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ALMOST ALL ENTRIES IN THE CHARACTER TABLE OF THE SYMMETRIC GROUP ARE MULTIPLES OF ANY GIVEN PRIME

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Abstract. We show that almost every entry in the character table of S_N is divisible by any fixed prime as $N \to \infty$. This proves a conjecture of Miller.

1. Introduction

In [11], Miller computed the character table of S_N for all N up to 38, and noticed that the proportion of entries not divisible by 2, 3, or 5 seemed to tend to zero. Based on this, he conjectured that, for every fixed prime q, almost every entry in the character table of S_N is divisible by q as $N \to \infty$. It has been known for a long time, due to work of McKay [9], that almost every character of S_N has even degree. Recently, Gluck [5] showed that the proportion of odd entries in a sparse but infinite set of columns of the character table tends to zero (see also the results of Malik, Stan, and Zaharescu [8] on zeros in certain columns of the character table), and Morotti [12] made further progress on Miller's conjecture for each fixed prime q. Even more recently, the first author [14] proved Miller's conjecture for $q = 2, 3, 5, 7, 11$, and 13.

In this paper, we completely resolve Miller's conjecture, proving it for all primes. We also give an explicit bound, uniform in q almost up to $\log N$, for the number of entries in the character table not divisible by q.

Theorem 1. *Let $p(N)$ denote the number of unrestricted partitions of N, so that $p(N)^2$ denotes the number entries in the character table for S_N. Let N be large, and let q be a prime with $q \leq (\log N)/(\log\log N)^2$. The number of entries in the character table of S_N that are not divisible by q is at most*

$$O\Big(p(N)^2 N^{-\frac{1}{12q}}\Big).$$

In particular, almost all entries in the character table for S_N are multiples of

$$\prod_{q \leq (\log N)/(\log\log N)^2} q.$$

Miller [11] also computed the density of entries in the character table of S_N divisible by $4, 8, 9, 25, 27$, and 125, as well as the density of zeros in the character table. From this, it looks like the density of entries divisible by any fixed prime power may go to 1, while the density of zeros may be approaching a positive constant less than 1. Our arguments do not apply to the problem of divisibility by higher prime powers. Regarding the number of zeros

in the character table, Proposition 1 below combined with the distribution of the largest part of a random partition yields that at least a proportion $C/\log N$ (for some positive constant C) of the character values must be zero, and it is unclear whether a positive proportion of the entries are zero. However, in the related setting of finite simple groups of Lie type with rank going to infinity, Larsen and Miller [6] have shown that almost every entry of the character table is zero.

When χ is chosen uniformly at random from the set of irreducible characters of S_N and σ is a uniformly random permutation, Miller [10] showed that $\chi(\sigma)$ almost always vanishes. Another natural variant is to choose the character χ randomly according to the Plancherel measure (which assigns to the irreducible representation ρ the weight $\dim(\rho)^2/N!$). If a conjugacy class C of S_N is chosen at random (with uniform measure from the $p(N)$ possibilities) then $\chi(C) = 0$ almost always. We give a brief indication of these results in §3.

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2. Plan of the proof

For any two partitions λ and μ of N, let χ_μ^λ denote the value of the character of S_N corresponding to the partition λ on the congruence class of permutations with cycle type μ. The basic idea of the proof of Theorem 1 is the same as that used in [14] to prove Miller's conjecture for $q \leq 13$: we will show that, for most partitions μ of N, one has $\chi_\mu^\lambda \equiv \chi_{\widetilde{\mu}}^\lambda \pmod q$ for some partition $\widetilde{\mu}$ of N that possesses a part so large that $\chi_{\widetilde{\mu}}^\lambda$ is forced to be zero for most partitions λ of N. To that end, our first proposition, which is a quantification of an argument in [12], states that if the partition μ of N has a large part, then for most partitions λ of N one has $\chi_\mu^\lambda = 0$.

Proposition 1. *Let $1 \leq A \leq \log N/\log\log N$ be a real number. Suppose that μ is a partition of N such that the largest part of μ is*

$$\geq \frac{\sqrt{6}}{2\pi}\sqrt{N}(\log N)\Big(1+\frac{1}{A}\Big).$$

Then the number of partitions λ of N with $\chi_\mu^\lambda \neq 0$ is at most

$$O\Big(p(N)\frac{\log N}{N^{\frac{1}{2A}}}\Big).$$

Erdős and Lehner [2] showed that a random partition of N has largest part of size $\frac{\sqrt{6}}{2\pi}\sqrt{N}\log N + O(\sqrt{N})$, so that the partitions μ considered in Proposition 1 are just a little bit atypical.

We will, as in [14], use repeated applications of the following lemma to move from our original partition μ to the partition $\widetilde{\mu}$ that we aim to show has a large part.

Lemma 1. *Let q be a prime. Suppose μ is a partition of N, and that ν is another partition of N obtained from μ by replacing q parts of the same size m by one part of size qm. Then for all partitions λ of N we have*

$$\chi^\lambda_\mu \equiv \chi^\lambda_\nu \pmod q.$$

This is a simple consequence of Frobenius's formula for computing character values, see for example Section 3 of [13], or Proposition 1 of [11].

Our second proposition says that, for a typical partition μ, the partition $\widetilde{\mu}$ obtained by repeatedly applying the procedure described in Lemma 1 until no part appears more than $q-1$ times has a part significantly larger than $\frac{\sqrt{6}}{2\pi}\sqrt{N}\log N$.

Proposition 2. *Let $q \le (\log N)/(\log\log N)^2$ be a prime. Starting with a partition μ of N, we repeatedly replace every occurrence of q parts of the same size m by one part of size qm until we arrive at a partition $\widetilde{\mu}$ where no part appears more than $q-1$ times. Then the largest part of $\widetilde{\mu}$ exceeds*

$$\frac{\sqrt{6}}{2\pi}\sqrt{N}(\log N)\Big(1+\frac{1}{5q}\Big),$$

except for at most

$$O\Big(p(N)\exp\Big(-N^{\frac{1}{15q}}\Big)\Big)$$

partitions μ.

If we consider only partitions of N where no part appears more than $q-1$ times, then a small variation of the Erdős–Lehner argument shows that such partitions typically have a largest part of size about $\frac{\sqrt{6N}}{2\pi}\frac{\sqrt{q}}{\sqrt{q-1}}\log N$. This suggests why a result like Proposition 2 may be expected. However, some care is needed, since the partitions $\widetilde{\mu}$ that are the result of our procedure may not look like a typical partition with no part appearing more than $q-1$ times (for example, the largest part in $\widetilde{\mu}$ will very likely be a multiple of q).

Theorem 1 is now a straightforward consequence of Propositions 1 and 2, which we will prove in Sections 3 and 5, respectively.

Deducing Theorem 1. We are given a prime $q \le (\log N)/(\log\log N)^2$, and wish to bound the number of partitions λ, μ with $\chi^\lambda_\mu \not\equiv 0 \pmod q$. Let $\widetilde{\mu}$ be as in Proposition 2. If the largest part of $\widetilde{\mu}$ is below $\frac{\sqrt{6N}}{2\pi}(\log N)(1+\frac{1}{5q})$ then Proposition 2 tells us that there are at most

$$O\Big(p(N)\exp(-N^{\frac{1}{15q}})\times p(N)\Big) = O\Big(p(N)^2\exp(-N^{\frac{1}{15q}})\Big)$$

choices for μ and λ.

On the other hand, if the largest part of $\widetilde{\mu}$ exceeds $\frac{\sqrt{6N}}{2\pi}(\log N)(1+\frac{1}{5q})$, then by Proposition 1 $\chi^\lambda_{\widetilde{\mu}} \neq 0$ for at most $O(p(N)(\log N)N^{-\frac{1}{10q}})$ partitions λ. Thus, in this situation, since $\chi^\lambda_\mu \equiv \chi^\lambda_{\widetilde{\mu}} \pmod q$ by Lemma 1, the number of partitions μ and λ with $\chi^\lambda_\mu \not\equiv 0 \pmod q$ is at most

$$O\Big(p(N)\times p(N)\frac{\log N}{N^{\frac{1}{10q}}}\Big) = O\Big(p(N)^2 N^{-\frac{1}{12q}}\Big).$$

Combining this (which is the bottleneck to improving Theorem 1 quantitatively) with our earlier bound, we conclude that there are at most

$$O\Big(p(N)^2 N^{-\frac{1}{12q}}\Big)$$

pairs μ, λ with $q \nmid \chi_\mu^\lambda$. This establishes the first assertion of the theorem, and the second assertion follows upon summing this bound over all $q \leq (\log N)/(\log\log N)^2$. □

3. Proof of Proposition 1

To prove Proposition 1, we will need the notion of a t-core partition. For any box b in the Young diagram of a partition, the *hook-length* of b is 1 plus the number of boxes directly to the right of b plus the number of boxes directly below b. For example, the Young diagram of $\lambda = (4, 2, 1)$ below has each box labeled with its hook-length.

6	4	2	1
3	1		
1			

Figure 1. Hook-lengths for $\lambda = (4, 2, 1)$.

A partition is called a *t-core* if none of the hook lengths of its Young diagram are divisible by t. For example, from Figure 1 one can see that $(4, 2, 1)$ is a 5-core.

Proof of Proposition 1. Let t denote the largest part of μ, so that $t \geq \frac{\sqrt{6N}}{2\pi}(\log N)(1 + 1/A)$ by assumption. If the partition λ is a t-core, then it follows from the Murnaghan–Nakayama rule (see Subsection 4.3 of [4]) that $\chi_\mu^\lambda = 0$. Now, from Lemma 5 of [12], we know that there are at most $(t+1)p(N-t)$ partitions λ that are not t-cores. Therefore, the number of partitions λ with $\chi_\mu^\lambda \neq 0$ is at most

$$(t+1)p(N-t) \ll \frac{t+1}{N-t+1}\exp\Big(\frac{2\pi}{\sqrt{6}}\sqrt{N-t}\Big) \leq \frac{t+1}{N-t+1}\exp\Big(\frac{2\pi}{\sqrt{6}}\sqrt{N} - \frac{\pi t}{\sqrt{6N}}\Big),$$

where in the first inequality we have used the famous Hardy–Ramanujan asymptotic formula

$$p(N) \sim \frac{1}{4N\sqrt{3}}\exp\Big(\frac{2\pi}{\sqrt{6}}\sqrt{N}\Big)$$

(see [15] for even more precise asymptotics).

For $N \geq t \geq \frac{\sqrt{6N}}{2\pi}(\log N)(1 + 1/A)$, the right side above is maximized at the lower end point $t = \frac{\sqrt{6N}}{2\pi}(\log N)(1 + 1/A)$, yielding the bound

$$\ll \frac{\sqrt{N}\log N}{N} N^{-\frac{1}{2}(1+\frac{1}{A})}\exp\Big(\frac{2\pi}{\sqrt{6}}\sqrt{N}\Big) \ll p(N)\frac{\log N}{N^{\frac{1}{2A}}}.$$

This establishes Proposition 1. □

This may be a convenient juncture to elaborate on the comments at the end of our Introduction on variations of our problem. If the representation corresponding to λ is chosen at random (with the uniform measure on all irreducible representations), then we have seen that λ almost surely a t-core if $t \geq \frac{\sqrt{6N}}{2\pi}(\log N)(1+1/A)$. A random element (chosen uniformly) σ of the group S_N will have a cycle of length $\geq N/\log N$ with very high probability. This is the basis of Miller's result [10] that $\chi^\lambda(\sigma) = 0$ almost always.

If the representation corresponding to λ is chosen with the Plancherel measure, then from the work of Vershik and Kerov [16] it follows that almost surely the largest part of λ and the total number of parts in λ are $\sim 2\sqrt{N}$, so that the maximal possible hook length is $\leq (4+\epsilon)\sqrt{N}$. On the other hand, by Erdős–Lehner the largest part of a typical partition μ is about $\frac{\sqrt{6N}}{2\pi}\log N$. It follows that if λ is chosen randomly according to the Plancherel measure and the conjugacy class corresponding to μ is chosen uniformly, then $\chi^\lambda_\mu = 0$ almost always.

4. Preliminaries for the proof of Proposition 2

Let $\widetilde{p}(j)$ denote the number of partitions of $j \geq 0$ into powers of q, with the convention that $\widetilde{p}(0) = 1$. We define the generating function $F_q(x)$ of $\widetilde{p}(j)$ by

$$F_q(x) := \sum_{j=0}^{\infty} \widetilde{p}(j) e^{-j/x} = \prod_{j=0}^{\infty} (1 - e^{-q^j/x})^{-1}$$

for a real number $x > 0$. Both $\widetilde{p}(j)$ and the generating function $F_q(x)$ have been studied extensively for fixed primes q, beginning with work of Mahler [7] and de Bruijn [1]. In our work we need only some simple estimates for these objects, but with uniformity in q.

Lemma 2. *In the range $0 < x \leq 1$, we have $F_q(x) = O(1)$. When $x \geq 1$ we have*

$$\frac{(\log x)^2}{2\log q} + \frac{1}{2}\log x + O(1) \leq \log F_q(x) \leq \frac{(\log x)^2}{2\log q} + \frac{1}{2}\log x + \frac{1}{8}\log q + O(1). \tag{1}$$

Proof. When $x \leq 1$, note that $F_q(x) \leq \prod_{j=0}^{\infty}(1-e^{-q^j})^{-1} \leq \prod_{j=0}^{\infty}(1-e^{-2^j})^{-1}$, so that $F_q(x) = O(1)$. Now suppose $x \geq 1$. Note that the terms in the product $\prod_{j=0}^{\infty}(1-e^{-q^j/x})^{-1}$ with $q^j > x$ multiply out to a quantity bounded by $\prod_{j=0}^{\infty}(1-e^{-q^j})^{-1}$, so that they are bounded by an absolute constant. For the terms with $q^j \leq x$, note that $\log(1-e^{-q^j/x})^{-1} =$

$\log(x/q^j) + O(q^j/x)$, so that

$$\log \prod_{j=0}^{\infty} (1 - e^{-q^j/x})^{-1} = \sum_{0 \le j \le \log x/\log q} \log \frac{x}{q^j} + O(1)$$
$$= (\log x)\Big(1 + \Big\lfloor \frac{\log x}{\log q} \Big\rfloor\Big) - \frac{\log q}{2} \Big\lfloor \frac{\log x}{\log q} \Big\rfloor \Big(1 + \Big\lfloor \frac{\log x}{\log q} \Big\rfloor\Big) + O(1).$$

The estimates in (1) follow at once. □

In the second lemma of this section, we will record some basic properties of $\widetilde{p}$.

Lemma 3. *The function $\widetilde{p}(k)$ is monotone non-decreasing in k. For all $r \ge 2$ we have*

$$\widetilde{p}(q^r) \ge \frac{q^{r(r-1)/2}}{(r-1)^{r-1}}.$$

Proof. Appending 1 to a partition of k into powers of q produces a partition of $k+1$ into powers of q. This shows that $\widetilde{p}(k)$ is monotone non-decreasing in k.

Suppose $r \ge 2$. For each $1 \le j \le r-1$, pick a non-negative integer k_j with $0 \le k_j \le q^{r-j}/(r-1)$. Each choice for the k_j's gives a partition counted in $\widetilde{p}(q^r)$ by using k_j copies of q^j, and then using $q^r - \sum_{j=1}^{r-1} k_j q^j$ copies of 1. Therefore

$$\widetilde{p}(q^r) \ge \prod_{j=1}^{r-1} \frac{q^{r-j}}{(r-1)} = \frac{q^{r(r-1)/2}}{(r-1)^{(r-1)}},$$

as desired. □

5. Proof of Proposition 2

Let us analyze the process of transforming a partition μ to a partition $\widetilde{\mu}$ as in Proposition 2. Consider an integer k coprime to q, and all parts in μ of the form kq^j with $j \ge 0$. If these parts sum to $k\ell$, then in the partition $\widetilde{\mu}$ we would have corresponding parts of the form kq^j also summing to $k\ell$ with the additional property that no part appears more than $q-1$ times. But this simply means that the number of parts of size kq^j in $\widetilde{\mu}$ equals the coefficient (or 'digit') of q^j in the base q expansion of ℓ. In particular, if $\ell \ge q^r$, then the partition $\widetilde{\mu}$ must have a part kq^j with some $j \ge r$ (since ℓ must have more than r digits in base q).

Next, suppose $\widetilde{\mu}$ has parts of the form kq^j summing to $k\ell$ with no part appearing more than $q-1$ times. From how many partitions μ could this $\widetilde{\mu}$ have arisen? Restricting our attention to these parts of the form kq^j, note that μ could have had any collection of parts kq^j that sum to $k\ell$; or in other words there are $\widetilde{p}(\ell)$ (the number of partitions of ℓ into powers of q) choices for parts of the form kq^j in μ.

Let $\mathcal{K}$ be a set of integers k with $(k,q) = 1$. We wish to count the number of partitions μ such that for $k \in \mathcal{K}$ the parts of the form kq^j in μ sum to $k\ell$ with $\ell < q^r$; call this quantity $p(N; \mathcal{K}, r)$. In other words, these are the partitions μ for which $\widetilde{\mu}$ does not have a part kq^j

with $j \geq r$ for all $k \in \mathcal{K}$. By our remarks above, the count of such partitions μ is the coefficient of z^N in the generating function

(2)
$$\prod_{\substack{(k,q)=1\\ k\notin\mathcal{K}}} (1+\widetilde{p}(1)z^k+\widetilde{p}(2)z^{2k}+\ldots)\prod_{k\in\mathcal{K}}\Big(\sum_{\ell=0}^{q^r-1}\widetilde{p}(\ell)z^{k\ell}\Big) = \prod_{\substack{(k,q)=1\\ k\notin\mathcal{K}}}\prod_{j=0}^{\infty}(1-z^{kq^j})^{-1}\prod_{k\in\mathcal{K}}\Big(\sum_{\ell=0}^{q^r-1}\widetilde{p}(\ell)z^{k\ell}\Big).$$

For example, if $\mathcal{K}=\emptyset$ then we are just counting all partitions of N, and the above generating function may be seen to be $\prod_{n=1}^{\infty}(1-z^n)^{-1}$.

Since the coefficients in the expansion of the generating function (2) are all positive, for any $0<z<1$ we have

$$\begin{aligned} p(N;\mathcal{K},r) &\leq z^{-N}\prod_{\substack{(k,q)=1\\ k\notin\mathcal{K}}}\prod_{j=0}^{\infty}(1-z^{kq^j})^{-1}\prod_{k\in\mathcal{K}}\Big(\sum_{\ell=0}^{q^r-1}\widetilde{p}(\ell)z^{k\ell}\Big)\\ &= \Big(z^{-N}\prod_{n=1}^{\infty}(1-z^n)^{-1}\Big)\prod_{k\in\mathcal{K}}\Big(\frac{\sum_{\ell=0}^{q^r-1}\widetilde{p}(\ell)z^{k\ell}}{\sum_{\ell=0}^{\infty}\widetilde{p}(\ell)z^{k\ell}}\Big). \end{aligned} \tag{3}$$

Here we shall take $z=e^{-1/x}$ with $x=\sqrt{N}/\sqrt{\zeta(2)}=\sqrt{6N}/\pi$. This choice of z is motivated by the fact that the asymptotic for the unrestricted partitions $p(N)$ arises from a contour integral computation integrating z over a circle with approximately this radius. For this choice of z, one has

$$z^{-N}\prod_{n=1}^{\infty}(1-z^n)^{-1}\sim\exp\Big(2\sqrt{N\zeta(2)}-\frac{1}{2}\log(\sqrt{24N})\Big)\ll N^{\frac{3}{4}}p(N)$$

(see Section VIII.6 of [3]). Thus, with this choice of x, we have

$$p(N;\mathcal{K},r)\ll N^{\frac{3}{4}}p(N)\prod_{k\in\mathcal{K}}\Big(\frac{\sum_{\ell=0}^{q^r-1}\widetilde{p}(\ell)e^{-k\ell/x}}{\sum_{\ell=0}^{\infty}\widetilde{p}(\ell)e^{-k\ell/x}}\Big). \tag{4}$$

We are now ready for the proof of Proposition 2.

Proof of Proposition 2. We apply the above considerations, taking

$$r=\Big\lfloor\frac{\log N}{2eq}\Big\rfloor$$

and $\mathcal{K}$ to be the set of integers $k\geq K$ with $(k,q)=1$ where

$$K=\frac{\sqrt{6N}}{2\pi q^r}(\log N)\Big(1+\frac{1}{5q}\Big).$$

Then $p(N;\mathcal{K},r)$ gives an upper bound for the number of partitions μ for which $\widetilde{\mu}$ has largest part below $Kq^r=\frac{\sqrt{6N}}{2\pi}(\log N)(1+1/(5q))$, which is the quantity we desire to bound. Thus

(4) furnishes here the upper bound

$$\ll N^{\frac{3}{4}}p(N)\exp\Big(-\sum_{\substack{k\geq K\\(k,q)=1}}\sum_{\ell\geq q^r}\frac{\widetilde{p}(\ell)e^{-\ell k/x}}{F_q(x/k)}\Big)=N^{\frac{3}{4}}p(N)\exp(-\Delta), \tag{5}$$

say, for this quantity.

It remains to show that Δ is suitably large. Since $\widetilde{p}(\ell)\geq\widetilde{p}(q^r)$ for $\ell\geq q^r$, and $F_q(x/k)\leq F_q(x/K)\leq F_q(2q^r/\log N)$ for $k\geq K$, we obtain

$$\begin{aligned}\Delta&\geq\frac{\widetilde{p}(q^r)}{F_q(2q^r/\log N)}\sum_{\substack{k\geq K\\(k,q)=1}}\sum_{\ell\geq q^r}e^{-\ell k/x}=\frac{\widetilde{p}(q^r)}{F_q(2q^r/\log N)}\sum_{\substack{k\geq K\\(k,q)=1}}\frac{e^{-q^rk/x}}{(1-e^{-k/x})}\\&\geq\frac{\widetilde{p}(q^r)}{F_q(2q^r/\log N)}\sum_{\substack{k\geq K\\(k,q)=1}}e^{-q^rk/x}\frac{x}{k}.\end{aligned} \tag{6}$$

Restricting just to the terms $K\leq k\leq K+x/q^r(\leq 2K)$ the sum over k above is

$$\geq\frac{x}{2K}e^{-Kq^r/x-1}\sum_{\substack{K\leq k\leq K+x/q^r\\(k,q)=1}}1\geq\frac{1}{20}\frac{x}{K}\frac{x}{q^r}e^{-Kq^r/x}\geq\Big(20N^{\frac{1}{10q}}\log N\Big)^{-1} \tag{7}$$

after a small calculation. Further, by Lemma 2 we have

$$\begin{aligned}\log F_q(2q^r/\log N)&\leq\frac{1}{2\log q}\Big(\log\frac{q^r}{\log\sqrt{N}}\Big)^2+\frac{1}{2}\log\frac{q^r}{\log\sqrt{N}}+\frac{1}{8}\log q+O(1)\\&\leq\frac{1}{2\log q}\Big(\log\frac{q^r}{\log\sqrt{N}}\Big)^2+\frac{r}{2}\log q+O(1),\end{aligned}$$

and combining this with the lower bound of Lemma 3 we obtain

$$\begin{aligned}\log\frac{\widetilde{p}(q^r)}{F_q(2q^r/\log N)}&\geq\frac{r^2}{2}\log q-\frac{1}{2\log q}\Big(\log\frac{q^r}{\log\sqrt{N}}\Big)^2-r\log(qr)+O(1)\\&=r\log\Big(\frac{\log\sqrt{N}}{qr}\Big)-\frac{1}{2\log q}(\log\log\sqrt{N})^2+O(1).\end{aligned}$$

Using this and (7) in (6), and recalling that $r=\lfloor(\log\sqrt{N})/(eq)\rfloor$, we conclude that

$$\Delta\gg N^{\frac{1}{2eq}-\frac{1}{10q}}\exp\Big(-\frac{1}{2\log q}(\log\log\sqrt{N})^2-\log\log N\Big)\gg N^{\frac{1}{12q}}.$$

Using this in (5), we conclude that the number of partitions μ for which $\widetilde{\mu}$ has largest part below $\frac{\sqrt{6N}}{2\pi}(\log N)(1+1/(5q))$ is

$$\ll p(N)N^{\frac{3}{4}}\exp(-\Delta)\leq p(N)\exp\big(-N^{\frac{1}{15q}}\big),$$

establishing Proposition 2. □

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