SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 23, No. 1, pp. 26–49

Truncation of Contact Defects in Reaction-Diffusion Systems*

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Abstract. Contact defects are time-periodic patterns in one space dimension that resemble spatially homogeneous oscillations with a defect embedded in their core region. For theoretical and numerical purposes, it is important to understand whether these defects persist when the domain is truncated to large spatial intervals, supplemented by appropriate boundary conditions. The present work shows that truncated contact defects exist and are unique on sufficiently large spatial intervals.

Key words. spatial dynamics, nonlinear waves, reaction-diffusion systems, defects, Lin's method

MSC codes. 35K57, 35Q92, 35B36

DOI. 10.1137/23M1546257

1. Introduction.

Solutions of reaction-diffusion systems exhibit a wide variety of patterns, which makes them ubiquitous in models of chemical, biological, and ecological systems [13]. In biological systems, Turing patterns are a potential mechanism for the emergence of stripes and spots on animal coats [22]. In chemistry, complex spatiotemporal patterns have been observed in several autocatalytic reactions, including the chlorite-iodide-malonic acid (CIMA) reaction [14] and the light-sensitive Belousov–Zhabotinsky (BZ) reaction [24]. The CIMA reaction exhibits one-dimensional defect patterns in which a stationary, spatially periodic core is connected to spatially homogeneous oscillations in the far field [14]; see Figure 1(a) for a similar pattern arising in numerical simulations of the Brusselator model of the CIMA reaction [14, 23]. In [18], similar patterns were proved to exist in the complex cubic-quintic Ginzburg–Landau equation. In the BZ reaction, two-dimensional spiral waves occur that exhibit one or more line defects [24]; we refer the reader to Figures 1(b)–(c) for similar patterns that arise in numerical simulations of the Rössler model [4]. These line defects are caused by the destabilization of a rigidly rotating spiral wave through a period-doubling bifurcation [4, 20, 24]: across the line defect, the phase of the spatiotemporal oscillations jumps by half a period. In [20, Theorem 10], the existence of one-dimensional line defects with a half-period phase jump across the defect was proved near period-doubling bifurcations of spatially homogeneous oscillations.

*Received by the editors January 10, 2023; accepted for publication (in revised form) August 10, 2023; published electronically January 3, 2024.
https://doi.org/10.1137/23M1546257
Funding: The first author was partially supported by the NSF under grant DMS-1714429 and by the Ministry of Education and Science of Bulgaria through the Scientific Programme "Enhancing the Research Capacity in Mathematical Sciences (PIKOM)," DO1-67/05.05.2022. The second author was partially supported by the NSF under grants DMS-1714429 and DMS-2106566.
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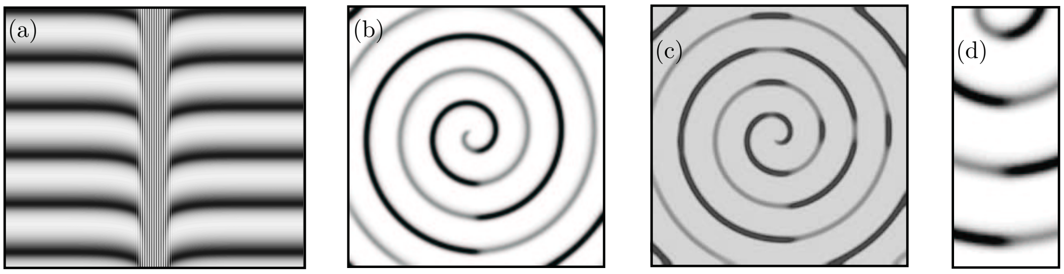

Figure 1. *Panel* (a) *shows a space-time plot (space plotted horizontally, and time vertically) of a time-periodic solution of the one-dimensional Brusselator model: the solution exhibits a spatially periodic core in the center of the domain and spatially homogeneous oscillations in the far field away from the core region. Panels* (b)–(c) *contain snapshots of planar spiral waves that exhibit similar line defects (images taken from* [19, 20]*). Panel* (d) *shows an enlarged version of the spatial region around the line defect from panel* (b)*: if we interpret the vertical direction as time, the solution resembles the space-time plot in panel* (a), *except that the oscillations to either side of the defect are out of phase by half the period.*

Our motivation for this paper comes from the solutions shown in panels (a) and (d) of Figure 1, which can be thought of as defects in the center of the domain that are embedded into a background of spatially homogeneous oscillations in the far field and which we refer to as contact defects. Our goal is to prove that a contact defect defined on the entire real line and positioned at, say, $x=0$ persists as a solution on a large bounded interval $(-L,L)$ with Neumann boundary conditions added at the end points. Our reason for proving this result is threefold. First, it validates numerical computations that are conducted on bounded intervals rather than the entire real line. Second, it allows us to transfer the existence and multiplicity results for contact defects on the real line that were proved in [18, 19] to defects on large bounded domains.

The third aspect arises from the long-time dynamics of the line defects shown in Figure 1(c). Over sufficiently long times, these line defects attract and annihilate each other in pairs until only one line defect remains; Figure 1(c) illustrates this behavior through the two pairs of collocated line defects that are about to merge and disappear. Once we established the persistence of line defects on bounded intervals with Neumann boundary conditions, we can concatenate several defects by reflecting them across $x=L$ or $x=-L$ and then attempt to understand their interaction properties. Thus, the persistence results in this paper serve also as a first step for the stability analysis of multiple line defects, which we plan to publish separately.

2. Main results.

We outline our setup and assumptions before describing our main results. Consider the reaction-diffusion system

$$u_t = Du_{xx} + f(u)$$

with $x \in \mathbb{R}$ and $u(x,t) \in \mathbb{R}^n$, where D is a constant, positive-definite diagonal matrix and $f \in C^\infty(\mathbb{R}^n)$ is a smooth nonlinearity. We are interested in time-periodic solutions and therefore introduce the normalized time variable $\tau = \omega t$, where $\omega > 0$ is the temporal frequency; this allows us to focus on solutions that are 2π-periodic in τ for an appropriate value of ω. In these coordinates, the reaction-diffusion system becomes

$$\omega u_\tau = D u_{xx} + f(u). \tag{2.1}$$

We assume that this equation admits a spatially homogeneous oscillation of the form $u(x,\tau) = u_{\mathrm{wt}}(\tau)$ for $\omega = \omega_0 > 0$, where $u_{\mathrm{wt}}(\tau)$ is an appropriate 2π-periodic profile that satisfies the ordinary differential equation (ODE)

$$\omega_0 u_\tau = f(u). \tag{2.2}$$

We now formalize the assumption that this solution exists and is asymptotically stable for the ODE (2.2).

Hypothesis 1. There are an $\omega_0 > 0$ and a 2π-periodic nonconstant function $u_{\mathrm{wt}}(\tau)$ that satisfy (2.2). Furthermore, $\lambda = 0$ is an algebraically simple Floquet exponent of the linearization

$$\omega_0 v_\tau = f_u(u_{\mathrm{wt}}(\tau))v \tag{2.3}$$

of (2.2) about $u_{\mathrm{wt}}(\tau)$, and all other Floquet exponents of (2.3) have strictly negative real part.

Next, we consider the PDE linearization

$$\omega_0 v_\tau = D v_{xx} + f_u(u_{\mathrm{wt}}(\tau))v$$

of (2.1) about the homogeneous oscillation $u_{\mathrm{wt}}(\tau)$ and the resulting system

$$\omega_0 \hat{v}_\tau = -k^2 D\hat{v} + f_u(u_{\mathrm{wt}}(\tau))\hat{v} \tag{2.4}$$

for the Fourier modes $\hat{v}(k,\tau)$ belonging to the spatial wavenumbers $k \in \mathbb{R}$. Note that (2.4) evaluated at $k=0$ agrees with (2.3). As shown in [18, section 3.3], we can continue the simple Floquet exponent $\lambda = 0$ of (2.3) as an even smooth function $\lambda_{\mathrm{lin}}(k)$ of temporal Floquet exponents of the ODEs (2.4) for $k \in \mathbb{R}$ near $k=0$. We refer to $\lambda_{\mathrm{lin}}(k)$ as the linear dispersion curve of the homogeneous oscillations.

It was also shown in [18, section 3.3] that Hypothesis 1 implies that there exists a unique even, smooth function $\omega_{\mathrm{nl}}(k)$ with $\omega_{\mathrm{nl}}(0) = \omega_0$ so that (2.1) has a traveling-wave solution of the form $u(x,\tau) = u_{\mathrm{wt}}(\tau - kx; k)$ for k close to zero, where $u_{\mathrm{wt}}(\tau;k)$ is 2π-periodic in τ and close to $u_{\mathrm{wt}}(\tau)$ in the C^0-norm, if and only if $\omega = \omega_{\mathrm{nl}}(k)$. We refer to the function $\omega_{\mathrm{nl}}(k)$, which relates the spatial wavenumber k to the temporal frequency ω, as the nonlinear dispersion curve.

Hypothesis 2. We assume that $\lambda''_{\mathrm{lin}}(0) < 0$ and $\omega''_{\mathrm{nl}}(0) \neq 0$. Furthermore, we assume that the function $\lambda_{\mathrm{lin}}(k)$ for k close to zero captures all Floquet exponents of (2.4) in the closed right half-plane.

We are interested in 2π-periodic solutions $u(x,\tau)$ of (2.1) with $\omega = \omega_0$ that are spatially asymptotic to the homogeneous oscillations $u_{\mathrm{wt}}(\tau)$ as $|x| \to \infty$. More precisely, we introduce the following definition.

Definition 2.1. *We say that a function* $u_{\rm d}(x,\tau)$ *is a contact defect with frequency* ω_0 *if it is* 2π*-periodic in* τ*, it satisfies the reaction-diffusion system* (2.1) *with* $\omega=\omega_0$*, and there are phase-correction functions* $\theta_\pm(x)$ *with* $\theta'_\pm(x)\to 0$ *as* $x\to\pm\infty$ *so that*

$$u_{\rm d}(x,\tau)-u_{\rm wt}(\tau+\theta_\pm(x))\to 0 \text{ as } x\to\pm\infty,$$

where convergence is uniform in τ *for the functions and their first derivatives with respect to* (x,τ).

It is worth noting that the phase-correction functions $\theta_\pm(x)$ will necessarily diverge logarithmically as $x\to\pm\infty$; see [19, section 3.1]. In particular, the phases of the defects do not converge.

Our goal is to prove that contact defects persist under domain truncation to a sufficiently large interval $[-L,L]$ with Neumann boundary conditions at $x=\pm L$. Before stating our persistence result, we describe our assumptions on the contact defect. To do so, we first reformulate the reaction-diffusion system as a spatial dynamical system and formulate our assumptions for the latter; this allows us to keep the discussion concise and makes it easier to connect the hypotheses more directly with the proofs in the later sections. We refer the reader to [18] for hypotheses formulated for (2.1) that imply our assumptions formulated for the spatial system (2.5) below.

We proceed as in [18] and capture solutions $u(x,\tau)$ of (2.1) that are 2π-periodic in τ as solutions of an appropriate first-order dynamical system that is posed on a function space of 2π-periodic functions of τ and for which the spatial variable x (instead of the time variable τ) is viewed as the evolution variable. We let $S^1:=\mathbb{R}/2\pi\mathbb{Z}$ and write $\mathbf{u}(x)=(u,u_x)(x,\cdot)\in Y:=H^1(S^1)\times H^{\frac12}(S^1)$ for each $x\in\mathbb{R}$ so that $\mathbf{u}(x)$ is for each fixed x a function of τ defined by $[\mathbf{u}(x)](\tau):=(u,u_x)(x,\tau)$. With these definitions, we consider the first-order spatial dynamical system

$$\mathbf{u}_x=\begin{pmatrix}u_x\\ v_x\end{pmatrix}=\begin{pmatrix}v\\ D^{-1}(\omega u_\tau-f(u))\end{pmatrix}=:F(u,v;\omega)=F(\mathbf{u};\omega), \tag{2.5}$$

where $\mathbf{u}(x)=(u,v)(x,\cdot)$ lies for each x in the dense subspace Y of $X:=H^{\frac12}(S^1)\times L^2(S^1)$. Thus, we exchange the evolution in time for evolution in the space variable x, hence the term "spatial dynamics." This method was pioneered by Kirchgässner [8, 9] and Mielke [12]; see also [2, 17, 18]. While the initial-value problem for (2.5) is ill-posed, many approaches from dynamical-systems theory, including invariant-manifold theory, continue to hold.

Before continuing, we discuss the equivariance and reversibility properties of (2.5). First, for $\alpha\in S^1$, we define the S^1-action

$$\mathcal{T}_\alpha: X\longrightarrow X,\quad \mathbf{u}\longmapsto\mathcal{T}_\alpha\mathbf{u},\qquad [\mathcal{T}_\alpha\mathbf{u}](\tau)=[\mathbf{u}](\tau+\alpha)$$

of operators on X that shift the τ variable. We denote by $\Gamma\mathbf{u}:=\{\mathcal{T}_\alpha\mathbf{u}:\alpha\in S^1\}$ the group orbit of an element $\mathbf{u}\in X$. Since the nonlinearity $F(\mathbf{u};\omega)$ in (2.5) does not depend explicitly on τ, a computation shows that it is equivariant under the S^1-action of $\mathcal{T}_\alpha$ so that

$$F(\mathcal{T}_\alpha\mathbf{u};\omega)=\mathcal{T}_\alpha F(\mathbf{u};\omega),\qquad \mathbf{u}\in X,\quad \alpha\in S^1.$$

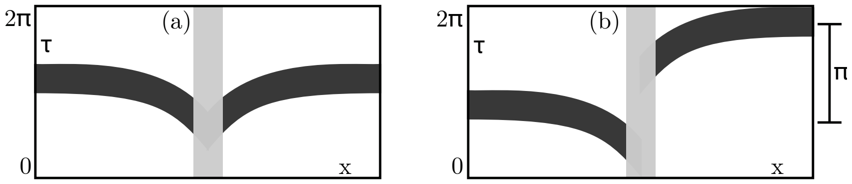

Figure 2. *Shown are space-time plots of time-periodic defects that are reversible under the reverser $\mathcal{R}_0$ in panel* (a) *and the reverser $\mathcal{R}_\pi$ in panel* (b). *Patterns are reversible under $\mathcal{R}_0$ if they are invariant under the reflection $x \mapsto -x$, while they are reversible under $\mathcal{R}_\pi$ if they are invariant under reflection in x followed by a translation in τ by π (corresponding to half the temporal period).*

In particular, $\tilde{\mathbf{u}}(x) := \mathcal{T}_\alpha \mathbf{u}(x)$ satisfies (2.5) if and only if $\mathbf{u}(x)$ satisfies (2.5). Second, (2.5) is reversible with respect to the reversers

$$[\mathcal{R}_0(u,v)](\tau) := (u(\tau), -v(\tau)), \qquad [\mathcal{R}_\pi(u,v)](\tau) := (u(\tau+\pi), -v(\tau+\pi)) = [\mathcal{R}_0\mathcal{T}_\pi(u,v)](\tau) \tag{2.6}$$

so that

$$F(\mathcal{R}_j\mathbf{u};\omega) = -\mathcal{R}_j F(\mathbf{u};\omega), \qquad \mathbf{u} \in X, \quad j = 0, \pi.$$

Given a reverser $\mathcal{R}$, we denote by $\operatorname{Fix}\mathcal{R} := \{\mathbf{u} \in X : \mathcal{R}\mathbf{u} = \mathbf{u}\}$ the space of its fixed points. Reversibility implies that $\tilde{\mathbf{u}}(x) := \mathcal{R}\mathbf{u}(-x)$ satisfies (2.5) if and only if $\mathbf{u}(x)$ satisfies (2.5). We say that a solution $\mathbf{u}(x)$ of (2.5) is reversible under a reverser $\mathcal{R}$ if $\mathbf{u}(0) \in \operatorname{Fix}\mathcal{R}$ so that $\mathbf{u}(x) := \mathcal{R}\mathbf{u}(-x)$ for all x. We refer the reader to Figure 2 for illustrations of solutions that are reversible under these reversers. Note that $\mathcal{T}_\alpha$ and $\mathcal{R}_j$ commute for all α and $j = 0, \pi$.

Returning to the homogeneous oscillation $u_{\mathrm{wt}}(\tau)$, it corresponds to the equilibrium $\mathbf{u}_{\mathrm{wt}} := (u_{\mathrm{wt}}, 0)$ of (2.5). Equivariance implies that $\Gamma\mathbf{u}_{\mathrm{wt}}$ is a circle of equilibria of (2.5). It was proved in [12, section 3] and [18, Theorem 5.1] that this circle of equilibria has center, center-stable, center-unstable, strong stable, and strong unstable invariant manifolds that respect the reversers $\mathcal{R}_{0,\pi}$. Recall that we assumed in Hypothesis 2 that the second derivatives $\lambda_{\mathrm{lin}}''(0)$ and $\omega_{\mathrm{nl}}''(0)$ of the linear and nonlinear dispersion relations λ_{lin} and ω_{nl}, respectively, do not vanish. Hypotheses 1 and 2 together with [2, section 8.1] imply that (i) the center manifold $W^{\mathrm{c}}(\Gamma\mathbf{u}_{\mathrm{wt}})$ of the group orbit of the homogeneous oscillations is two-dimensional; (ii) the dynamics on the center manifold is given by the ODE

$$\varphi_x = \kappa, \qquad \kappa_x = \frac{2}{\lambda_{\mathrm{lin}}''(0)}(\omega - \omega_0) + \frac{-\omega_{\mathrm{nl}}''(0)}{\lambda_{\mathrm{lin}}''(0)}\kappa^2 + g(\kappa,\omega), \tag{2.7}$$

where $(\varphi,\kappa) \in S^1 \times \mathbb{R}$ and $(\varphi, 0)$ parametrizes $\Gamma\mathbf{u}_{\mathrm{wt}}$; and (iii) the reversers $\mathcal{R}_j$ act as $\mathcal{R}_j(\varphi,\kappa) = (\varphi + j, -\kappa)$ for $j = 0, \pi$. Reversibility of (2.7) under $\mathcal{R}_0$ and the estimates provided in [2, (8.15)] imply that $g(\kappa,\omega)$ is even in κ with

$$g(\kappa,\omega) = g(-\kappa,\omega) = \mathrm{O}(|\omega-\omega_0|^2 + |\omega-\omega_0|\kappa^2 + \kappa^4) \tag{2.8}$$

for all (κ,ω) near $(0,\omega_0)$.

Note that the equation for κ, which decouples from the equation for φ, exhibits a nondegenerate saddle-node bifurcation that is unfolded generically as we vary ω near ω_0. In particular, if $\omega''_{\mathrm{nl}}(0) < 0$, the center manifold does not contain any equilibria for $\omega > \omega_0$, and the same statement holds if we reverse both inequality signs.

We now describe our assumptions on the existence of a contact defect of (2.1).

Hypothesis 3. We assume that $\mathbf{u}_{\mathrm{d}}(x)$ is a solution of (2.5) that satisfies

(i) $\mathbf{u}_{\mathrm{d}}(x) \in W^{\mathrm{cs}}(\Gamma \mathbf{u}_{\mathrm{wt}}) \setminus W^{\mathrm{ss}}(\Gamma \mathbf{u}_{\mathrm{wt}})$, and the distance $d(\mathbf{u}_{\mathrm{d}}(x), \Gamma \mathbf{u}_{\mathrm{wt}})$ in Y goes to zero as $x \to \infty$;

(ii) $\mathbf{u}_{\mathrm{d}}(0) \in \operatorname{Fix} \mathcal{R}$ for either $\mathcal{R} = \mathcal{R}_0$ or $\mathcal{R} = \mathcal{R}_\pi$; and

(iii) $W^{\mathrm{cs}}(\Gamma \mathbf{u}_{\mathrm{wt}}) \pitchfork W^{\mathrm{cu}}(\Gamma \mathbf{u}_{\mathrm{wt}})$ at $\mathbf{u}_{\mathrm{d}}(0)$.

Note that Hypothesis 3(i)–(ii) and reversibility of (2.5) imply that $\mathbf{u}_{\mathrm{d}}(x) \in W^{\mathrm{cu}}(\Gamma \mathbf{u}_{\mathrm{wt}})$, and Hypothesis 3(iii) is therefore meaningful. We also note that the intersection of the tangent spaces of $W^{\mathrm{cs}}(\Gamma \mathbf{u}_{\mathrm{wt}})$ and $W^{\mathrm{cu}}(\Gamma \mathbf{u}_{\mathrm{wt}})$ at $\mathbf{u}_{\mathrm{d}}(0)$ is at least two-dimensional since it contains $\partial_x \mathbf{u}_{\mathrm{d}}(0)$ and $\partial_\tau \mathbf{u}_{\mathrm{d}}(0)$, which are linearly independent. The transversality assumed in Hypothesis 3(iii) implies that this intersection is two-dimensional; see [18, Proposition 5.3]. We can now state our main result.

Theorem 1. *Assume that Hypotheses 1–3 hold for the reverser $\mathcal{R}$. Then there exist positive constants L_0, C and a function $\epsilon^* : [L_0, \infty) \to \mathbb{R}$ so that the following is true for each $L \geq L_0$. First, (2.5) with $\omega = \omega_0 - \operatorname{Sign}(\omega''_{\mathrm{nl}}(0))\epsilon^*(L)^2$ has an $\mathcal{R}$-reversible solution $\mathbf{u}_L(x) : [-L; L] \to X$ that is uniformly at most C/L^2 away from $\Gamma \mathbf{u}_{\mathrm{d}}(x)$ and satisfies Neumann boundary conditions $\mathbf{u}_L(\pm L) \in \operatorname{Fix} \mathcal{R}_0$. Furthermore, if $\mathbf{u}_L$ and $\tilde{\mathbf{u}}_L$ are two such solutions, then there exists an $\alpha \in S^1$ such that $\mathcal{T}_\alpha \mathbf{u}_L(x) = \tilde{\mathbf{u}}_L(x)$ for all x. Finally, the function $\epsilon^*(L)$ is C^2 and satisfies the estimates*

$$\epsilon^*(L) = \frac{\pi \lambda''_{\mathrm{lin}}(0)}{2\sqrt{2|\omega''_{\mathrm{nl}}(0)|}L} + \mathrm{O}\left(\frac{1}{L^2}\right), \qquad \frac{\mathrm{d}\epsilon^*}{\mathrm{d}L}(L) = -\frac{\pi \lambda''_{\mathrm{lin}}(0)}{2\sqrt{2|\omega''_{\mathrm{nl}}(0)|}L^2} + \mathrm{O}\left(\frac{1}{L^3}\right). \tag{2.9}$$

If $\mathbf{u}_{\mathrm{d}}(x)$ is reversible under $\mathcal{R} = \mathcal{R}_0$, then the truncated contact defect satisfies Neumann boundary conditions at $x = 0, \pm L$; we then necessarily have $\mathbf{u}_L(-L) = \mathbf{u}_L(L)$ and can extend the truncated defect smoothly as a spatially $2L$-periodic solution to $x \in \mathbb{R}$. If $\mathbf{u}_{\mathrm{d}}(x)$ is reversible under $\mathcal{R} = \mathcal{R}_\pi$, then we can extend it smoothly as a spatially $4L$-periodic solution to $x \in \mathbb{R}$ by reflecting first across $x = L$ and then at $x = 3L$. Theorem 1 can therefore be viewed as a result on the existence of periodic orbits with large periods near a given homoclinic orbit to a circle of saddle-node equilibria. We will provide more details on this viewpoint in section 3.1 when we outline our proof of Theorem 1. For our proof, we will need the following auxiliary result on passage times near nondegenerate saddle-node bifurcations, which may be of independent interest.

Theorem 2. *Consider the one-dimensional ODE*

$$\kappa_x = \epsilon^2 + \kappa^2 + g(\kappa, \epsilon^2) \tag{2.10}$$

with parameter $\omega = \epsilon^2$ and assume that the function $g(\kappa, \omega)$ is C^k for some $k \geq 4$ and satisfies $g(0,0) = g_\kappa(0,0) = g_{\kappa\kappa}(0,0) = g_\omega(0,0) = g_{\kappa\omega}(0,0) = 0$. The following statements are then true:

(i) *There exist positive constants* ϵ_1, δ_1 *and a function* $\ell = \ell(\epsilon,\delta)$ *defined for* $(\epsilon,\delta) \in (0,\epsilon_1]\times[\delta_1/2, 2\delta_1]$ *such that the solution of* (2.10) *with* $\kappa(0) = -\delta$ *satisfies* $\kappa(\ell(\epsilon,\delta)) = 0$. *Furthermore, there are* C^{k-1} *functions* a, b *so that* $\ell(\epsilon,\delta) = \frac{\pi}{2\epsilon} + a(\epsilon)\log\epsilon + b(\epsilon,\delta)$.

(ii) *There exists an* $L_0 > 0$ *and a unique function* $\epsilon^*(L,\delta) : (L_0,\infty)\times[\delta_1/2,\delta_1] \to (0,\epsilon_1)$ *such that* $L = \ell(\epsilon^*(L,\delta),\delta)$ *for all* $L \geq L_0$.

(iii) *For each fixed* $\beta \in [0,1)$, *the function* ϵ^* *is* $C^{1,\beta}$, *and there is a* $C^{1,\beta}$ *function* $r(z,\delta)$ *such that*

$$\epsilon^*(L,\delta) = \frac{\pi}{2L} + r\left(\frac{1}{L},\delta\right) = \frac{\pi}{2L} + \mathrm{O}\left(\frac{1}{L^{1+\beta}}\right),$$
$$\frac{\partial\epsilon^*}{\partial L}(L,\delta) = -\frac{\pi}{2L^2} - \frac{1}{L^2} r_z\left(\frac{1}{L},\delta\right) = -\frac{\pi}{2L^2} + \mathrm{O}\left(\frac{1}{L^{2+\beta}}\right). \tag{2.11}$$

(iv) *If* $g(-\kappa,\omega) = g(\kappa,\omega)$ *for all* (κ,ω), *then* $r(z,\delta) \in C^k$, *and the estimates in* (iii) *hold with* $\beta = 1$.

(v) *Analogous statements hold for the problem* $\kappa(0) = 0$ *and* $\kappa(\ell(\epsilon,\delta)) = \delta$.

Theorem 2 provides expansions of the travel time from $\kappa = -\delta$ to $\kappa = 0$ (and similarly from $\kappa = \delta$ backwards in time to $\kappa = 0$) in the unfolding of a nondegenerate saddle-node bifurcation at $\kappa = 0$: Theorem 2(i) shows that the travel times typically contain logarithmic terms $\log\epsilon$ and are therefore not differentiable in ϵ regardless of how smooth the right-hand side is. In contrast, Fontich and Sardanyes [3] showed that the travel time from $\kappa = -\delta$ to $\kappa = \delta$ for the unfolding of possibly degenerate saddle-node bifurcations is analytic in ϵ for analytic right-hand sides. These two results are reconciled by noting that the logarithmic terms in the travel times from $\kappa = -\delta$ to $\kappa = 0$ and from $\kappa = 0$ to $\kappa = \delta$ cancel, yielding a smooth expression for the travel times from $\kappa = -\delta$ to $\kappa = \delta$. Finally, we remark that Kuehn [10] showed that travel times may exhibit many different scaling laws when the right-hand side depends only continuously on ω.

We will prove Theorems 1 and 2 in sections 3 and 4, respectively, and end with a brief discussion in section 5.

3. Existence of truncated contact defects.

In this section, we prove Theorem 1. Throughout this section, we assume that Hypotheses 1–3 are met. First, we provide intuition into why this theorem should hold by appealing to a finite-dimensional analogue and outline our strategy for proving this theorem in two steps. We then provide the details of the proof.

Recall from the discussion after (2.7) that the center manifold $W^c(\Gamma\mathbf{u}_{\mathrm{wt}})$ of the homogeneous oscillations $\mathbf{u}_{\mathrm{wt}}$ does not contain any equilibria for $\omega > \omega_0$ when $\omega''_{\mathrm{nl}}(0) < 0$ (and that the same statement holds if we reverse both inequality signs). For clarity, and without loss of generality, we assume throughout this section that $\omega''_{\mathrm{nl}}(0) < 0$ so that the equilibria disappear for $\omega > \omega_0$.

3.1. Outline of the proof.

We first discuss our intuition into the underlying geometry and refer the reader to Figure 3 for illustrations. Equivariance shows that $\mathcal{T}_\alpha\mathbf{u}_{\mathrm{d}}(x)$ is, for each $\alpha \in S^1$, a solution that converges to the set $\Gamma\mathbf{u}_{\mathrm{wt}}$ as $|x| \to \infty$. It follows from Hypothesis 3(i) that the set $\{\mathbf{u} = \mathcal{T}_\alpha\mathbf{u}_{\mathrm{d}}(x) : \alpha \in S^1,\ x \in \mathbb{R}\} \cup \Gamma\mathbf{u}_{\mathrm{wt}}$, which consists of the defect solution, its time translates, and the S^1-group orbit of the homogeneous oscillations, is a smooth invariant torus.

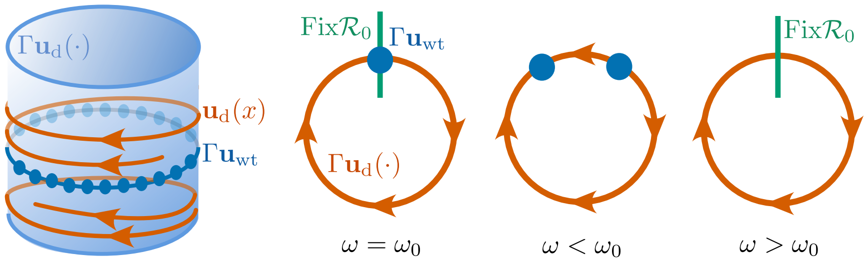

Figure 3. *The left panel illustrates part of the invariant torus formed by the contact defect and its time translates together with the circle $\Gamma\mathbf{u}_{\mathrm{wt}}$ of equilibria. The remaining three panels illustrate the dynamics on the invariant torus after factoring out the S^1-action $\mathcal{T}_\alpha$ for $\omega \gtreqless \omega_0$. The rightmost panel shows that for each given $\omega > \omega_0$ there is an $L \gg 1$ so that the solution $\mathbf{u}(x)$ on the circle satisfies $\mathbf{u}(\pm L) \in \mathrm{Fix}\,\mathcal{R}_0$, and Theorem 2 shows that we can invert the relationship between $\omega - \omega_0$ and L.*

Hypothesis 3(iii) can be used to show that this invariant torus is normally hyperbolic. If (2.5) were finite-dimensional, we could appeal to existing results, for instance, those described in [5, 11], to conclude that the invariant torus persists as a smooth torus for all ω near ω_0. Assuming that the same results hold for (2.5), we can then focus our analysis on these tori. Factoring out the S^1-action $\mathcal{T}_\alpha$ on these tori, we end up with an ω-dependent dynamical system on a circle. Note also that the equilibrium $\mathbf{u}_{\mathrm{wt}} = (u_{\mathrm{wt}}, 0)$ automatically lies in $\mathrm{Fix}\,\mathcal{R}_0$. The solution we want to construct should reach $\mathrm{Fix}\,\mathcal{R}_0$ in finite time. Hence, we need to focus on the case where the equilibria on the circle disappear, which happens for $\omega > \omega_0$. As outlined in the rightmost panel of Figure 3, for each given $\omega > \omega_0$ there is indeed a unique $L = L(\omega)$ so that the solution $\mathbf{u}(x)$ on the circle satisfies $\mathbf{u}(\pm L) \in \mathrm{Fix}\,\mathcal{R}_0$ as needed. Theorem 2 shows that we can invert this relationship so that for each large L there is a unique $\omega = \omega(L) > \omega_0$ for which a defect solution exists that satisfies Neumann boundary conditions at $x = \pm L$. For finite-dimensional Galerkin approximations of (2.5), these arguments were made rigorous by the first author in [7]. As mentioned above, this proof relies on the persistence of normally hyperbolic invariant manifolds; however, these persistence results have not yet been generalized to the case of infinite-dimensional ill-posed spatial dynamics problems of the form (2.5). Thus, while this approach provides geometric intuition, we need to find a different strategy to prove Theorem 1.

Figure 4 illustrates the two main steps of our proof of Theorem 1. In the first step, we will show that, using an appropriate boundary-value problem formulation of the ill-posed spatial dynamical system (2.5) for ω near ω_0, we can transport the space $\mathrm{Fix}\,\mathcal{R}$ near $\mathbf{u}_{\mathrm{d}}(0)$ from $x = 0$ to $x = L_0$ to yield the space $M_{\mathcal{R}}(L_0)$ near $\mathbf{u}_{\mathrm{d}}(L_0)$. In the second step, we prove that for each $\omega > \omega_0$ there is a unique $\ell = \ell(\omega)$ so that there is a solution $\mathbf{u}_{\mathrm{N}}(x)$ with $\mathbf{u}_{\mathrm{N}}(0) \in M_{\mathcal{R}}(L_0)$ and $\mathbf{u}_{\mathrm{N}}(\ell) \in \mathrm{Fix}\,\mathcal{R}_0$.

3.2. Dynamics near the homogeneous oscillations.

We describe results and notation for the dynamics of (2.5), given by

$$\mathbf{u}_x = \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} v \\ D^{-1}(\omega u_\tau - f(u)) \end{pmatrix} =: F(u, v; \omega) = F(\mathbf{u}; \omega), \qquad \mathbf{u}(x) \in Y = H^1(S^1) \times H^{\frac{1}{2}}(S^1), \tag{3.1}$$

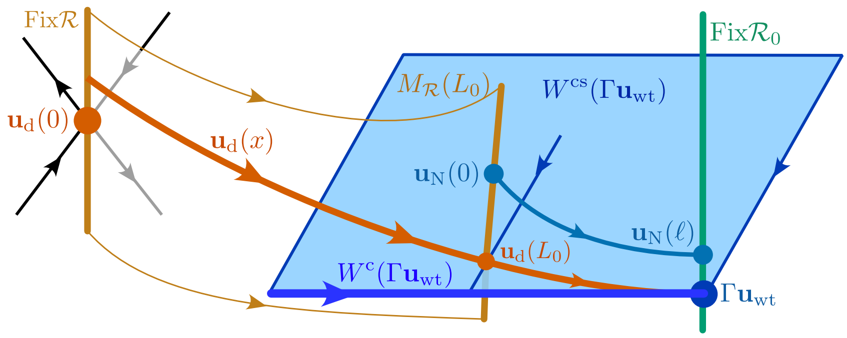

Figure 4. *We illustrate the proof of Theorem 1. In step 1, we use (2.5) for ω near ω_0 to transport* Fix$\,\mathcal{R}$ *at $x=0$ to the manifold $M_{\mathcal{R}}(L_0)$ at $x=L_0$, where L_0 is so large that $\mathbf{u}_{\rm d}(L_0)$ lies in the center-stable manifold $W^{\rm cs}(\Gamma\mathbf{u}_{\rm wt})$. In step 2, we focus on values of ω near ω_0 in the region where the equilibria on the center manifold $W^{\rm c}(\Gamma\mathbf{u}_{\rm wt})$ have disappeared. For each such ω, we construct a solution $\mathbf{u}_{\rm N}(x)$ that lies in $M_{\mathcal{R}}(L_0)$ for $x=0$ and in* Fix$\,\mathcal{R}_0$ *for $x=\ell$ for some large $\ell=\ell(\omega)\gg 1$.*

near the S^1-orbit $\Gamma\mathbf{u}_{\rm wt}$ of the equilibria associated with the homogeneous oscillations. Throughout, we will denote by $\delta_0>0$ a small positive constant so that existence of the invariant manifolds and fibers that we will review below holds within a neighborhood of $\Gamma\mathbf{u}_{\rm wt}$ of radius δ_0. We may decrease δ_0 later, but its final size will depend only on the quantities in Hypotheses 1–3 and not on the length L of the interval on which we will construct the truncated defect.

We begin by summarizing the consequences of the results in [18, section 3.4] or [2, sections 4.1 and 8.1] for (3.1), whose assumptions are met due to our Hypotheses 1–2. The linearized operator $F_{\mathbf{u}}(\mathbf{u}_{\rm wt};\omega_0)$ has an eigenvalue at $\lambda=0$ with geometric multiplicity one and algebraic multiplicity two, and there is an $\eta>0$ so that the remainder of the spectrum is discrete and satisfies $|\mathrm{Re}\lambda|\geq 4\eta$ for all $\lambda\neq 0$ in the spectrum. We denote the associated spectral projections for the unstable, stable, and center parts of the spectrum by $P_0^j(\mathbf{u}_{\rm wt})$ for $j=\mathrm{c},\mathrm{s},\mathrm{u}$ and note that reversibility and equivariance imply that

$$\mathcal{R}_{0,\pi}\,\mathrm{Rg}\,P_0^{\rm u}(\mathbf{u}_{\rm wt})=\mathrm{Rg}\,P_0^{\rm s}(\mathbf{u}_{\rm wt}),\quad \mathcal{T}_\alpha P_0^j(\mathbf{u}_{\rm wt})=P_0^j(\mathcal{T}_\alpha\mathbf{u}_{\rm wt})\mathcal{T}_\alpha,\quad \mathcal{T}_\alpha\,\mathrm{Rg}\,P_0^j(\mathbf{u}_{\rm wt})=\mathrm{Rg}\,P_0^j(\mathcal{T}_\alpha\mathbf{u}_{\rm wt}) \tag{3.2}$$

for all $\alpha\in S^1$ and all j. The two-dimensional center space of $F_{\mathbf{u}}(\mathbf{u}_{\rm wt};\omega_0)$ belonging to the eigenvalue $\lambda=0$ is spanned by the eigenvector $\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{T}_\alpha\mathbf{u}_{\rm wt}|_{\alpha=0}=\partial_\tau\mathbf{u}_{\rm wt}$, which corresponds to the tangent vector of $\Gamma\mathbf{u}_{\rm wt}$, and a generalized eigenvector that we denote by $\mathbf{v}_{\rm wt}$. By [12, section 3] or [2, section 8.1], the circle $\Gamma\mathbf{u}_{\rm wt}$ of equilibria of (3.1) has a two-dimensional locally invariant manifold

$$W^{\rm c}(\Gamma\mathbf{u}_{\rm wt},\omega)=\Gamma\tilde{W}^{\rm c}(\Gamma\mathbf{u}_{\rm wt},\omega)=\Gamma\left\{\mathbf{u}=\mathbf{u}_{\rm wt}+\kappa\mathbf{v}_{\rm wt}+h(\kappa,\omega):|\kappa|<\delta_0\right\}, \tag{3.3}$$

where h is C^k for each fixed finite k and maps into the space $\mathrm{Rg}\,P_0^{\rm s}(\mathbf{u}_{\rm wt})\oplus\mathrm{Rg}\,P_0^{\rm u}(\mathbf{u}_{\rm wt})$. The vector field on the center manifold is given by (2.7),

$$\varphi_x=\kappa,\quad \kappa_x=\frac{2}{\lambda_{\rm lin}''(0)}(\omega-\omega_0)+\frac{-\omega_{\rm nl}''(0)}{\lambda_{\rm lin}''(0)}\kappa^2+g(\kappa,\omega),\quad g(\kappa,\omega)=\mathrm{O}(|\omega-\omega_0|^2+|\omega-\omega_0|\kappa^2+\kappa^4), \tag{3.4}$$

where the reverser $\mathcal{R}_0$ leaves the center manifold invariant and acts as $\mathcal{R}_0(\varphi,\kappa)=(\varphi,-\kappa)$. In particular, $W^{\rm c}(\Gamma\mathbf{u}_{\rm wt},\omega)\cap\operatorname{Fix}\mathcal{R}_0$ is parametrized by $\{(\varphi,\kappa):\kappa=0\}$. Using (3.2) and

$$X=\mathbb{R}\partial_\tau\mathbf{u}_{\rm wt}\oplus\mathbb{R}\mathbf{v}_{\rm wt}\oplus\operatorname{Rg}P_0^{\rm s}(\mathbf{u}_{\rm wt})\oplus\operatorname{Rg}P_0^{\rm u}(\mathbf{u}_{\rm wt})$$

we conclude that

$$\operatorname{Fix}\mathcal{R}_0=\{\mathbf{u}=\alpha\partial_\tau\mathbf{u}_{\rm wt}+\mathbf{b}^{\rm s}+\mathcal{R}_0\mathbf{b}^{\rm s}:\alpha\in\mathbb{R},\ \mathbf{b}^{\rm s}\in\operatorname{Rg}P_0^{\rm s}(\mathbf{u}_{\rm wt})\}\,. \tag{3.5}$$

Since the truncated contact defect we want to construct will need to reach the space $\operatorname{Fix}\mathcal{R}_0$ of Neumann boundary conditions in a large but finite time, we will first characterize the time it takes for solutions $\kappa(x)$ on the center manifold to reach $\kappa=0$.

Lemma 3.1. *For each $\delta_2>0$, there is an ϵ_0 and a unique function $L(\epsilon,\kappa_0)$ defined for $0<\epsilon<\epsilon_0$ and $\kappa_0\le-\delta_2$ so that the following is true for $\omega=\omega_0+\epsilon^2$. The ODE (3.4) for $\kappa(x)$ has a solution that satisfies $\kappa(0)=\kappa_0$ and $\kappa(\ell)=0$ for some $\ell>0$ if and only if $\ell=L(\epsilon,\kappa_0)$. Furthermore,*

$$L(\epsilon,\kappa_0)=\frac{\pi\lambda''_{\rm lin}(0)}{2\sqrt{-2\omega''_{\rm nl}(0)}\epsilon}+r(\epsilon,\kappa_0),$$

where r depends smoothly on (ϵ,κ_0), and we have $\kappa(L(\epsilon,\kappa_0)+\tilde{x})=2\epsilon^2\tilde{x}/\lambda''_{\rm lin}(0)+\mathrm{O}(\epsilon^4\tilde{x}^2)$ for all $\tilde{x}$ with $|\tilde{x}|\le\frac{1}{3}$.

Proof. First, since $g(\kappa,\omega)$ is even in κ, [6, Corollary 1] shows that there is a C^k-smooth near-identity diffeomorphism $(\tilde\kappa,\tilde\omega)=(H(\kappa,\omega),h(\omega))$ with $H(0,\omega_0)=h(\omega_0)=0$ that brings (3.4) into the normal form

$$\tilde\kappa_x=\frac{2}{\lambda''_{\rm lin}(0)}\tilde\omega+\frac{-\omega''_{\rm nl}(0)}{\lambda''_{\rm lin}(0)}\tilde\kappa^2.$$

Setting $\tilde\omega=\tilde\epsilon^2$, the statements in the lemma can be verified directly for the normal form, as its solutions can be found explicitly using separation of variables. We note that these results follow also from Theorem 2 (proved in Lemmas 4.1 and 4.2 in section 4), which is valid for the general case when g is not necessarily even. ∎

We now discuss the center-stable manifold $W^{\rm cs}(\mathbf{u}_{\rm wt},\omega)$, which for $\omega=\omega_0$ contains all solutions $\mathbf{u}(x)$ of (3.1) whose distance to $\Gamma\mathbf{u}_{\rm wt}$ converges to zero as $x\to\infty$. Using the strong stable fibers $\mathcal{F}^{\rm ss}(\mathbf{p},\omega)$ belonging to base points $\mathbf{p}\in W^{\rm c}(\mathbf{u}_{\rm wt},\omega)$, we can write

$$W^{\rm cs}(\mathbf{u}_{\rm wt},\omega)=\bigcup_{\mathbf{p}\in W^{\rm c}(\mathbf{u}_{\rm wt},\omega)}\mathcal{F}^{\rm ss}(\mathbf{p},\omega)=\bigcup_{\mathbf{p}\in\tilde{W}^{\rm c}(\mathbf{u}_{\rm wt},\omega)}\Gamma\mathcal{F}^{\rm ss}(\mathbf{p},\omega).$$

We know from [18, Theorem 5.1] that $\mathcal{F}^{\rm ss}(\mathbf{p},\omega)$ is a smooth manifold that depends smoothly on $(\mathbf{p},\omega)$ and whose tangent space converges to $\operatorname{Rg}P_0^{\rm s}(\mathbf{u}_{\rm wt})$ as $(\mathbf{p},\omega)$ approaches $(\mathbf{u}_{\rm wt},\omega_0)$. We parametrize each strong stable fiber for base points in $\tilde{W}^{\rm c}(\Gamma\mathbf{u}_{\rm wt},\omega)$ via

$$\begin{aligned}\mathcal{F}^{\rm ss}(\mathbf{p},\omega)&=\bigcup_{\mathbf{b}^{\rm s}\in\operatorname{Rg}P_0^{\rm s}(\mathbf{u}_{\rm wt})}\mathcal{F}^{\rm ss}(\mathbf{p},\mathbf{b}^{\rm s},\omega),\\ \mathcal{F}^{\rm ss}(\mathbf{p},\mathbf{b}^{\rm s},\omega)&=\mathbf{p}+\mathbf{b}^{\rm s}+h^{\rm u}(\mathbf{p},\mathbf{b}^{\rm s},\omega),\quad|h^{\rm u}(\mathbf{p},\mathbf{b}^{\rm s},\omega)|\le C\delta_0|\mathbf{b}^{\rm s}|,\end{aligned}\tag{3.6}$$

where h^{u} maps into $\mathrm{Rg}\, P_0^{\mathrm{u}}(\mathbf{u}_{\mathrm{wt}})$ and the positive constant C does not depend on δ_0. The defect solution $\mathbf{u}_{\mathrm{d}}(x)$ lies in the center-stable manifold $W^{\mathrm{cs}}(\mathbf{u}_{\mathrm{wt}}, \omega_0)$ and therefore has the representation

$$\mathbf{u}_{\mathrm{d}}(x) = \mathcal{T}_{\varphi_{\mathrm{d}}(x)} \left(\mathbf{u}_{\mathrm{wt}} + \kappa_{\mathrm{d}}(x) \mathbf{v}_{\mathrm{wt}} + h(\kappa_{\mathrm{d}}(x), \omega_0) + \mathrm{O}(\mathrm{e}^{-3\eta x}) \right) \tag{3.7}$$

for $x \geq L_0$ for an appropriate $L_0 \gg 1$, where $(\varphi_{\mathrm{d}}, \kappa_{\mathrm{d}})(x)$ satisfy (3.4) with $\omega = \omega_0$ and the exponentially decaying term accounts for the contribution from the strong stable fiber that $\mathbf{u}_{\mathrm{d}}(x)$ belongs to.

3.3. Step 1: Transporting the fixed-point space of the reverser along the defect.

Throughout this section, we denote by $\mathcal{R}$ the reverser from Hypothesis 3 for which $\mathbf{u}_{\mathrm{d}}(0) \in \mathrm{Fix}\, \mathcal{R}$. Our goal is to transport $\mathrm{Fix}\, \mathcal{R}$ near the solution $\mathbf{u}_{\mathrm{d}}(x)$ from $x = 0$ to $x = L_0$ for some $L_0 \gg 1$ so large that $\mathbf{u}_{\mathrm{d}}(L_0)$ has distance less than δ_0 from $\Gamma \mathbf{u}_{\mathrm{wt}}$. To accomplish this, we will exploit that the linearization

$$\mathbf{v}_x = F_{\mathbf{u}}(\mathbf{u}_{\mathrm{d}}(x); \omega_0) \mathbf{v} \tag{3.8}$$

of (3.1) about the defect $\mathbf{u}_{\mathrm{d}}(x)$ has exponential trichotomies, which will allow us to decompose the underlying space into three complementary subspaces that consist of, respectively, initial conditions of solutions that decay exponentially in forward time, decay exponentially in backward time, or grow only moderately.

Lemma 3.2. *Assume that Hypotheses 1–3 are met. Then the linearization (3.8) of (3.1) about $\mathbf{u}_{\mathrm{d}}(x)$ has an exponential trichotomy on $\mathbb{R}^+$. More precisely, there exist strongly continuous families $\{\Phi^{\mathrm{s}}(x,y)\}_{x \geq y \geq 0}$, $\{\Phi^{\mathrm{c}}(x,y)\}_{x,y \geq 0}$, and $\{\Phi^{\mathrm{u}}(x,y)\}_{y \geq x \geq 0}$ of operators in $L(X)$ with the following properties:*

(i) *For $j = \mathrm{c,s,u}$ and each $\mathbf{u}_0 \in X$, the function $\Phi^j(x,y)\mathbf{u}_0$ satisfies (3.8) for all x, y for which this term is defined, and we have $\Phi^j(x,y)\Phi^j(y,z) = \Phi^j(x,z)$ for $j = \mathrm{c,s,u}$ whenever these terms are defined.*

(ii) *There is a constant $C > 0$ so that $\|\Phi^{\mathrm{s}}(x,y)\| + \|\Phi^{\mathrm{u}}(y,x)\| \leq C\mathrm{e}^{-3\eta|x-y|}$ for all $x \geq y \geq 0$ and $\|\Phi^{\mathrm{c}}(x,y))\| \leq C\mathrm{e}^{\eta|x-y|}$ for $x, y \geq 0$, where $4\eta > 0$ is the spectral gap defined in section 3.2.*

(iii) *For $j = \mathrm{c,s,u}$, the operators $P_{\mathrm{d}}^j(x) := \Phi^j(x,x)$ are projections with $\|P_{\mathrm{d}}^j(x) - P_0^j(\mathcal{T}_{\varphi_{\mathrm{d}}(x)}\mathbf{u}_{\mathrm{wt}})\|_{L(X)} \to 0$ as $x \to \infty$ and their sum is the identity in $L(X)$.*

(iv) *The projections $P_{\mathrm{d}}^j(x)$ can be chosen so that $\mathrm{Rg}\, P_{\mathrm{d}}^{\mathrm{c}}(x) = \mathrm{Span}\{\partial_x \mathbf{u}_{\mathrm{d}}(x), \partial_\tau \mathbf{u}_{\mathrm{d}}(x)\}$ and $\mathcal{R}\, \mathrm{Rg}\, P_{\mathrm{d}}^{\mathrm{s}}(0) = \mathrm{Rg}\, P_{\mathrm{d}}^{\mathrm{u}}(0)$. Furthermore,*

$$X = \mathrm{Fix}\, \mathcal{R} \oplus \mathbb{R}\partial_x \mathbf{u}_{\mathrm{d}}(0) \oplus \mathrm{Rg}\, P_{\mathrm{d}}^{\mathrm{s}}(0), \tag{3.9}$$

and we have that $\mathbf{v} = \mathbf{a}^{\mathrm{s}} + \mathbf{a}^{\mathrm{u}}$ with $\mathbf{a}^j \in \mathrm{Rg}\, P_{\mathrm{d}}^j(0)$ for $j = \mathrm{s,u}$ lies in $\mathrm{Fix}\, \mathcal{R}$ if and only if $\mathbf{a}^{\mathrm{s}} = \mathcal{R}\mathbf{a}^{\mathrm{u}}$.

Proof. The existence of center-stable and unstable exponential dichotomies and their convergence properties for $x \to \infty$ were established in [15, 17] and also in [18, Appendices A.1 and A.2]. The arguments in [18, section 4.1] show how exponential weights with rates $\pm 2\eta$ can be used to construct exponential trichotomies from exponential dichotomies, and we also refer the reader to [21, section 5.2] for proofs of similar statements in the more complicated case of planar spiral waves. To establish (iv), we note that it follows from [15, 16] that we can pick any complement of the range of the center-stable projection as the range of $P^{\mathrm{u}}_{\mathrm{d}}(0)$; reversibility and Hypothesis 3(iii) show that $\mathcal{R}\,\mathrm{Rg}\,P^{\mathrm{s}}_{\mathrm{d}}(0)$ is such a complement. A similar argument establishes the claim about the range of the center projection. Using $\Gamma\mathbf{u}_{\mathrm{d}}(0)\subset\mathrm{Fix}\,\mathcal{R}$, the fact that the sum of the projections is the identity in X, and the characterization of the ranges of the projections we just established, we conclude that (3.9) holds. The final claim in (iv) follows from

$$\mathbf{a}^{\mathrm{s}}+\mathbf{a}^{\mathrm{u}}=\mathbf{v}\overset{\mathbf{v}\in\mathrm{Fix}\,\mathcal{R}}{=}\mathcal{R}\mathbf{v}=\mathcal{R}\mathbf{a}^{\mathrm{s}}+\mathcal{R}\mathbf{a}^{\mathrm{u}}$$

combined with $\mathcal{R}\mathbf{a}^{\mathrm{s}}\in\mathrm{Rg}\,P^{\mathrm{u}}_{\mathrm{d}}(0)$ and $\mathcal{R}\mathbf{a}^{\mathrm{u}}\in\mathrm{Rg}\,P^{\mathrm{s}}_{\mathrm{d}}(0)$, which implies that $\mathbf{a}^{\mathrm{s}}=\mathcal{R}\mathbf{a}^{\mathrm{u}}$. ∎

The next lemma shows that we can transport $\mathrm{Fix}\,\mathcal{R}$ along the defect solution $\mathbf{u}_{\mathrm{d}}(x)$ from $x=0$ to $x=L_0$ for each sufficiently large $L_0\gg 1$.

Lemma 3.3. *There are positive constants C_0,C_1,C_2,L_0^* so that for each $L_0\geq L_0^*$ the following is true. For all ω with $|\omega-\omega_0|\leq C_1\mathrm{e}^{-2\eta L_0}$, we have*

$$\begin{aligned}M_{\mathcal{R}}(L_0,\omega):=&\Big\{\tilde{\mathbf{u}}\in X:\exists\ \text{solution }\mathbf{u}(x)\text{ of (3.1) with }\mathbf{u}(0)\in\mathrm{Fix}\,\mathcal{R},\mathbf{u}(L_0)=\tilde{\mathbf{u}},\\&\sup_{0\leq x\leq L_0}|\mathbf{u}(x)-\mathbf{u}_{\mathrm{d}}(x)|\leq C_0\mathrm{e}^{-\eta L_0}\Big\}\\=&\,\Gamma\tilde{M}_{\mathcal{R}}(L_0,\omega)=\Gamma\Big\{\tilde{\mathbf{u}}=\mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{u}_{\mathrm{d}}(L_0)+\mathbf{a}^{\mathrm{u}}+h^{\mathrm{cs}}_{\mathcal{R}}(\mathbf{a}^{\mathrm{u}},L_0,\omega),\\&\mathbf{a}^{\mathrm{u}}\in\mathrm{Rg}\,P^{\mathrm{u}}_0(\mathrm{u}_{\mathrm{wt}}),|\mathbf{a}^{\mathrm{u}}|\leq C_1\mathrm{e}^{-\eta L_0}\Big\},\end{aligned}$$

where $h^{\mathrm{cs}}_{\mathcal{R}}$ is smooth, maps into $\mathrm{Rg}\,P^{\mathrm{cs}}_0(\mathrm{u}_{\mathrm{wt}})$, and satisfies $|h^{\mathrm{cs}}_{\mathcal{R}}(\mathbf{a}^{\mathrm{u}},L_0,\omega)|\leq C_2(\delta_0|\mathbf{a}^{\mathrm{u}}|+|\mathbf{a}^{\mathrm{u}}|^2+|\omega-\omega_0|)$.

Proof. We exploit the existence of exponential trichotomies to solve (3.1) near the defect solution $\mathbf{u}_{\mathrm{d}}(x)$. To do so, we write $\mathbf{u}=\mathbf{u}_{\mathrm{d}}(x)+\mathbf{v}$ and $\omega=\omega_0+\bar{\omega}$ so that $\mathbf{u}$ satisfies (3.1) if and only if $\mathbf{v}$ satisfies

$$\mathbf{v}_x=F_{\mathbf{u}}(\mathbf{u}_{\mathrm{d}}(x),\omega_0)\mathbf{v}+\underbrace{F(\mathbf{u}_{\mathrm{d}}(x)+\mathbf{v},\omega_0+\bar{\omega})-F(\mathbf{u}_{\mathrm{d}}(x),\omega_0)-F_{\mathbf{u}}(\mathbf{u}_{\mathrm{d}}(x),\omega_0)\mathbf{v}}_{=:G(\mathbf{v},\bar{\omega})},$$
$$G(\mathbf{v},\bar{\omega})=\mathrm{O}(|\mathbf{v}|^2+|\bar{\omega}|).$$

Solutions to this equation can be found as solutions of the fixed-point problem

(3.10)
$$\begin{aligned}\mathbf{v}(x)=&\,\Phi^{\mathrm{s}}(x,0)\mathbf{a}^{\mathrm{s}}+\Phi^{\mathrm{u}}(x,L_0)\mathbf{a}^{\mathrm{u}}+\int_0^x\Phi^{\mathrm{cs}}(x,y)G(\mathbf{v}(y),\bar{\omega})\,\mathrm{d}y+\int_{L_0}^x\Phi^{\mathrm{u}}(x,y)G(\mathbf{v}(y),\bar{\omega})\,\mathrm{d}y\\=:&\,\mathcal{G}(\mathbf{v},\mathbf{a}^{\mathrm{s}},\mathbf{a}^{\mathrm{u}},\bar{\omega})(x)\end{aligned}$$

with $x\in[0,L_0]$, $\mathbf{a}^{\mathrm{s}}\in\mathrm{Rg}\,P^{\mathrm{s}}_{\mathrm{d}}(0)$, and $\mathbf{a}^{\mathrm{u}}\in\mathrm{Rg}\,P^{\mathrm{u}}_{\mathrm{d}}(L_0)$, where $\mathcal{G}(\cdot,\mathbf{a}^{\mathrm{s}},\mathbf{a}^{\mathrm{u}},\bar{\omega})$ maps $C^0([0,L_0],X)$, equipped with the supremum norm $\|\cdot\|$, into itself for each $(\mathbf{a}^{\mathrm{s}},\mathbf{a}^{\mathrm{u}},\bar{\omega})$. The estimates from Lemma 3.2 show that there is a constant C_0 that does not depend on L_0 such that

$$\|\mathcal{G}(\mathbf{v},\mathbf{a}^{\mathrm{s}},\mathbf{a}^{\mathrm{u}},\bar\omega)\| \le C_0\left(|\mathbf{a}^{\mathrm{s}}|+|\mathbf{a}^{\mathrm{u}}|+\mathrm{e}^{\eta L_0}(\|\mathbf{v}\|^2+|\bar\omega|)\right), \quad \|\mathcal{G}_{\mathbf{v}}(\mathbf{v},\mathbf{a}^{\mathrm{s}},\mathbf{a}^{\mathrm{u}},\bar\omega)\| \le C_0\mathrm{e}^{\eta L_0}\|\mathbf{v}\|.$$

Setting $C_1 := \frac{1}{12C_0^2}$ and $C_3 := \frac{1}{2C_0}$, we find that

$$\|\mathcal{G}(\mathbf{v},\mathbf{a}^{\mathrm{s}},\mathbf{a}^{\mathrm{u}},\bar\omega)\| \le C_3\mathrm{e}^{-\eta L_0}, \qquad \|\mathcal{G}_{\mathbf{v}}(\mathbf{v},\mathbf{a}^{\mathrm{s}},\mathbf{a}^{\mathrm{u}},\bar\omega)\| \le \frac{1}{2}$$

for all $(\mathbf{v},\mathbf{a}^{\mathrm{s}},\mathbf{a}^{\mathrm{u}},\bar\omega)$ with $|\mathbf{a}^{\mathrm{s}}|,|\mathbf{a}^{\mathrm{u}}| \le C_1\mathrm{e}^{-\eta L_0}$, $|\bar\omega| \le C_1\mathrm{e}^{-2\eta L_0}$, and $\|\mathbf{v}\| \le C_3\mathrm{e}^{-\eta L_0}$. The uniform contraction mapping principle now guarantees the existence of a unique fixed point $\mathbf{v}$ of (3.10) that depends smoothly on $(\mathbf{a}^{\mathrm{s}},\mathbf{a}^{\mathrm{u}},\bar\omega)$ provided these satisfy the estimates noted above, and we have $\|\mathbf{v}\| \le 2C_0(|\mathbf{a}^{\mathrm{s}}|+|\mathbf{a}^{\mathrm{u}}|+\mathrm{e}^{\eta L_0}|\bar\omega|)$. We are interested in solutions with $\mathbf{v}(0) \in \operatorname{Fix}\mathcal{R}$. Since

$$\mathbf{v}(0) = \mathbf{a}^{\mathrm{s}} + \Phi^{\mathrm{u}}(0,L_0)\mathbf{a}^{\mathrm{u}} + \int_{L_0}^{0}\Phi^{\mathrm{u}}(0,y)G(\mathbf{v}(y),\bar\omega)\,\mathrm{d}y,$$

we conclude from Lemma 3.2(iv) that $\mathbf{v}(0) \in \operatorname{Fix}\mathcal{R}$ if and only if $\mathbf{a}^{\mathrm{s}}$ satisfies

$$\mathbf{a}^{\mathrm{s}} = \mathcal{R}\left(\Phi^{\mathrm{u}}(0,L_0)\mathbf{a}^{\mathrm{u}} + \int_{L_0}^{0}\Phi^{\mathrm{u}}(0,y)G(\mathbf{v}(y),\bar\omega)\,\mathrm{d}y\right) = \mathrm{O}\left(\mathrm{e}^{-3\eta L_0}|\mathbf{a}^{\mathrm{u}}| + |\mathbf{a}^{\mathrm{s}}|^2 + |\mathbf{a}^{\mathrm{u}}|^2 + |\bar\omega|\right),$$

which we can solve uniquely for $\mathbf{a}^{\mathrm{s}}$ as a function of $(\mathbf{a}^{\mathrm{u}},\bar\omega)$ with $|\mathbf{a}^{\mathrm{s}}| \le C_2(\mathrm{e}^{-3\eta L_0}|\mathbf{a}^{\mathrm{u}}|+|\mathbf{a}^{\mathrm{u}}|^2+|\bar\omega|)$ for some C_2 that does not depend on L_0. We substitute this function for $\mathbf{a}^{\mathrm{s}}$ into the expression for $\mathbf{v}(L_0)$ in (3.10) to obtain

$$\mathbf{v}(L_0) = \mathbf{a}^{\mathrm{u}} + H^{\mathrm{cs}}_{\mathcal{R}}(\mathbf{a}^{\mathrm{u}},L_0,\bar\omega), \quad \mathbf{a}^{\mathrm{u}} \in \operatorname{Rg}P^{\mathrm{u}}_{\mathrm{d}}(L_0), \quad H^{\mathrm{cs}}_{\mathcal{R}}(\mathbf{a}^{\mathrm{u}},L_0,\bar\omega) = \mathrm{O}\left(\mathrm{e}^{-3\eta L_0}|\mathbf{a}^{\mathrm{u}}| + |\mathbf{a}^{\mathrm{u}}|^2 + |\bar\omega|\right),$$

where $H^{\mathrm{cs}}_{\mathcal{R}}$ maps into $\operatorname{Rg}P^{\mathrm{cs}}_{\mathrm{d}}(L_0)$, so that

$$\mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{v}(L_0) = \mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{a}^{\mathrm{u}} + \mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}H^{\mathrm{cs}}_{\mathcal{R}}(\mathbf{a}^{\mathrm{u}},L_0,\bar\omega) = \tilde{\mathbf{a}}^{\mathrm{u}} + \underbrace{\mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}H^{\mathrm{cs}}_{\mathcal{R}}(\mathcal{T}_{\varphi_{\mathrm{d}}(L_0)}\tilde{\mathbf{a}}^{\mathrm{u}},L_0,\bar\omega)}_{=:\tilde h^{\mathrm{cs}}_{\mathcal{R}}(\tilde{\mathbf{a}}^{\mathrm{u}},L_0,\bar\omega)}$$

with $\tilde{\mathbf{a}}^{\mathrm{u}} \in \mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\operatorname{Rg}P^{\mathrm{u}}_{\mathrm{d}}(L_0)$ and

$$\tilde h^{\mathrm{cs}}_{\mathcal{R}}(\tilde{\mathbf{a}}^{\mathrm{u}},L_0,\bar\omega) = \mathrm{O}\left(\mathrm{e}^{-3\eta L_0}|\tilde{\mathbf{a}}^{\mathrm{u}}| + |\tilde{\mathbf{a}}^{\mathrm{u}}|^2 + |\bar\omega|\right) \in \mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\operatorname{Rg}P^{\mathrm{cs}}_{\mathrm{d}}(L_0).$$

Lemma 3.2(iii) and (3.2) imply that $\mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\operatorname{Rg}P^{j}_{\mathrm{d}}(L_0)$ is δ_0-close to $\operatorname{Rg}P^{j}_{0}(\mathbf{u}_{\mathrm{wt}})$ for $j = \mathrm{cs},\mathrm{u}$, and we can therefore also parametrize $\mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{v}(L_0)$ in the form

$$\mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{v}(L_0) = \check{\mathbf{a}}^{\mathrm{u}} + \check h^{\mathrm{cs}}_{\mathcal{R}}(\check{\mathbf{a}}^{\mathrm{u}},L_0,\omega_0+\bar\omega)$$

with

$$\check{\mathbf{a}}^{\mathrm{u}} \in \operatorname{Rg}P^{\mathrm{u}}_{0}(\mathbf{u}_{\mathrm{wt}}, \qquad \check h^{\mathrm{cs}}_{\mathcal{R}}(\check{\mathbf{a}}^{\mathrm{u}},L_0,\omega) = \mathrm{O}\left(\delta_0|\check{\mathbf{a}}^{\mathrm{u}}| + |\check{\mathbf{a}}^{\mathrm{u}}|^2 + |\omega-\omega_0|\right) \in \operatorname{Rg}P^{\mathrm{cs}}_{0}(\mathbf{u}_{\mathrm{wt}}).$$

Substituting this expression together with $\omega = \omega_0 + \bar\omega$ into the equation

$$\begin{aligned}\tilde{\mathbf{u}} &:= \mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{u}(L_0) = \mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{u}_{\mathrm{d}}(L_0) + \mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{v}(L_0)\\ &= \mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{u}_{\mathrm{d}}(L_0) + \check{\mathbf{a}}^{\mathrm{u}} + \check h^{\mathrm{cs}}_{\mathcal{R}}(\check{\mathbf{a}}^{\mathrm{u}},L_0,\omega) \in \tilde M_{\mathcal{R}}(L_0,\omega)\end{aligned}$$

and omitting the superscripts ˇ completes the proof. ■

Next, we show that $\mathrm{Fix}\,\mathcal{R}$ transported along the defect solution $\mathbf{u}_{\mathrm{d}}$ intersects $W^{\mathrm{cs}}(\Gamma\mathbf{u}_{\mathrm{wt}},\omega)$ in a unique group orbit for each ω near ω_0.

Lemma 3.4. *Fix $L_0 \geq L_0^*$. Then there exists a $\delta_1(L_0)>0$ so that for each ω with $|\omega-\omega_0| \leq \delta_1(L_0)$ we have*

$$M_{\mathcal{R}}(L_0,\omega)\cap W^{\mathrm{cs}}(\Gamma\mathbf{u}_{\mathrm{wt}},\omega)=\Gamma\mathcal{F}^{\mathrm{ss}}(\mathbf{p}_{\mathcal{R}}(\omega),\mathbf{b}^{\mathrm{s}}_{\mathcal{R}}(\omega),\omega)$$

for smooth functions $\mathbf{p}_{\mathcal{R}}(\omega)\in\tilde{W}^{\mathrm{c}}(\Gamma\mathbf{u}_{\mathrm{wt}},\omega)$ and $\mathbf{b}^{\mathrm{s}}_{\mathcal{R}}(\omega)\in\mathrm{Rg}\,P_0^{\mathrm{s}}(\mathbf{u}_{\mathrm{wt}})$ with the property that $\mathbf{u}_{\mathrm{d}}(L_0)=\mathcal{T}_{\varphi_{\mathrm{d}}(L_0)}\mathcal{F}^{\mathrm{ss}}(\mathbf{p}_{\mathcal{R}}(\omega_0),\mathbf{b}^{\mathrm{s}}_{\mathcal{R}}(\omega_0),\omega_0)$.

Proof. Equivariance shows that it suffices to construct intersections of $\tilde{M}_{\mathcal{R}}(L_0,\omega)$ and $W^{\mathrm{cs}}(\Gamma\mathbf{u}_{\mathrm{wt}},\omega)$. We first construct these intersections for $\omega=\omega_0$. Lemma 3.3 shows that $\tilde{M}_{\mathcal{R}}(L_0,\omega_0)$ has the representation

$$\tilde{M}_{\mathcal{R}}(L_0,\omega_0)=\left\{\tilde{\mathbf{u}}=\mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{u}_{\mathrm{d}}(L_0)+\mathbf{a}^{\mathrm{u}}+\mathrm{O}(\delta_0|\mathbf{a}^{\mathrm{u}}|+|\mathbf{a}^{\mathrm{u}}|^2):\mathbf{a}^{\mathrm{u}}\in\mathrm{Rg}\,P_0^{\mathrm{u}}(\mathbf{u}_{\mathrm{wt}})\right\}.$$

Next, we parametrize $W^{\mathrm{cs}}(\Gamma\mathbf{u}_{\mathrm{wt}},\omega_0)$. Equation (3.7) shows that we can choose $\mathbf{p}_{\mathcal{R}}(\omega_0)\in\tilde{W}^{\mathrm{c}}(\Gamma\mathbf{u}_{\mathrm{wt}},\omega_0)$ and $\mathbf{b}^{\mathrm{s}}_{\mathcal{R}}(\omega_0)\in\mathrm{Rg}\,P_0^{\mathrm{s}}(\mathbf{u}_{\mathrm{wt}})$ so that

$$\mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{u}_{\mathrm{d}}(L_0)=\mathcal{F}^{\mathrm{ss}}(\mathbf{p}_{\mathcal{R}}(\omega_0),\mathbf{b}^{\mathrm{s}}_{\mathcal{R}}(\omega_0),\omega_0).$$

We write $\mathbf{p}=\mathbf{p}_{\mathcal{R}}(\omega_0)+\tilde{\mathbf{p}}$ and $\mathbf{b}^{\mathrm{s}}=\mathbf{b}^{\mathrm{s}}_{\mathcal{R}}(\omega_0)+\tilde{\mathbf{b}}$ and use the representation (3.6) of the strong stable fibers to get

$$\begin{aligned}\mathcal{F}^{\mathrm{ss}}(\mathbf{p},\mathbf{b}^{\mathrm{s}},\omega_0)&=\mathbf{p}_{\mathcal{R}}(\omega_0)+\tilde{\mathbf{p}}+\mathbf{b}^{\mathrm{s}}_{\mathcal{R}}(\omega_0)+\tilde{\mathbf{b}}+h^{\mathrm{u}}(\mathbf{p}_{\mathcal{R}}(\omega_0)+\tilde{\mathbf{p}},\mathbf{b}^{\mathrm{s}}_{\mathcal{R}}(\omega_0)+\tilde{\mathbf{b}},\omega_0)\\&=\mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{u}_{\mathrm{d}}(L_0)+\tilde{\mathbf{p}}+\tilde{\mathbf{b}}^{\mathrm{s}}+\tilde{h}^{\mathrm{u}}(\tilde{\mathbf{p}},\tilde{\mathbf{b}}),\end{aligned}$$

where $|\tilde{h}^{\mathrm{u}}(\tilde{\mathbf{p}},\tilde{\mathbf{b}})|\leq C\delta_0(|\tilde{\mathbf{p}}|+|\tilde{\mathbf{b}}|)$. The union of these fibers for small $(\tilde{\mathbf{p}},\tilde{\mathbf{b}})$ parametrizes a neighborhood of $\mathcal{T}_{-\varphi_{\mathrm{d}}(L_0)}\mathbf{u}_{\mathrm{d}}(L_0)$ in $W^{\mathrm{cs}}(\Gamma\mathbf{u}_{\mathrm{wt}},\omega_0)$. Comparing the expressions for $\tilde{M}_{\mathcal{R}}(L_0,\omega_0)$ and $\mathcal{T}_{\varphi_{\mathrm{d}}(L_0)}\mathcal{F}^{\mathrm{ss}}(\mathbf{p},\mathbf{b}^{\mathrm{s}},\omega_0)$, we see that intersections are determined by the equation

$$\mathbf{a}^{\mathrm{u}}+\mathrm{O}(\delta_0|\mathbf{a}^{\mathrm{u}}|+|\mathbf{a}^{\mathrm{u}}|^2)=\tilde{\mathbf{p}}+\tilde{\mathbf{b}}+\tilde{h}^{\mathrm{u}}(\tilde{\mathbf{p}},\tilde{\mathbf{b}}),$$

which we can solve uniquely for $(\tilde{\mathbf{p}},\tilde{\mathbf{b}},\tilde{\mathbf{a}}^{\mathrm{u}})$ by the implicit function theorem since $\tilde{\mathbf{p}}$ spans the center directions, $\tilde{\mathbf{b}}\in\mathrm{Rg}\,P_0^{\mathrm{s}}(\mathbf{u}_{\mathrm{wt}})$, and $\tilde{\mathbf{a}}^{\mathrm{u}}\in\mathrm{Rg}\,P_0^{\mathrm{u}}(\mathbf{u}_{\mathrm{wt}})$. The same chain of arguments can also be applied if we now vary ω near ω_0 and treat the additional terms as small perturbations. ■

Finally, we provide a parametrization of $M_{\mathcal{R}}(L_0,\omega)$ near the group orbit of the intersection with $W^{\mathrm{cs}}(\Gamma\mathbf{u}_{\mathrm{wt}},\omega)$.

Lemma 3.5. *There is a constant $C>0$ such that for each fixed $L_0\geq L_0^*$ there exists a $\delta_1(L_0)>0$ so that for each ω with $|\omega-\omega_0|\leq\delta_1(L_0)$ we have*

$$M_{\mathcal{R}}(L_0,\omega)=\Gamma\left\{\mathcal{F}^{\mathrm{ss}}(\mathbf{p}_{\mathcal{R}}(\omega),\mathbf{b}^{\mathrm{s}}_{\mathcal{R}}(\omega),\omega)+\mathbf{a}^{\mathrm{u}}+H^{\mathrm{cs}}_{\mathcal{R}}(\mathbf{a}^{\mathrm{u}},\omega):\mathbf{a}^{\mathrm{u}}\in\mathrm{Rg}\,P_0^{\mathrm{u}}(\mathbf{u}_{\mathrm{wt}})\right\},\tag{3.11}$$

where $|H^{\mathrm{cs}}_{\mathcal{R}}(\mathbf{a}^{\mathrm{u}},\omega)|\leq C\delta_0|\mathbf{a}^{\mathrm{u}}|$ uniformly in ω.

Proof. Lemma 3.3 gives

$$\tilde{M}_{\mathcal{R}}(L_0,\omega) = \left\{\mathcal{T}_{-\varphi_{\rm d}(L_0)}\mathbf{u}_{\rm d}(L_0) + \mathbf{a}^{\rm u} + h^{\rm cs}_{\mathcal{R}}(\mathbf{a}^{\rm u}, L_0, \omega), \mathbf{a}^{\rm u} \in \operatorname{Rg} P^{\rm u}_0(\mathbf{u}_{\rm wt})\right\},$$

where $|h^{\rm cs}_{\mathcal{R}}(\mathbf{a}^{\rm u}, L_0, \omega)| \le C_2(\delta_0|\mathbf{a}^{\rm u}| + |\mathbf{a}^{\rm u}|^2 + |\omega-\omega_0|)$. Lemma 3.4 shows that by construction

$$\mathcal{F}^{\rm ss}(\mathbf{p}_{\mathcal{R}}(\omega), \mathbf{b}^{\rm s}_{\mathcal{R}}(\omega), \omega) \in \tilde{M}_{\mathcal{R}}(L_0,\omega)$$

and therefore

$$\mathcal{F}^{\rm ss}(\mathbf{p}_{\mathcal{R}}(\omega), \mathbf{b}^{\rm s}_{\mathcal{R}}(\omega), \omega) = \mathcal{T}_{-\varphi_{\rm d}(L_0)}\mathbf{u}_{\rm d}(L_0) + \mathbf{a}^{\rm u}_0(\omega) + h^{\rm cs}_{\mathcal{R}}(\mathbf{a}^{\rm u}_0(\omega), L_0, \omega) \tag{3.12}$$

for a unique $\mathbf{a}^{\rm u}_0(\omega)$. To establish the parametrization (3.11), we need to find for each $\mathbf{a}^{\rm u}$ an element $\mathbf{v}$ so that

$$\mathcal{T}_{-\varphi_{\rm d}(L_0)}\mathbf{u}_{\rm d}(L_0) + \mathbf{a}^{\rm u} + h^{\rm cs}_{\mathcal{R}}(\mathbf{a}^{\rm u}, L_0, \omega) = \mathcal{F}^{\rm ss}(\mathbf{p}_{\mathcal{R}}(\omega), \mathbf{b}^{\rm s}_{\mathcal{R}}(\omega), \omega) + \mathbf{v}.$$

Substituting (3.12), we arrive at

$$\mathcal{T}_{-\varphi_{\rm d}(L_0)}\mathbf{u}_{\rm d}(L_0) + \mathbf{a}^{\rm u} + h^{\rm cs}_{\mathcal{R}}(\mathbf{a}^{\rm u}, L_0, \omega) = \mathcal{T}_{-\varphi_{\rm d}(L_0)}\mathbf{u}_{\rm d}(L_0) + \mathbf{a}^{\rm u}_0(\omega) + h^{\rm cs}_{\mathcal{R}}(\mathbf{a}^{\rm u}_0(\omega), L_0, \omega) + \mathbf{v}$$

so that $\mathbf{v}$ needs to be chosen according to

$$\mathbf{v} = \mathbf{a}^{\rm u} - \mathbf{a}^{\rm u}_0(\omega) + h^{\rm cs}_{\mathcal{R}}(\mathbf{a}^{\rm u}, L_0, \omega) - h^{\rm cs}_{\mathcal{R}}(\mathbf{a}^{\rm u}_0(\omega), L_0, \omega).$$

We write $\mathbf{a}^{\rm u} = \mathbf{a}^{\rm u}_0(\omega) + \tilde{\mathbf{a}}^{\rm u}$ to get

$$\mathbf{v} = \tilde{\mathbf{a}}^{\rm u} + \underbrace{h^{\rm cs}_{\mathcal{R}}(\mathbf{a}^{\rm u}_0(\omega) + \tilde{\mathbf{a}}^{\rm u}, L_0, \omega) - h^{\rm cs}_{\mathcal{R}}(\mathbf{a}^{\rm u}_0(\omega), L_0, \omega)}_{|\cdot| \le C\delta_0|\tilde{\mathbf{a}}^{\rm u}|}$$

as required. ■

3.4. Step 2: Constructing the truncated defect solution. In the second step of our proof, we construct solutions that start in $\tilde{M}_{\mathcal{R}}(L_0,\omega)$ and end in the space $\operatorname{Fix}\mathcal{R}_0$ of Neumann boundary conditions. We remark that our arguments below no longer involve the defect solution, since we can rely on the parametrization provided in Lemma 3.5. Recall from Lemma 3.4 that $\mathbf{p}_{\mathcal{R}}(\omega) \in \tilde{W}^{\rm c}(\Gamma\mathbf{u}_{\rm wt}, \omega)$ is the base point of the intersection $\mathcal{F}^{\rm ss}(\mathbf{p}_{\mathcal{R}}(\omega), \mathbf{b}^{\rm s}_{\mathcal{R}}(\omega), \omega)$ of $\tilde{M}_{\mathcal{R}}(L_0,\omega)$ and $W^{\rm cs}(\Gamma\mathbf{u}_{\rm wt}, \omega)$. Let $\kappa_{\mathcal{R}}(\omega)$ be the center-manifold coordinate κ of $\mathbf{p}_{\mathcal{R}}(\omega)$ that we introduced in (3.3) and denote by $\mathbf{p}^*(x;\epsilon,\tilde{\kappa})$ the solution on $\tilde{W}^{\rm c}(\mathbf{u}_{\rm wt}, \omega_0+\epsilon^2)$ that belongs to the solution $\kappa(x)$ with $\kappa(0) = \kappa_0 = \kappa_{\rm R}(\omega) + \tilde{\kappa}$ and $\kappa(L(\epsilon,\tilde{\kappa})) = 0$ that we constructed in Lemma 3.1. Next, we also vary the strong stable coordinate of the intersection $\mathcal{F}^{\rm ss}(\mathbf{p}_{\mathcal{R}}(\omega), \mathbf{b}^{\rm s}_{\mathcal{R}}(\omega), \omega)|_{\omega=\omega_0+\epsilon^2}$ and consider the points

$$\mathbf{u}^*(0;\epsilon,\tilde{\kappa},\tilde{\mathbf{b}}^{\rm s}) := \mathcal{F}^{\rm ss}(\mathbf{p}^*(0;\epsilon,\tilde{\kappa}), \mathbf{b}_{\mathcal{R}}(\omega_0+\epsilon^2) + \tilde{\mathbf{b}}^{\rm s}, \omega_0+\epsilon^2), \qquad \tilde{\mathbf{b}}^{\rm s} \in \operatorname{Rg} P^{\rm s}_0(\mathbf{u}_{\rm wt}),$$

in the center-stable manifold. Note that these points parametrize an open neighborhood of

$$\mathbf{u}^*(0;\epsilon,0,0) = \mathcal{F}^{\rm ss}(\mathbf{p}_{\mathcal{R}}(\omega), \mathbf{b}^{\rm s}_{\mathcal{R}}(\omega), \omega)|_{\omega=\omega_0+\epsilon^2} \tag{3.13}$$

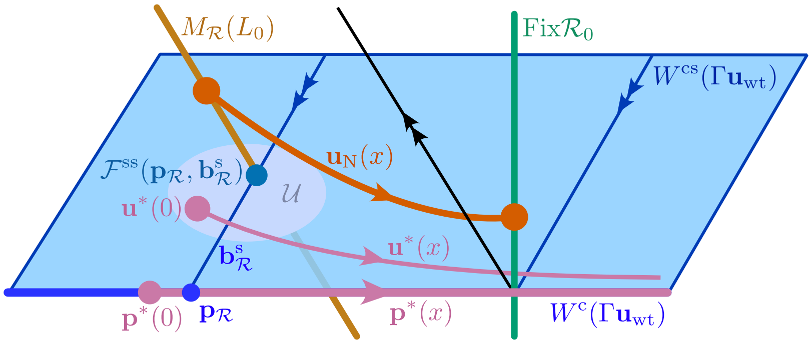

Figure 5. *We illustrate the strategy behind step 2. The manifold $M_{\mathcal{R}}(L_0)$ intersects $W^{\rm cu}(\Gamma\mathbf{u}_{\rm wt})$ in the point $\mathcal{F}^{\rm ss}(\mathbf{p}_{\mathcal{R}},\mathbf{b}^{\rm s}_{\mathcal{R}})$. Varying the initial condition $\mathrm{p}^*(0)$ near the base point $\mathbf{p}_{\mathcal{R}}$ in $W^{\rm c}(\Gamma\mathbf{u}_{\rm wt})$ and the strong stable fiber coordinate $\mathbf{b}^{\rm s}$ near $\mathbf{b}^{\rm s}_{\mathcal{R}}$ creates an open neighborhood $\mathcal{U}$ of initial conditions $\mathbf{u}^*(0)$ in $W^{\rm cu}(\Gamma\mathbf{u}_{\rm wt})$ with associated solutions $\mathbf{u}^*(x)$, which we use to construct the desired solution $\mathrm{u}_{\rm N}(x)$ that starts in $M_{\mathcal{R}}(L_0)$ and ends in $\mathrm{Fix}\,\mathcal{R}_0$.*

inside the center-stable manifold upon varying $(\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})$. We denote the associated solutions, which are guaranteed to exist for $x\geq 0$ by the construction of the center-stable manifold, by $\mathbf{u}^*(x;\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})$ and recall that the base point of this solution in the center manifold is $\mathbf{p}^*(x;\epsilon,\tilde\kappa)$. The properties of strong stable fibers guarantee that

$$\mathbf{u}^*(x;\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})=\mathbf{p}^*(x;\epsilon,\tilde\kappa)+\mathrm{O}(\mathrm{e}^{-3\eta x}) \tag{3.14}$$

as $x\to\infty$. Figure 5 illustrates these solutions. For future reference, we will denote the φ-coordinate of $\mathbf{p}^*$ in the center manifold by φ^*.

Our goal is now to use the variation-of-constants formula to construct solutions near $\mathbf{u}^*(x;\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})$ that start in $M_{\mathcal{R}}(L_0)$ and end in $\mathrm{Fix}\,\mathcal{R}_0$. The roughness theorem for dichotomies for (3.1), which was established in [15, 17, 18], shows that the linearization of (3.1) about $\mathbf{u}^*(x;\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})$ admits exponential dichotomies $\Phi^{\rm cs,u}_*(x,y)$ with constants and rates given by Lemma 3.2 and that the operators $\Phi^{\rm cs,u}_*(x,y)$ depend smoothly on $(\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})$. We can now seek solutions of (3.1) near $\mathbf{u}^*(x;\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})$.

Lemma 3.6. *For each $(\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s},\tilde x)$ and $\mathbf{a}^{\rm u}\in\mathcal{T}_{\varphi^*(\ell;\epsilon,\tilde\kappa)}\,\mathrm{Rg}\,P^{\rm u}_0(\mathbf{u}_{\rm wt})$ with $|\epsilon|,|\tilde\kappa|,|\tilde{\mathbf{b}}^{\rm s}|,|\mathbf{a}^{\rm u}|<\delta_2$, $\epsilon>0$, and $|\tilde x|\leq\frac13$, equation (3.1) with $\omega=\omega_0+\epsilon^2$ has a unique solution $\mathbf{u}(x)$ on $[0,L(\epsilon,\tilde\kappa)+\tilde x]$ with*

$$\mathbf{u}(0)=\mathbf{u}^*(0;\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})+\mathrm{O}(\mathrm{e}^{-\eta/\epsilon}|\mathbf{a}^{\rm u}|),\quad \mathbf{u}(\ell)=\mathbf{u}^*(\ell;\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})+\mathbf{a}^{\rm u}+\mathrm{o}(1)|\mathbf{a}^{\rm u}|+\mathrm{O}(|\mathbf{a}^{\rm u}|^2), \tag{3.15}$$

where we used the notation $\ell=L(\epsilon,\tilde\kappa)+\tilde x$, and $\mathrm{o}(1)$ is a term that converges to zero as $\epsilon\to 0$.

Proof. We seek solutions of (3.1) of the form

$$\mathbf{u}(x)=\mathbf{u}^*(x;\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})+\mathbf{v}(x),\qquad 0\leq x\leq\ell,\quad \ell:=L(\epsilon,\tilde\kappa)+\tilde x.$$

Using that $\mathbf{u}^*(x;\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})$ is a solution of (3.1) for all $(\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})$, we see that $\mathbf{v}(x)$ satisfies an equation of the form

$$\mathbf{v}_x=F_{\mathbf{u}}(\mathbf{u}^*(x;\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s}),\omega_0+\epsilon^2)\mathbf{v}+G(\mathbf{v},\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s}),\qquad G(\mathbf{v},\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\rm s})=\mathrm{O}(|\mathbf{v}|^2).$$

By [18, Lemma A.1 and Theorem A.4] and [17, Lemma 5.2], the linear part of this equation admits an exponential dichotomy with center-stable and unstable projections, and the range of the unstable projection at $x=\ell$ is δ_0-close to $\mathcal{T}_{\varphi^*(\ell;\epsilon,\tilde\kappa)}\,\mathrm{Rg}\,P_0^{\mathrm{u}}(\mathbf{u}_{\mathrm{wt}})$, where φ^* denotes the φ-coordinate of $\mathbf{p}^*$. Omitting the dependence of G on $(\epsilon,\tilde\kappa,\tilde{\mathbf{b}}^{\mathrm{s}})$, we can rewrite the full nonlinear equation for $\mathbf{v}$ as the fixed-point equation

$$\mathbf{v}(x)=\Phi_*^{\mathrm{u}}(x,\ell)\mathbf{a}^{\mathrm{u}}+\int_0^x\Phi_*^{\mathrm{cs}}(x,y)G(\mathbf{v}(y))\,\mathrm{d}y+\int_\ell^x\Phi_*^{\mathrm{u}}(x,y)G(\mathbf{v}(y))\,\mathrm{d}y,\qquad 0\le x\le\ell. \tag{3.16}$$

The fact that the function $G(\mathbf{v})$ vanishes at $\mathbf{v}=0$ allows us to use the exponentially weighted norms

$$\|\mathbf{v}\|_\eta:=\sup_{0\le x\le\ell}\mathrm{e}^{2\eta(\ell-x)}|\mathbf{v}(x)|.$$

Using this norm, we have

$$\begin{aligned}\|\mathrm{RHS\ of\ (3.16)}\|_\eta&\le C|\mathbf{a}^{\mathrm{u}}|+C\mathrm{e}^{2\eta(\ell-x)}\left[\int_0^x\mathrm{e}^{\eta(x-y)}\mathrm{e}^{-4\eta(\ell-y)}\,\mathrm{d}y+\int_x^\ell\mathrm{e}^{-3\eta(y-x)}\mathrm{e}^{-4\eta(\ell-y)}\,\mathrm{d}y\right]\|\mathbf{v}\|_\eta^2\\&\le C|\mathbf{a}^{\mathrm{u}}|+C\mathrm{e}^{2\eta(\ell-x)}\left[\mathrm{e}^{-4\eta(\ell-x)}+\mathrm{e}^{\eta x}\mathrm{e}^{-4\eta\ell}+\mathrm{e}^{-3\eta(\ell-x)}+\mathrm{e}^{-4\eta(\ell-x)}\right]\|\mathbf{v}\|_\eta^2\\&\le C(|\mathbf{a}^{\mathrm{u}}|+\|\mathbf{v}\|_\eta^2),\end{aligned}$$

and a similar calculation shows that the derivative of the right-hand side of (3.16) with respect to $\mathbf{v}$ in the exponential weighted norm can be bounded by $\frac12$ for $\|\mathbf{v}\|_\eta$ sufficiently small. The uniform contraction principle therefore guarantees existence, uniqueness, and smooth dependence on data of a fixed point of (3.16). Evaluating (3.16) at $x=0$ and $x=\ell$, noting that $\ell\ge\frac{1}{\epsilon}$ by Lemma 3.1, and using the exponential weights for $\mathbf{v}(x)$ establishes the estimates (3.15) and completes the proof. ∎

Final step of our proof. It remains to prove that the solution $\mathbf{u}(x)$ we constructed in Lemma 3.6 satisfies the two conditions

$$\mathbf{u}(0)\in\tilde M_{\mathcal{R}}(L_0,\omega_0+\epsilon^2)\qquad\text{and}\qquad\mathbf{u}(\ell)\in\mathrm{Fix}\,\mathcal{R}_0. \tag{3.17}$$

For the first condition, we use (3.15) for $\mathbf{u}(0)$ together with (3.11) and (3.13) for $\tilde M_{\mathcal{R}}(L_0,\omega_0+\epsilon^2)$ to arrive at the equation

$$\mathbf{u}^*(0;\epsilon,\tilde\kappa,\mathbf{b}_0^{\mathrm{s}})+\mathrm{O}(\mathrm{e}^{-\eta/\epsilon}|\mathbf{a}_1^{\mathrm{u}}|)=\mathbf{u}^*(0;\epsilon,0,0)+\mathbf{a}_0^{\mathrm{u}}+\mathrm{O}(\delta_0)|\mathbf{a}_0^{\mathrm{u}}|, \tag{3.18}$$

where $\mathbf{b}_0^{\mathrm{s}}\in\mathrm{Rg}\,P_0^{\mathrm{s}}(\mathbf{u}_{\mathrm{wt}})$ and $\mathbf{a}_0^{\mathrm{u}}\in\mathrm{Rg}\,P_0^{\mathrm{u}}(\mathbf{u}_{\mathrm{wt}})$, and the $\mathrm{O}(\delta_0)$ term goes to zero as δ_0 goes to zero uniformly in ϵ. Expanding (3.18) in $(\tilde\kappa,\mathbf{b}_0^{\mathrm{s}})$ near zero, we obtain the equation

$$\partial_{\tilde\kappa}\mathbf{u}^*(0;\epsilon,0,0)\tilde\kappa+\partial_{\mathbf{b}_0^{\mathrm{s}}}\mathbf{u}^*(0;\epsilon,0,0)\mathbf{b}_0^{\mathrm{s}}+\mathrm{O}(\mathrm{e}^{-\eta/\epsilon}|\mathbf{a}_1^{\mathrm{u}}|+|\tilde\kappa|^2+|\mathbf{b}_0^{\mathrm{s}}|^2)=\mathbf{a}_0^{\mathrm{u}}+\mathrm{O}(\delta_0)|\mathbf{a}_0^{\mathrm{u}}|,$$

which we can solve uniquely for

$$(\tilde\kappa,\mathbf{a}_0^{\mathrm{u}},\mathbf{b}_0^{\mathrm{s}})=\mathrm{O}(\mathrm{e}^{-\eta/\epsilon}|\mathbf{a}_1^{\mathrm{u}}|) \tag{3.19}$$

as a function of $\mathbf{a}_1^{\mathrm{u}}$. With $(\tilde{\kappa}, \mathbf{a}_0^{\mathrm{u}}, \mathbf{b}_0^{\mathrm{s}})$ now determined, it remains to satisfy the second condition in (3.17), which is given by $\mathbf{u}(\ell) \in \operatorname{Fix} \mathcal{R}_0$. We first recall from (3.15) that

$$\mathbf{u}(\ell) = \mathbf{u}^*(\ell; \epsilon, \tilde{\kappa}, \mathbf{b}_0^{\mathrm{s}}) + \mathbf{a}_1^{\mathrm{u}} + \mathrm{o}(1)|\mathbf{a}_1^{\mathrm{u}}| + \mathrm{O}(|\mathbf{a}_1^{\mathrm{u}}|^2). \tag{3.20}$$

We now exploit the estimates (3.19) for $(\tilde{\kappa}, \mathbf{a}_0^{\mathrm{u}}, \mathbf{b}_0^{\mathrm{s}})$ to simplify the expression for $\mathbf{u}^*(\ell; \epsilon, \tilde{\kappa}, \mathbf{b}_0^{\mathrm{s}})$ as follows:

$$\begin{aligned}
\mathbf{u}^*(\ell; \epsilon, \tilde{\kappa}, \mathbf{b}_0^{\mathrm{s}}) &= \mathbf{u}^*(L(\epsilon, \tilde{\kappa}) + \tilde{x}; \epsilon, \tilde{\kappa}, \mathbf{b}_0^{\mathrm{s}}) = \mathbf{u}^*(L(\epsilon, \mathrm{O}(\mathrm{e}^{-\eta/\epsilon})) + \tilde{x}; \epsilon, \mathrm{O}(\mathrm{e}^{-\eta/\epsilon}), \mathrm{O}(\mathrm{e}^{-\eta/\epsilon})) \\
&= \mathbf{u}^*(L(\epsilon, 0) + \tilde{x}; \epsilon, 0, 0) + \mathrm{O}(\mathrm{e}^{-\eta/\epsilon}) = \mathbf{p}^*(L(\epsilon, 0) + \tilde{x}; \epsilon, 0) + \mathrm{O}(\mathrm{e}^{-\eta/\epsilon}) \\
&= \mathbf{p}^*(L(\epsilon, 0); \epsilon, 0) + \frac{2\epsilon^2 \tilde{x}}{\lambda_{\mathrm{lin}}''(0)} \mathcal{T}_{\varphi^*(L(\epsilon,0);\epsilon,0)} \mathbf{v}_{\mathrm{wt}} + \mathrm{O}(\mathrm{e}^{-\eta/\epsilon} + \epsilon^4 \tilde{x}^2),
\end{aligned}$$

where we used (3.14) and Lemma 3.1 and recall that we defined $\mathbf{v}_{\mathrm{wt}}$ in the paragraph after (3.2) and denoted the φ-coordinate of $\mathbf{p}^*$ in the center manifold by φ^*. Using that $\mathbf{p}^*(L(\epsilon, 0); \epsilon, 0) \in \operatorname{Fix} \mathcal{R}_0$ by Lemma 3.1 and that $\operatorname{Fix} \mathcal{R}_0$ is a linear space, equation (3.20) shows that it suffices to show that

$$\mathcal{T}_{\varphi^*(L(\epsilon,0);\epsilon,0)} \left(\frac{2\epsilon^2 \tilde{x}}{\lambda_{\mathrm{lin}}''(0)} \mathbf{v}_{\mathrm{wt}} + \mathrm{O}(\mathrm{e}^{-\eta/\epsilon} + \epsilon^4 \tilde{x}^2) + \mathbf{a}_1^{\mathrm{u}} + \mathrm{o}(1)|\mathbf{a}_1^{\mathrm{u}}| + \mathrm{O}(|\mathbf{a}_1^{\mathrm{u}}|^2) \right) \in \operatorname{Fix} \mathcal{R}_0,$$

where $\mathbf{a}_1^{\mathrm{u}} \in \operatorname{Rg} P_0^{\mathrm{u}}(\mathbf{u}_{\mathrm{wt}})$ by Lemma 3.6 and (3.20). Exploiting equivariance and using the representation (3.5) for $\operatorname{Fix} \mathcal{R}_0$, we arrive at the equation

$$\frac{2\epsilon^2 \tilde{x}}{\lambda_{\mathrm{lin}}''(0)} \mathbf{v}_{\mathrm{wt}} + \mathbf{a}_1^{\mathrm{u}} + \mathrm{o}(1)|\mathbf{a}_1^{\mathrm{u}}| + \mathrm{O}(\mathrm{e}^{-\eta/\epsilon} + \epsilon^4 \tilde{x}^2 + |\mathbf{a}_1^{\mathrm{u}}|^2) = \alpha \partial_\tau \mathbf{u}_{\mathrm{wt}} + \mathbf{b}_1^{\mathrm{s}} + \mathcal{R}_0 \mathbf{b}_1^{\mathrm{s}}.$$

Projecting this equation onto $\mathbb{R}\partial_\tau \mathbf{u}_{\mathrm{wt}} \oplus \operatorname{Rg} P_0^{\mathrm{s}}(\mathbf{u}_{\mathrm{wt}}) \oplus \operatorname{Rg} P_0^{\mathrm{u}}(\mathbf{u}_{\mathrm{wt}})$ along $\mathbb{R}\mathbf{v}_{\mathrm{wt}}$ allows us to solve uniquely for $(\alpha, \mathbf{a}_1^{\mathrm{u}}, \mathbf{b}_1^{\mathrm{s}}) = \mathrm{O}(\mathrm{e}^{-\eta/\epsilon} + \epsilon^4 \tilde{x}^2)$ as a function of $(\epsilon, \tilde{x})$, and the remaining equation in the $\mathbf{v}_{\mathrm{wt}}$-direction becomes

$$\frac{2\epsilon^2 \tilde{x}}{\lambda_{\mathrm{lin}}''(0)} + \mathrm{O}(\mathrm{e}^{-\eta/\epsilon} + \epsilon^4 \tilde{x}^2) = 0,$$

which we can solve uniquely for $\tilde{x} = \mathrm{O}(\mathrm{e}^{-\eta/2\epsilon})$. This proves that there are unique functions $(\tilde{\kappa}, \tilde{x}, \mathbf{a}_{0,1}^{\mathrm{u}}, \mathbf{b}_{0,1}^{\mathrm{s}}) = (\tilde{\kappa}^*, \tilde{x}^*, \mathbf{a}_{0,1}^{\mathrm{u},*}, \mathbf{b}_{0,1}^{\mathrm{s},*})(\epsilon) = \mathrm{O}(\mathrm{e}^{-\eta/2\epsilon})$ so that the matching conditions (3.17) are satisfied for each $\epsilon > 0$.

It remains to express the parameter ϵ as a function of the interval length L. Lemma 3.1 and the preceding discussion show that the total interval length L over which the truncated defect solution exists is given by

$$\begin{aligned}
L = L(\epsilon, \tilde{\kappa}^*(\epsilon)) + \tilde{x}^*(\epsilon) + L_0 &= \frac{\pi \lambda_{\mathrm{lin}}''(0)}{2\sqrt{-2\omega_{\mathrm{nl}}''(0)\epsilon}} + r(\epsilon, \tilde{\kappa}^*(\epsilon)) + \tilde{x}(\epsilon) + L_0 \\
&= \frac{\pi \lambda_{\mathrm{lin}}''(0)}{2\sqrt{-2\omega_{\mathrm{nl}}''(0)\epsilon}} + r(\epsilon, 0) + L_0 + \mathrm{O}(\mathrm{e}^{-\eta/2\epsilon}).
\end{aligned}$$

Smoothness of $r(\epsilon, 0)$ and the remainder terms allows us to solve this equation for $\epsilon = \epsilon^*(L)$ with

$$\epsilon^*(L) = \frac{\pi \lambda_{\mathrm{lin}}''(0)}{2\sqrt{-2\omega_{\mathrm{nl}}''(0)L}} + \mathrm{O}\left(\frac{1}{L^2}\right),$$

as claimed. This completes the proof of Theorem 1.

4. Passage times near saddle-node equilibria. In this section, we prove Theorem 2. Our goal is to find expansions of the passage times through $\kappa = 0$ for solutions of the one-dimensional ODE

$$\kappa_x = \epsilon^2 + \kappa^2 + g(\kappa, \epsilon^2), \tag{4.1}$$

where the function $g(\kappa, \omega) = \mathrm{O}(\omega^2 + \kappa^3)$ is sufficiently smooth. To simplify the analysis, we will first bring (4.1) into an appropriate normal form.

Lemma 4.1. *Consider*

$$\kappa_x = \omega + \kappa^2 + g(\kappa, \omega) \tag{4.2}$$

and assume that the function $g(\kappa, \omega)$ is C^k for some $k \geq 4$ and satisfies $g(0,0) = g_\kappa(0,0) = g_{\kappa\kappa}(0,0) = g_\omega(0,0) = g_{\kappa\omega}(0,0) = 0$. There is then a C^k near-identity diffeomorphism $(u, \mu) = (\kappa + H(\kappa, \omega), \omega + h(\omega))$ with $H(0,0) = h(0) = 0$ and $H_{(\kappa,\omega)}(0,0) = h_\omega(0) = 0$ and C^k functions $c_{0,1}(\mu)$ so that (4.2) becomes

$$u_x = \mu + (1 + \mu c_0(\mu))u^2 + c_1(\mu)u^3. \tag{4.3}$$

Furthermore, if $g(\kappa, \omega)$ is even in κ for all ω, then we have $c_0(\mu) = c_1(\mu) = 0$ for all μ.

Proof. First, [6, Corollary 1] shows that there is a C^k-smooth near-identity transformation that brings (4.2) into the normal form

$$v_x = (\nu + v^2)(1 + c_1(\nu)v), \tag{4.4}$$

where $c_1(\nu)$ is C^k. If $g(\kappa, \omega)$ is even in κ, then $c_1(\nu)$ vanishes identically by symmetry, and (4.4) is already of the form (4.3) with $c_{0,1}(\mu) = 0$ as claimed. If g is not even in κ, (4.4) is cubic in v, and a calculation shows that the transformation $u := v - \nu c_1(\nu) d(\nu)$ and $\mu := \nu + \nu^2 c_1(\nu)^2 d(\nu)(1 + d(\nu) + \nu d(\nu)^2 c_1(\nu)^2)$ turns (4.4) into (4.3), where the function $d = d(\nu)$ is the unique C^k solution of $1 + 2d + 3\nu c_1(\nu) d^2 = 0$ with $d(0) = -\frac{1}{2}$, which exists by the implicit function theorem. ■

We now focus first on the normal form (4.3) and discuss extensions to the original equation (4.1) afterward. For given δ with $0 < \delta \ll 1$, we consider the boundary-value problem

$$u_x = \epsilon^2 + (1 + \epsilon^2 c_0(\epsilon^2))u^2 + c_1(\epsilon^2)u^3, \quad u(0) = 0, \quad u(L) = \delta, \tag{4.5}$$

where the functions $c_{0,1}(\mu)$ are C^k. In this situation, we say that L is the travel time from $u = 0$ to $u = \delta$. Our goal is to find expansions of L in terms of ϵ uniformly in $0 < \epsilon \ll 1$.

Lemma 4.2. *Assume that $c_{0,1}(\mu)$ are C^k for some $k \geq 4$. Then there exist numbers $\epsilon_1, \delta_1 > 0$ so that the following statements hold with $Q := (0, \epsilon_1] \times [\delta_1/2, 2\delta_1]$:*

(i) *Equation (4.5) has a solution for $(\epsilon, \delta) \in Q$ if and only if $L = L_+(\epsilon, \delta)$ for a unique $L_+(\epsilon, \delta)$.*

(ii) *There are functions $a \in C^k([0, \epsilon_1])$ and $b \in C^k(\overline{Q})$ with $a(0) = b(0, \delta) = 0$ for all δ so that*

$$\epsilon L_+(\epsilon, \delta) = \frac{\pi}{2} + a(\epsilon)\log\epsilon + b(\epsilon, \delta).$$

In particular, $\epsilon L_+(\epsilon, \delta) \to 0$ uniformly in δ as $\epsilon \to 0$.

(iii) *If $c_{0,1}(\mu) = 0$ for all μ, then $a(\epsilon) = 0$ for all ϵ, and we have $\epsilon L_+(\epsilon, \delta) \in C^k(\overline{Q})$.*

Proof. Note that (i) follows immediately from positivity of the right-hand side of the ODE in (4.5) for small data together with the existence and uniqueness theorem of solutions of ODEs. To prove (ii), we use the identity

$$L_+(\epsilon,\delta) = \int_0^\delta \frac{1}{\epsilon^2 + (1+\epsilon^2 c_0(\epsilon^2))u^2 + c_1(\epsilon^2)u^3}\,\mathrm{d}u.$$

The idea is to use partial fractions to evaluate the integral. However, ϵ is small and $c_1(\epsilon^2)$ may vanish, which obstructs the decomposition into partial fractions. To remedy this issue, we multiply the equation by ϵ and use the substitution $u=\epsilon/v$ to get

$$\begin{aligned}\epsilon L_+(\epsilon,\delta) &= \int_{\epsilon/\delta}^\infty \frac{\epsilon}{\epsilon^2 + \frac{\epsilon^2}{v^2}(1+\epsilon^2 c_0(\epsilon^2)) + \frac{\epsilon^3}{v^3}c_1(\epsilon^2)}\frac{\epsilon}{v^2}\,\mathrm{d}v\\ &= \int_1^\infty \frac{v}{v^3 + v(1+\epsilon^2 c_0(\epsilon^2)) + \epsilon c_1(\epsilon^2)}\,\mathrm{d}v\\ &\quad + \int_{\epsilon/\delta}^1 \frac{v}{v^3 + v(1+\epsilon^2 c_0(\epsilon^2)) + \epsilon c_1(\epsilon^2)}\,\mathrm{d}v =: I_1 + I_2.\end{aligned}$$

The dominated convergence theorem implies that $\epsilon L_+(\epsilon,\delta)|_{\epsilon=0} = \frac{\pi}{2}$ and that the integral I_1 is C^k in ϵ. To evaluate I_2, we use partial fractions. For simplicity, we will omit the argument ϵ^2 of the functions $c_{0,1}$ for the remainder of this proof. We denote by $v_{1,2,3}$ the roots of $v^3 + v(1+\epsilon^2 c_0) + \epsilon c_1$ so that $v_1 = -\epsilon c_1(1+\mathrm{O}(\epsilon^2))$, $v_2 = \mathrm{i} + \mathrm{O}(\epsilon c_1 + \epsilon^2 c_0)$, and $v_3 = \bar v_2$. Hence, setting

$$A_j(\epsilon) := \frac{v_j(\epsilon)}{3v_j(\epsilon)^2 + 1 + \epsilon^2 c_0(\epsilon^2)}, \quad j=1,2,3, \tag{4.6}$$

we have

$$\frac{v}{v^3 + v(1+\epsilon^2 c_0) + \epsilon c_1} = \frac{A_1(\epsilon)}{v - v_1(\epsilon)} + \frac{A_2(\epsilon)}{v - v_2(\epsilon)} + \frac{A_3(\epsilon)}{v - v_3(\epsilon)}.$$

We claim that the integral

$$\tilde I_2(\epsilon,\delta) := \int_{\epsilon/\delta}^1 \left[\frac{A_2(\epsilon)}{v - v_2(\epsilon)} + \frac{A_3(\epsilon)}{v - v_3(\epsilon)}\right]\mathrm{d}v \tag{4.7}$$

is C^k in (ϵ,δ). Informally, combining the two terms in $\tilde I_2$ into a single term and using that $v_{2,3} = \pm\mathrm{i}$ to leading order, we obtain an integrand of the form $\frac{Bv+C}{v^2+1}$ whose integral is $B\log(v^2+1) + C\arctan v$, which is smooth up to $v=0$. To turn this into a rigorous argument, we use that $v_3 = \bar v_2$ and $A_3 = \bar A_2$ to simplify the integrand of $\tilde I_2$ as follows:

$$\begin{aligned}\frac{A_2}{v-v_2} + \frac{A_3}{v-v_3} &= \frac{A_2(v-v_3) + A_3(v-v_2)}{(v-v_2)(v-v_3)} = \frac{2v\operatorname{Re}A_2 - 2\operatorname{Re}(A_2 v_3)}{v^2 - 2v\operatorname{Re}v_2 + (\operatorname{Re}v_2)^2 + (\operatorname{Im}v_2)^2}\\ &= \frac{1}{\operatorname{Im}v_2}\frac{\frac{v-\operatorname{Re}v_2}{\operatorname{Im}v_2}B(\epsilon) + C(\epsilon)}{(\frac{v-\operatorname{Re}v_2}{\operatorname{Im}v_2})^2 + 1},\end{aligned}$$

where $B(\epsilon), C(\epsilon)$ are C^k functions of ϵ, which can be computed explicitly from $v_{2,3}(\epsilon)$. Hence,

$$\begin{aligned}\tilde{I}_2(\epsilon,\delta) &= \int_{\epsilon/\delta}^{1} \frac{1}{\operatorname{Im} v_2} \frac{\frac{v-\operatorname{Re} v_2}{\operatorname{Im} v_2} B(\epsilon) + C(\epsilon)}{(\frac{v-\operatorname{Re} v_2}{\operatorname{Im} v_2})^2 + 1} \, \mathrm{d}v \\ &= \left[\frac{B(\epsilon)}{2} \log \left(\left(\frac{v - \operatorname{Re} v_2(\epsilon)}{\operatorname{Im} v_2(\epsilon)} \right)^2 + 1 \right) + C(\epsilon) \arctan \left(\frac{v - \operatorname{Re} v_2(\epsilon)}{\operatorname{Im} v_2(\epsilon)} \right) \right]_{\epsilon/\delta}^{1},\end{aligned}$$

which is C^k in (ϵ,δ) up to $\epsilon=0$ as claimed, since $\operatorname{Im} v_2(\epsilon) = 1 + \mathrm{O}(\epsilon)$.

The smoothness properties of I_2 are therefore determined by the remaining integral that involves A_1. We write $v_1(\epsilon) = -\epsilon c_1(\epsilon^2)(1+\mathrm{O}(\epsilon^2)) =: -\epsilon c_2(\epsilon^2)$, where c_2 is C^k, and then obtain

(4.8)
$$\begin{aligned}\int_{\epsilon/\delta}^{1} \frac{A_1}{v - v_1} \, \mathrm{d}v = A_1 \left[\log(1-v_1) - \log\left(\frac{\epsilon}{\delta} - v_1\right) \right] &= \frac{v_1(\log(1-v_1) - \log \epsilon - \log(c_2 + 1/\delta))}{3v_1^2 + 1 + \epsilon^2 c_0} \\ &=: \underbrace{\frac{-v_1(\epsilon)}{3v_1(\epsilon)^2 + 1 + \epsilon^2 c_0(\epsilon^2)}}_{=:a(\epsilon)} \log \epsilon + \tilde{b}(\epsilon,\delta),\end{aligned}$$

where $\tilde{b}(\epsilon,\delta)$ is C^k provided we choose δ_1 so small that $c_2(\epsilon) + 1/\delta_1 \geq 1$. In summary, we proved that

$$\epsilon L_+(\epsilon,\delta) = I_1 + I_2 = \frac{\pi}{2} + a(\epsilon)\log\epsilon + b(\epsilon,\delta),$$

where $a(\epsilon)$ is C^k with $a(0)=0$, and where $b(\epsilon,\delta)$ contains the C^k remainder terms and satisfies $b(0,\delta)=0$. This proves part (ii). Part (iii) follows from the observation that $a(\epsilon)$ vanishes identically whenever the function c_1 vanishes identically. ∎

We briefly collect a few consequences of the preceding lemma.

Lemma 4.3. *Assume that $c_{0,1}(\mu)$ are C^k for some $k \geq 4$. Then the travel time of the ODE in (4.5) from $u=-\delta$ to $u=0$ is given by $L = L_-(\epsilon,\delta)$, and we have the expansion*

$$\epsilon L_-(\epsilon,\delta) = \frac{\pi}{2} - a(\epsilon)\log\epsilon + \tilde{b}(\epsilon,\delta),$$

where $a(\epsilon)$ is the function from Lemma 4.2, and $\tilde{b}$ is C^k. Furthermore, the travel time $L_-(\epsilon,\delta) + L_+(\epsilon,\delta)$ from $u=-\delta$ to $u=\delta$ satisfies

$$\epsilon(L_-(\epsilon,\delta) + L_+(\epsilon,\delta)) = \pi + b(\epsilon,\delta) + \tilde{b}(\epsilon,\delta),$$

where the right-hand side is C^k in (ϵ,δ).

Proof. The statement about L_- follows by applying Lemma 4.2 to the system obtained by substituting $(u,x) \to (-u,-x)$, which leaves c_0 the same and replaces c_1 by its negative. The claim about $L_- + L_+$ follows by adding the expressions for ϵL_+ and ϵL_-. ∎

Our next step is to show that we can solve the equation $L = L_+(\epsilon,\delta)$ for ϵ as a function of L.

Lemma 4.4. *Assume that $c_{0,1}(\mu)$ are C^k for some $k \geq 4$. Then there is a unique function $r(z,\delta)$ so that $L = L_+(\epsilon,\delta)$ if and only if $\epsilon = \epsilon_*(\frac{1}{L}) = \frac{\pi}{2L} + r(\frac{1}{L},\delta)$, and the function $r(z,\delta)$ is in $C^{1,\beta}$ for each fixed $0 \leq \beta < 1$ with $r(z,\delta) = \mathrm{O}(z^{1+\beta})$. If the functions $c_{0,1}$ vanish identically, then $r \in C^k$ with $r(z,\delta) = \mathrm{O}(z^2)$.*

Proof. Setting $L := L_+(\epsilon,\delta)$, we proved in Lemma 4.2 that

$$\epsilon L = \frac{\pi}{2} + a(\epsilon)\log\epsilon + b(\epsilon,\delta),$$

where $a, b \in C^k$ with $a(0) = b(0,\delta) = 0$ for all δ. Writing $z = 1/L$ and $a(\epsilon) = \epsilon\tilde{a}(\epsilon)$, we conclude that

$$z = \frac{\epsilon}{\frac{\pi}{2} + \epsilon\tilde{a}(\epsilon)\log\epsilon + b(\epsilon,\delta)} =: f(\epsilon,\delta), \tag{4.9}$$

where $\tilde{a} \in C^{k-1}$. It is not difficult to check that for each fixed $0 \leq \beta < 1$ the functions $f(\epsilon,\delta)$ and $f_\epsilon(\epsilon,\delta)$ are C^β (since $\epsilon\log\epsilon$ is C^β) on $[0,\epsilon_1]$ and satisfy $f(0,\delta) = 0$ and $f_\epsilon(0,\delta) = \frac{2}{\pi}$ for all δ. Hence we can apply the implicit function theorem to solve (4.9) for $\epsilon = \epsilon_*(z)$ as a function of $z = 1/L$ and conclude that

$$\epsilon_*(z) = \frac{\pi z}{2} + r(z,\delta)$$

for an appropriate $C^{1,\beta}$ function $r(z,\delta)$ with $r(z,\delta) = \mathrm{O}(z^{1+\beta})$. When $c_{0,1}$ vanish, we know that $a(\epsilon)$ vanishes identically, and we see from (4.9) that $r \in C^k$ with $r(z,\delta) = \mathrm{O}(z^2)$. ■

Finally, we justify that the preceding results extend directly from the normal form (4.3) to the original equation (4.2). We recall that the boundary-value problems

$$\kappa_x = \omega + \kappa^2 + g(\kappa,\omega), \quad \omega = \tilde{\epsilon}^2, \quad \kappa(0) = 0, \quad \kappa(\tilde{L}) = \tilde{\delta} \tag{4.10}$$

and

$$u_x = \mu + (1 + \mu c_0(\mu))u^2 + c_1(\mu)u^3, \quad \mu = \epsilon^2, \quad u(0) = 0, \quad u(L) = \delta \tag{4.11}$$

are related via the transformation $(u,\mu) = (H(\kappa,\omega), h(\omega))$ described in Lemma 4.1, which guarantees that ϵ and $\tilde{\epsilon}$ are also related by an invertible smooth transformation of the form $\epsilon = \tilde{h}(\tilde{\epsilon})$. We then obtain the relationship

$$\tilde{L}_+(\tilde{\epsilon},\tilde{\delta}) = L_+(\tilde{h}(\tilde{\epsilon}), H(\tilde{\delta},\tilde{\epsilon}^2))$$

between the travel times $\tilde{L}_+$ and L_+ for (4.10) and (4.11), respectively, which allows us to transfer the results from this section to (4.10) as claimed in Theorem 2.

5. Discussion.

The present work establishes the existence and uniqueness of truncated contact defects in reaction-diffusion systems. A forthcoming paper will address the issue of spectral stability of the truncated contact defect solutions we constructed here under the assumption that the contact defect on the whole line is spectrally stable: it turns out that $\mathcal{R}_0$-reversible truncated contact defects are spectrally stable when periodic boundary conditions are used, while reversible truncated contact defects are always spectrally unstable under

Neumann boundary conditions, regardless of which of the two reversers $\mathcal{R}_{0,\pi}$ is present, since the eigenvalue corresponding to the approximate eigenfunction $\partial_x \mathbf{u}_L$ becomes positive. These results will, in particular, explain why these defects pairwise attract each other. We believe that nonlinear stability of contact defects and their truncation are difficult to establish due to the logarithmically diverging phase correction. Already in the case of source defects (whose spectra are, from an Evans function viewpoint, more regular than those of contact defects; see [18, Figure 6.1] and [19]), the proof of nonlinear stability is highly nontrivial [1].

There are three other types of generic defect solutions, namely, sources, sinks, and transmission defects. These defects have very different truncation properties, and we refer the reader to [18, section 6.8] for a brief discussion of these different cases and to [21] for recent results on the truncation of planar spiral-wave sources. In experiments, source-sink pairs often interact with each other. These interactions have not been analyzed in the literature, though a brief discussion of anticipated behaviors can be found in [18, section 6.9].

Acknowledgment. We are grateful to the referees for many helpful and constructive suggestions that helped improve the presentation of this paper.

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