ in (C-21) is smaller than or equal to that in the desired bound (C-13) provided that $\alpha \geq \frac{1}{3}$. Proceeding similarly integrating by parts, using the operator L if needed, we can bound all the other terms in (C-20) by the bound which is the same or better than

$$\mathcal{O}(\lambda^{\alpha(n-3)/2+3/2})\|\psi\|_{H^2(M^{\mathrm{int}})}.$$

Thus, using this and in view of (C-18)–(C-21), we conclude that

$$\left\|\sum_{\alpha,\beta=1}^{n-1}(g^{\alpha\beta}(x)-\delta^{\alpha\beta})D_{x_\alpha}D_{x_\beta}v_0\right\|_{H^{-2}(M^{\mathrm{int}})}=\mathcal{O}(\lambda^{\alpha(n-3)/2+3/2}), \tag{C-22}$$

provided that $\frac{1}{3}\leq\alpha\leq\frac{1}{2}$. Finally, as $R(x, D_{x'})$ is a differential operator of order 1 in x', similarly, we get

$$\|f(x)D_{x_n}v_0+R(x,D_{x'})v_0\|_{H^{-2}(M^{\mathrm{int}})}=\mathcal{O}(\lambda^{\alpha(n-1)/2+3/2}), \tag{C-23}$$

which is better than the desired bound (C-13). Hence, combining (C-17), (C-22), and (C-23), we get (C-13).

Now it follows from (C-12) and (C-13) that

$$\|r\|_{L^2(M)}=\mathcal{O}(\lambda^{\alpha(n-3)/2+3/2}),\quad \tfrac{1}{3}\leq\alpha\leq\tfrac{1}{2}, \tag{C-24}$$

as $\lambda\to 0$. Notice that the bound for r in L^2 is better than the bound for v_0 in L^2; see (C-9).

Letting

$$u_1=v_0+r,\quad u_2=\overline{v_0+r}, \tag{C-25}$$

in (C-1) and multiplying (C-1) by $\lambda^{-\alpha(n-1)-1}$, we get

$$0=\lambda^{-\alpha(n-1)-1}\int_M V(v_0+r)(\bar v_0+\bar r)\,dV_g=\lambda^{-\alpha(n-1)-1}(I_1+I_2+I_3). \tag{C-26}$$

Here

$$I_1=\int_M V|v_0|^2\,dV_g,\quad I_2=\int_M V(v_0\bar r+\bar v_0 r)\,dV_g,\quad \text{and}\quad I_3=\int_M V|r|^2\,dV_g.$$

Using (C-9) and (C-24), we obtain

$$\lambda^{-\alpha(n-1)-1}|I_2|\leq\mathcal{O}(\lambda^{-\alpha(n-1)-1})\|v_0\|_{L^2(M)}\|r\|_{L^2(M)}=\mathcal{O}(\lambda^{1-\alpha}), \tag{C-27}$$

and

$$\lambda^{-\alpha(n-1)-1}|I_3| \le \mathcal{O}(\lambda^{-\alpha(n-1)-1})\|r\|^2_{L^2(M)} = \mathcal{O}(\lambda^{2-2\alpha}), \tag{C-28}$$

as $\lambda \to 0$. Using (C-7), (C-6), the fact that V is continuous, and making the change of variables $y' = x'/\lambda^\alpha$ and $y_n = x_n/\lambda$, we get

$$\begin{aligned}\lim_{\lambda\to 0} \lambda^{-\alpha(n-1)-1} I_1 &= \lim_{\lambda\to 0} \int_{\mathbb{R}^{n-1}} \int_0^\infty V(\lambda^\alpha y', \lambda y_n)\eta^2(y', \lambda^{1-\alpha} y_n) e^{-2y_n} |g(\lambda^\alpha y', \lambda y_n)|^{1/2}\, dy'\, dy_n \\ &= V(0)|g(0)|^{1/2} \int_0^{+\infty} e^{-2y_n}\, dy_n = \frac{1}{2} V(0). \end{aligned} \tag{C-29}$$

Passing to the limit $\lambda \to 0$ in (C-26) and using (C-27)–(C-29), we obtain $V(0) = 0$, showing that $V|_{\partial M} = 0$. Notice that here we can consider any α, $\frac{1}{3} \le \alpha \le \frac{1}{2}$.

Next we would like to prove that $\partial_\nu V|_{\partial M} = 0$. To that end, as before, we let $x_0 \in \partial M$ and consider boundary normal coordinates centered at x_0. As $V \in C^{1,1}$ and $V(x', 0) = 0$, using the fundamental theorem of calculus and integrating by parts, we have for x near $x_0 = 0$,

$$\begin{aligned} V(x', x_n) &= \int_0^1 \frac{d}{dt} V(x', tx_n)\, d(t-1) = V'_{x_n}(x', 0)x_n + \int_0^1 (1-t)\frac{d^2}{dt^2} V(x', tx_n) \\ &= V'_{x_n}(x', 0)x_n + \int_0^1 (1-t) V''_{x_n x_n}(x', tx_n) x_n^2\, dt = V'_{x_n}(x', 0)x_n + \mathcal{O}(x_n^2). \end{aligned} \tag{C-30}$$

Now substituting u_1 and u_2 as given by (C-25) into (C-1), multiplying (C-1) by $\lambda^{-\alpha(n-1)-2}$, and then using (C-30), we get

$$0 = \lambda^{-\alpha(n-1)-2} \int_M V(v_0 + r)(\bar{v}_0 + \bar{r})\, dV_g = \lambda^{-\alpha(n-1)-2}(I_{1,1} + I_{1,2} + I_2 + I_3). \tag{C-31}$$

Here

$$\begin{aligned} I_{1,1} &= \int_M V'_{x_n}(x', 0)x_n |v_0|^2\, dV_g, \quad I_{1,2} = \int_M \mathcal{O}(x_n^2)|v_0|^2\, dV_g, \\ I_2 &= \int_M V(v_0\bar{r} + \bar{v}_0 r)\, dV_g, \qquad I_3 = \int_M V|r|^2\, dV_g. \end{aligned}$$

Using (C-7) and (C-6), making the change of variables $y' = x'/\lambda^\alpha$ and $y_n = x_n/\lambda$, and using that V'_{x_n} is continuous, we obtain

$$\begin{aligned}\lim_{\lambda\to 0} \lambda^{-\alpha(n-1)-2} I_{1,1} &= \lim_{\lambda\to 0} \int_{\mathbb{R}^{n-1}} \int_0^\infty V'_{x_n}(\lambda^\alpha y', 0)\eta^2(y', \lambda^{1-\alpha} y_n) y_n e^{-2y_n} |g(\lambda^\alpha y', \lambda y_n)|^{1/2}\, dy'\, dy_n \\ &= V'_{x_n}(0)|g(0)|^{1/2} \int_0^{+\infty} y_n e^{-2y_n}\, dy_n = \frac{1}{4} V'_{x_n}(0). \end{aligned} \tag{C-32}$$

Using (C-7), we get

$$\lambda^{-\alpha(n-1)-2}|I_{1,2}| \le \mathcal{O}(\lambda^{-\alpha(n-1)-2}) \int_{|x'|\le c\lambda^\alpha,\, x_n \ge 0} x_n^2 e^{-2x_n/\lambda}\, dx'\, dx_n = \mathcal{O}(\lambda). \tag{C-33}$$

Using (C-24), we see that

$$\lambda^{-\alpha(n-1)-2}|I_3| \leq \mathcal{O}(\lambda^{-\alpha(n-1)-2})\|r\|^2_{L^2(M)} = \mathcal{O}(\lambda^{1-2\alpha}) = o(1), \tag{C-34}$$

as $\lambda \to 0$, provided that $\alpha < \frac{1}{2}$.

In view of (C-7) and (C-30), we have

$$\|Vv_0\|_{L^2(M)} = \left(\int_{|x|\leq c\lambda^\alpha, x_n \geq 0} \mathcal{O}(x_n^2)e^{-2x_n/\lambda}\, dx'\, dx_n\right)^{1/2} = \mathcal{O}(\lambda^{\alpha(n-1)/2+3/2}),$$

and therefore, using (C-24), we obtain

$$\lambda^{-\alpha(n-1)-2}|I_2| \leq \mathcal{O}(\lambda^{-\alpha(n-1)-2})\|r\|_{L^2(M)}\|Vv_0\|_{L^2(M)} = \mathcal{O}(\lambda^{1-\alpha}). \tag{C-35}$$

Passing to the limit $\lambda \to 0$ in (C-31), and using (C-32), (C-33), (C-23), and (C-35), we get $V'_{x_n}(0) = 0$ provided that α is a fixed number satisfying $\frac{1}{3} \leq \alpha < \frac{1}{2}$. This shows that $\partial_\nu V|_{\partial M} = 0$. □

In order to prove Proposition 1.6, we shall need the following boundary determination result.

Proposition C.3. *Let (M, g) be a conformally transversally anisotropic manifold of dimension $n \geq 3$. Let $A \in C^{1,1}(M, T^*M)$ be a 1-form. If*

$$\int_M \langle A, du_1\rangle_g u_2\, dV_g = 0, \tag{C-36}$$

for all harmonic functions $u_1, u_2 \in C^\infty(M)$, then $A|_{\partial M} = 0$ and $\partial_\nu A|_{\partial M} = 0$.

Proof. First by Proposition C.1, we see that (C-36) holds for all harmonic functions $u_2 \in L^2(M)$. To prove this result, we shall test the integral identity (C-36) with harmonic functions $u_2 \in L^2(M)$, constructed in Proposition C.2, of the form

$$u_2 = \overline{v_0 + r}. \tag{C-37}$$

Since for u_1 we need estimates in $H^1(M^{\text{int}})$, we shall construct u_1 following [Brown 2001; Brown and Salo 2006]; see also [Krupchyk and Uhlmann 2018, Appendix A]. We let

$$u_1 = v_0 + r_1, \tag{C-38}$$

where $r_1 \in H^1_0(M^{\text{int}})$ is a solution to the Dirichlet problem

$$\begin{cases} -\Delta_g r_1 = \Delta_g v_0 \quad \text{in } M, \\ r_1|_{\partial M} = 0. \end{cases} \tag{C-39}$$

Note that by boundary elliptic regularity, $r_1 \in C^\infty(M)$, and therefore, $u_1 \in C^\infty(M)$.

Applying the Lax–Milgram lemma to (C-39), we get

$$\|r_1\|_{H^1_0(M^{\text{int}})} \leq C\|\Delta_g v_0\|_{H^{-1}(M^{\text{int}})}. \tag{C-40}$$

Similarly to the bound (C-13), one can show that

$$\|\Delta_g v_0\|_{H^{-1}(M^{\text{int}})} = \mathcal{O}(\lambda^{\alpha(n-3)/2+1/2}), \quad \tfrac{1}{3} \leq \alpha \leq \tfrac{1}{2};$$

see also [Krupchyk and Uhlmann 2018, Appendix A]. This bound together with (C-40) implies that

$$\|r_1\|_{H^1(M^{\mathrm{int}})} = \mathcal{O}(\lambda^{\alpha(n-3)/2+1/2}), \quad \tfrac{1}{3} \le \alpha \le \tfrac{1}{2}, \tag{C-41}$$

as $\lambda \to 0$.

We shall also need the bound

$$\|dv_0\|_{L^2(M)} = \mathcal{O}(\lambda^{\alpha(n-1)/2-1/2}), \tag{C-42}$$

as $\lambda \to 0$, which is in view of (C-7) implied by the estimate

$$\|dv_0\|_{L^2(M)} \le \mathcal{O}(1)\bigg(\int_{|x|\le c\lambda^\alpha,\, x_n \ge 0} \lambda^{-2} e^{-2x_n/\lambda}\, dx'\, dx_n\bigg)^{1/2} = \mathcal{O}(\lambda^{\alpha(n-1)/2-1/2}).$$

Now substituting u_1 and u_2 given by (C-38) and (C-37), respectively, into (C-36) and multiplying (C-36) by $\lambda^{-\alpha(n-1)}$, we get

$$0 = \lambda^{-\alpha(n-1)} \int_M \langle A, dv_0 + dr_1\rangle_g (\bar v_0 + \bar r)\, dV_g = \lambda^{-\alpha(n-1)}(I_1 + I_2 + I_3), \tag{C-43}$$

where

$$I_1 = \int_M \langle A, dv_0\rangle_g \bar v_0\, dV_g, \quad I_2 = \int_M \langle A, dr_1\rangle_g (\bar v_0 + \bar r)\, dV_g, \quad \text{and} \quad I_3 = \int_M \langle A, dv_0\rangle_g \bar r\, dV_g.$$

First using (C-7), we write

$$I_1 = I_{1,1} + I_{1,2},$$

where

$$I_{1,1} = i\lambda^{-1} \int_M \langle A, \tau'\cdot dx' + i\, dx_n\rangle_g \eta^2\bigg(\frac{x}{\lambda^\alpha}\bigg) e^{-2x_n/\lambda}\, dV_g,$$
$$I_{1,2} = \lambda^{-\alpha} \int_M \bigg\langle A, (d\eta)\bigg(\frac{x}{\lambda^\alpha}\bigg)\bigg\rangle_g \eta\bigg(\frac{x}{\lambda^\alpha}\bigg) e^{-2x_n/\lambda}\, dV_g.$$

Using (C-2), and making the change of variables $y' = x'/\lambda^\alpha$ and $y_n = x_n/\lambda$, we get

$$\begin{aligned}
\lim_{\lambda\to 0} \lambda^{-\alpha(n-1)} I_{1,1} &= i \lim_{\lambda\to 0} \int_{\mathbb{R}^{n-1}} \int_0^{+\infty} |g(\lambda^\alpha y', \lambda y_n)|^{1/2} \eta^2(y', \lambda^{1-\alpha} y_n) e^{-2y_n} \\
&\qquad \times \bigg(\sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(\lambda^\alpha y', \lambda y_n) A_\alpha(\lambda^\alpha y', \lambda y_n)\tau'_\beta + A_n(\lambda^\alpha y', \lambda y_n) i\bigg)\, dy'\, dy_n \\
&= i\bigg(\sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(0) A_\alpha(0)\tau'_\beta + A_n(0) i\bigg) |g(0)|^{1/2} \int_0^{+\infty} e^{-2y_n}\, dy_n \\
&= \frac{i}{2}\langle A(0), (\tau', i)\rangle.
\end{aligned} \tag{C-44}$$

Estimating similarly as in (C-8), we get

$$\lambda^{-\alpha(n-1)} |I_{1,2}| \le \mathcal{O}(\lambda^{-\alpha n}) \bigg\| (d\eta)\bigg(\frac{x}{\lambda^\alpha}\bigg)\bigg\|_{L^2(M)} \bigg\| \eta\bigg(\frac{x}{\lambda^\alpha}\bigg) e^{-2x_n/\lambda}\bigg\|_{L^2(M)} = \mathcal{O}(\lambda^{(1-\alpha)/2}). \tag{C-45}$$

Using (C-9), (C-24), and (C-41), we see that

$$\lambda^{-\alpha(n-1)}|I_2| \le \mathcal{O}(\lambda^{-\alpha(n-1)})\|dr_1\|_{L^2(M)}\|v_0+r\|_{L^2(M)} = \mathcal{O}(\lambda^{1-\alpha}). \tag{C-46}$$

Finally, using (C-42) and (C-24), we obtain

$$\lambda^{-\alpha(n-1)}|I_3| \le \mathcal{O}(\lambda^{-\alpha(n-1)})\|dv_0\|_{L^2(M)}\|r\|_{L^2(M)} = \mathcal{O}(\lambda^{1-\alpha}). \tag{C-47}$$

Passing to the limit $\lambda \to 0$ in (C-43) and using (C-44)–(C-47), we conclude that $\langle A(0), (\tau', i)\rangle = 0$. Now changing τ' to $-\tau'$, we see that $A_n(0) = 0$, and therefore, $\langle A'(0), \tau'\rangle = 0$, where $A' = (A_1, \dots, A_{n-1})$. As $\tau' \in \mathbb{R}^{n-1}$ is an arbitrary tangent vector to ∂M at $x_0 = 0$, we get $A'(0) = 0$. This shows that $A|_{\partial M} = 0$.

Next we shall show that $\partial_\nu A|_{\partial M} = 0$. To that end, as before, we let $x_0 \in \partial M$ and consider the boundary normal coordinates centered at x_0. Applying computations similar to (C-30) to each component of A, we get

$$A(x', x_n) = (A'_{1x_n}, \dots, A'_{nx_n})(x', 0)x_n + \mathcal{O}(x_n^2) = \partial_{x_n} A(x', 0)x_n + \mathcal{O}(x_n^2). \tag{C-48}$$

Substituting u_1 and u_2 given by (C-38) and (C-37) into (C-36), and multiplying (C-36) by $\lambda^{-\alpha(n-1)-1}$, we have in view of (C-48),

$$0 = \lambda^{-\alpha(n-1)-1}\int_M \langle A, dv_0 + dr_1\rangle_g(\bar v_0 + \bar r)\,dV_g = \lambda^{-\alpha(n-1)-1}(I_{1,1} + I_{1,2} + I_2 + I_3 + I_4), \tag{C-49}$$

where

$$I_{1,1} = \int_M \langle \partial_{x_n} A(x', 0)x_n, dv_0\rangle_g \bar v_0\,dV_g, \quad I_{1,2} = \int_M \langle \mathcal{O}(x_n^2), dv_0\rangle_g \bar v_0\,dV_g,$$
$$I_2 = \int_M \langle A, dr_1\rangle_g \bar v_0\,dV_g, \quad I_3 = \int_M \langle A, dr_1\rangle_g \bar r\,dV_g, \quad I_4 = \int_M \langle A, dv_0\rangle_g \bar r\,dV_g.$$

In view of (C-7) we write

$$I_{1,11} = i\lambda^{-1}\int_M \langle \partial_{x_n} A(x', 0)x_n, \tau'\cdot dx' + i\,dx_n\rangle_g \eta^2\left(\frac{x}{\lambda^\alpha}\right)e^{-2x_n/\lambda}\,dV_g,$$
$$I_{1,12} = \lambda^{-\alpha}\int_M \left\langle \partial_{x_n} A(x', 0)x_n, (d\eta)\left(\frac{x}{\lambda^\alpha}\right)\right\rangle_g \eta\left(\frac{x}{\lambda^\alpha}\right)e^{-2x_n/\lambda}\,dV_g.$$

Using (C-2), and making the change of variables $y' = x'/\lambda^\alpha$ and $y_n = x_n/\lambda$, we get

$$\begin{aligned}\lim_{\lambda\to 0}\lambda^{-\alpha(n-1)-1}I_{1,11} &= i\lim_{\lambda\to 0}\int_{\mathbb{R}^{n-1}}\int_0^{+\infty}|g(\lambda^\alpha y', \lambda y_n)|^{1/2} y_n\eta^2(y', \lambda^{1-\alpha}y_n)e^{-2y_n}\\ &\qquad\times\left(\sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(\lambda^\alpha y', \lambda y_n)\partial_{x_n}A_\alpha(\lambda^\alpha y', 0)\tau'_\beta + \partial_{x_n}A_n(\lambda^\alpha y', 0)i\right)dy'\,dy_n\\ &= i\left(\sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(0)\partial_{x_n}A_\alpha(0)\tau'_\beta + \partial_{x_n}A_n(0)i\right)|g(0)|^{1/2}\int_0^{+\infty} y_n e^{-2y_n}\,dy_n\\ &= \frac{i}{4}\langle \partial_{x_n}A(0), (\tau', i)\rangle.\end{aligned} \tag{C-50}$$

Estimating similarly as in (C-8), we get

$$\lambda^{-\alpha(n-1)-1}|I_{1,12}| \leq \mathcal{O}(\lambda^{-\alpha n-1})\left\|(d\eta)\left(\frac{x}{\lambda^{\alpha}}\right)\right\|_{L^2(M)}\left\|x_n\eta\left(\frac{x}{\lambda^{\alpha}}\right)e^{-2x_n/\lambda}\right\|_{L^2(M)} = \mathcal{O}(\lambda^{(1-\alpha)/2}). \tag{C-51}$$

Using (C-42) and estimating similarly as in (C-8), we obtain

$$\lambda^{-\alpha(n-1)-1}|I_{1,2}| \leq \mathcal{O}(\lambda^{-\alpha(n-1)-1})\|dv_0\|_{L^2(M)}\|x_n^2 v_0\|_{L^2(M)} = \mathcal{O}(\lambda). \tag{C-52}$$

Using (C-41) and (C-48), we get

$$\lambda^{-\alpha(n-1)-1}|I_2| \leq \mathcal{O}(\lambda^{-\alpha(n-1)-1})\|dr_1\|_{L^2(M)}\|x_n v_0\|_{L^2(M)} = \mathcal{O}(\lambda^{1-\alpha}). \tag{C-53}$$

Using (C-41) and (C-24), we have

$$\lambda^{-\alpha(n-1)-1}|I_3| \leq \mathcal{O}(\lambda^{-\alpha(n-1)-1})\|dr_1\|_{L^2(M)}\|r\|_{L^2(M)} = \mathcal{O}(\lambda^{1-2\alpha}) = o(1), \tag{C-54}$$

as $\lambda \to 0$, provided that $\alpha < \frac{1}{2}$.

Using (C-48), (C-24), and the fact that

$$\|x_n dv_0\|_{L^2(M)} = \mathcal{O}(\lambda^{\alpha(n-1)/2+1/2}),$$

we obtain

$$\lambda^{-\alpha(n-1)-1}|I_4| \leq \mathcal{O}(\lambda^{-\alpha(n-1)-1})\|x_n dv_0\|_{L^2(M)}\|r\|_{L^2(M)} = \mathcal{O}(\lambda^{1-\alpha}). \tag{C-55}$$

Let us fix $\frac{1}{3} \leq \alpha < \frac{1}{2}$. Passing to the limit $\lambda \to 0$ in (C-49) and using (C-50)–(C-55), we conclude that $\langle \partial_{x_n} A(0), (\tau', i)\rangle = 0$, and therefore, $\partial_{x_n} A(0) = 0$. This shows that $\partial_\nu A|_{\partial M} = 0$. □

Finally, in order to prove Theorem 1.3 we shall need the following boundary determination result.

Proposition C.4. *Let (M, g) be a conformally transversally anisotropic manifold of dimension $n \geq 3$. Let $A \in C^{1,1}(M, T^*M)$ be a 1-form and $V \in C^{1,1}(M)$. If*

$$\int_M (4i\langle A, d(u_1u_2u_3)\rangle_g u_4 - (3id^*(A) + V)u_1u_2u_3u_4)\, dV_g = 0 \tag{C-56}$$

for all harmonic functions $u_j \in C^{2,\alpha}(M)$, $j = 1, \ldots 4$, then $A|_{\partial M} = 0$ and $\partial_\nu A|_{\partial M} = 0$.

Proof. We also have

$$\int_M (4i\langle A, d(u_2u_3u_4)\rangle_g u_1 - (3id^*(A) + V)u_1u_2u_3u_4)\, dV_g = 0. \tag{C-57}$$

Subtracting (C-57) from (C-56), we get

$$\int_M \langle A, d(u_1u_2u_3)\rangle_g u_4\, dV_g - \int_M \langle A, d(u_2u_3u_4)\rangle_g u_1\, dV_g = 0. \tag{C-58}$$

Letting $u_3 = u_4 = 1$, (C-58) gives

$$\int_M \langle A, du_1\rangle_g u_2\, dV_g = 0 \tag{C-59}$$

for all harmonic functions $u_1, u_2 \in C^{2,\alpha}(M)$, and therefore for all harmonic functions $u_1, u_2 \in C^\infty(M)$. The result follows by an application of Proposition C.3. □

When proving Proposition 1.6, we shall also need the following standard density result.

Proposition C.5. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary. The set of harmonic functions in M^{int} that are smooth up to the boundary is dense in the space of $C^{2,\alpha}(M)$-harmonic functions, $0 < \alpha < 1$, in the $C^{2,\beta}(M)$ topology, for $0 < \beta < \alpha$.*

Proof. The proof follows along the lines of the proof of Proposition C.1. Indeed, let $u \in C^{2,\alpha}(M)$ be harmonic in M^{int} and let $f = u|_{\partial M} \in C^{2,\alpha}(\partial M)$. Let $0 < \beta < \alpha$, and by density, there exists $f_j \in C^\infty(\partial M)$ such that $\| f_j - f \|_{C^{2,\beta}(\partial M)} \to 0$, as $j \to \infty$; see [Hörmander 1976, Theorem A.10]. The Dirichlet problem

$$\begin{cases} -\Delta_g u_j = 0 & \text{in } M^{\text{int}}, \\ u_j|_{\partial M} = f_j, \end{cases}$$

has a unique solution $u_j \in C^{2,\alpha}(M)$, and by elliptic regularity, we have $u_j \in C^\infty(M)$. Using the fact that $C^{2,\alpha}(M) \subset C^{2,\beta}(M)$ and the following bound for the solution to the Dirichlet problem for the Laplacian, see [Gilbarg and Trudinger 1983, Section 6.3, p. 109],

$$\| u_j - u \|_{C^{2,\beta}(M)} \leq C \| f_j - f \|_{C^{2,\beta}(\partial M)} \to 0,$$

we get the claim. □

Appendix D: Some facts about nontangential geodesics

When proving Proposition 1.6, in order to avoid the use of stationary and nonstationary phase arguments on the boundary of the manifold, we shall need the following result concerning nontangential geodesics which was kindly proven for us by Gabriel Paternain.

Proposition D.1. *Let (M_0, g_0) be a smooth compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary, and let γ be a unit speed nontangential geodesic on M_0 between boundary points. Then for each point $y_0 = \gamma(t_0) \in M_0^{\text{int}}$, except for finitely many, there exists a small neighborhood*

$$W \subset S_{y_0} M_0 = \{ w \in T_{y_0} M_0 : |w|_g = 1 \}$$

of $w_0 = \dot\gamma(t_0)$ such that for every $w \in W$, $w \neq w_0$, the unit speed geodesic η on M_0 passing through (y_0, w) is also nontangential between boundary points, and γ and η do not intersect each other at the boundary of M_0.

Proof. Let us first notice that the property of a geodesic being nontangential is stable under small perturbations of the initial conditions, in view of the C^∞-dependence of the geodesic flow on the initial conditions. Let $y_0 = \gamma(t_0) \in M_0^{\text{int}}$. Reparametrizing the geodesic γ if necessary, we may assume that $\gamma : [-S_1, S_2] \to M_0$, $0 < S_1, S_2 < \infty$, is such that $\gamma(0) = y_0$ and $\dot\gamma(0) = w_0$. Let us consider the map

$$F_{y_0} : \text{neigh}(w_0, S_{y_0} M_0) \to \text{neigh}(\gamma(S_2), \partial M_0), \quad F_{y_0}(w) = \pi(\varphi_{\tau(y_0, w)}(y_0, w)), \tag{D-1}$$

where $\tau(y_0, w)$ is the exit time of the geodesic $\gamma_{y_0, w}$ through (y_0, w), $\varphi_t : SM_0 \to SM_0$, $t \in \mathbb{R}$, is the geodesic flow, given by

$$\varphi_t(y, w) = (\gamma_{y,w}(t), \dot\gamma_{y,w}(t)), \tag{D-2}$$

and $\pi : SM_0 \to M_0$, $\pi(y, w) = y$ is the canonical projection.

The exit time $\tau(y_0, w)$ depends smoothly on w, in view of the implicit function theorem and the fact that the geodesic γ is nontangential. The map F_{y_0} is therefore smooth, and we have $F_{y_0}(w_0) = \gamma(S_2)$.

Let us now compute the differential of F_{y_0} at w_0 acting on a vector $\eta \in T_{w_0} S_{y_0} M_0$. To that end, consider a curve $w : (-a, a) \to S_{y_0} M_0$ such that $w(0) = w_0$ and $\dot{w}(0) = \eta$, and by the chain rule, we get

$$\begin{aligned} F'_{y_0}(w_0)\eta &= \frac{d}{ds}\bigg|_{s=0} F_{y_0}(w(s)) = \frac{d}{ds}\bigg|_{s=0} \pi(\varphi_{\tau(y_0,w(s))}(y_0, w(s))) \\ &= d\pi(\varphi_{\tau(y_0,w_0)}(y_0, w_0))\bigg(\frac{d}{dt}\bigg|_{t=\tau(y_0,w_0)} \varphi_t(y_0, w_0)\frac{\partial \tau}{\partial w}(y_0, w_0)\cdot\eta + \frac{\partial \varphi_{\tau(y_0,w_0)}}{\partial w}(y_0, w_0)\eta\bigg). \end{aligned} \tag{D-3}$$

To proceed, we recall some facts about the geometry of the tangent bundle following [Paternain 1999]. First, letting

$$V(y, w) = \ker(d\pi(y, w)) \subset T_{(y,w)} SM_0$$

be the vertical fiber of TSM_0 at (y, w), see [Paternain 1999, Section 1.3.1], we have the splitting

$$T_{(y,w)} SM_0 = H(y, w) \oplus V(y, w),$$

where $H(y, w)$ is the horizontal fiber of TSM_0 at (y, w); see [Paternain 1999, Section 1.3, p. 13]. Both $V(y, w)$ and $H(y, w)$ can be identified with $S_y M_0$, and for $\xi \in T_{(y,w)} SM_0$, we write $\xi = (\xi^h, \xi^v)$, where $\xi^h, \xi^v \in S_y M_0$ are the corresponding horizontal and vertical parts of ξ. Let $X : SM_0 \to TSM_0$ be the geodesic vector field given by

$$X(\varphi_t(y, w)) = \frac{d}{dt}\varphi_t(y, w). \tag{D-4}$$

It follows from [Paternain 1999, Section 1.3, p. 13] that we have

$$X(y, w) = (w, 0). \tag{D-5}$$

Now in view of the above splitting, we have $(0, \eta) \in V(y_0, w_0)$, and therefore, we get

$$\frac{\partial \varphi_{\tau(y_0,w_0)}}{\partial w}(y_0, w_0)\eta = d\varphi_{\tau(y_0,w_0)}(y_0, w_0)(0, \eta). \tag{D-6}$$

Using the fact that $\tau(y_0, w_0) = S_2$, (D-2), and (D-4)–(D-6), we obtain from (D-3) that

$$\begin{aligned} F'_{y_0}(w_0)\eta &= d\pi(\gamma(S_2), \dot{\gamma}(S_2))\bigg(X(\gamma(S_2), \dot{\gamma}(S_2))\frac{\partial \tau}{\partial w}(y_0, w_0)\cdot\eta + d\varphi_{\tau(y_0,w_0)}(y_0, w_0)(0, \eta)\bigg) \\ &= \dot{\gamma}(S_2)\frac{\partial \tau}{\partial w}(y_0, w_0)\cdot\eta + d\pi(\gamma(S_2), \dot{\gamma}(S_2))(d\varphi_{S_2}(y_0, w_0)(0, \eta)). \end{aligned} \tag{D-7}$$

Now by [Paternain 1999, Lemma 1.40], see also [Ilmavirta 2020, Theorem 11.2], for the differential of the geodesic flow we get that

$$d\varphi_{S_2}(y_0, w_0)(0, \eta) = (J_{(0,\eta)}(S_2), \dot{J}_{(0,\eta)}(S_2)), \tag{D-8}$$

where $J_{(0,\eta)}$ is the Jacobi field along the geodesic $t \mapsto \pi(\varphi_t(y_0, w_0)) = \gamma(t)$ with the initial conditions

$$J_{(0,\eta)}(0) = 0, \quad \dot{J}_{(0,\eta)} = \eta. \tag{D-9}$$

Using [Ilmavirta 2020, Exercise 5.9], (D-9), and the fact that $\eta \in T_{w_0} S_{y_0} M_0$, we have

$$\langle \dot\gamma(S_2), J_{(0,\eta)}(S_2)\rangle = \langle \dot\gamma(0), J_{(0,\eta)}(0)\rangle + S_2\langle \dot\gamma(0), \dot J_{(0,\eta)}(0)\rangle = S_2\langle w_0, \eta\rangle = 0, \tag{D-10}$$

showing that the Jacobi field $J_{(0,\eta)}$ is normal to γ. It follows from (D-7) and (D-8) that

$$F'_{y_0}(w_0)\eta = \dot\gamma(S_2)\frac{\partial\tau}{\partial w}(y_0, w_0)\cdot\eta + J_{(0,\eta)}(S_2). \tag{D-11}$$

Using (D-11) and the orthogonally (D-10), we see that if $F'_{y_0}(w_0)$ has a nontrivial kernel, then there exists $\eta \neq 0$ such $J_{(0,\eta)}(S_2) = 0$, and therefore, the points y_0 and $\gamma(S_2)$ are conjugate points along γ; see [Ilmavirta 2020, Definition 7.3]. Thus, $F'_{y_0}(w_0)$ is bijective as long as y_0 is not a conjugate point to $\gamma(S_2)$ along γ.

By the inverse function theorem, F_{y_0} is a local diffeomorphism if y_0 is not a conjugate point to $\gamma(S_2)$ along γ.

Hence, if y_0 is not a conjugate point to $\gamma(S_2)$ and $\gamma(-S_1)$ along γ, there exists a small neighborhood $W \subset S_{y_0}M_0$ of w_0 such that for every $w \in W,\ w \neq w_0$, the unit speed geodesic $\eta : [-T_1, T_2] \to M_0$, $0 < T_1, T_2 < \infty$, such that $\eta(0) = y_0$ and $\dot\eta(0) = w$ is also nontangential between boundary points, and γ and η do not intersect each other at the boundary of M_0. Using the fact that γ can only self-intersect at y_0 finitely many times, see [Kenig and Salo 2013, Lemma 7.2], by choosing W sufficiently small so that the corresponding finitely many tangent vectors of γ and their negatives do not belong to W, we achieve that the geodesics η and γ are distinct and are not reverses of each other.

To conclude the proof, we recall from [do Carmo 1992, p. 248] that

$$\{p \in \gamma([-S_1, S_2]) : p \text{ is conjugate to } \gamma(-S_1) \text{ or } \gamma(S_2)\}$$

is discrete, and since M_0 is compact, it is finite. This completes the proof of the claim. □

When proving Proposition 1.6 in the simplified setting, we shall need some basic facts about nontangential geodesics. These facts are known, see [Dos Santos Ferreira et al. 2020, Section 3], and are presented here for completeness and the convenience of the reader.

Proposition D.2. *Let (M_0, g_0) be a smooth compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary.*

(i) *Let γ be a unit speed non-self-intersecting nontangential geodesic on M_0, and let $y_0 = \gamma(t_0) \in M_0^{\text{int}}$. Then there exists a small neighborhood W of $w_0 = \dot\gamma(t_0)$ in $S_{y_0}M_0$ such that for every $w \in W$, the unit speed geodesic $\gamma_{y_0,w}$ passing through (y_0, w) is nontangential between boundary points and does not have self-intersections.*

(ii) *Let γ and η be unit speed non-self-intersecting nontangential geodesics on M_0 with the only point of intersection $y_0 = \gamma(t_0) = \eta(s_0) \in M_0^{\text{int}}$. Then there exists a small neighborhood W of $w_0 = \dot\gamma(t_0)$ in $S_{y_0}M_0$ such that for every $w \in W$, the unit speed geodesic $\gamma_{y_0,w}$ passing through (y_0, w) is nontangential between boundary points, does not have self-intersections, and intersects η at the point y_0 only.*

Proof. Here we follow [Dos Santos Ferreira et al. 2020, Section 3]. Let us prove (i). Reparametrizing the geodesic γ if necessary, we may assume that $\gamma : [-S_1, S_2] \to M_0,\ 0 < S_1, S_2 < \infty$, is such that $\gamma(0) = y_0$ and $\dot\gamma(0) = w_0$. First the property of a geodesic being nontangential is stable under small perturbations of the initial conditions, in view of C^∞-dependence of the geodesic flow on the initial conditions. Assume the contrary: there is a sequence $w_k \to w_0$ in $S_{y_0}M_0$ as $k \to \infty$ such that there are times $t_k < s_k$ when the corresponding geodesic $\gamma_{y_0,w_k} : [-S_1(k), S_2(k)] \to M_0$ with $\gamma_{y_0,w_k}(0) = y_0,\ \dot\gamma_{y_0,w_k}(0) = w_k$ self-intersects:

$$a_k := \gamma_{y_0,w_k}(t_k) = \gamma_{y_0,w_k}(s_k). \tag{D-12}$$

Note that the sequences $-S_1(k)$ and $S_2(k)$ approach $-S_1$ and S_2, respectively, as $k \to \infty$. Therefore, the sequences t_k and s_k are bounded, and passing to subsequences, we may assume that $t_k \to t_0$ and $s_k \to s_0$. Letting $k \to \infty$ in (D-12), we get $\gamma(t_0) = \gamma(s_0)$. Since γ does not have self-intersections we obtain $t_0 = s_0$.

As all geodesics γ_{y_0,w_k} are nontangential, it follows from (D-12) that $a_k \in M_0^{\text{int}}$. As M_0 is compact, it has a positive injectivity radius $\text{Inj}(M_0) > 0$. Here we have extended M_0 to a closed manifold to speak about the injectivity radius and the boundary will not cause any problems as $a_k \in M_0^{\text{int}}$. Now (D-12) implies that

$$s_k \geq t_k + 2\,\text{Inj}(M_0),$$

and therefore, $s_0 - t_0 \geq 2\,\text{Inj}(M_0) > 0$, which is a contradiction. Hence, (i) follows.

To prove (ii), first reparametrizing the geodesics γ and η if necessary, we may assume that the map $\gamma : [-S_1, S_2] \to M_0,\ 0 < S_1, S_2 < \infty$, is such that $\gamma(0) = y_0$ and $\dot\gamma(0) = w_0$, and $\eta : [-T_1, T_2] \to M_0$, $0 < T_1, T_2 < \infty$, is such that $\eta(0) = y_0$. By (i), there exists a small neighborhood W of w_0 in $S_{y_0}M_0$ such that for every $w \in W$, the unit speed geodesic $\gamma_{y_0,w}$ such that $\gamma_{y_0,w}(0) = y_0$ and $\dot\gamma_{y_0,w}(0) = w$ is nontangential between boundary points and does not have self-intersections. We shall show that the neighborhood W can be made smaller so that every $\gamma_{y_0,w}$ intersects η at the point y_0 only. Let us assume the opposite: there is a sequence $w_k \to w_0$ in $S_{y_0}M_0$ as $k \to \infty$ such that there are times $t_k \neq 0,\ s_k \neq 0$ when the corresponding geodesic γ_{y_0,w_k} intersects η:

$$\gamma_{y_0,w_k}(t_k) = \eta(s_k). \tag{D-13}$$

Note that here we used that γ_{y_0,w_k} and η do not have self-intersections. We also have

$$\gamma_{y_0,w_k}(0) = \eta(0) = y_0. \tag{D-14}$$

Passing to subsequences, we have that $t_k \to t_0$ and $s_k \to s_0$. Thus, it follows from (D-13) that $\gamma(t_0) = \eta(s_0)$, and therefore, as γ and η do not self-intersect and y_0 is the only point of their intersection, we get $t_0 = s_0 = 0$. In view of (D-13) we have

$$\gamma_{y_0,w_k}(t_k) = \eta(s_k) \to \eta(0) = y_0 \in M_0^{\text{int}},$$

and thus, for k sufficiently large, $\gamma_{y_0,w_k}(t_k) = \eta(s_k) \in M_0^{\text{int}}$. This together with (D-14) gives

$$|t_k| > \text{Inj}(M_0) > 0 \quad \text{and} \quad |s_k| > \text{Inj}(M_0) > 0$$

for k sufficiently large, otherwise the geodesics γ_{y_0,w_k} and η would intersect at a geodesic ball centered at y_0, which is a contradiction. Thus, (ii) follows. □

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References

[Alessandrini 1990] G. Alessandrini, "Singular solutions of elliptic equations and the determination of conductivity by boundary measurements", *J. Differential Equations* **84**:2 (1990), 252–272. MR Zbl

[Alessandrini et al. 2018] G. Alessandrini, M. V. de Hoop, R. Gaburro, and E. Sincich, "Lipschitz stability for a piecewise linear Schrödinger potential from local Cauchy data", *Asymptot. Anal.* **108**:3 (2018), 115–149. MR Zbl

[Anikonov 1978] Y. E. Anikonov, Некоторые методы исследования многомерных обратных задач для дифференциалных уравнений, Nauka, Novosibirsk, Russia, 1978. MR Zbl

[Brown 2001] R. M. Brown, "Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result", *J. Inverse Ill-Posed Probl.* **9**:6 (2001), 567–574. MR Zbl

[Brown and Salo 2006] R. M. Brown and M. Salo, "Identifiability at the boundary for first-order terms", *Appl. Anal.* **85**:6-7 (2006), 735–749. MR Zbl

[do Carmo 1992] M. P. do Carmo, *Riemannian geometry*, Birkhäuser, Boston, 1992. MR Zbl

[Caro and Garcia 2017] P. Caro and A. Garcia, "The Calderón problem with corrupted data", *Inverse Problems* **33**:8 (2017), art. id. 085001. MR Zbl

[Caro and Meroño 2020] P. Caro and C. J. Meroño, "The observational limit of wave packets with noisy measurements", *SIAM J. Math. Anal.* **52**:5 (2020), 5196–5212. MR Zbl

[Cârstea and Feizmohammadi 2021] C. I. Cârstea and A. Feizmohammadi, "An inverse boundary value problem for certain anisotropic quasilinear elliptic equations", *J. Differential Equations* **284** (2021), 318–349. MR Zbl

[Cârstea et al. 2019] C. I. Cârstea, G. Nakamura, and M. Vashisth, "Reconstruction for the coefficients of a quasilinear elliptic partial differential equation", *Appl. Math. Lett.* **98** (2019), 121–127. MR Zbl

[Cekić 2017] M. Cekić, "Calderón problem for connections", *Comm. Partial Differential Equations* **42**:11 (2017), 1781–1836. MR Zbl

[Choe et al. 2004] B. R. Choe, H. Koo, and H. Yi, "Projections for harmonic Bergman spaces and applications", *J. Funct. Anal.* **216**:2 (2004), 388–421. MR Zbl

[Dos Santos Ferreira et al. 2009] D. Dos Santos Ferreira, C. E. Kenig, M. Salo, and G. Uhlmann, "Limiting Carleman weights and anisotropic inverse problems", *Invent. Math.* **178**:1 (2009), 119–171. MR Zbl

[Dos Santos Ferreira et al. 2016] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas, and M. Salo, "The Calderón problem in transversally anisotropic geometries", *J. Eur. Math. Soc.* **18**:11 (2016), 2579–2626. MR Zbl

[Dos Santos Ferreira et al. 2020] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas, T. Liimatainen, and M. Salo, "The linearized Calderón problem in transversally anisotropic geometries", *Int. Math. Res. Not.* **2020**:22 (2020), 8729–8765. MR Zbl

[Eskin 2011] G. Eskin, *Lectures on linear partial differential equations*, Grad. Stud. Math. **123**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl

[Feizmohammadi and Oksanen 2020] A. Feizmohammadi and L. Oksanen, "An inverse problem for a semi-linear elliptic equation in Riemannian geometries", *J. Differential Equations* **269**:6 (2020), 4683–4719. MR Zbl

[Feizmohammadi et al. 2021] A. Feizmohammadi, M. Lassas, and L. Oksanen, "Inverse problems for nonlinear hyperbolic equations with disjoint sources and receivers", *Forum Math. Pi* **9** (2021), art. id. e10. MR Zbl

[García and Zhang 2016] A. García and G. Zhang, "Reconstruction from boundary measurements for less regular conductivities", *Inverse Problems* **32**:11 (2016), art. id. 115015. MR Zbl

[Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Math. Wissenschaften **224**, Springer, 1983. MR Zbl

[Guillarmou 2017] C. Guillarmou, "Lens rigidity for manifolds with hyperbolic trapped sets", *J. Amer. Math. Soc.* **30**:2 (2017), 561–599. MR Zbl

[Guillarmou and Tzou 2011] C. Guillarmou and L. Tzou, "Calderón inverse problem with partial data on Riemann surfaces", *Duke Math. J.* **158**:1 (2011), 83–120. MR Zbl

[Guillarmou et al. 2021] C. Guillarmou, M. Mazzucchelli, and L. Tzou, "Boundary and lens rigidity for non-convex manifolds", *Amer. J. Math.* **143**:2 (2021), 533–575. MR Zbl

[Hervas and Sun 2002] D. Hervas and Z. Sun, "An inverse boundary value problem for quasilinear elliptic equations", *Comm. Partial Differential Equations* **27**:11-12 (2002), 2449–2490. MR Zbl

[Hörmander 1976] L. Hörmander, "The boundary problems of physical geodesy", *Arch. Ration. Mech. Anal.* **62**:1 (1976), 1–52. MR Zbl

[Ilmavirta 2020] J. Ilmavirta, "Geometry of geodesics", preprint, 2020. arXiv 2008.00073

[Isakov and Nachman 1995] V. Isakov and A. I. Nachman, "Global uniqueness for a two-dimensional semilinear elliptic inverse problem", *Trans. Amer. Math. Soc.* **347**:9 (1995), 3375–3390. MR Zbl

[Isakov and Sylvester 1994] V. Isakov and J. Sylvester, "Global uniqueness for a semilinear elliptic inverse problem", *Comm. Pure Appl. Math.* **47**:10 (1994), 1403–1410. MR Zbl

[Isozaki 2004] H. Isozaki, "Inverse spectral problems on hyperbolic manifolds and their applications to inverse boundary value problems in Euclidean space", *Amer. J. Math.* **126**:6 (2004), 1261–1313. MR Zbl

[Kang and Nakamura 2002] K. Kang and G. Nakamura, "Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map", *Inverse Problems* **18**:4 (2002), 1079–1088. MR Zbl

[Kenig and Salo 2013] C. Kenig and M. Salo, "The Calderón problem with partial data on manifolds and applications", *Anal. PDE* **6**:8 (2013), 2003–2048. MR Zbl

[Kohn and Vogelius 1984] R. Kohn and M. Vogelius, "Determining conductivity by boundary measurements", *Comm. Pure Appl. Math.* **37**:3 (1984), 289–298. MR Zbl

[Krupchyk and Uhlmann 2014] K. Krupchyk and G. Uhlmann, "Uniqueness in an inverse boundary problem for a magnetic Schrödinger operator with a bounded magnetic potential", *Comm. Math. Phys.* **327**:3 (2014), 993–1009. MR Zbl

[Krupchyk and Uhlmann 2018] K. Krupchyk and G. Uhlmann, "Inverse problems for magnetic Schrödinger operators in transversally anisotropic geometries", *Comm. Math. Phys.* **361**:2 (2018), 525–582. MR Zbl

[Krupchyk and Uhlmann 2020a] K. Krupchyk and G. Uhlmann, "Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities", *Math. Res. Lett.* **27**:6 (2020), 1801–1824. MR Zbl

[Krupchyk and Uhlmann 2020b] K. Krupchyk and G. Uhlmann, "A remark on partial data inverse problems for semilinear elliptic equations", *Proc. Amer. Math. Soc.* **148**:2 (2020), 681–685. MR Zbl

[Kurylev et al. 2018] Y. Kurylev, M. Lassas, and G. Uhlmann, "Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations", *Invent. Math.* **212**:3 (2018), 781–857. MR Zbl

[Lai and Zhou 2020] R.-Y. Lai and T. Zhou, "Partial data inverse problems for nonlinear magnetic Schrödinger equations", 2020. To appear in *Math. Res. Lett.* arXiv 2007.02475

[Lassas and Uhlmann 2001] M. Lassas and G. Uhlmann, "On determining a Riemannian manifold from the Dirichlet-to-Neumann map", *Ann. Sci. École Norm. Sup.* (4) **34**:5 (2001), 771–787. MR Zbl

[Lassas et al. 2018] M. Lassas, G. Uhlmann, and Y. Wang, "Inverse problems for semilinear wave equations on Lorentzian manifolds", *Comm. Math. Phys.* **360**:2 (2018), 555–609. MR Zbl

[Lassas et al. 2021a] M. Lassas, T. Liimatainen, Y.-H. Lin, and M. Salo, "Inverse problems for elliptic equations with power type nonlinearities", *J. Math. Pures Appl.* (9) **145** (2021), 44–82. MR Zbl

[Lassas et al. 2021b] M. Lassas, T. Liimatainen, Y.-H. Lin, and M. Salo, "Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations", *Rev. Mat. Iberoam.* **37**:4 (2021), 1553–1580. MR Zbl

[Lee and Uhlmann 1989] J. M. Lee and G. Uhlmann, "Determining anisotropic real-analytic conductivities by boundary measurements", *Comm. Pure Appl. Math.* **42**:8 (1989), 1097–1112. MR Zbl

[Muhometov 1977] R. G. Muhometov, "The problem of recovery of a two-dimensional Riemannian metric and integral geometry", *Dokl. Akad. Nauk SSSR* **232**:1 (1977), 32–35. In Russian; translated in *Soviet Math. Dokl.* **18**:1 (1977), 27–31. MR Zbl

[Nakamura et al. 1995] G. Nakamura, Z. Q. Sun, and G. Uhlmann, "Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field", *Math. Ann.* **303**:3 (1995), 377–388. MR Zbl

[Paternain 1999] G. P. Paternain, *Geodesic flows*, Progr. Math. **180**, Birkhäuser, Boston, 1999. MR Zbl

[Petersen 2006] P. Petersen, *Riemannian geometry*, 2nd ed., Grad. Texts in Math. **171**, Springer, 2006. MR Zbl

[Pöschel and Trubowitz 1987] J. Pöschel and E. Trubowitz, *Inverse spectral theory*, Pure Appl. Math. **130**, Academic Press, Boston, 1987. MR Zbl

[Salo 2017] M. Salo, "The Calderón problem and normal forms", preprint, 2017. arXiv 1702.02136

[Salo and Tzou 2009] M. Salo and L. Tzou, "Carleman estimates and inverse problems for Dirac operators", *Math. Ann.* **344**:1 (2009), 161–184. MR Zbl

[Stefanov et al. 2018] P. Stefanov, G. Uhlmann, and A. Vasy, "Inverting the local geodesic X-ray transform on tensors", *J. Anal. Math.* **136**:1 (2018), 151–208. MR Zbl

[Sun 1996] Z. Sun, "On a quasilinear inverse boundary value problem", *Math. Z.* **221**:2 (1996), 293–305. MR Zbl

[Sun 2004] Z. Sun, "Inverse boundary value problems for a class of semilinear elliptic equations", *Adv. Appl. Math.* **32**:4 (2004), 791–800. MR Zbl

[Sun 2010] Z. Sun, "An inverse boundary-value problem for semilinear elliptic equations", *Electron. J. Differential Equations* **2010** (2010), art. id. 37. MR Zbl

[Sun and Uhlmann 1997] Z. Sun and G. Uhlmann, "Inverse problems in quasilinear anisotropic media", *Amer. J. Math.* **119**:4 (1997), 771–797. MR Zbl

[Sylvester and Uhlmann 1987] J. Sylvester and G. Uhlmann, "A global uniqueness theorem for an inverse boundary value problem", *Ann. of Math.* (2) **125**:1 (1987), 153–169. MR Zbl

[Sylvester and Uhlmann 1988] J. Sylvester and G. Uhlmann, "Inverse boundary value problems at the boundary: continuous dependence", *Comm. Pure Appl. Math.* **41**:2 (1988), 197–219. MR Zbl

[Uhlmann and Vasy 2016] G. Uhlmann and A. Vasy, "The inverse problem for the local geodesic ray transform", *Invent. Math.* **205**:1 (2016), 83–120. MR Zbl

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