Optimal Patrolling Strategies for Trees and Complete Networks

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Abstract

We present solutions to a continuous patrolling game played on network. In this zero-sum game, an Attacker chooses a time and place to attack a network for a fixed amount of time. A Patroller patrols the network with the aim of intercepting the attack with maximum probability. Our main result is the proof of a recent conjecture on the optimal patrolling strategy for trees. The conjecture asserts that a particular patrolling strategy called the *E-patrolling strategy* is optimal for all tree networks. The conjecture was previously known to be true in a limited class of special cases. The E-patrolling strategy has the advantage of being straightforward to calculate and implement. We prove the conjecture by presenting ε-optimal strategies for the Attacker which provide upper bounds for the value of the game that come arbitrarily close to the lower bound provided by the E-patrolling strategy. We also solve the patrolling game in some cases for complete networks.

Keywords: game theory, patrolling, zero-sum games, networks

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1 Introduction

In the continuous patrolling game, introduced by Alpern *et al.* (2016), an Attacker picks a point on a network Q and a time interval of fixed duration during which to carry out an attack. A Patroller moves on the network at unit speed and intercepts the attack (and wins the game) if she reaches the attacked point during the attack interval. Alpern *et al.* (2022) proposed a mixed strategy for the Patroller, called the *E-patrolling strategy*, which was shown to be optimal for certain classes of tree networks. In Conjecture 1 of that paper, they suggested that the E-patrolling strategy was optimal for all trees. We refer to this conjecture as the *tree patrolling conjecture*. In this paper we settle the tree patrolling conjecture by proving that the E-patrolling strategy is optimal for all tree networks. We also solve the game in certain cases for complete networks (those for which every pair of nodes is connected by precisely one arc).

The key idea we use to prove the conjecture for trees is that as long as the Attacker randomizes over a large enough time period, there are mixed strategies that are arbitrarily close to being optimal that simply pick the time of the attack uniformly over that period. This means that we need only specify a distribution over the network Q. We define a mixed strategy for the Attacker that is played over a large time interval $[0, T]$ and show that for any given $\varepsilon > 0$, this strategy is ε-optimal for large enough T.

Most work in the area of patrolling games focuses on discrete models, such as Alpern *et al.* (2018), Alpern *et al.* (2011), Lin *et al.* (2013), Pita *et al.* (2008), Yolmeh and Baykal-Gürsoy (2018) and Zoroa *et al.* (2012). A disadvantage of discrete models is that in many real world examples of patrolling, an attack or infiltration can occur anywhere continuously along a border, boundary or network. Discrete models also assume that attacks occur at discrete times, but of course it is more realistic to model time as continuous. This was the motivation behind the continuous patrolling game introduced by Alpern *et al.* (2016). As well as the recent work of Alpern *et al.* (2022) on the game, Garrec (2019) has also made some important contributions, including establishing that the game has a value and optimal (or ε-optimal) strategies. Lin (2019) studied a different continuous patrolling game on a perimeter.

The layout of the paper is as follows. In Section 2, we recall the definition of the continuous patrolling game and give some background on previous work on the game. We also describe the tree patrolling conjecture precisely. In Section 3, we work towards defining a decomposition of any tree Q which we call its *subtree decomposition*. This decomposition consists of a set of subtrees of length at most $\alpha/2$ containing all the leaf nodes and another connected set we call the *core*. We also define the concept of the *density* of a subset of a network, which, for a given Attacker strategy, is defined as the ratio of the probability the attack takes place in that subset to the length of the subset. This definition is analogous to the concept of *search density*, which is well known in the field of *search games*. The concept originates from the work of Gal (1979), but has been used more recently in, for example, Alpern and Lidbetter (2013), Fokkink *et al.* (2019) and Hermans *et al.* (2022). The ideas of density and the subtree decomposition are crucial for us to define in Section 4 the Attacker strategy that we proceed to show is ε-optimal. In Section 5 we solve the game on complete networks for some values of α. In Section 6 we conclude.

The significance of our main result on trees lies in the fact that the E-patrolling strategy is intuitive and easy to implement. Roughly speaking, the Patroller repeatedly tours the network, but performs extra tours of subtrees of the network that are close to the leaf nodes.

2 Background and Definitions

In this section we make some definitions and give some more background to the continuous patrolling game. We finish the section by stating the tree patrolling conjecture precisely.

We start by defining a network Q in a little more detail, though we refer the reader to Alpern *et al.* (2022) for a precise definition. A network Q is given by a multigraph whose arcs can be viewed as open intervals. The length of an arc a is denoted $\lambda(a)$, and λ is extended to define a measure on Q. At each end of an arc is a node, and we refer to points of Q that are not nodes as *regular*. We also define a metric d on Q, where $d(x, y)$ is the length of the shortest path between two points $x, y \in Q$.

In the continuous patrolling game on Q, the Attacker picks a point $x \in Q$ and a time $t \geq 0$ at which to start the attack. The attack lasts for time α, where $\alpha > 0$ is some parameter of the problem known to both players, and is no greater than the minimum tour time of Q. The Patroller picks a patrol of the network, which is given by a unit speed path $S : [0, \infty) \to Q$. If the patrol intercepts the attack, then the Patroller wins the game. More precisely, the payoff of the game is equal to 1 if $x \in S([t, t + \alpha])$, otherwise the payoff is 0. The Patroller is the maximizer and the Attacker is the minimizer.

As mentioned in the Introduction, the continuous patrolling game was introduced in Alpern *et al.* (2016). Garrec (2019) later proved that this zero-sum game has a value; moreover that the Patroller has optimal mixed strategies and the Attacker has ε-optimal mixed strategies (that is strategies that ensure the expected payoff is within ε of the value of the game, for any $\varepsilon > 0$). Garrec also found optimal strategies in the game in some special cases, as did Alpern *et al.* (2016).

Alpern *et al.* (2022) solved the game in some further special cases. Firstly, they gave a solution for arbitrary networks as long as α is shorter than the length of any arc of the network. Secondly, they gave a solution for tree networks when α is such that a particular condition called the *Leaf Condition* is satisfied. They defined a patrolling strategy called the E-patrolling strategy, and showed that it is optimal for trees that satisfy the Leaf Condition. They conjectured that the E-patrolling strategy is optimal for all tree networks (the tree patrolling conjecture). They verified their conjecture for a class of star networks consisting of one long arc and an arbitrary number of short arcs of equal length. They also verified it for one particular example of a tree network that is not a star and does not satisfy the Leaf Condition.

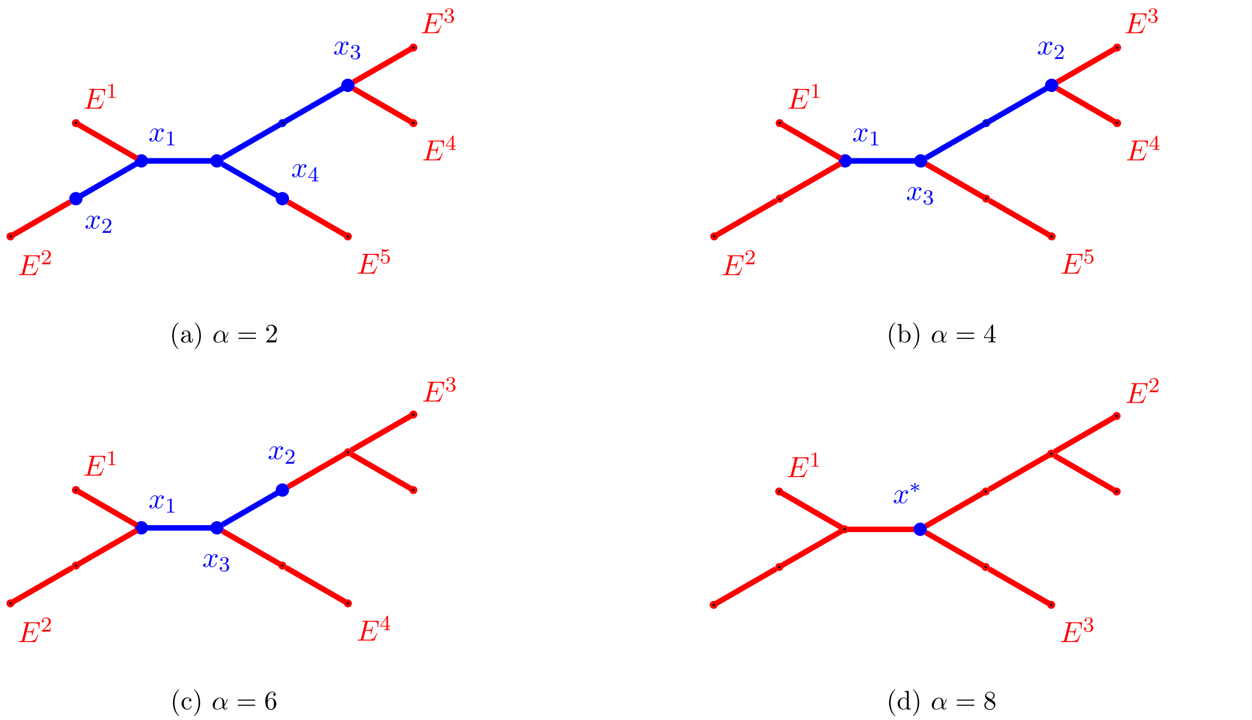
Generally speaking, the Leaf Condition is satisfied when α is particular small and, in the case of star networks, also when it is particularly large. This leaves a sizeable gap of values of α for which the optimality of the E-patrolling strategy was unproven. In Section 4, we settle the tree patrolling conjecture.

Of crucial importance to stating and proving the tree patrolling conjecture, we must first define the *extremity set* E for a tree network Q.

Let Q be a tree network of length μ. For any set of points Y, we denote Y^c for $Q - Y$ and $\overline{Y}$ for the topological closure of Y. If x is a regular point of Q, then $Q - \{x\}$ has two components $Q_1(x)$ and $Q_2(x)$ such that $\lambda(Q_1(x)) + \lambda(Q_2(x)) = \mu$, and $\min_{i=1,2} Q_i(x) \leq \mu/2$. If x is a node of degree n ($n \geq 3$), then $Q - \{x\}$ has n components.

Definition 1 Let Q be a tree. The extremity set $E \equiv E(Q, \alpha)$ is defined as the set of all regular points $x \in Q$ such that $\min_{i=1,2} \lambda(Q_i(x)) < \alpha/2$.

Although it is convenient to define E as an open set, we will largely work with its topological closure $\overline{E}$. In Figure 1 we depict the set $\overline{E}$ in red for various values of α on a specific tree network Q of length $\mu = 10$. Note that $\overline{E} = Q$ for $\alpha \geq 8$, and it is easy to see that in fact for any tree network Q, we have $\overline{E} = Q$ for all $\alpha > \mu$.



(a) $\alpha = 2$ (b) $\alpha = 4$ (c) $\alpha = 6$ (d) $\alpha = 8$

Figure 1: The components of $\overline{E}$ are shown in red and the core E^0 is shown in blue for $\alpha = 2, 4, 6, 8$. The local roots x^*, x_i $(i \geq 1)$ are labeled as blue points.

We make a number of observations about $\overline{E}$, which we state without proof.

Proposition 2 Let Q be a tree. Then

(i) $\overline{E}(Q, \alpha_1) \subseteq \overline{E}(Q, \alpha_2)$ for any $\alpha_1 \leq \alpha_2$;

(ii) there exists an unique α^* such that $\overline{E}(Q, \alpha^*) = Q$ and $\overline{E}(Q, \alpha) \neq Q$ for any $\alpha < \alpha^*$;

(iii) if $\overline{E}(Q, \alpha) \neq Q$, then the boundary of each maximal connected component X of $\overline{E}$ is a single point x, which we call the **local root** of X. When x is removed, the remaining disjoint components of X are subtrees of measure at most $\alpha/2$. We will also refer to x as the local root of these subtrees.

We have labeled the local roots $x_1, x_2, \ldots$ in Figure 1. Both the location and number of local roots may change as α changes. In the case $\alpha = 2$, the set $\overline{E}$ has four maximal connected components, and four corresponding local roots, x_1, x_2, x_3 and x_4. When $\alpha = 4$ or 6, the set $\overline{E}$ has only three maximal connected components with local roots x_1, x_2 and x_3. When $\alpha = 8$, the set $\overline{E}$ has only one maximal connected component. In this case, we have labeled the local root x^*, to be defined later in Subsection 3.1.

Alpern *et al.* (2022) showed that the E-patrolling strategy guarantees that the value of the continuous patrolling game on trees is at most $\alpha/(\mu+\lambda(E))$. Roughly speaking, the E-patrolling strategy repeatedly performs a tour of the tree, adding extra tours of each of the components of $\overline{E}$. For the details of the construction of the E-patrolling strategy, we refer the reader to Alpern *et al.* (2022). We give here an example of the E-patrolling strategy for the network of Figure 1(b) ($\alpha = 4$). For $i = 1, 2, 3$, let C_i be a minimum length tour of the maximal connected component of $\overline{E}$ with the local root x_i. For example, C_1 is a minimum length tour of subtree $E^1 \cup E^2$. Let S^E be a tour which starts at x_3, travels to x_1 and follows C_1 twice, then goes back to x_3 and performs C_3 twice, then moves to x_2 and performs C_2 twice, and finally returns to x_3. Then, the E-patrolling strategy of the tree is a strategy which repeatedly performs S^E with the starting point chosen uniformly at random.

Conjecture 1 of Alpern *et al.* (2022) was as follows.

Conjecture 3 (Tree patrolling conjecture) If Q is a tree network, then for any α the E-patrolling strategy is optimal and the value of the game is $v^* \equiv \alpha/(\mu+\lambda(E))$.

We will settle the tree patrolling conjecture in Section 4.

3 Subtree Decomposition and Density

In this section we introduce the notion of the *local root of* Q and the *subtree decomposition* of a tree network in Subsection 3.1 and the idea of *density* in Subsection 3.2.

3.1 Subtree Decomposition

In order to define the subtree decomposition of a tree network, we first introduce a new subset of Q here called the *core* of Q, defined as the closure of the complement of $\overline{E}$ and denoted $E^0 = E^0(Q, \alpha)$. The core is connected and closed. The reason for this rather awkward definition is that E is only defined on regular points, but informally we can think of the core as the complement of the extremity set. The core is depicted in blue in Figure 1 for each value of α. As α increases, the extremity set grows while the core shrinks. Notice that when $\alpha \geq 8$, the set $\overline{E}$ is equal to Q and $E^0 = \emptyset$.

Thus, for $\alpha < \alpha^*$, any tree network Q can be expressed as the disjoint union of the core and a set of subtrees each of length at most $\alpha/2$ (see Proposition 2, part (iii)). This is the subtree decomposition of Q. It is easy to see that the core cannot contain any leaf nodes of Q. In the remainder of this subsection we will show that for $\alpha \geq \alpha^*$, we can form a decomposition of Q with similar properties.

If $\alpha \geq \alpha^*$, the set $\overline{E}$ has only one connected component, which is equal to Q. In this case, we define the *local root of* Q.

Definition 4 Let Q be a tree and let $\alpha_1, \alpha_2, \ldots$ be a sequence of increasing positive numbers converging to α^*. The local root of Q is the set $\cap_{n=1}^{\infty} E^0(Q, \alpha_n)$.

It is easy to show that the local root of Q is specified independently of the choice of sequence $(\alpha_n)_{n=1}^{\infty}$, and is in fact equal to $\cap_{0<\alpha<\alpha^*} E^0(Q, \alpha)$. The fact that the local root is non-empty follows from Cantor's intersection theorem, since it is the intersection of a sequence of non-empty, non-increasing, closed sets, by Proposition 2, part (i). In fact, we will show in Proposition 5 that the local root of Q is a singleton, and without ambiguity, we will call its unique member the local root of Q and denote it by x^*. The local root of the tree Q is labeled in Figure 1.

Proposition 5 Let Q be a tree. Then,

(i) The local root of Q is a singleton, x^*.

(ii) Each of the maximal connected components of $Q - \{x^*\}$ has measure at most $\alpha^*/2$.

Proof. For (i), let $(\alpha_n)_{n=1}^{\infty}$ be an increasing sequence converging to α^* and let f be the real function defined by $f(\alpha) = \lambda(E^0(Q, \alpha))$. Then f is a continuous, and it follows that

$$\lambda(E^0(Q, \alpha_n)) = f(\alpha_n) \to f(\alpha^*) = \lambda(E^0(Q, \alpha^*)) = 0.$$

Now suppose the local root of Q contains two points x and y with $x \neq y$, and let $\varepsilon = d(x, y)$. Let N be such that $f(\alpha_N) < \varepsilon$. Since $E^0(Q, \alpha_N)$ is connected and contains both x and y, it must contain the path from x to y. Therefore, its measure must be at least ε, contradicting $f(\alpha_N) < \varepsilon$. So the local root of Q is a singleton, x^*.

To prove (ii), assume for a contradiction that $Q - \{x^*\}$ has a component Q_1 with $\lambda(Q_1) > \alpha^*/2$. First suppose that x^* is a regular point. In this case, $Q - \{x^*\}$ only has two components, and by definition of α^* (Proposition 2, part (ii)), the other component Q_2 must satisfy $\lambda(Q_2) < \alpha^*/2$. Let $\alpha' = \lambda(Q_2) + \alpha^*/2 < \alpha^*$. Since $\lambda(Q_2) < \alpha'/2$, we must have $x^* \in E(Q, \alpha')$ by definition of the extremity set. But by definition of x^* and because $\alpha' < \alpha^*$, we must have $x^* \in E^0(Q, \alpha')$, which is a contradiction, since $E(Q, \alpha') \cap E^0(Q, \alpha') = \emptyset$

Now suppose x^* is a node and let Q' be the subtree $Q_1 \cup \{x^*\}$. Then, $\lambda(Q') = \lambda(Q_1) > \alpha^*/2$. Let z be a regular point on the arc incident to x^* in Q' such that $d(x^*, z) < \lambda(Q_1) - \alpha^*/2$. It is easy to see that one component $Q_1(z)$ of $Q - \{z\}$ is a subset of Q_1 and the other component $Q_2(z)$ contains x^*. We have $\lambda(Q_1(z)) = \lambda(Q_1) - d(x^*, z) > \alpha^*/2$. So, $\lambda(Q_2(z)) < \alpha^*/2$, by definition of α^*. Let $\alpha'' = \lambda(Q_2(z)) + \alpha^*/2 < \alpha^*$. Since $\lambda(Q_2(z)) < \alpha''/2$, we must have $x^* \in Q_2(z) \subset \overline{E}(Q, \alpha'')$ by definition of the extremity set. Because x^* is not in the boundary of $Q_2(z)$, it is obvious that $x^* \notin E^0(Q, \alpha'')$. But by definition of x^* and because $\alpha'' < \alpha^*$, we must have $x^* \in E^0(Q, \alpha'')$, which is again a contradiction. □

The local root x^* of Q is labeled on part (d) of Figure 1. Note that it does not depend on α.

Proposition 5 implies that the tree Q can be expressed as a disjoint union of its local root and a set of subtrees of length at most $\alpha/2$. Combining this with the decomposition described earlier in this subsection for $\alpha < \alpha^*$, we have shown the following.

Proposition 6 (Subtree decomposition of Q) For any tree Q and any attack time α, we can express Q as a union of its core E^0 and a set of closed subtrees $E^1, \ldots, E^k$ whose union is $\overline{E}$ such that $\lambda(E^i) \leq \alpha/2$ for each $i = 1, \ldots, k$ and $\sum_{i=1}^{k} \lambda(E^i) = \lambda(E)$.

3.2 Density

In this subsection we introduce the concept of *density*.

Suppose a measure P on Q is fixed. For any measurable $A \subseteq Q$, we define the *density* $\rho_P(A) \equiv \rho(A)$ by $\rho(A) = P(A)/\lambda(A)$.

Suppose Q is a tree with a distinguished point O, called its *root*. We say a point $y \in Q$ is *above* a point (or arc) x if the unique path from O to y contains x. We write Q_x for the subtree of Q containing x and all points above x. We call a node x a *branch node* if it is not a leaf node. For a branch node x of Q, we call the *branches at* x the collection of maximal disjoint components of $Q_x - \{x\}$.

We state the definition of the *Equal Branch Density* (EBD) distribution, as given in Alpern and Lidbetter (2013), Alpern (2010) and Alpern and Lidbetter (2014).

Definition 7 For a tree Q with root O, the *Equal Branch Density* (EBD) distribution is the unique measure h on the leaf nodes of Q (not including O) such that at every branch node x all the branches at x have the same density ρ_h.

In Figure 2, we illustrate the EBD distribution h on a tree Q of length 10 with root O and four leaf nodes, A, B, C, D. The length of the three branches at O are $\lambda(OA) = 3$, $\lambda(OBC) = 5$, and $\lambda(OD) = 2$. So, the measure h on all leaf nodes of each branch are $h(OA) = 3/10$, $h(OD) = 2/10$, and $h(OBC) = 5/10$. The branch OA has one leaf node A, so the measure h on leaf node A is $h(A) = 3/10$. Similarly, $h(D) = 2/10$. For the branch OBC, the two branches at branch node x have length $\lambda(xB) = 1$, $\lambda(xC) = 2$. To ensure the two branches xB and xC have the same density, we set $h(B) = \frac{1}{3}h(OBC) = \frac{5}{30}$ and $h(C) = \frac{2}{3}h(OBC) = \frac{10}{30}$.

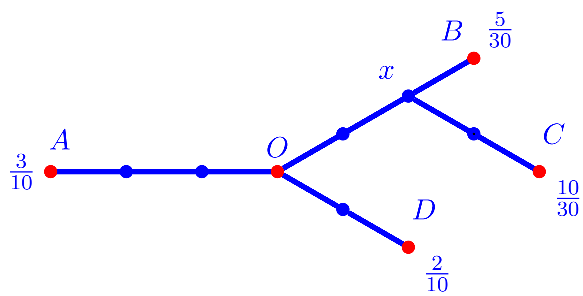


Figure 2: The EBD distribution on the tree Q with the root O.

We state here an important property of the EBD distribution, which is a consequence of Lemma 6 of Alpern and Lidbetter (2013).

Lemma 8 The EBD distribution h on a rooted tree Q has the property that for any subtree Z with root x contained in Q_x, we have $\rho_h(Z) \leq \rho_h(Q_x)$.

4 Proof of the Tree Patrolling Conjecture

We begin this section by constructing an Attacker strategy in Subsection 4.1, which we call the *tree attack strategy*. In Subsection 4.2, we will show that this strategy is ε-optimal.

4.1 The Tree Attack Strategy

The tree attack strategy is actually a collection of strategies, and is defined in terms of a parameter $T > 0$, which we can think of as the length of some long time interval.

Definition 9 (tree attack strategy) Let Q be a tree network, and let $E^0, E^1, \ldots, E^k$ be its subtree decomposition. Let x_j be the local root of E^j for $j = 1, .., k$. Let h^j be the EBD measure on E^j. For $T > 0$, the tree attack strategy (with parameter T) begins at a time chosen uniformly at random from the interval $[0, T]$. The location of the attack is given by the measure e, defined below.

(i) With probability $e(E^0) \equiv \lambda(E^0)/(\mu + \lambda(E))$, a point of E^0 chosen uniformly at random.

(ii) With probability $e(E^j) \equiv 2\lambda(E^j)/(\mu + \lambda(E))$, a point of E^j chosen according to the EBD distribution h^j, for $j = 1, \ldots, k$.

The tree attack strategy is well defined. Indeed, the total probability $e(Q)$ of attack is given by

$$e(Q) = \sum_{j=0}^{k} e(E^j) = \frac{\lambda(E^0)}{\mu + \lambda(E)} + \sum_{j=1}^{k} \frac{2\lambda(E^j)}{\mu + \lambda(E)} = \frac{\lambda(E^0)}{\mu + \lambda(E)} + \frac{2\lambda(E)}{\mu + \lambda(E)} = \frac{\mu + \lambda(E)}{\mu + \lambda(E)} = 1.$$

We illustrate the tree attack strategy by revisiting the network Q with length $\mu = 10$ from Figure 1. We illustrate the attack probability at the leaf nodes and in E^0 in Figure 3 for different values of α.

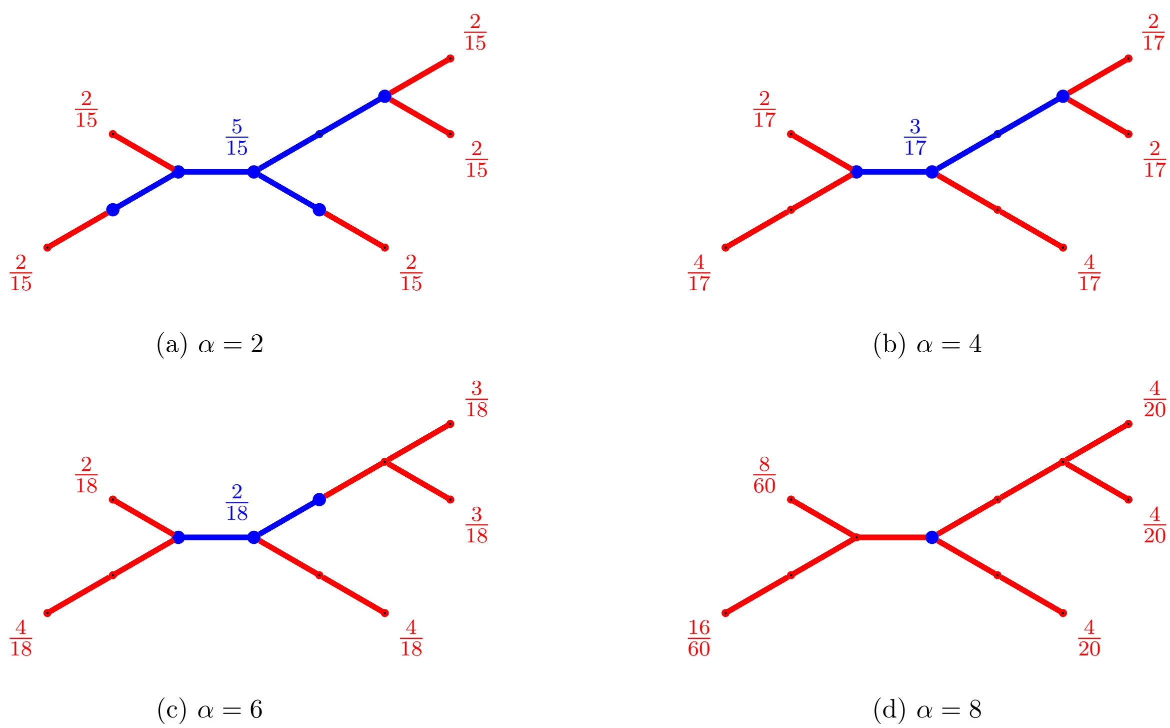


Figure 3: The tree attack strategy on the tree network Q.

Observe that the density $\rho_e(E^j) \equiv \rho(E^j)$ for any $j = 1, \ldots, k$ is

$$\rho(E^j) = \frac{e(E^j)}{\lambda(E^j)} = \frac{2\lambda(E^j)}{\mu + \lambda(E)} \frac{1}{\lambda(E^j)} = \frac{2}{\mu + \lambda(E)}.$$

So, by Lemma 8, for any subtree Z of E^j such that $x_j \in Z$,

$$\rho(Z) \leq \rho(E^j) = \frac{2}{\mu + \lambda(E)}. \tag{1}$$

4.2 ε-Optimality of the Tree Attack Strategy

Before proving the tree patrolling conjecture, we extend a lemma from Alpern *et al.* (2022) concerning the *uniform attack strategy*. This is the strategy for the attacker that begins the attack at an arbitrary time M (for example $M = 0$) at a point of the network chosen uniformly at random. Alpern *et al.* (2022) showed that this strategy ensures the attack will be intercepted with probability at most α/μ (this was also shown in Alpern *et al.* (2016) and Garrec (2019)).

Lemma 10 Let Z be a connected subset of a network Q. Consider an attack strategy that chooses a point of Z uniformly at random to carry out the attack, and starts the attack at some time t, which may be fixed or a random variable. Then for any Patroller strategy, the probability that attack is intercepted is at most $\alpha/\lambda(Z)$.

Proof. The lemma is trivially true if $\alpha \geq \lambda(Z)$, so assume that $\alpha < \lambda(Z)$. First suppose t is fixed. Then Lemma 1 of Alpern *et al.* (2022) applied to the network Z says that probability of interception is at most $\alpha/\lambda(Z)$.

Now suppose t is a random variable. Then from the previous paragraph, the probability the attack is intercepted, conditional on the attack starting at fixed time $t = t_0$ is at most $\alpha/\lambda(Z)$. It follows that the unconditional probability of interception is also at most $\alpha/\lambda(Z)$. □

We are now ready to prove the tree patrolling conjecture.

Theorem 11 Let Q be a tree of length μ. Then for any $\varepsilon > 0$, there exists a value of T such that the tree attack strategy (with parameter T) cannot be intercepted with probability greater than $\alpha/(\mu + \lambda(E)) + \varepsilon \equiv v^* + \varepsilon$. Hence, the value of the continuous patrolling game on Q is v^* and the E-patrolling strategy is optimal.

Proof. Let $\varepsilon > 0$ be given, and suppose the Attacker uses the tree attack strategy (with parameter T), for some T, where the precise value of T will be specified later. Consider an arbitrary patrol S, and let $0 = t_0 < t_1 < \cdots < t_m = T + \alpha$ be the coursest partition of $[0, T+\alpha]$ such that S is confined to a single set E^j ($j = 0, 1, \ldots, k$) during each time interval $[t_i, t_{i+1}]$. For $i = 1, \ldots, m$, let $I_i = [t_{i-1}, t_i]$, let $\delta_i = t_i - t_{i-1}$ and let $Z_i = S(I_i)$.

We will show that the probability $P(S)$ that S intercepts the tree attack strategy is at most $v^* + \varepsilon$. To do so, we will calculate an upper bound for the probability P_i that S intercepts the attack during each of the intervals I_i for each $i = 1, \ldots, m$, and we will show that the sum of these upper bounds is no more than $v^* + \varepsilon$.

First suppose $m = 1$. In this case, the patrol just stays in one component E^j during the whole time $[0, T+\alpha] \equiv I_1$. If $j \neq 0$, the interception probability P_i is at most

$$e(E^j) = 2\lambda(E^j)/(\mu + \lambda(E)) \leq \alpha/(\mu + \lambda(E)) = v^*,$$

since $\lambda(E^j) \leq \alpha/2$. If $j = 0$, then by Lemma 10, then the interception probability P_i satisfies

$$P_i \leq \frac{\alpha}{\lambda(E^0)} \cdot e(E^0) = \frac{\alpha}{\lambda(E^0)} \cdot \frac{\lambda(E^0)}{\mu + \lambda(E)} = v^*.$$

Now suppose $m \geq 2$, and we calculate an upper bound of interception probability P_i in three cases:

(i) $Z_i \subseteq E^0$;

(ii) $i = 2, .., m-1$ and $Z_i \subseteq E^j$ for some $j = 1, \ldots, k$;

(iii) $i = 1$ or m and $Z_i \subseteq E^j$ for some $j = 1, \ldots, k$.

Starting with case (i), when $Z_i \subseteq E^0$, we observe that at any time $y \in I_i$ the patrol S can intercept the attack at point $S(y)$ starting in time $[\max(0, y-\alpha), y]$. Note that $y - \max(0, y-\alpha) \leq \alpha$. Since S moves at unit speed, we have that $\lambda(S(I_i)) \leq \delta_i$. So, the conditional probability that $S(I_i)$ intercepts the attack given it takes place in E^0 is at most $(\alpha\delta_i)/(T\lambda(E^0))$. This gives the bound

$$P_i \leq \frac{\alpha\delta_i}{T\lambda(E^0)} \cdot e(E^0) = \frac{\delta_i v^*}{T}. \tag{2}$$

Second, in the case that $i = 2, .., m-1$ and $Z_i \subseteq E^j$ for some $j = 1, \ldots, k$, the patrol must perform a tour with the startpoint and endpoint x_j. Because the length of this tour is at least $2\lambda(Z_i)$, the patrol can spend at most time $\delta_i - 2\lambda(Z_i) \geq 0$ at leaf nodes of E^j. Therefore, P_i satisfies

$$P_i \leq \frac{1}{T} e(Z_i)(\alpha + \delta_i - 2\lambda(Z_i)).$$

By (1), $\rho(Z_i) = e(Z_i)/\lambda(Z_i) \leq \rho(E^j) = 2/(\mu + \lambda(E))$. Applying this to the inequality above and rearranging,

$$\begin{aligned} P_i &\leq \frac{1}{T} e(Z_i)(\alpha + \delta_i - 2\lambda(Z_i)) \\ &\leq \frac{1}{T} \frac{2\lambda(Z_i)}{\mu + \lambda(E)} (\alpha + \delta_i - 2\lambda(Z_i)) \\ &= \frac{1}{T} \frac{\alpha}{\mu + \lambda(E)} \left(\delta_i - \big(\delta_i - 2\lambda(Z_i)\big) \left(1 - \frac{2\lambda(Z_i)}{\alpha}\right)\right). \end{aligned}$$

As already observed, $\delta_i - 2\lambda(Z_i) \geq 0$. Also, $\lambda(Z_i) \leq \lambda(E^j) \leq \alpha/2$, by definition of the subtree decomposition, so $\big(\delta_i - 2\lambda(Z_i)\big)\big(1 - \frac{2\lambda(Z_i)}{\alpha}\big) \geq 0$. Consequently,

$$P_i \leq \frac{1}{T} \frac{\alpha}{\mu + \lambda(E)} \delta_i = \frac{\delta_i v^*}{T}. \tag{3}$$

Third, we consider the case that $i = 1$ or m and $Z_i \subseteq E^j$ for some $j = 1, \dots, k$. This case is different from the second case since it is not necessary for the patrol to perform a tour in E^j. For example, the patrol may start at a leaf node in Z_1, stay within Z_1 for sometime then move directly to Z_2. Therefore, the time the patrol can stay at leaf nodes in E^j is at most $\delta_i - \lambda(Z_i) \geq 0$, and the interception probability P_i satisfies

$$P_i \leq \frac{1}{T} e(Z_i)(\alpha + \delta_i - \lambda(Z_i)).$$

The condition $\rho(Z_i) \leq \alpha/(\mu + \lambda(E))$ still holds since Z_i contains x_j, and must therefore be a subtree of E^j. Applying this to the inequality above and rearranging,

$$\begin{aligned} P_i &\leq \frac{1}{T} \frac{2\lambda(Z_i)}{\mu + \lambda(E)} (\alpha + \delta_i - \lambda(Z_i)) \\ &= \frac{1}{T} \frac{\alpha}{\mu + \lambda(E)} \left(\delta_i + 2\lambda(Z_i) - \left(1 - \frac{2\lambda(Z_i)}{\alpha}\right) \delta_i - \frac{2(\lambda(Z_i))^2}{\alpha} \right). \end{aligned}$$

Since $\lambda(Z_i) \leq \lambda(E^j) \leq \alpha/2$, we have $(1 - 2\lambda(Z_i)/\alpha)\delta_i \geq 0$ and

$$P_i \leq \frac{1}{T} \frac{\alpha}{\mu + \lambda(E)} \big(\delta_i + 2\lambda(Z_i)\big) \leq \frac{1}{T} \frac{\alpha}{\mu + \lambda(E)} (\delta_i + \alpha) = \frac{(\delta_i + \alpha) v^*}{T}. \tag{4}$$

Combining inequalities (2) - (4), we obtain

$$P(S) \leq \sum_{i=1}^{m} P_i \leq \frac{2\alpha v^*}{T} + \sum_{i=1}^{m} \frac{\delta_i v^*}{T} = \frac{2\alpha v^*}{T} + \frac{v^*}{T}(T + \alpha) \leq v^* + \varepsilon,$$

where we choose $T = 3\alpha/\varepsilon$.

We have shown that the tree attack strategy cannot be intercepted with probability greater than $v^* + \varepsilon$, so that the value of the game is at most $v^* + \varepsilon$. Combining this with the lower bound of v^* from Alpern *et al.* (2022) given by the E-patrolling strategy, the rest of the theorem follows. □

5 Solving the Game for Complete Networks

In this section, we study the game on complete networks. We begin this section by introducing some standard definitions and the concept of a k-factorization of complete networks in Subsection 5.1. In Subsection 5.2, we introduce a Patroller strategy which we call the *complete network patrolling strategy* and show that this strategy is optimal for some values of α.

5.1 k-factorization of Complete Networks

In this section, we just consider simple networks (i.e networks that do not contain any loops and for which there is at most one arc connecting any pair of nodes). A *k-regular network* is a simple network all of whose nodes have degree k ($k \geq 1$). A *complete network* is a k-regular network on m ($m \geq 2$) nodes where $k = m - 1$. We denote a complete network with n nodes by K_n. Note that since the arcs of K_n may have different lengths, it is not uniquely defined.

We denote the set of arcs of a network Q by $E(Q)$ and the set of nodes of Q by $V(Q)$.

Definition 12 Let Q be a k-regular network. A *k-factorization* of Q is a set of sub-networks $F = \{F_1, \ldots, F_z\}$ such that

(i) for all $i = 1, \ldots, z$, the sub-network F_i is a k-regular network with $V(F_i) = V(Q)$,

(ii) $E(Q) = \cup_{i=1}^{z} E(F_i)$ and

(iii) for any $1 \leq i \neq j \leq z$, we have $E(F_i) \cap E(F_j) = \emptyset$.

In particular, a 1-factorization of Q is a set of arc-disjoint perfect matchings whose union is $E(Q)$. In other words, a 1-factorization is an arc-coloring of a network where each color class consists of a perfect matching. In Figure 4, we illustrate a 1-factorization of a complete network K_4 on four nodes with three color classes, red, blue, and green.

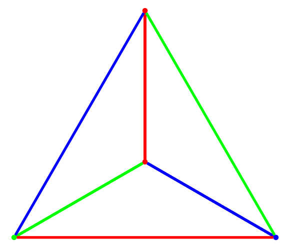


Figure 4: 1-factorization of a complete network K_4.

It is obvious that if Q has a 1-factorization, the number of nodes of Q must be even. It is well-known that every complete network on $2n$ $(n = 1, \ldots)$ nodes admits a 1-factorization. Csaba *et al.* (2016) showed that every k-regular network on $2n$ nodes has a 1-factorization if $k \geq 2\lceil n/2 \rceil - 1$. For a small number of nodes $2n \leq 4$, the network K_{2n} has a unique 1-factorization. When $2n \geq 6$, the network K_{2n} has many 1-factorizations (see Zinoviev, 2014). For example, K_8 has 6240 distinct 1-factorizations.

5.2 A Patrolling Strategy for Complete Networks

In this subsection, we introduce a Patroller strategy for the complete network K_{2n} on $2n$ $(n = 1, 2, \ldots)$ nodes and prove this strategy is optimal for some values of α. Note that a complete network on an odd number of nodes is Eulerian. The solution for Eulerian networks was presented in Garrec (2019) and Alpern *et al.* (2022).

Suppose the complete network K_{2n} has a 1-factorization $F = \{F_1, \ldots, F_{2n-1}\}$. We first observe that for any $i = 1, \ldots, 2n-1$, the sub-network $Q_i = K_{2n} - F_i$ is a k-regular network where $k = 2n-2$. Therefore, Q_i is Eulerian for all i. We define the complete network patrolling strategy below.

Definition 13 (complete network patrolling strategy) Let $F = \{F_1, \ldots, F_{2n-1}\}$ be a 1-factorization of a complete network K_{2n}. For $i = 1, \ldots, 2n-1$, let $Q_i = K_{2n} - F_i$ and let S_i be an Eulerian tour of Q_i starting at a randomly chosen point. The complete network patrolling strategy S^F is a patrol such that the Patroller chooses S_i with probability $s_i = \lambda(Q_i)/((2n-2)\mu)$.

Note that S^F is well defined, since

$$\sum_{i=1}^{2n-1} s_i = \frac{\sum_{i=1}^{2n-1} \lambda(Q_i)}{(2n-2)\mu} = \frac{\sum_{i=1}^{2n-1} \big(\lambda(K_{2n}) - \lambda(F_i)\big)}{(2n-2)\mu} = \frac{\sum_{i=1}^{2n-1} \lambda(K_{2n}) - \sum_{i=1}^{2n-1} \lambda(F_i)}{(2n-2)\mu} = \frac{(2n-1)\mu - \mu}{(2n-2)\mu} = 1.$$

For a 1-factorization F, let $\delta(F) = \max_{1 \le i \le 2n-1} \lambda(F_i)$. We have the following result.

Proposition 14 Let F be a 1-factorization of the complete network K_{2n} for some $n = 1, 2, \ldots$. For $\alpha \le \mu - \delta(F)$, the strategy S^F is optimal for the Patroller and the uniform attack strategy is optimal for the Attacker on K_{2n}. The value of the game is $V = \alpha/\mu$.

Proof. Consider an arbitrary attack taking place at some point $x \in Q$. We will show that the patrol S^F can intercept this attack with probability at least α/μ.

Let $P(S_i)$ be the probability that S_i intercepts the attack, for $i = 1, \ldots, 2n-1$. For each i, we have $\lambda(Q_i) = \mu - \lambda(F_i) \ge \mu - \delta(F) \ge \alpha$. It follows from Corollary 1 of Alpern *et al.* (2022) that $P(S_i) = \alpha/\lambda(Q_i)$.

If x is a node, it is easy to see that $x \in Q_i$ for all i. So, the patrol S^F will intercept the attack with probability

$$P(S^F) = \sum_{i=1}^{2n-1} s_i P(S_i) = \sum_{i=1}^{2n-1} \frac{\lambda(Q_i)}{(2n-2)\mu} \frac{\alpha}{\lambda(Q_i)} = \frac{2n-1}{2n-2} \frac{\alpha}{\mu} \ge \frac{\alpha}{\mu}.$$

If x is not a node, there exists a unique $j \in [2n-1]$ such that $x \in F_j$ (where $[m]$ denotes the set $\{1, \ldots, m\}$). So, $x \notin Q_j$ and $x \in Q_i$ for all $i \ne j$. The probability the patrol S^F intercepts the attack is

$$P(S^F) = \sum_{i=1}^{2n-1} s_i P(S_i) = \sum_{\substack{i \in [2n-1] \\ i \ne j}} \frac{\lambda(Q_i)}{(2n-2)\mu} \frac{\alpha}{\lambda(Q_i)} = \frac{2n-2}{2n-2} \frac{\alpha}{\mu} = \frac{\alpha}{\mu}.$$

We have shown that $V \ge \alpha/\mu$. But by Lemma 1 of Alpern *et al.* (2022) (or Lemma 10 of this paper), the uniform attack strategy guarantees that $V \le \alpha/\mu$ for any network. We conclude that the strategy S^F and the uniform attack strategy are optimal and the value of the game is $V = \alpha/\mu$. □

As mentioned in Subsection 5.1, when the number of nodes $2n \ge 6$, the network K_{2n} has many 1-factorizations. Let F^* be a 1-factorization of K_{2n} such that $\delta^* = \delta(F^*) \le \delta(F)$ for any 1-factorization F of Q. We then have the following stronger result.

Proposition 15 For $\alpha \le \mu - \delta^*$, the strategy S^{F^*} and the uniform attack strategy are optimal. The value of the game is $V = \alpha/\mu$.

In comparison with the recent work of Alpern *et al.* (2022), the complete network patrolling strategy S^F helps us solve the game for a significantly larger range of α. Alpern *et al.* (2022)

introduced a patrolling strategy for networks without leaf arcs and proved it is optimal for $\alpha \leq g$ where g is the girth of the network, defined as the minimum length of a circuit in the network. For a complete network K_{2n} ($n \geq 2$), the girth g is very small compared to $\mu - \delta(F)$, for any 1-factorization F. In fact, we have $\mu - \delta(F) \geq \frac{n(n-1)}{2} g$. Indeed, by Theorem 1 of Alspach and Gavlas (2001), any sub-network $Q_i = K_{2n} - F_i$ ($F_i \in F$) can be decomposed into circuits C_4 of four arcs. Since $|E(Q_i)| = 2n(n-1)$, a C_4-decomposition of Q_i has $n(n-1)/2$ circuits and the length of any circuit is not less than g by definition of g. So, $\lambda(Q_i) \geq \frac{n(n-1)}{2} g$. Therefore, $\mu - \delta(F) \geq \min_i \lambda(Q_i) \geq \frac{n(n-1)}{2} g$ for any 1-factorization F.

In summary, the patrolling strategy of Alpern *et al.* (2022) is known to be optimal for values of α in $(0, g]$, whereas the complete network patrolling strategy is known to be optimal for values of α in $(0, \frac{n(n-1)}{2} g]$, an interval that is $O(n^2)$ longer.

Notice that if Q is a network with unit length arcs (i.e every arc is of length 1), then $\delta(F) = n$ for all 1-factorizations F, so that $\delta^* = n$. Thus, for $\alpha \leq \mu - n$, the value of the game is α/μ. In fact, this bound can be tight. In other words, for some networks, for $\alpha > \mu - \delta^*$, the value of the game is strictly less than α/μ.

Proposition 16 Consider an attack strategy for the network K_4 with unit length arcs which attacks at a random point with a start time chosen uniformly at random from the interval $[0, 6-\alpha]$. For $\mu - n = 4 < \alpha \leq 6$, this attack strategy guarantees an interception probability of stricty less than α/μ.

The proof of Proposition 16 is in the Appendix.

Remark 17

Proposition 14 can be extended to general k-regular networks Q on $2n$ ($n \geq 2$) nodes. We consider the case $k \geq n+1$ and k is odd. From Subsection 5.1, we know Q admits a 1-factorization $F = \{F_1, \ldots, F_k\}$. Also, for all $i = 1, \ldots, k$, the sub-network $Q_i = Q - F_i$ is Eulerian. It is well known that a network Q all of whose nodes have degree at least $|V(Q)|/2$ is connected. So Q_i must be connected because all its nodes are of degree $k - 1 \geq n$. Let S_i be an Eulerian tour of Q_i which starts at a random point. Let S be a patrolling strategy which chooses S_i with probability $\lambda(Q_i)/\sum_{j=1}^{k} \lambda(Q_j)$. Similarly to the proof of Proposition 14, it is easy to show that the strategy S is optimal and the value of the game is α/μ for $\alpha \leq \mu - \delta(F)$.

In general, if we know a k-regular network has an m-factorization, we can generalize Proposition 14 as follows.

Theorem 18 Let Q be a k-regular network on $2n$ vertices such that $n \geq 2$ and k is odd. Assume Q admits an m- factorization F_m for some odd m such that $k \geq n + m$. Then, the value of the game is $V = \alpha/\mu$ for $\alpha \leq \mu - \delta(F)$.

Proof. Observe that $|F_m| = k/m = r$. Let $F_m = \{F_1, \ldots, F_r\}$ and $Q_i = Q - F_i$ for $F_i \in F$. Then, for all $i = 1, \ldots, r$, the sub-network Q_i is a k'-regular network where $k' = k - m$. Since $k \geq n + m$, we have $k' \geq n = V(Q_i)/2$ and Q_i is connected. Moreover, k' is even because k and m are odd. Therefore, Q_i is Eulerian for all $i = 1, \ldots, r$.

Let S_i be an Eulerian tour of Q_i which starts at a point chosen randomly. Let S^{F_m} be a patrolling strategy which picks S_i with probability $s_i = \lambda(S_i)/\sum_{j=1}^{r} \lambda(S_j)$. Then, similarly to the proof of Proposition 14, it can be shown that for $\alpha \leq \mu - \delta(F) \leq \min_i \lambda(Q_i)$, the patrol S^{F_m} can intercept any attack with probability at least α/μ and $V \geq \alpha/\mu$. Since the uniform attack strategy can guarantee $V \leq \alpha/\mu$ (Alpern *et al.*, 2022), for $\alpha \leq \mu - \delta(F)$, we conclude the value of the game is $V = \alpha/\mu$, the uniform attack strategy is optimal for the Attacker and the patrol S^{F_m} is optimal for the Patroller. □

6 Conclusion

We have settled a conjecture posed by Alpern *et al.* (2022) and thus shown that for tree networks, an easily implementable patrolling strategy is optimal in the continuous patrolling game. Although we have found ε-optimal attack strategies, we believe that optimal attack strategies exist in all cases, and it may be of interest to refine the tree attack strategy defined in this paper to obtain optimal strategies.

We have also solved the game for complete networks as long as α is sufficiently small, significantly increasing the range of values of α for which a solution is known. The solution to the continuous patrolling game remains open for many classes of networks for larger values of α. For example, for a network with two nodes connected by an odd number of arcs, Garrec (2019) gave a solution for the particular case of three unit arcs when $\alpha \leq 2$. Alpern *et al.* (2022) solved the game for any arbitrary number of arcs for $\alpha \leq \mu - D$ where D is the length of the longest arc. The solution for this network for $\alpha > \mu - D$ is still open. For future research, we could also consider some networks which were well studied in the discrete setting but have not been studied in the continuous setting, such as bipartite networks.

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Appendix: proof of Proposition 16

Proof. Let w be an arbitrary patrol. Let $I_1 = [0, 6-\alpha]$, $I_2 = [6-\alpha, \alpha]$, $I_3 = [\alpha, 6]$ and $I_4 = [6, \infty)$. For $i = 1, \ldots, 4$, let $P(I_i)$ be the interception probability that $w(I_i)$ contributes to $P(w)$. Since the attack time is chosen uniformly at random in the interval $[0, 6-\alpha]$, all attacks are finished by time 6 and $P(I_4) = 0$.

We observe that $P(I_1) \leq (6-\alpha)/12$. Indeed, during I_1, the patrol can walk for length at most $6-\alpha$ without any point being revisited (see Figure 5) and that walk gives interception probability

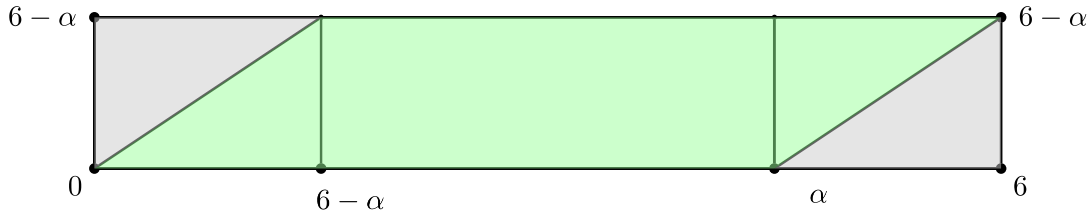

Figure 5: The maximum attacks that patrol w can intercept in the time interval $[0, 6]$. The starting time of the attack $[0, \alpha - 6]$ is shown by the vertical lines. All attacks in the green area can be intercepted and all attacks in the grey area will not be intercepted.

$(6 - \alpha)/(2\mu) = (6 - \alpha)/12$. Similarly, we have $P(I_2) \leq (2\alpha - 6)/6$ and $P(I_3) \leq (6 - \alpha)/12$ (see Figure 5). Then,

$$P(w) = P(I_1) + P(I_2) + P(I_3) \leq \frac{6-\alpha}{12} + \frac{2\alpha-6}{6} + \frac{6-\alpha}{12} = \frac{\alpha}{6}.$$

So, $P(w) = \alpha/6$ if and only if all $P(I_i)$ meet their bounds. In other words, in time $[0, 6]$ the patrol w must satisfy: (i) the patrol always walks with speed 1, and (ii) if any point x is revisited, then $T_{j+1}(x) - T_j(x) \geq \alpha$ where $T_j(x)$ $(j = 1, \ldots)$ is the j^{th} time x is visited.

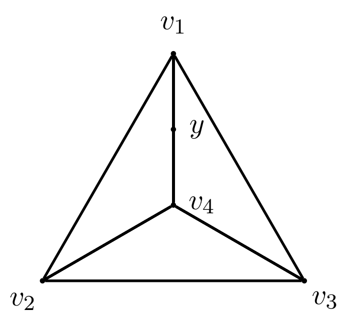

Figure 6: Unit length arc network K_4.

We claim that there is no patrol satisfying both (i) and (ii). First, assume at time 0, the patrol stays at a node. Since $\alpha > 4$, $w([0, 4])$ must be a path consisting of 4 distinct adjacent arcs. Without loss of generality, we consider 3 possible paths for $w([0, 4])$: $w_1 = (v_1, v_2, v_3, v_4, v_2)$, $w_2 = (v_1, v_2, v_3, v_4, v_1)$, $w_3 = (v_1, v_2, v_3, v_1, v_4)$ (see Figure 6). For w_1, to continue, the patrol can go to v_3 or v_1; however, both ways will immediately violate the condition (ii). For w_2, the patrols must continue by going from v_1 to v_3. Then, at v_3, there is no way to continue without violating the condition (ii). With the same analysis, w_3 cannot be completed such that the condition (ii) still holds.

Second, we consider the case that the patrol starts at a regular point y. We assume $y \in (v_1, v_4)$ and the patrol first travels from y to v_1 at time $t = d(y, v_1) < 1$. Since $t + 3 < \alpha$, $w([0, t+3])$ cannot contain the same arc twice. It is enough to examine three possible cases for $w([0, t+3])$:

- Case 1: $w' = (y, v_1, v_2, v_3, v_1)$
- Case 2: $w'' = (y, v_1, v_2, v_3, v_4)$
- Case 3: $w'' = (y, v_1, v_2, v_4, v_3)$

Similar to the previous analysis, it is easy to see that in all cases condition (ii) cannot be satisfied.

For $\alpha = 6$, it is easy to see that $V < 1$ since there is no tour which cover all arcs in time $[0, 6]$. So, for $4 < \alpha$, the attack cannot be intercepted with probability α/μ and the value of the game is $V < \alpha/\mu$. □