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Existence of Partially Quadratic Lyapunov Functions That Can Certify The Local Asymptotic Stability of Nonlinear Systems

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Abstract— **This paper proposes a method for certifying the local asymptotic stability of a given nonlinear Ordinary Differential Equation (ODE) by using Sum-of-Squares (SOS) programming to search for a partially quadratic Lyapunov Function (LF). The proposed method is particularly well suited to the stability analysis of ODEs with high dimensional state spaces. This is due to the fact that partially quadratic LFs are parametrized by fewer decision variables when compared with general SOS LFs. The main contribution of this paper is using the Center Manifold Theorem to show that partially quadratic LFs that certify the local asymptotic stability of a given ODE exist under certain conditions.**

I. Introduction

There is an abundance and diversity of applications found throughout science where a dynamical system is modelled as a nonlinear Ordinary Differential Equation (ODE). ODEs are at the core of many topics ranging from chaos theory, with the Lorenz equation [1], population dynamics [2], power systems [3] and many more. Understanding the long term properties of solutions to general ODEs is therefore of critical importance. Arguably the most fundamental and sort after long term property is that of local stability. A system described by an ODE is said to be locally asymptotically stable if solutions initialized near an equilibrium point remain near this equilibrium point for all time and furthermore converge towards this equilibrium point as time increases.

This paper considers the following problem: Given an ODE and its equilibrium point, certify whether or not this ODE is locally asymptotically stable. To solve this problem we take the approach that is perhaps the most universally used technique, Lyapunov's second method. This method certifies the stability of ODEs by finding a function satisfying certain properties called a Lyapunov Function (LF).

A common approach to numerically searching for LFs has been to use Sum-of-Square (SOS) programming [4]. Unfortunately, searching for SOS LFs is known to scale poorly with respect to the state space dimension of the system [5]. One approach to improving the scalability of SOS has been to decompose large scale systems into lower dimensional subsystems. Several methods exist that show that if a suitable decomposition can be found then the stability of the lower dimensional subsystems imply the stability of the original large scale system [6]–[8]. Unfortunately, these methods often lack generality assuming that the system has a certain structure that allows for such decompositions.

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Recently there has been significant interest in improving the scalability of SOS methods by searching for "separable" or "structured" LFs of the form,

$$V(x) = \sup_{1\leq i\leq n} V_i(x_i) \quad \text{or} \quad V(x) = \sum_{i=1}^{n} V_i(x_i).$$

Such separable LFs can be found in the works of [9]–[11] and in the related reachable set computation problems [12], [13]. These works demonstrate that searching for "structured" LFs improve numerical performance. It has been shown in [14] that monotone systems over compact state spaces posses max-separable LFs. However, in general, it is unknown for what class systems possess such "structured" LFs.

Inspired by the works of [15], [16] that use the Center Manifold Theorem to construct converse LFs with certain structure, in this paper we propose a new approach for certifying the local asymptotic stability of general high state space nonlinear ODEs by searching for partially quadratic LFs of the form,

$$V(x_1, x_2) = J(x_1) + x_2^\top H(x_1) + x_2^\top P x_2,$$

where $J : \mathbb{R}^k \to \mathbb{R}$, $H : \mathbb{R}^k \to \mathbb{R}^{(n-k)}$, $P \in \mathbb{R}^{(n-k)\times(n-k)}$ and $k \in \{1, \ldots, n\}$. Such LFs are partially quadratic since a subset of the state space variables, $x_2 \in \mathbb{R}^{(n-k)}$, appear in V with degree at most two. The main contribution of this paper is to provide several conditions under which it can be shown that partially quadratic LFs exist.

II. Notation

We denote a ball with radius $R > 0$ centred at the origin by $B_R(0) = \{x \in \mathbb{R}^n : x^\top x < R^2\}$. Let $C(X, Y)$ be the space of continuous functions with domain $X \subset \mathbb{R}^n$ and image $Y \subset \mathbb{R}^n$. We denote the set of differentiable functions by $C^i(X, Y) := \{f \in C(X, Y) : \Pi_{k=1}^n \frac{\partial^{\alpha_k} f}{\partial x_k^{\alpha_k}} \in C(X, Y)\ \forall \alpha \in \mathbb{N}^n$ such that $\sum_{j=1}^n \alpha_j \leq i\}$. For $V \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ we denote ∇V as the $n \times m$ matrix function such that $(\nabla V(x))_{i,j} = \frac{\partial V_j}{\partial x_i}(x)$. For $d \in \mathbb{N}$ and $x \in \mathbb{R}^n$ we denote $z_d(x)$ to be the vector of monomial basis in n-dimensions with maximum degree $d \in \mathbb{N}$. We denote the space of scalar valued polynomials $p : \mathbb{R}^n \to \mathbb{R}$ with degree at most $d \in \mathbb{N}$ by $\mathbb{R}_d[x]$. We say $p \in \mathbb{R}_d[x]$ is a Sum-of-Squares (SOS) polynomial if there exists $p_i \in \mathbb{R}_d[x]$ such that $p(x) = \sum_{i=1}^k (p_i(x))^2$. We denote Σ_{2d} to be the set of $2d$-degree SOS polynomials.

III. ODEs and Solution Maps

Consider a nonlinear Ordinary Differential Equation (ODE) of the form

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the vector field. Note that, throughout this paper we will assume $f(0) = 0$, implying the origin is an equilibrium point of the ODE (1).

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a) The Solution Map of ODEs: Given $D \subset \mathbb{R}^n$, and $I \subset [0, \infty)$ we say any function $\phi_f : D \times I \to \mathbb{R}^n$ satisfying

$$\frac{\partial \phi_f(x,t)}{\partial t} = f(\phi_f(x,t)),\ \phi_f(x,0) = x \text{ for } (x,t) \in D \times I, \tag{2}$$

is a solution map of the ODE (1) over $D \times I$. For simplicity throughout the paper we will assume there exists a unique solution map to the ODE (1) over all $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Note, if the vector field, f, is Lipschitz continuous then the solution map exists for some finite time interval, furthermore, this finite time interval can be arbitrarily extended if the solution map does not leave some compact set, see [15].

b) Stability of Nonlinear ODEs: We now use the solution map of the ODE (1) to define several notions of stability.

Definition 1: The equilibrium point $x = 0$ of ODE (1) is,

- Stable if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $||\phi_f(x,t)||_2 < \varepsilon$ for all $x \in B_\delta(0)$ and $t \geq 0$.
- Asymptotically stable if it is stable and there exists $\delta > 0$ such that $\lim_{t\to\infty} ||\phi_f(x,t)||_2 = 0$ for all $x \in B_\delta(0)$.
- Exponentially stable if there exists $\lambda, \mu > 0$ such that $||\phi_f(x,t)||_2 < \mu e^{-\lambda t}||x||_2$ for all $x \in B_\delta(0)$ and $t \geq 0$.

Given an ODE, if origin is an asymptotically stable equilibrium point of the ODE then we will say that the ODE is **locally asymptotically stable**.

c) Certifying the Stability of Nonlinear ODEs: In general there is no analytical expression for the solution map of a nonlinear ODE. Hence, directly certifying whether a nonlinear ODE is locally asymptotically stable by first finding the solution map, ϕ_f, and then showing $\lim_{t\to\infty} ||\phi_f(x,t)||_2 = 0$ over some set $B_\delta(0)$ is challenging. Fortunately, there exists several methods that can certify the local asymptotic stability of an ODE without first finding the solution map. Arguably, the most important of these methods, that we now state next, are Lyapunov's first and second methods.

Lemma 1 (Lyapunov's First Method): Consider an ODE (1) defined by some vector field $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $f(0) = 0$. Let

$$A := \begin{bmatrix} \frac{\partial f_1(0)}{\partial x_1} & \cdots & \frac{\partial f_1(0)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(0)}{\partial x_1} & \cdots & \frac{\partial f_n(0)}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{3}$$

It follows that,

- If all the real parts of the eigenvalues of A are negative then the ODE is locally asymptotically stable.
- If there exists an eigenvalue of A whose real part is positive then the ODE is **not** locally asymptotically stable.

From Lem. 1 we see that in the case when $A \in \mathbb{R}^{n\times n}$, given in Eq. (3), has an eigenvalue that is purely imaginative we are unable to use Lyapunov's first method to certify whether the associated ODE is locally asymptotically stable or not. For this case we can still certify local asymptotic stability using Lyapunov's Second Method, stated next.

Theorem 1 (Lyapunov's Second Method [17]): Consider an ODE (1) defined by some $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $f(0) = 0$. The ODE is locally asymptomatically stable if and only if there exists $R > 0$ and $V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ that satisfies,

$$V(0) = 0, \quad V(x) > 0 \text{ for all } x \in B_R(0)/\{0\}, \tag{4}$$

$$\nabla V(x)^\top f(x) < 0 \text{ for all } x \in B_R(0)/\{0\}. \tag{5}$$

In the special case of asymptotically stable linear systems, $\dot{x}(t) = Ax(t)$, it is well known that there exists a quadratic LF, $V(x) = x^\top P x$ where $P > 0$, and the Lyapunov condition of Thm. 1 reduces to the Matrix Equation (6) as shown in the next theorem.

Theorem 2 ([18]): For any symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$, the Lyapunov matrix equation,

$$A^\top P + PA = -Q, \tag{6}$$

has a unique symmetric positive definite solution $P \in \mathbb{R}^{n\times n}$ if every eigenvalue of $A \in \mathbb{R}^{n\times n}$ has strictly negative real part.

d) Coordinate changes for block diagonalization of linearization matrix: In order to state the main result in Thm. 3, that there exists a converse partially quadratic LF, we must first make a coordinate change to the ODE (1). This coordinate change will allow us to write the ODE as two coupled ODEs whose state variables will either appear quadratically or non-quadratically in our converse LF.

Since we are concerned with certifying whether the ODE (1) is locally stable, WLOG, we now assume that the associated linearization matrix, $A \in \mathbb{R}^{n\times n}$, given in Eq. (3), has $k \in \mathbb{N}$ purely imaginary eigenvalues and that the remaining eigenvalues of A have negative real parts. We assume this WLOG because in the case where all of the eigenvalues of A have negative real parts (i.e $k = 0$) we can certify that the ODE is locally stable by Lem. 1. Alternatively, if any of the eigenvalues of A have positive real part then by Lem. 1 we can certify that the ODE is not locally asymptotically stable. In both of these cases there would be no need to find a LF.

Now, for a matrix, $A \in \mathbb{R}^{n\times n}$, that has eigenvalues that are either purely imaginary or have negative real parts, Lemma 3 (found in the Appendix) shows that there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ for which

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{7}$$

where A_1 has only purely imaginary eigenvalues and A_2 has eigenvalues with only negative real part.

Note that for any vector field f with associated linearization matrix, $A \in \mathbb{R}^{n\times n}$, given in Eq. (3), it follows that $f(x) = Ax + \tilde{g}(x)$, where $\tilde{g}(x) := f(x) - Ax$ is such that $\frac{\partial}{\partial x_i}\tilde{g}(0) = 0$. Thus given an ODE (1), defined by a vector field f, WLOG we assume $f(x) = Ax + \tilde{g}(x)$ for some function $\tilde{g}$ such that $\frac{\partial}{\partial x_i}\tilde{g}(0) = 0$. Then, by making the coordinate change $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Tx$ to ODE (1) we can consider the equivalent nonlinear ODE:

$$\dot{z_1}(t) = A_1 z_1(t) + g_1(z_1(t), z_2(t)) \tag{8}$$

$$\dot{z_2}(t) = A_2 z_2(t) + g_2(z_1(t), z_2(t)), \tag{9}$$

where $A_1 \in \mathbb{R}^{k\times k}$ has purely imaginary eigenvalues, $A_2 \in \mathbb{R}^{(n-k)\times(n-k)}$ has eigenvalues with only negative real

part, $g_1 \in C^1(\mathbb{R}^k \times \mathbb{R}^{n-k}, \mathbb{R}^k)$ is such that $\frac{\partial}{\partial x_i} g_1(0) = 0$ for $i \in \{1, \ldots, n\}$, $g_2 \in C^1(\mathbb{R}^k \times \mathbb{R}^{n-k}, \mathbb{R}^{n-k})$ is such that $\frac{\partial}{\partial x_i} g_2(0) = 0$ for $i \in \{1, \ldots, n\}$, $k = \sum_{\lambda \in S} Dim\Big(Ker(\lambda I - A)^{m(\lambda)}\Big)$, $m(\lambda)$ is the algebraic multiplicity of eigenvalue λ, and $S \subset \mathbb{C}$ is the set of distinct eigenvalues of A with zero real part.

IV. Converse Partially Quadratic LFs

We now use the Center Manifold Theorem (Thm. 4 found in the Appendix) to prove the main result of the paper, Thm. 3, that shows that under certain conditions there exists a partially quadratic LF. Before stating Thm. 3 we first give a preliminary result. This preliminary result shows that the conditions required in our main result are satisfied by some commonly encountered systems. Note that similar conditions appear in [19].

Lemma 2: Consider an ODE (1) defined by a vector field f. Suppose one or more of the following statements holds:

- The ODE is locally exponentially stable (Def. 1).
- The ODE is a gradient system. That is its vector field is of the form $f(y) = -\nabla V(y)$, where $V : \mathbb{R}^n \to \mathbb{R}$ is some function that satisfies $\nabla V(0) = 0$, $V(y) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(y) = 0$ if and only if $x = 0$.
- The ODE is locally asymptotically stable with a one dimensional state space.

Then there exists a radius, $R > 0$, and a LF, $W \in C^1(\mathbb{R}^k, \mathbb{R})$, satisfying

$$\begin{aligned} &W(y) \geq 0 \text{ for all } y \in B_R(0), \\ &W(y) = 0 \text{ if and only if } y = 0, \\ &\nabla W(y)^\top f(y) < -c_1 \alpha(||y||_2)^2 \text{ for all } y \in B_R(0), \\ &||\nabla W(y)||_2 < c_2 \alpha(||y||_2) \text{ for all } y \in B_R(0), \end{aligned} \tag{10}$$

where $c_1, c_2 \in [0, \infty)$ and $\alpha : [0, \infty) \to [0, \infty)$ is such that $\alpha(0) = 0$.

Proof: Suppose the ODE is locally exponentially stable. Then by Corollary 77 from Page 245 in [20] there exists a LF that satisfies Eq. (10) with $\alpha(y) := y$.

Next, suppose the ODE is a gradient system. Then $W(y) := V(y)$ satisfies Eq. (10) with $\alpha(y) := ||\nabla V(y)||_2$.

Finally, suppose the ODE has state space dimension equal to one. By defining $V(y) = -\int_0^y f(x)dx$ we see that the ODE is a gradient system. Hence, Eq. (10) is satisfied. ■

We now show that for the ODE given in Eqs. (8) and (9) if Eq. (10) holds for the associated reduced ODE (30) then there exists a partially quadratic LF of the form given in Eq. (15) that can certify the local asymptotic stability of the ODE.

Theorem 3 (Existence of converse partially quadratic LFs): Consider an ODE given by Eqs. (8) and (9) and the associated reduced ODE (30). Suppose there exists a function $W \in C^1(\mathbb{R}^k, \mathbb{R})$ satisfying Eq. (10) for the vector field f of the reduced ODE (30). Then, the ODE given by Eqs. (8) and (9) is locally asymptotically stable if and only if there exists a matrix $P > 0$, a scalar $R > 0$ and functions $J \in C^1(\mathbb{R}^k, \mathbb{R})$ and $H \in C^1(\mathbb{R}^k, \mathbb{R}^{(n-k)})$ such that

$$PA_2 + A_2^\top P = -I, \tag{11}$$

and

$$V(z_1, z_2) > 0 \text{ for all } (z_1, z_2) \in B_R(0)/\{0\}, \tag{12}$$

$$V(0, 0) = 0, \tag{13}$$

$$\nabla V(z_1, z_2)^\top \begin{bmatrix} A_1 z_1 + g_1(z_1, z_2) \\ A_2 z_2 + g_2(z_1, z_2) \end{bmatrix} < 0 \quad \text{for all } (z_1, z_2) \in B_R(0)/\{0\}. \tag{14}$$

$$\text{where } V(z_1, z_2) := J(z_1) + z_2^\top H(z_1) + z_2^\top P z_2. \tag{15}$$

Proof: First suppose that there exists a matrix $P > 0$, a scalar $R > 0$ and functions $J \in C^1(\mathbb{R}^k, \mathbb{R})$ and $H \in C^1(\mathbb{R}^k, \mathbb{R}^{(n-k)})$ such that Eqs. (11), (12), (13) and (14) hold, where V is given by Eq. (15). Now, it follows that V is a LF for the ODE given by Eqs. (8) and (9) and hence this ODE is locally asymptotically stable by Thm. 1.

On the other hand let us now suppose the ODE given by Eqs. (8) and (9) is locally asymptotically stable. Consider the following function,

$$V(z_1, z_2) = W(z_1) + (z_2 - \eta(z_1))^\top P (z_2 - \eta(z_1)),$$

where W satisfies Eq. (10) for some radius $R_1 > 0$ and for the vector field of the reduced ODE (30), given by $f(y) = A_1 y(t) + g_1(y(t), \eta(y(t)))$, where η satisfies PDE (29) for some radius $R_2 > 0$ (known to exist by Thm. 4), and $P > 0$ is such that

$$PA_2 + A_2^\top P = -I. \tag{16}$$

Note that such a $P > 0$ exists by Thm. 2 since A_2 is defined in Eq. (9) to have eigenvalues with only negative real part.

Now, it clearly follows by multiplying out the quadratic terms in V that,

$$V(z_1, z_2) = W(z_1) + \eta(z_1)^\top P \eta(z_1) - 2 z_2^\top P \eta(z_1) + z_2^\top P z_2. \tag{17}$$

Hence, V satisfies Eq. (15) with $J(z_1) = W(z_1) + \eta(z_1)^\top P \eta(z_1)$ and $H(z_1) = -2P\eta(z_1)$.

We next show that V satisfies Eqs. (12) and (13). The function V comprises of the sum of two positive terms and thus it is clear $V(z_1, z_2) \geq 0$ for all $(z_1, z_2) \in B_\delta(0)$. Clearly $V(z_1, z_2) = 0$ if and only if both of these positive terms are zero. Now, $W(z_1) = 0$ if and only if $z_1 = 0$ and $(z_2 - \eta(z_1))^\top P (z_2 - \eta(z_1)) = 0$ if and only if $z_2 = \eta(z_1)$. If $z_1 = 0$ and $z_2 = \eta(z_1)$ then $z_2 = \eta(0) = 0$ (note that $\eta(0) = 0$ by Theorem 4). Therefore $V(z_1, z_2) = 0$ if and only if $(z_1, z_2) = 0$.

We next show that V satisfies Eq. (14). First note that g_1 and g_2 defined in Eqs. (8) and (9) are such that $\nabla g_1(0, 0) = 0$ and $\nabla g_2(0, 0) = 0$. Then by Lem. 4 (found in the Appendix) it follows that for $\varepsilon := \frac{1}{2} \min\{\frac{2c_1}{c_2^2}, \frac{2}{1 + 4\lambda_{max}}\} > 0$, where $\lambda_{max} > 0$ is the largest eigenvalue of $P > 0$, there exists $R_3 > 0$ such that

$$\begin{aligned} &||g_1(u_1, u_2) - g_1(v_1, v_2)||_2 < \varepsilon ||(u_1, u_2) - (v_1, v_2)||_2 \\ &||g_2(u_1, u_2) - g_2(v_1, v_2)||_2 < \varepsilon ||(u_1, u_2) - (v_1, v_2)||_2 \\ &\qquad \text{for all } (u_1, u_2), (u_1, u_2) \in B_{R_3}(0). \end{aligned} \tag{18}$$

It now follows from application of Eq. (18) that,

$$\begin{aligned}
&\nabla V(z_1,z_2)^\top \begin{bmatrix} A_1z_1+g_1(z_1,z_2) \\ A_2z_2+g_2(z_1,z_2)\end{bmatrix} \qquad (19)\\
&=\nabla W(z_1)^\top(A_1z_1+g_1(z_1,z_2))\\
&\quad-2(z_2-\eta(z_1))^\top P\nabla\eta(z_1)^\top(A_1z_1+g_1(z_1,z_2))\\
&\quad+2(z_2-\eta(z_1))^\top P(A_2z_2+g_2(z_1,z_2))\\
&=\nabla W(z_1)^\top(A_1z_1+g_1(z_1,\eta(z_1)))\\
&\quad+\nabla W(z_1)^\top(g_1(z_1,z_2)-g_1(z_1,\eta(z_1)))\\
&\quad-2(z_2-\eta(z_1))^\top P(A_2\eta(z_1)+g_2(z_1,\eta(z_1)))\\
&\quad+(z_2-\eta(z_1))^\top(PA_2+A_2^\top P)(z_2-\eta(z_1))\\
&\quad+2(z_2-\eta(z_1))^\top P(A_2\eta(z_1)+g_2(z_1,z_2))\\
&<-c_1\alpha(||z_1||_2)^2+||\nabla W(z_1)||_2||g_1(z_1,z_2)-g_1(z_1,\eta(z_1))||_2\\
&\quad-||z_2-\eta(z_1)||_2^2+2(z_2-\eta(z_1))^\top P(g_2(z_1,z_2)-g_2(z_1,\eta(z_1)))\\
&\le-c_1\alpha(||z_1||_2)^2+\left(c_2\sqrt{\varepsilon}\alpha(||z_1||_2)\right)\left(\sqrt{\varepsilon}||z_2-\eta(z_1)||_2\right)\\
&\quad-||z_2-\eta(z_1)||_2^2+2\varepsilon\lambda_{max}||z_2-\eta(z_1)||_2^2\\
&\le-\left(c_1-\frac{c_2^2\varepsilon}{2}\right)\alpha(||z_1||_2)^2-\left(1-\frac{\varepsilon(1+4\lambda_{max})}{2}\right)||z_2-\eta(z_1)||_2^2\\
&<0 \text{ for all } (z_1,z_2)\in B_R(0)/\{0\},
\end{aligned}$$

where $R=\min\{R_1,R_2,R_3\}>0$ and $\lambda_{max}>0$ is the largest eigenvalue of $P>0$.

The second equality from Eq. (19) follows from the application of PDE (29), found in Thm. 4 from the Appendix. The first inequality of Eq. (19) follows from Eqs. (10) and (16) and the Cauchy Schwarz inequality. The second inequality of Eq. (19) follows from Eq. (18). The third inequality of Eq. (19) using the inequality $xy\le\frac{x^2+y^2}{2}$ for all $x,y\in\mathbb{R}$. The fourth and final inequality in Eq. (19) follows since $\varepsilon:=\frac{1}{2}\min\{\frac{2c_1}{c_2^2},\frac{2}{1+4\lambda_{max}}\}$ and hence $c_1-\frac{c_2^2\varepsilon}{2}>0$ and $1-\frac{\varepsilon(1+4\lambda_{max})}{2}>0$.

Therefore it follows that V, given in Eq. (17), satisfies Eqs. (12), (13) and (14) for $R{=}\min\{R_1,R_2,R_3\}{>}0$. ∎

If there exists a function W satisfying Eq. (10) for the reduced ODE (30) then Thm. 3 shows that the ODE given in Eqs. (8) and (9) is locally asymptotically stable if and only if there exists a partially quadratic LF. Further to this Lem. 2 provides some sufficient conditions that guarantee the existence of such a function W. In the next corollary we combine these results to show the existence of partially quadratic LFs for systems whose linearization matrix has only one purely imaginary eigenvalue. This corollary provides the theoretical justification for the search of partially quadratic LFs to certify local asymptotic stability in our numerical examples in Sec. VI.

Corollary 1: Consider an ODE (1) with associated linearization matrix $A\in\mathbb{R}^{n\times n}$ defined in Eq. (3). Suppose there is a single purely imaginary eigenvalue of A and all other eigenvalues of A have negative real parts. Then the ODE is stable if and only if there exists a partially quadratic LF of the form given in Eq. (15) that satisfies Eqs (12), (13) and (14).

Proof: Follows using Prop. 1, Lem. 2 and Thm. 1. ∎

Note, the proof of the existence of partially quadratic LFs given in Thm. 3 is non-constructive, being based on the center manifold, $z_2=\eta(z_1)$, for which in general there is no analytical formula. In the special case where the center manifold is analytically known the proof becomes constructive. In the following illustrative example we use Eq. (17) to construct a partially quadratic LF without any computation. Later, in Sec. V we will consider high dimensional systems for which the center manifold is not known and hence such a LF cannot be constructively found. For such systems we will use numerical methods to search for partially quadratic LFs to certify local asymptotic stability.

a) An Illustrative Example: Consider the following ODE,

$$\begin{aligned}\dot{x}_1(t)&=-x_1(t)x_2(t)\\ \dot{x}_2(t)&=-x_2(t)+x_1(t)^2-2x_2(t)^2.\end{aligned}\qquad(20)$$

The associated linearization matrix, found in Eq. (3), is given by $A=\begin{bmatrix}0&0\\0&-1\end{bmatrix}$. The matrix $A\in\mathbb{R}^{2\times2}$ has eigenvalues 0 and -1 and thus ODE (20) cannot be certified as locally asymptotically stable using Lyapunov's first method (Lem. 1).

Note that without any coordinate transformations ODE (20) is already in the form of ODE given by Eqs. (8) and (9) with $A_1=0$ and $A_2=-1$. Thus setting $P=0.5>0$ it follows that $PA_2+A_2^\top P=-I$. It is shown in [21] that $\eta(y)=y^2$ gives the center manifold. Hence, the reduced ODE (30) associated with ODE (20) is,

$$\dot{y}(t)=-y(t)^3. \qquad (21)$$

ODE (21) has a one dimensional state space so by Lem. 2 it follows that there exists a function W satisfying Eq. (10). Specifically, if we let $W(y)=\frac{y^4}{4}$ it can be shown W satisfies Eq. (10) with $\alpha(y)=y^3$. The proof of Thm. 3 then shows that the function given in Eq. (17) is a LF. For this ODE this then implies that the following function is a LF to ODE (20),

$$V(x_1,x_2)=x_1^4/4+0.5(x_2-x_1^2)^2. \qquad (22)$$

Clearly, the LF given in Eq. (22) is partially quadratic since the x_2 terms appear with degree at most 2 while the x_1 terms can have degree greater 2. We have plotted the largest set of initial conditions that this LF can certify as asymptotically stable as the green region in Fig. 1a.

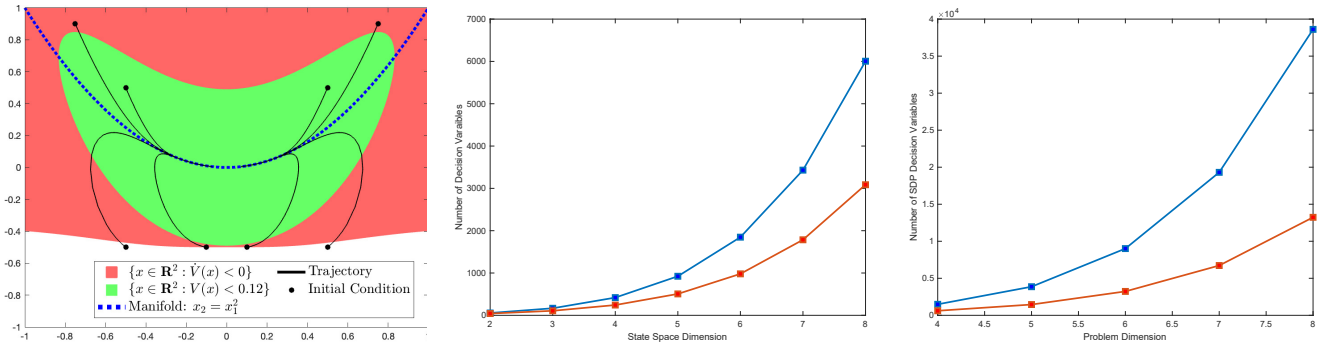
V. Using SOS to Certify Local Stability

Consider the problem of certifying the local stability of an ODE (1), defined by some vector field f. This problem can be solved by using Lyapunov's second method (Thm. 1). In cases where the vector field, f, is polynomial we can search for such a LF using SOS programming [4], [22]. We can find such LFs by solving the following $2d$-degree SOS feasibility problem:

Find: $V\in\mathbb{R}_{2d}[x], s_1,s_2,s_3\in\Sigma_{2d}$ such that,

$$\begin{aligned}&V(0)=0,\quad V(x)=s_1(x)+\varepsilon x^\top x \text{ for } x\in\mathbb{R}^n,\qquad(23)\\ &-\nabla V(x)^\top f(x)-s_2(x)(R^2-||x||_2^2)=s_3(x) \text{ for } x\in\mathbb{R}^n,\end{aligned}$$

where $R>0$ and $\varepsilon>0$. Note that $R>0$ is included in Opt. (23) so we only enforce V to be a LF locally (over the



(a) (b) (c)

Fig. 1: (a) Graph showing that the LF given in Eq. (22) certifies the local stability of ODE (20). The center manifold, $y = x^2$, is also plotted as the dotted blue line. Several trajectories of the ODE for various initial conditions are plotted as the black curves. (b) Graph showing that for ODE (25) the number of decision variables in the underlying SDP problem of Opt. (23), plotted as the blue curve, is larger than that of Opt. (24), plotted as the red curve. (c) Graph showing that for ODE (26) the number of decision variables in the underlying SDP problem of Opt. (23), plotted as the blue curve, is larger than that of Opt. (24), plotted as the red curve.

ball $B_R(0)$). Also note that $\varepsilon > 0$ is included in Opt. (23) to avoid the trivial solution $V(x) \equiv 0$. Typically $R > 0$ and $\varepsilon > 0$ are selected to be small, for instance $R = \varepsilon = 0.1$.

If ODE (1) is of the form given in Eqs. (8) and (9) then Thm. 3 indicates that we can certify local stability by searching for a partially quadratic LF of the form given in Eq. (15). This motivates the following $2d$-degree SOS feasibility problem:

$$\begin{aligned}&\text{Find: } J \in \mathbb{R}_{2d}[x_1], H_i \in \mathbb{R}_{2d}[x_1], P \in \mathbb{R}^{n\times n}, s_1, s_2, s_3 \in \Sigma_{2d}\\ &\text{such that, } V(0,0) = 0, \qquad (24)\\ &V(x_1,x_2) = s_1(x_1,x_2) + \varepsilon(x_1,x_2)^\top (x_1,x_2) \text{ for } x \in \mathbb{R}^n,\\ &-\nabla V(x_1,x_2)^\top \begin{bmatrix} A_1x_1 + g_1(x_1,x_2) \\ A_2x_2 + g_2(x_1,x_2)\end{bmatrix}\\ &\quad - s_2(x_1,x_2)(R^2 - ||(x_1,x_2)||_2^2) = s_3(x_1,x_2) \text{ for } x \in \mathbb{R}^n,\end{aligned}$$

where $V(x_1,x_2) = J(x_1) + x_2^T \begin{bmatrix} H_1(x_1) \\ \vdots \\ H_{n-k}(x_1)\end{bmatrix} + x_2^\top P x_2, \varepsilon > 0$ and $R > 0$.

Note, Opt. (24) can certify the local asymptotic stability of ODEs of the form given in Eqs. (8) and (9). General ODEs can be converted to be of this form using a coordinate change given in Eq. (7). This coordinate change can be numerically found using Matlab functions `jordan` and `cdf2rdf`.

Searching for partially quadratic LFs by solving Opt. (24) as opposed to searching for fully non-quadratic LFs by solving Opt. (23) results in computational savings due to the reduction in decision variables. These computational savings will be demonstrated through several numerical examples in the next section.

VI. Numerical Examples

Example 1 (The Generalized Lotka–Volterra equations): The competition of different groups (species, resources, etc) can be modelled by the following ODE,

$$\begin{aligned}&\dot{x}_i(t) = x_i(t) g(x(t)), \qquad (25)\\ &\text{where } g(x) = r + Bx, \quad r \in \mathbb{R}^n \text{ and } B \in R^{n\times n}.\end{aligned}$$

Clearly, ODE (25) has an equilibrium point at $0 \in \mathbb{R}^n$. The corresponding linearization matrix $A \in \mathbb{R}^{n\times n}$, given in Eq. (3), is such that $A_{i,j} = \begin{cases} r_i \text{ if } i = j \\ 0 \text{ otherwise.}\end{cases}$ Let us consider randomly generated values for r and B where $r_1 = 0$ and $r_i < 0$ for all $i \in \{2, \ldots, n\}$. Hence A has eigenvalues $\{r_1, \ldots, r_n\}$ which in this case are either purely imaginative or have negative real part. Thus we are unable to determine the stability of $x = 0$ by Lyapunov's first method. Cor. 1 shows this system is asymptotically stable iff there exists a partially quadratic LF. Solving Opts. (23) and (24) for $n = 2$ to 8 at $d = 6$, $R = 0.01$ and $\varepsilon = 0.00001$, allows us to find feasible LFs in each case. In Fig. 1b we have plotted the difference in the number of decision variables associated with each optimization problem. For $n = 8$ it took Yalmip [23] and Mosek 213s to solve Opt. (23) and 157s to solve Opt. (24).

Example 2 (Stable linear systems with nonlinear interconnection): Let us consider the following ODE system,

$$\begin{aligned}\dot{z}_1(t) &= g(z_1(t), z_2(t), z_3(t)), \qquad (26)\\ \dot{z}_2(t) &= Q_1 z_2(t), \quad \dot{z}_3(t) = Q_2 z_3(t),\end{aligned}$$

where $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is such that $\nabla g(0,0,0) = 0$ and where all the eigenvalues of $Q_1 \in \mathbb{R}^{n_1 \times n_1}$ and $Q_2 \in \mathbb{R}^{n_2 \times n_2}$ have negative real part. The linearization matrix, given in Eq. (3), for this system is then $A := \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_1 & 0 \\ 0 & 0 & Q_2\end{bmatrix}$. This matrix is already in the block diagonal form of Eq. (7) with $A_1 = 0$ and $A_2 = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2\end{bmatrix}$. Since A_1 is one dimensional by Cor. 1 the system is locally asymptotically stable iff there exists a partially quadratic LF for this system.

For simplicity we will consider the case $g(z_1, z_2, z_3) = z_1^2 + z_2^\top z_2 + z_3^\top z_3$, $Q_1 = -\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$ and $Q_2 = -I \in \mathbb{R}^{n\times n}$. For this ODE, $d = 8$, $R = 0.05$ and $\varepsilon = 0.0001$ we solve Opts. (23) and (24) for $n = 1$ to 5, finding a LF in each case. Fig. 1c shows how the number of decision variables grows for each problem. For $n = 5$ it took Yalmip [23] and Mosek 12645s to solve Opt. (23) and 10398s to solve Opt. (24).

VII. Conclusion

We have proposed conditions for which there exists a partially quadratic LF that can certify the local asymptotic stability of nonlinear ODEs. The existence proof was non-constructive, relying on the existence of the center manifold.

However, knowledge of the existence of partially quadratic LFs allows us to tighten our search of LFs, providing computational savings. This paper opens up many directions of future work such as investigating the conditions for which there exists a SOS partially quadratic LF and the conditions under which the proposed methods can be extended to global stability analysis.

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APPENDIX

Lemma 3: Suppose the matrix $A \in \mathbb{R}^{n\times n}$ has distinct eigenvalues $\{\lambda_1, ..., \lambda_p\} \subset \mathbb{C}$ for some $1 < p \leq n$. If sets $S_1, S_2 \subset \mathbb{C}$ are such that $S_1 \cup S_2 = \{\lambda_1, ..., \lambda_p\}$, $S_1 \cap S_2 = \emptyset$, and if $\lambda \in S_i$ then $\bar{\lambda} \in S_i$ for $i = 1, 2$. Then there exists a non-singular matrix $T \in \mathbb{R}^{n\times n}$ such that

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where the set of eigenvalues of $A_1 \in \mathbb{R}^{k\times k}$ is equal to S_1, the set of eigenvalues of $A_2 \in \mathbb{R}^{(n-k)\times(n-k)}$ is equal to S_2, $k = \sum_{\lambda\in S_1} Dim\left(Ker\left((\lambda I - A)^{m(\lambda)}\right)\right)$ and $m(\lambda)$ is the algebraic multiplicity of eigenvalue λ.

Proof: Apply Theorem 4.2 Page 257 from [24]. ■

Lemma 4: Consider $V \in C^1(\mathbb{R}^n, \mathbb{R})$. Let $K := \sup_{x\in B_R(0)} ||\nabla V(x)||_2 < \infty$. Then

$$||V(x) - V(y)||_2 \leq K||x - y||_2 \text{ for all } x, y \in B_R(0). \tag{27}$$

Furthermore, if $\nabla V(0) = 0$ then for any $\delta > 0$ there exists $R > 0$ such that

$$||V(x) - V(y)||_2 \leq \delta||x - y||_2 \text{ for all } x, y \in B_R(0). \tag{28}$$

Proof: By the Mean Value Theorem for any $x, y \in B_R(0)$ there exists $c \in (0, 1)$ such that

$$||V(x) - V(y)||_2 \leq ||\nabla V(cx + (1 - c)y)||_2||x - y||_2.$$

Then letting $K := \sup_{x\in B_R(0)} ||\nabla V(x)||_2$ it follows that $||\nabla V(cx + (1 - c)y)||_2 \leq K$ for all $x, y \in B_R(0)$ and $c \in (0, 1)$. Hence, Eq. (27) follows.

Now suppose $\nabla V(0) = 0$. Since $V \in C^1(\mathbb{R}^n, \mathbb{R})$ it follows that $F \in C(\mathbb{R}^n, \mathbb{R})$, where $F(x) := ||\nabla V(x)||_2$. Then for any $\delta > 0$ there exists $R > 0$ such that $|F(0) - F(x)| < \delta/2$ for all $||x||_2 < R^2$. Thus it follows,

$$||\nabla V(x)||_2 < \delta/2 \text{ for all } x \in B_R(0),$$

implying $K := \sup_{x\in B_R(0)} ||\nabla V(x)||_2 < \delta$. Hence, Eq. (28) holds. ■

Theorem 4 (The Center Manifold Theorem [15]): Consider an ODE given by the Eqs. (8) and (9) where $\frac{\partial}{\partial x_i} g_j(0) = 0$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$. There exists $R > 0$ and a function $\eta : C^\infty(B_R(0), \mathbb{R}^{n-k})$ such that

- The function η is such that $\eta(0) = 0$ and $\frac{\partial}{\partial y_i}\eta(0) = 0$.
- The function η solves the following system of PDEs,

$$A_2\eta(y) + g_2(x, \eta(y)) - \nabla\eta(y)^\top(A_1 y + g_1(y, \eta(y))) = 0$$
$$\text{for all } y \in \{x \in \mathbb{R}^{n-k} : ||x||_2 < R\}. \tag{29}$$

Proposition 1 ([15]): The ODE given by the Eqs. (8) and (9) where $\frac{\partial}{\partial x_i} g_j(0) = 0$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$ is locally asymptotically stable if and only if the following ODE is locally asymptotically stable,

$$\dot{z}_1(t) = A_1 z_1(t) + g_1(z_1(t), \eta(z_1(t))), \tag{30}$$

where η solves the PDE (29) (known to exist by Theorem 4).