

# Representation of linear PDEs with spatial integral terms as Partial Integral Equations

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**Abstract**—In this paper, we present the Partial Integral Equation (PIE) representation of linear Partial Differential Equations (PDEs) in one spatial dimension, where the PDE has spatial integral terms appearing in the dynamics and the boundary conditions. The PIE representation is obtained by performing a change of variable where every PDE state is replaced by its highest, well-defined derivative using the Fundamental Theorem of Calculus to obtain a new equation (a PIE). We show that this conversion from PDE representation to PIE representation can be written in terms of explicit maps from the PDE parameters to PIE parameters. Lastly, we present numerical examples to demonstrate the application of the PIE representation by performing stability analysis of PDEs via convex optimization methods.

## I. INTRODUCTION

While Partial Differential Equations (PDE) models typically invoke the idea of differential equations with derivative terms, there is an extended class of PDE models with spatial integral terms — typically due to some saturation limitations or property conservation. Such PDEs appear when modeling phenomena such as population distribution across age [1], entropy variation in thermoelastic materials [2], chemical reactions [3], etc. In contrast to previous examples, some PDEs do not originally have spatial integral terms but the dynamics are altered due to the presence of a sensor (e.g., contact-based point measurements may be spatial averages over a small interval) or a controller (e.g. closed-loop PDEs coupled with a backstepping controller or observer [4] have spatial integral terms).

Our goal is to develop computational tools for analysis/control of such PDEs — i.e., solve problems such as stability analysis, input-output  $L_2$ -gain bounds, estimator design, etc. While there exist many works on the analysis and control of PDEs, most of them cannot be applied to develop computational tools because those approaches have one or more of the following issues: are ad-hoc, are designed for a specific PDE, or are conservative. In this paper, we will look at convex optimization methods to provide certificates of stability PDEs with spatial integral terms.

Before discussing the problems with existing methods on the stability analysis and our proposed approach, let us briefly look at the type of PDEs considered in this paper to demonstrate potential applications of this work.

*Example 1:* The McKendrick PDE is a linear PDE model used to model population distribution over a range of age

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(the spatial dimension) [1]. Such PDE models can be used to predict or influence population growth via legislative policies. The McKendrick PDE model has the form

$$\begin{aligned}\dot{x}(t, s) &= -\partial_s x(t, s) + f(s)x(t, s), \\ x(t, 0) &= \int_0^1 h(s)x(t, s)ds,\end{aligned}$$

where  $x$  is the density of the population of age  $s$  at any given time  $t$ . The boundary condition can be considered an approximation of the number of newborns at time  $t$  and is usually modeled as an integral boundary condition (a weighted average of the population density at different ages).

*Example 2:* Next, we look at an example where a simple PDE without integral terms, coupled with a boundary observer, becomes a PDE with spatial integral terms. Let us take the reaction-diffusion PDE model with a boundary observer. The equations can be written as:

$$\begin{aligned}\dot{x}(t, s) &= \lambda x(t, s) + \partial_s^2 x(t, s), \\ \dot{\hat{x}}(t, s) &= \lambda \hat{x}(t, s) + \partial_s^2 \hat{x}(t, s) + l(s)(\partial_s x(t, 1) - \partial_s \hat{x}(t, 1)), \\ \partial_s x(t, 0) &= 0, \quad x(t, 1) = 0, \quad \partial_s \hat{x}(t, 0) = 0, \quad \hat{x}(t, 1) = 0,\end{aligned}$$

where  $x$  is the distributed PDE state,  $\hat{x}$  is the observer state, and  $l$  is the observer gain. A physical implementation of a boundary sensor leads to the appearance of integral terms because point measurements such as  $y(t) = \partial_s x(t, 1)$  (the sensor measurement that drives the observer dynamics) may not be exact depending on the type of the sensing mechanism. In such cases, one might expect a distributed measurement which leads to a PDE of the form

$$\begin{aligned}\dot{\hat{x}}(t, s) &= \lambda \hat{x}(t, s) + \partial_s^2 \hat{x}(t, s) \\ &+ \int_0^1 l(s)w(\theta)(\partial_s x(t, \theta) - \partial_s \hat{x}(t, \theta))d\theta.\end{aligned}$$

where  $w$  is a Gaussian function centered at  $s = 1$ .

Various methods have been developed for analyzing such PDEs such as early-lumping methods in [5], [6] and late-lumping methods in [4]. Since both lumping methods use some type of discretization or projection, at various stages of analysis, they typically require a suitable choice of discretization scheme or bases to project the system. The discretization-based approaches result in an Ordinary Differential Equation (ODE) approximation of the PDE and stability proved for the ODE may not imply stability of the PDE. To avoid this disparity in stability, discretization schemes must be carefully chosen and “high enough” order to get meaningful analysis results — may have high computational costs and may require different schemes for different PDEs. Projection-based approaches also approximate the PDE by

an ODE, however, these ODEs are likely to reflect the PDE behavior more closely than the ODEs obtained through discretization because one can use the eigenfunctions of the space of solutions as bases for projection — thereby, using the dominant modes in analysis. However, a closed form of the eigenfunctions for PDEs with integral terms cannot typically be found and must be approximated numerically resulting in additional numerical errors (apart from truncation errors). These challenges make the development of a general computational tool for analysis and control difficult to solve. We note that some mathematical analysis methods were presented, specifically for PDEs with integral terms, in [7], [8] whereas [9] used computational tools like LMIs for analysis. However, they did not consider PDEs with integral terms at the boundary which is also addressed in this paper.

Our approach for the analysis/control of PDEs involves the use of Partial Integral Equation (PIE) representation. Ever since the PIE representation of linear PDEs was introduced in [10], various analysis and control methods for PIEs have been developed (see [11]). These analysis/control methods are based on convex-optimization problems that can be solved using Linear Matrix Inequalities (LMIs) and do not involve any approximation. If we can find a PIE representation for PDEs with integral terms, then we can use analysis/control methods for the PIE to solve the analysis/control problem of such PDEs. Thus, this paper aims to find the conditions under which a PDE with integral terms has a PIE representation and then find the PIE representation if it exists. To achieve these goals, we first present a standard parametric representation for the PDEs with integral terms that allows integral terms in the dynamics and the boundary. Then, based on this standard representation, we find a sufficient condition for the existence of a PIE representation and then show that the two representations are equivalent by finding a bijective map from the solution of the PDE to the solution of the corresponding PIE.

To summarize the main contribution of this paper, we provide a test for the existence of a PIE representation for a given PDE with integral terms. Then, we also present explicit maps from the parameters of a PDE with integral terms to the parameters of the PIE where the PIE, so obtained, will have stability properties identical to that of the PDE. Then, we present the stability test for PDE with integral terms as an operator-valued optimization problem. While the equivalence of properties of a PDE and the corresponding PIE can be easily extended to input-output properties (see [12]), that topic is beyond the scope of this paper and will not be discussed here.

## II. NOTATION

We use the notation  $0_{m \times n}$  to represent the zero matrix of dimension  $m \times n$  and  $0_n := 0_{n \times n}$ . Similarly,  $I_n$  is the identity matrix of dimension  $n \times n$ . When the dimensions are clear from the context, we use 0 and  $I$  for the zero and identity matrix.  $\mathbb{R}_+$  is the set of non-negative real numbers.  $L_2^n[a, b]$  is the Hilbert space of  $n$ -dimensional vector-valued Lebesgue square-integrable functions defined on the interval

$[a, b]$  and is equipped with the standard inner product. For a suitably differentiable function,  $\mathbf{x}$  of spatial variable  $s$ , we use  $\partial_s^j \mathbf{x}$  to denote the  $j$ -th order partial derivative  $\frac{\partial^j \mathbf{x}}{\partial s^j}$ . For a suitably differentiable function of time and possibly space, we denote  $\dot{\mathbf{x}}(t) = \frac{\partial}{\partial t} \mathbf{x}(t)$ . We use  $W_k^n$  to denote the Sobolev spaces

$$W_k^n[a, b] := \{\mathbf{u} \in L_2^n[a, b] \mid \partial_s^l \mathbf{u} \in L_2^n[a, b] \text{ for all } l \leq k\}.$$

with the canonical Sobolev inner product denoted by  $\langle \cdot, \cdot \rangle_{W_k^n}$ . For brevity, we omit the domain  $[a, b]$  and write  $L_2^n$  or  $W_k^n$  when clear from the context.

## III. PI OPERATORS AND PIE REPRESENTATION

PIEs are defined by bounded, linear maps from  $L_2^n \rightarrow L_2^n$  called Partial Integral (PI) operators, which are parameterized by separable functions. To quickly recall, a separable function is defined as follows.

**Definition 1** (Separable Function). *We say a function  $R : [a, b]^2 \rightarrow \mathbb{R}^{p \times q}$ , is separable if there exist  $r \in \mathbb{N}$  and functions  $F \in L_\infty^{r \times p}[a, b]$ ,  $G \in L_\infty^{r \times q}[a, b]$  such that  $R(s, \theta) = F(s)^T G(\theta)$  where  $L_\infty$  is the Banach space of essentially bounded measurable matrix-valued functions.*

Using the above definition, we can define 3-PI operators (with three parameters) as follow.

**Definition 2** (3-PI operators,  $\Pi_3$ ). *Given  $R_0 \in L_\infty^{p \times q}[a, b]$  and separable functions  $R_1, R_2 : [a, b]^2 \rightarrow \mathbb{R}^{p \times q}$ , we define the operator  $\mathcal{P}_{\{R_i\}}$  for  $\mathbf{v} \in L_2$  as*

$$\begin{aligned} (\mathcal{P}_{\{R_i\}} \mathbf{v})(s) := R_0(s) \mathbf{v}(s) + \int_a^s R_1(s, \theta) \mathbf{v}(\theta) d\theta \\ + \int_s^b R_2(s, \theta) \mathbf{v}(\theta) d\theta. \end{aligned} \quad (1)$$

Furthermore, we say an operator,  $\mathcal{P}$ , is 3-PI of dimension  $p \times q$ , denoted  $\mathcal{P} \in [\Pi_3]_{p,q} \subset \mathcal{L}(L_2^q, L_2^p)$ , if there exist functions  $R_0 \in L_\infty^{p \times q}$  and separable functions  $R_1, R_2$  such that  $\mathcal{P} = \mathcal{P}_{\{R_i\}}$ .

Proving that the set of 3-PI operators form a \*-algebra is beyond the scope of this paper; however, the proof can be found in [11]. Additionally, we define  $[\Pi_3]_{n,n}^+$  as the set of positive definite 3-PI operators where the positivity is defined with respect to  $L_2$ -inner product, i.e.,  $[\Pi_3]_{n,n}^+ := \{\mathcal{P} \in [\Pi_3]_{n,n} \mid \langle x, \mathcal{P}x \rangle_{L_2} > 0 \ \forall x \in L_2 \text{ and } x \neq 0\}$ .

### A. Partial Integral Equations

A PIE is an extension of the state-space representation of ODEs to spatially-distributed states on  $L_2$ . A PIE model is parameterized by 3-PI operators as

$$\mathcal{T} \dot{\mathbf{x}}_f(t) = \mathcal{A} \mathbf{x}_f(t), \quad \mathbf{x}_f(0) = \mathbf{x}_f^0 \in L_2^n[a, b], \quad (2)$$

where  $\mathcal{T} \in [\Pi_3]_{n,n}$  and  $\mathcal{A} \in [\Pi_3]_{n,n}$  are 3-PI operators. Unlike a PDE, a PIE does not allow for spatial derivatives — only a first-order derivative w.r.t. time. In particular, the state of the PIE model,  $\mathbf{x}_f \in L_2[a, b]$ , is not differentiable; consequently, no boundary conditions are possible in a PIE model. Finally, given a PIE, we require the time derivative of the solution w.r.t. the  $\mathcal{T}$ -norm, which is defined as

$$\|\mathbf{x}_f\|_{\mathcal{T}} := \|\mathcal{T} \mathbf{x}_f\|_{L_2}, \quad \text{for } \mathbf{x}_f \in L_2. \quad (3)$$

We require the map  $\mathcal{T}$  in a PIE model (if the PIE model corresponds to a PDE) to be bijective, and hence one can easily prove that  $\|\cdot\|_{\mathcal{T}}$  is a norm [11]. The solution, if it exists, of a PIE must satisfy the following requirements.

**Definition 3** (Solution of a PIE). *For given inputs  $\mathbf{x}_f^0 \in L_2^n$ , we say that  $\{\mathbf{x}_f\}$  satisfies the PIE defined by  $\{\mathcal{T}, \mathcal{A}\}$  with initial condition  $\mathbf{x}_f^0$  if: a)  $\mathbf{x}_f(t) \in L_2^n[a, b]$  for all  $t \geq 0$ ; b)  $\mathbf{x}_f$  is Frechét differentiable with respect to the  $\mathcal{T}$ -norm almost everywhere on  $\mathbb{R}_+$ ; c)  $\mathbf{x}_f(0) = \mathbf{x}_f^0$ ; and d) Eq. (2) is satisfied for almost all  $t \in \mathbb{R}_+$ .*

#### IV. A PARAMETRIC FORM OF PDES WITH INTEGRAL TERMS

Now that we have defined the “target” representation, we need to define the class of PDEs that will be converted to this target representation. To define the PDE, we first categorize the parameters of a PDE into three groups based on the type of constraint they appear in – namely the continuity constraints, the in-domain dynamics, and the boundary conditions. The continuity constraints specify the existence of partial derivatives and boundary values for each state as required by the in-domain dynamics and boundary conditions. The boundary conditions are represented as a real-valued algebraic constraint that maps the distributed state to a vector of boundary values. The in-domain dynamics (or generating equation) specify the time derivative of the state at every point in the interior of the domain and allow for both integral and derivative operators in the spatial variable  $s$ . The following subsections highlight the parameters required to define the above three types of constraints in a PDE.

1) *The continuity constraint:* Given a PDE state,  $\mathbf{x}(t, \cdot)$ , the continuity constraint can be uniquely defined by the parameter  $n = \{n_0, n_1, n_2\}$ , which partitions and orders the PDE states  $\mathbf{x}$  by differentiability as follows.

$$\mathbf{x}(t, \cdot) = \begin{bmatrix} \mathbf{x}_0(t, \cdot) \\ \mathbf{x}_1(t, \cdot) \\ \mathbf{x}_2(t, \cdot) \end{bmatrix} \in \begin{bmatrix} W_0^{n_0} \\ W_1^{n_1} \\ W_2^{n_2} \end{bmatrix} =: W^n.$$

Given such an  $n \in \mathbb{N}^3$ , we can identify all well-defined partial derivatives of  $\mathbf{x}$ .

**Notation:** For convenience, we define the vector of all continuous partial derivatives of the PDE state  $\mathbf{x}$  as permitted by the continuity constraint as  $\mathbf{x}_c$ , the vector of all partial derivatives as  $\mathbf{x}_D$  and the list of all possible boundary values of  $\mathbf{x}$  as  $\mathbf{x}_b$ , i.e.,

$$\mathbf{x}_c(t, \cdot) = \begin{bmatrix} \mathbf{x}_1(t, \cdot) \\ \mathbf{x}_2(t, \cdot) \\ \partial_s \mathbf{x}_2(t, \cdot) \end{bmatrix}, \quad \mathbf{x}_D(t, \cdot) = \begin{bmatrix} \mathbf{x}_0(t, \cdot) \\ \mathbf{x}_1(t, \cdot) \\ \mathbf{x}_2(t, \cdot) \\ \partial_s \mathbf{x}_1(t, \cdot) \end{bmatrix}. \quad (4)$$

$$\mathbf{x}_b(t) = \begin{bmatrix} \mathbf{x}_c(t, a) \\ \mathbf{x}_c(t, b) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0(t, a) \\ \mathbf{x}_1(t, a) \\ \mathbf{x}_2(t, a) \\ \partial_s \mathbf{x}_1(t, a) \\ \partial_s^2 \mathbf{x}_2(t, a) \end{bmatrix}.$$

Additionally, we define  $n_x := \sum_{i=0}^2 n_i$  and  $n_S = \sum_{i=0}^2 i \cdot n_i$  given the continuity constraint parameter  $n = \{n_0, n_1, n_2\}$ .

2) *Boundary Conditions:* Given an  $n = \{n_0, n_1, n_2\}$ , we now parameterize a generalized class of boundary conditions consisting of a combination of boundary values and integrals of the PDE state. Specifically, the boundary conditions

are parameterized by square integrable functions  $B_I(s) \in \mathbb{R}^{n_{BC} \times (n_x + n_S)}$  and  $B \in \mathbb{R}^{n_{BC} \times 2n_S}$  as

$$0 = \int_a^b B_I(s) \mathbf{x}_D(t, s) ds - B \mathbf{x}_b(t) \quad (5)$$

where  $n_{BC}$  is the number of specified boundary conditions. For reasons of well-posedness, as discussed in Section V, we typically require  $n_{BC} = n_S$ .

These boundary conditions and the continuity constraints collectively define the domain of the PDE – which specifies a set of acceptable solutions  $\mathbf{x}(t) \in X$  for the PDE – as

$$X = \left\{ \mathbf{x} \in W^n[a, b] : B \mathbf{x}_b(t) = \int_a^b B_I(s) \mathbf{x}_D(t, s) ds \right\}. \quad (6)$$

**Notation:** We collect all the parameters which define the boundary-valued constraint in Eq. (5) and define  $\mathbf{G}_b$  which represents the labeled tuple of system parameters as  $\mathbf{G}_b = \{B, B_I\}$ .

3) *Dynamics of the PDE:* Finally, we may now define the dynamics of the PDE which is parameterized by the functions  $A_0(s)$ ,  $A_1(s, \theta)$ , and  $A_2(s, \theta) \in \mathbb{R}^{n_x \times (n_x + n_S)}$  as

$$\dot{\mathbf{x}}(t, s) = A_0(s) \mathbf{x}_D(t, s) + \int_a^s A_1(s, \theta) \mathbf{x}_D(t, \theta) d\theta + \int_s^b A_2(s, \theta) \mathbf{x}_D(t, \theta) d\theta, \quad (7)$$

with the constraint  $\mathbf{x}(t) \in X$ .

**Notation:** We collect all the parameters from the generating equation (Eq. (7)) and define  $\mathbf{G}_p$  which represents the labelled tuple of system parameters as  $\mathbf{G}_p = \{A_0, A_1, A_2\}$ . When the shorthand notations  $\mathbf{G}_p$ ,  $\mathbf{G}_b$ , and  $n$  are used to denote a given set of system parameters, it is presumed that all parameters have appropriate dimensions. Now, we define the notion of a solution to the above-described PDE.

**Definition 4** (Solution of a PDE). *For given  $\mathbf{x}^0 \in X$ , we say that  $\mathbf{x}$  satisfies the PDE defined by  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$  with initial condition  $\mathbf{x}^0$  if: a)  $\mathbf{x}(t) \in X$  for all  $t \geq 0$ ; b)  $\mathbf{x}$  is Frechét differentiable with respect to the  $L_2$ -norm almost everywhere on  $\mathbb{R}_+$ ; c)  $\mathbf{x}(0) = \mathbf{x}^0$ ; and d) Eq. (7) is satisfied for almost all  $t \geq 0$ .*

#### V. REPRESENTING A PDE AS A PIE

Recall that our goal is to find a PIE form of PDEs defined in Section IV so that the computational tools developed for PIES can be used in analysis/control. However, first, we have to verify that a PIE representation exists, i.e., we need to verify that for any well-posed PDE of the form given in Section IV with the initial condition  $\mathbf{x}^0 \in X$ , there exists a corresponding PIE with corresponding initial condition  $\mathbf{x}_f^0 \in L_2^n$  whose solution can be used to construct a solution to the PDE. For this purpose, we introduce the notion of *admissibility* and conditions for admissibility as follows.

##### A. Admissibility of the Boundary Conditions

The idea of admissibility imposes a notion of well-posedness on  $X$ , the domain of the PDE defined by the continuity constraints and the BCs, which, when satisfied, guarantees the existence of a PIE form for the PDE.

**Definition 5** (Admissible Boundary Conditions). *Given the parameters  $\{n, \mathbf{G}_b\}$ , we say the pair  $\{n, \mathbf{G}_b\}$  is **admissible** if  $B_T$  is invertible where*

$$B_T := B \begin{bmatrix} T(0) \\ T(b-a) \end{bmatrix} - \int_a^b B_I(s) U_2 T(s-a) ds, \quad (8)$$

and the functions  $T$ , and  $U_2$  are given by

$$T(s) = \begin{bmatrix} I_{n_1} & sI_{n_2} \\ 0 & I_{n_2} \end{bmatrix},$$

$$U_{2i} = \begin{bmatrix} 0_{n_i \times n_{i+1:2}} \\ I_{n_{i+1:2}} \end{bmatrix}, \quad U_2 = \begin{bmatrix} \text{diag}(U_{21}, U_{22}) \\ 0_{n_2 \times n_S} \end{bmatrix}.$$

Since  $B_T \in \mathbb{R}^{n_{BC} \times n_S}$  is invertible only when it is a square matrix, naturally, we require  $n_{BC} = n_S$ , i.e., when we have  $n_S$  differentiable states, we need  $n_S$  boundary conditions for a well-posed solution. Note that the test for invertibility of  $B_T$  depends only on the boundary condition parameters and not the dynamics or the initial condition. Hence, this test only guarantees existence, which is not the same as ‘well-posedness of a PDE’ a notion that requires both the ‘existence’ and ‘uniqueness’ of the solution.

### B. PIE representation of a PDE: $\mathcal{T}$ and $\mathcal{A}$

In this subsection, a PIE form is proposed for a PDE with admissible  $\{n, \mathbf{G}_b\}$ . Then, we will see that the solutions of the PDE and the corresponding PIE are equivalent, in a sense, the PDE solution exists if and only if the PIE solution exists.

$$U_{1i} = \text{col}(I_{n_i}, 0_{n_{i+1:2} \times n_i}) \quad Q(s) = \begin{bmatrix} 0 & I_{n_1} & 0 \\ 0 & 0 & sI_{n_2} \\ 0 & 0 & I_{n_2} \end{bmatrix}$$

$$U_1 = \text{diag}(U_{10}, U_{11}, U_{12})$$

$$B_U(s) = B_I(s)U_1 - B \text{col}(0, Q(b-s))$$

$$B_Q(s) = B_T^{-1} \left( B_U(s) + \int_s^b B_I(\theta) U_2 Q(\theta-s) d\theta \right)$$

$$G_2(s, \theta) = \text{col}(0, [I_{n_1} \ (s-a)I_{n_2}] B_Q(\theta))$$

$$G_1(s, \theta) = \text{col}(0, [0_{n_0} \ I_{n_1} \ 0_{n_2}]) + G_2(s, \theta)$$

$$R_b(s, \theta) = U_2 T(s-a) B_Q(\theta) \quad G_0 = \text{diag}(I_{n_0}, 0_{(n_S-n_0)})$$

$$R_a(s, \theta) = R_b(s, \theta) + U_2 Q(s-\theta) \quad \hat{A}_0(s) = A_0(s)U_1$$

$$\hat{A}_1(s, \theta) = A_0(s)R_a(s, \theta) + A_1(s, \theta)U_1$$

$$+ \int_a^\theta A_1(s, \beta)R_b(\beta, \theta) d\beta + \int_\theta^s A_1(s, \beta)R_a(\beta, \theta) d\beta$$

$$+ \int_s^b A_2(s, \beta)R_a(\beta, \theta) d\beta$$

$$\hat{A}_2(s, \theta) = A_0(s)R_b(s, \theta) + A_2(s, \theta)U_1$$

$$+ \int_a^s A_1(s, \beta)R_b(\beta, \theta) d\beta + \int_s^\theta A_2(s, \beta)R_b(\beta, \theta) d\beta$$

$$+ \int_\theta^b A_2(s, \beta)R_a(\beta, \theta) d\beta$$

$$\mathcal{T} = \mathcal{P}_{\{G_i\}} \quad \mathcal{A} = \mathcal{P}_{\{\hat{A}_i\}}$$

Fig. 1. Definitions based on PDE defined on an interval  $[a, b]$  with system parameters given by  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$ .

**Theorem 1.** *Given a set of PDE parameters  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$  with  $\{n, \mathbf{G}_b\}$  admissible, let the PI operators  $\{\mathcal{T}, \mathcal{A}\}$  be as defined in Figure 1. Then, for any  $\mathbf{x}^0 \in X$  (as defined in Equation (6)),  $\mathbf{x}$  satisfies the PDE defined by  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$  with initial condition  $\mathbf{x}^0$  if and only if  $\mathcal{D}\mathbf{x}$  satisfies the PIE defined by  $\{\mathcal{T}, \mathcal{A}\}$  with initial condition  $\mathcal{D}\mathbf{x}^0 \in L_2^{n_x}$  where  $\mathcal{D}\mathbf{x} = \text{col}(\partial_s^0 \mathbf{x}_0, \partial_s \mathbf{x}_1, \partial_s^2 \mathbf{x}_2)$ .*

*Proof.* The proof is simply a matter of using the definitions of  $\{\mathcal{T}, \mathcal{A}\}$  and the definition of solutions for the PDE and PIE. The proof is similar to the proof of Theorem 12 in [11] with  $n = \{n_0, n_1, n_2\}$ ,  $\hat{\mathcal{T}} = \mathcal{T}$ ,  $\hat{\mathcal{A}} = \mathcal{A}$  and other unused parameters set to empty sets or zeros.  $\square$

The above result provides a map from the PDE solution,  $\mathbf{x}$ , to the solution of the corresponding PIE,  $\mathcal{D}\mathbf{x}$  and explicit formulae to obtain the PIE representation given a PDE. An inverse map, given by  $\mathcal{T}$ , also exists that maps the solution of the PIE back to the PDE solution as shown below.

**Theorem 2.** *Given an  $n$ , and  $\mathbf{G}_b$  with  $\{n, \mathbf{G}_b\}$  admissible, let  $\mathcal{T}$  be as defined in fig. 1,  $X$  as defined in Eq. (6) and  $\mathcal{D} := \text{diag}(\partial_s^0 I_{n_0}, \partial_s I_{n_1}, \partial_s^2 I_{n_2})$ . Then we have the following.*

- (a) *If  $\mathbf{x} \in X$ , then  $\mathcal{D}\mathbf{x} \in L_2^{n_x}$  and  $\mathbf{x} = \mathcal{T}\mathcal{D}\mathbf{x}$ .*
- (b) *If  $\hat{\mathbf{x}} \in L_2^{n_x}$ , then  $\hat{\mathcal{T}}\hat{\mathbf{x}} \in X$  and  $\hat{\mathbf{x}} = \mathcal{D}\hat{\mathcal{T}}\hat{\mathbf{x}}$ .*

*Proof.* This can be verified by substituting the definitions of  $\mathcal{D}$  and  $\mathcal{T}$ . The proof is similar to the proof of Theorem 10 in [11] with  $n = \{n_0, n_1, n_2\}$ ,  $\hat{\mathcal{T}} = \mathcal{T}$  and other unused parameters set to empty sets or zeros.  $\square$

This bijective mapping ensures the existence of both solutions when one of them exists. Given the PDE parameters  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$  we now have a set of formulae to find the corresponding PIE parameters  $\{\mathcal{T}, \mathcal{A}\}$ .

## VI. EQUIVALENCE OF REPRESENTATIONS: PDE AND PIE

In this section, we show that the solutions to the two representations (PDE and PIE) have equivalent stability properties and present a solvable optimization problem to prove stability. However, we have to define a notion of stability for the two representations.

### A. Definitions of stability

We define the notion of stability of a PDE based on the stability of  $\mathbf{x}$  that satisfies the PDE for given initial conditions where the stability is defined w.r.t. the standard Sobolev norm  $H$  which is defined as  $\|\mathbf{x}\|_H = \sum_{i=0}^2 \|\mathbf{x}_i\|_{W_i}$ .

**Definition 6** (Exponential Stability of a PDE). *We say a PDE defined by  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$  is exponentially stable, if there exists constants  $M, \alpha > 0$  such that for any  $\mathbf{x}^0 \in X$ , if  $\mathbf{x}$  satisfies the PDE defined by  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$  with initial condition  $\mathbf{x}^0$  then  $\|\mathbf{x}(t)\|_H \leq M \|\mathbf{x}^0\|_H e^{-\alpha t}$  for all  $t \geq 0$ .*

Similar to the stability of a PDE, we can define the stability of a PIE system based on stability  $\mathbf{x}_f$  that satisfies the PIE for some initial conditions and zero inputs. Unlike the stability of PDE, the stability of a PIE is defined with respect to the  $L_2$ -norm since solutions of PIE need not have spatial continuity.

**Definition 7** (Exponential Stability of a PIE). *We say a PIE defined by  $\{\mathcal{T}, \mathcal{A}\}$  is exponentially stable, if there exists constants  $M, \alpha > 0$  such that for any  $\mathbf{x}_f^0 \in L_2^n$ , if  $\mathbf{x}_f$  satisfies the PIE defined by  $\{\mathcal{T}, \mathcal{A}\}$  with initial condition  $\mathbf{x}_f^0$ , then  $\|\mathbf{x}_f(t)\|_{L_2} \leq M \|\mathbf{x}_f^0\|_{L_2} e^{-\alpha t}$  for all  $t \geq 0$ .*

The goal now is to show that for any  $\mathbf{x}$  that satisfies the PDE and  $\mathbf{x}_f$  that satisfies the PIE, if  $\mathbf{x} = \mathcal{T}\mathbf{x}_f$  with  $\mathcal{T}$  invertible, then  $\mathbf{x}$  decays exponential if and only if  $\mathbf{x}_f$  decays exponentially. The main hurdle in proving this is that these two notions of stability are defined using different norms ( $\|\cdot\|_H$  and  $\|\cdot\|_{L_2}$ ). For any  $\mathbf{x}$  and  $\mathbf{x}_f$  such that  $\mathbf{x} = \mathcal{T}\mathbf{x}_f$ , we know that  $\|\mathbf{x}\|_H \leq c$  implies  $\|\mathcal{T}\mathbf{x}_f\|_{L_2} < c$ , but the converse is typically not true. To prove the converse implication we use a new norm on the space  $X$ , denoted by  $\|\cdot\|_X$ , to show that:

- 1)  $\mathcal{T}$  is a norm-preserving bijection from  $L_2$  to  $X$  (when equipped with  $\|\cdot\|_X$ ).
  - $X$  is closed under  $\|\cdot\|_X$  (and  $\mathcal{T}$  is unitary)
- 2)  $\|\cdot\|_H$  is equivalent to  $\|\cdot\|_X$  on the subspace  $X$  (thus stability of PDE w.r.t.  $\|\cdot\|_H$  is equivalent to stability of PIE w.r.t.  $\|\cdot\|_T$ )
- 3) A PDE  $\{n, \mathbf{G}_p, \mathbf{G}_b\}$  is exponentially stable if and only if the corresponding PIE defined by  $\{\mathcal{T}, \mathcal{A}\}$  is exponentially stable.

### B. The map $\mathcal{T}$ are unitary

First, we would like to show that  $X$  is complete w.r.t. the norm  $\|\cdot\|_X$ . Previously, in Theorem 2, we showed that  $\mathcal{T}$  is invertible. Therefore, we must show that  $\mathcal{T}$  preserves the inner product. However, we first define the new  $X$ -inner product as  $\langle \mathbf{x}, \mathbf{y} \rangle_X := \sum_{i=0}^2 \langle \partial_s^i \mathbf{x}_i, \partial_s^i \mathbf{y}_i \rangle_{L_2} = \langle \mathcal{D}\mathbf{x}, \mathcal{D}\mathbf{y} \rangle_{L_2}$ .

**Theorem 3.** *Suppose  $\{n, \mathbf{G}_b\}$  is admissible,  $\mathcal{T}$  is as defined in Figure 1. Then, for any  $\mathbf{x}, \mathbf{y} \in L_2^{n \times}$  we have  $\langle \mathcal{T}\mathbf{x}, \mathcal{T}\mathbf{y} \rangle_X = \langle \mathbf{x}, \mathbf{y} \rangle_{L_2}$ .*

*Proof.* The proof follows directly from the definition of the  $X$  inner product and the map  $\mathcal{T}$ . The proof is similar to the proof of Theorem 18 in [11] with  $n = \{n_0, n_1, n_2\}$ ,  $\hat{\mathcal{T}} = \mathcal{T}$  and other unused parameters set to empty sets or zeros.  $\square$

Next, we see that norms induced by the inner products  $\langle \cdot, \cdot \rangle_X$  and  $\langle \cdot, \cdot \rangle_H$  on  $X$  are equivalent and, consequently, notions of stability w.r.t. these norms will be equivalent.

**Lemma 4.** *Suppose pair  $\{n, \mathbf{G}_b\}$  is admissible. Then, for any  $\mathbf{x} \in X$ ,  $\|\mathbf{x}\|_X \leq \|\mathbf{x}\|_H$  and there exists a constant  $c_0 > 0$  such that  $\|\mathbf{x}\|_H \leq c_0 \|\mathbf{x}\|_X$ .*

*Proof.* The proof is same as the proof of Lemma 17 in [11] with  $n = \{n_0, n_1, n_2\}$  and other unused parameters set to empty sets or zeros.  $\square$

Having established that  $X$ -norm can be upper bounded by  $H$ -norm, we can now prove the equivalence of the stability of PDE and PIE because the spaces  $X$  and  $L_2$ -norm are isometric.

**Theorem 5.** *Given PDE system parameters  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$  with  $\{n, \mathbf{G}_b\}$  admissible, suppose  $\{\mathcal{T}, \mathcal{A}\}$  are as defined in Figure 1. Then, the PDE defined by  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$  is exponentially stable if and only if the PIE defined by  $\{\mathcal{T}, \mathcal{A}\}$  is exponentially stable.*

*Proof.* The proof is a direct application of the stability definitions. The proof is same as the proof of Theorem 22 in [11] with  $n = \{n_0, n_1, n_2\}$  and other unused parameters set to empty sets or zeros.  $\square$

Using the above results, we propose the following optimization problem to test the stability of a PDE with integral terms.

**Theorem 6.** *Given a set of PDE parameters  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$ , suppose there exist  $\alpha, \delta > 0$ , and matrix-valued polynomials  $R_0, R_1, R_2, H_0, H_1, H_2$  such that  $\mathcal{P}_{\{R_i\}}, \mathcal{P}_{\{H_i\}} \in [\Pi_3]_{n_x, n_x}^+$ ,  $\mathcal{P}_{\{R_i\}} \geq \alpha I$ ,  $\mathcal{P}_{\{H_i\}} \geq \delta \mathcal{T}^* \mathcal{T}$  and  $\mathcal{P}_{\{H_i\}} = -(\mathcal{T}^* \mathcal{P}_{\{R_i\}} \mathcal{A} + \mathcal{A}^* \mathcal{P}_{\{R_i\}} \mathcal{T})$  where  $\{\mathcal{T}, \mathcal{A}\}$  are as defined in Figure 1. Then, the PDE defined by  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$  is exponentially stable.*

*Proof.* Suppose  $R_i$  and  $H_i$  are as stated above. Suppose  $\mathbf{x}$  solves the PDE defined by  $\{n, \mathbf{G}_b, \mathbf{G}_p\}$  for some initial condition  $\mathbf{x}^0 \in X$ . Then  $\mathbf{x}_f := \mathcal{D}\mathbf{x}$  solves the PIE defined by  $\{\mathcal{T}, \mathcal{A}\}$  for initial condition  $x^0$ . Let the Lyapunov function candidate be  $V(\mathbf{x}_f) = \langle \mathcal{T}\mathbf{x}_f, \mathcal{P}_{\{R_i\}} \mathcal{T}\mathbf{x}_f \rangle_{L_2}$ . Then  $V(\mathbf{x}_f) \geq \alpha \|\mathcal{T}\mathbf{x}_f\|_{L_2}^2$  for all  $\mathbf{x}_f \in L_2$ . Taking the derivative of  $V$  with respect to time along the solution trajectories of the PIE, we have

$$\begin{aligned} \dot{V}(t) &= \langle \mathcal{T}\dot{\mathbf{x}}_f(t), \mathcal{P}_{\{R_i\}} \mathcal{T}\mathbf{x}_f(t) \rangle_{L_2} + \langle \mathcal{T}\mathbf{x}_f(t), \mathcal{P}_{\{R_i\}} \mathcal{T}\dot{\mathbf{x}}_f(t) \rangle_{L_2} \\ &= \langle \mathcal{A}\mathbf{x}_f(t), \mathcal{P}_{\{R_i\}} \mathcal{T}\mathbf{x}_f(t) \rangle_{L_2} + \langle \mathcal{T}\mathbf{x}_f(t), \mathcal{P}_{\{R_i\}} \mathcal{A}\mathbf{x}_f(t) \rangle_{L_2} \\ &= \langle \mathbf{x}_f(t), (\mathcal{A}^* \mathcal{P}_{\{R_i\}} \mathcal{T} + \mathcal{T}^* \mathcal{P}_{\{R_i\}} \mathcal{T}) \mathbf{x}_f(t) \rangle_{L_2} \\ &= -\langle \mathbf{x}_f(t), \mathcal{P}_{\{H_i\}} \mathbf{x}_f(t) \rangle_{L_2} \leq -\delta \|\mathcal{T}\mathbf{x}_f(t)\|_{L_2}^2. \end{aligned}$$

Then, from Gronwall-Bellman inequality,

$$\|\mathbf{x}_f(t)\|_{L_2}^2 \leq \frac{k}{\alpha} \|\mathcal{D}\mathbf{x}^0\|_{L_2}^2 \exp(-\delta \xi t)$$

where  $k = \|\mathcal{T}^* \mathcal{P}_{\{R_i\}} \mathcal{T}\|_{L(L_2)}$  and  $\xi = \|\mathcal{T}\|_{L(L_2)}$ . Since the initial condition was an arbitrary function, the above inequality is satisfied for any  $\mathcal{D}\mathbf{x}^0 \in L_2$ , hence the PIE defined by  $\{\mathcal{T}, \mathcal{A}\}$  is exponentially stable and, from Theorem 5, the PDE is exponentially stable.  $\square$

See [13], for a parametric form of  $R_i$  and  $H_i$  that allows the use of Linear Matrix Inequalities to enforce positivity constraint. Once the positivity constraint is rewritten as LMI constraints, we can use an SDP solver to find  $R_i$  and  $H_i$  that satisfy the constraints of the above theorem.

## VII. NUMERICAL EXAMPLE

This section presents numerical tests for the stability of the two PDEs introduced earlier. The steps involved in setting up the computational problem — defining the PDE

parameters, converting to PIE form, setting up the operator-valued optimization problem, and converting the operator-valued optimization problem to an LMI feasibility test — are all performed using the PIETOOLS toolbox for MATLAB.

### A. Population Dynamics

Recall the McKendrick PDE model for the population dynamics introduced in Example 1 of Section I

$$\begin{aligned}\dot{x}(t, s) &= -\partial_s x(t, s) + f(s)x(t, s), \quad s \in [0, 1], \\ x(t, 0) &= \int_0^1 h(s)x(t, s)ds.\end{aligned}$$

We will select the kernel in the integral term of the boundary condition as  $h(s) = (1 - s)s$ , which implies that the population outside some normalized limits  $[0, 1]$  does not contribute to the birth of newborns. We will employ a constant mortality rate  $f(s) = c$  and vary the  $c \in \mathbb{R}$  to find the mortality rate,  $c_0$  below which the population would go extinct (i.e.,  $\lim_{t \rightarrow \infty} x(t, \cdot) = 0$ ).

By testing the stability of the above PDE for various  $c$  values (using the method of bijection), we determined that for mortality rates greater than  $-0.740625$ , the population goes extinct, i.e., the population will survive when the population growth rate  $f(s) > 0.740625$ .

### B. Observer-based control of reaction-diffusion equation

For the second test, recall the reaction-diffusion PDE from Example 2 of Section I. We know that the observer gain,  $l$ ,

$$l(s) = -\sqrt{\lambda} \frac{I_1\left(\sqrt{\lambda(1-s^2)}\right)}{\sqrt{1-s^2}},$$

provides a stable boundary observer [14]. Since we are interested in the practical implementation, we will approximate the gains  $l$  by a polynomial of a fixed order  $n$  denoted by  $l_n$ . Furthermore, the boundary measurements are replaced by an integral. While a typical approach is to replace point measurements by an integral with a Gaussian kernel centered at the point, in this example specifically, we can use the Fundamental Theorem of Calculus to rewrite the point measurement  $\partial_s x(t, 1)$  exactly as

$$\partial_s x(t, 1) = \partial_s x(t, 0) + \int_0^1 \partial_s^2 x(t, s)ds = \int_0^1 \partial_s^2 x(t, s)ds.$$

Likewise, we replace the point measurement value  $\partial_s \hat{x}(t, 1)$  by an integral to get the closed-loop observer PDE as

$$\begin{aligned}\dot{x}(t, s) &= \lambda x(t, s) + \partial_s^2 x(t, s), \quad s \in [0, 1], \\ \dot{\hat{x}}(t, s) &= \lambda \hat{x}(t, s) + \partial_s^2 \hat{x}(t, s) \\ &+ \int_0^1 l(s) (\partial_s^2 x(t, \theta) - \partial_s^2 \hat{x}(t, \theta)) d\theta, \\ x(t, 0) &= 0, \quad x(t, 1) = 0, \quad \hat{x}(t, 0) = 0, \quad \hat{x}(t, 1) = 0.\end{aligned}$$

Then, using the conversion formulae presented in fig. 1, we can find the PIE representation for the closed-loop PDE where  $l$  replaced by  $l_n$  which is the  $n^{th}$  order polynomial approximation of  $l$ . For  $\lambda \leq 5$ , we can prove that 1<sup>st</sup>-order polynomial approximation (a straight line) is sufficient to guarantee the stability of the closed-loop PDE system. However, as  $\lambda$  becomes larger higher order polynomial

approximation of  $l$  ( $l_4$  for  $\lambda = 6$  and so on) was necessary to prove the stability.

## VIII. CONCLUSION

A standard parametric form for linear PDEs in one spatial dimension with spatial integral terms and a sufficient criterion to guarantee the existence of a PIE representation for such PDEs were presented. We also revisited results showing the equivalence of solutions and stability properties of the two representations. Finally, using the equivalence in the stability properties, we formulated the test for the stability of PDEs as an optimization problem that can be solved using SDP solvers and demonstrated the application using numerical examples.

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