

GENERALIZATIONS OF THE SCHRÖDINGER MAXIMAL OPERATOR: BUILDING ARITHMETIC COUNTEREXAMPLES

By

RENA CHU AND LILLIAN B. PIERCE

Dedicated to Peter Sarnak,

in appreciation for many years of inspiring discussions and encouragement

Abstract. Let $T_t^{P_2}f(x)$ denote the solution to the linear Schrödinger equation at time t , with initial value function f , where $P_2(\xi) = |\xi|^2$. In 1980, Carleson asked for the minimal regularity of f that is required for the pointwise a.e. convergence of $T_t^{P_2}f(x)$ to $f(x)$ as $t \rightarrow 0$. This was recently resolved by work of Bourgain, and Du and Zhang. This paper considers more general dispersive equations, and constructs counterexamples to pointwise a.e. convergence for a new class of real polynomial symbols P of arbitrary degree, motivated by a broad question: what occurs for symbols lying in a generic class? We construct the counterexamples using number-theoretic methods, in particular the Weil bound for exponential sums, and the theory of Dwork-regular forms. This is the first case in which counterexamples are constructed for indecomposable forms, moving beyond special regimes where P has some diagonal structure.

1 Introduction

Given a polynomial $P(\xi) \in \mathbb{R}[\xi_1, \dots, \xi_n]$ of degree $k \geq 2$, the operator

$$(1.1) \quad T_t^P f(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i(\xi \cdot x + P(\xi)t)} d\xi,$$

initially defined for f of Schwartz class on $\mathbb{R}^n$, gives a solution to the linear PDE

$$(1.2) \quad \begin{cases} \partial_t u - i\mathcal{P}(D)u = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x, 0) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

Here $D = \frac{1}{i}(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ and $\mathcal{P}(D)$ is defined according to its real symbol by

$$\mathcal{P}(D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} P(\xi) \hat{f}(\xi) d\xi.$$

When $P(\xi) = |\xi|^2$, in which case (1.2) is the linear Schrödinger equation, Carleson famously asked [Car80, Eqn. (14) p. 24]: what is the smallest $s > 0$ such

that

$$(1.3) \quad \lim_{t \rightarrow 0} T_t^P f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n, \text{ for all } f \in H^s(\mathbb{R}^n).$$

This question was resolved for dimension $n = 1$ quite swiftly by [Car80, DK82], which established that (1.3) holds if and only if $s \geq 1/4$. In higher dimensions, there is a long history of work on necessary and sufficient conditions for the Schrödinger pointwise convergence problem, including [Cow83, Car85, Sjö87, Veg88, Bou95, MVV96, TV00, Lee06, LR17, DGL17, DGLZ18, LR19]. For several decades it was expected that $s = 1/4$ might be the critical threshold in all dimensions, until Bourgain pushed the necessary condition on s above $1/4$ in [Bou13]. It was very recently resolved (up to the endpoint) by Bourgain [Bou16], who showed that $s \geq 1/4 + \delta(n)$ with $\delta(n) = (n - 1)/(4(n + 1))$ is necessary, while Du and Zhang [DZ19] showed that $s > 1/4 + \delta(n)$ is sufficient.

Bourgain's counterexample construction was interesting: it cleverly employed Gauss sums to force $\sup_{0 < t < 1} |T_t^P f(x)|$ to be large (from which a violation of (1.3) can be deduced) for test functions f defined using exponential sums. Recently, in [ACP23] we expanded this idea into a more flexible method for producing counterexamples to pointwise convergence results of the form (1.3) for the initial value problem (1.2), using the Weil bound for complete exponential sums. In that initial paper, we demonstrated the new method for symbols of the form $P(X_1, \dots, X_n) = X_1^k + \dots + X_n^k$ for any degree $k \geq 3$, and we proved that

$$s \geq 1/4 + \delta(n, k)$$

is necessary for (1.3) to hold, for $\delta(n, k) = (n - 1)/(4((k - 1)n + 1))$. Subsequently [EPV22] adapted the method of [ACP23] to achieve a result of the same strength, for any polynomial whose leading form (homogeneous part of highest degree) takes the special shape

$$(1.4) \quad P_k(X_1, \dots, X_n) = X_1^k + Q_k(X_2, \dots, X_n),$$

where $Q_k \in \mathbb{Q}[X_2, \dots, X_n]$ is a nonsingular form of degree k that is independent of X_1 . For degree 2 forms, the special shape (1.4) does not entail a loss of generality, since any quadratic form can be diagonalized over $\mathbb{R}$, and as we will explain, the underlying problem allows for such changes of coordinates. However, for $k \geq 3$, forms P_k of the shape (1.4) are quite sparse among degree k forms in $\mathbb{Q}[X_1, \dots, X_n]$ (in a sense we quantify in §8), and it is well-known that in arithmetic problems, a form with some diagonal structure is generally easier to handle. We are motivated by the question: what is the minimal regularity required for (1.3) when the real polynomial symbol P has leading form belonging to a generic class of degree k forms in $\mathbb{Q}[X_1, \dots, X_n]$?

For any fixed real symbol P , the key to proving or disproving pointwise convergence as in (1.3) is the associated maximal operator

$$(1.5) \quad f \mapsto \sup_{0 < t < 1} |T_t^P f|.$$

For a given s , to prove that pointwise convergence (1.3) holds for all $f \in H^s(\mathbb{R}^n)$, it suffices to prove (for example) that the maximal operator maps $H^s(\mathbb{R}^n)$ to $L_{\text{loc}}^2(\mathbb{R}^n)$. In the other direction, to prove that convergence (1.3) fails for some functions in $H^s(\mathbb{R}^n)$, it suffices to prove that the maximal operator is unbounded from $H^s(\mathbb{R}^n)$ to $L_{\text{loc}}^1(\mathbb{R}^n)$; see for example [Pie20, Appendix A] for a summary of these standard arguments. Thus we state our main result in terms of showing the maximal operator (1.5) is unbounded from $H^s(\mathbb{R}^n)$ to $L_{\text{loc}}^1(\mathbb{R}^n)$ for s in a certain range.

For any fixed $n \geq 2$ and degree $k \geq 2$, nonsingular forms are generic among degree k forms in $\mathbb{Q}[X_1, \dots, X_n]$. Given a value $s > 0$, the truth (or falsity) of a bound of the form

$$(1.6) \quad \left\| \sup_{0 < t < 1} |T_t^P f| \right\|_{L_{\text{loc}}^1(\mathbb{R}^n)} \leq C_s \|f\|_{H^s(\mathbb{R}^n)} \quad \text{for all } f \in H^s(\mathbb{R}^n)$$

is invariant under $\text{GL}_n(\mathbb{R})$ -action on the polynomial P (see §3.5). Thus if one wishes to understand this putative bound for an arbitrary polynomial P with nonsingular leading form $P_k \in \mathbb{Q}[X_1, \dots, X_n]$, it is no loss of generality to first apply a $\text{GL}_n(\mathbb{Q})$ change of variable to put P_k in a convenient form. We heavily exploit the following property: for every nonsingular form in $L[X_1, \dots, X_n]$ for an infinite field L , there is a $\text{GL}_n(L)$ change of variables under which the form becomes **Dwork-regular** (see the definition in (1.8)). Thus in the study of generic forms, it is no loss of generality to focus on Dwork-regular forms, and we do so here.

The fact that diagonalization, so convenient for quadratic forms, is out of reach for most higher-degree forms, is a dominant theme in the study of symmetric tensors (which, roughly speaking, generalize the symmetric matrix associated to a quadratic form). This has led to the development of many notions of rank for degree k forms, including the Schmidt rank (or h -index), Waring rank (symmetric tensor rank), slicing rank, relative rank, the property of decomposability, and more. Each such notion of rank is motivated by specific applications in algebraic invariant theory, number theory, algebraic geometry, computational complexity, etc. Similarly, our present work leads to a new notion of rank, which we now define.

Definition (Intertwining rank). A variable X_i intertwines with X_j (with $i \neq j$) in a form P_k of degree $k \geq 2$ if $(\partial^2 / \partial X_i \partial X_j) P_k \not\equiv 0$. By convention, X_i intertwines with itself. The intertwining rank $r(X_i)$ of X_i in P_k is the number of variables with which X_i intertwines. The **intertwining rank** of the form P_k is $\min_{1 \leq i \leq n} r(X_i)$.

For example, $X_1^3 + X_2^3 + X_3^3 + X_4^3$ has intertwining rank 1, while $X_1^3 + X_1X_2^2 + X_2X_3X_4$ has intertwining rank 2. Our main result is:

Theorem 1.1. *Fix $n \geq 2$ and $k \geq 2$. Let $P \in \mathbb{R}[X_1, \dots, X_n]$ be a polynomial whose leading form $P_k \in \mathbb{Q}[X_1, \dots, X_n]$ is Dwork-regular in $X_1, \dots, X_n$ over $\mathbb{Q}$ and has intertwining rank r . Suppose there is a constant C_s such that for all $f \in H^s(\mathbb{R}^n)$,*

$$(1.7) \quad \left\| \sup_{0 < t < 1} |T_t^P f| \right\|_{L^1(B_n(0,1))} \leq C_s \|f\|_{H^s(\mathbb{R}^n)}.$$

Then $s \geq \frac{1}{4} + \delta(n, k, r)$ with

$$\delta(n, k, r) = \frac{n - r}{4((k - 1)(n - (r - 1)) + 1)}.$$

We now briefly situate Theorem 1.1 with respect to previous literature, and then we explain the context of Dwork-regular forms, describe more precisely the notion of “generic” forms, and illustrate that a strength of the theorem is its application to indecomposable forms.

1.1 Relation to previous literature on convergence problems. As an immediate consequence of Theorem 1.1, pointwise convergence as in (1.3) fails for some $f \in H^s(\mathbb{R}^n)$ for the initial value problem (1.2) defined by P , for each $s < 1/4 + \delta(n, k, r)$ (following the standard arguments recorded in [Pie20, Appendix A]). Our present work also adapts (in a trivial way) to dimension $n = r = 1$ (see Remark 5.1), but we omit the details, since in 1 dimension, (1.3) holds for all polynomials P of degree $k \geq 2$ if $s \geq 1/4$ and fails if $s < 1/4$, by [KPV91, Cor. 2.6], [DK82, KR83].

The threshold $1/4$ is a common sticking point of many methods in the literature relating to the convergence problem (1.3); see, e.g., the survey in [ACP23, §1.2]. The main content of Theorem 1.1 is for $n \geq 3$, $k \geq 3$, and $2 \leq r < n$. In all dimensions, counterexamples to (1.3) and (1.7) for all $s < 1/4$, for any real polynomial symbol (with leading form of any intertwining rank $r \geq 1$), are due to Sjölin [Sjö98]. Theorem 1.1 is the first result to go beyond $1/4$ for intertwining rank $r \geq 2$, for all dimensions $n \geq 3$. The strength of our result decreases as r increases, and subsides to the requirement $s \geq 1/4$ when $r = n$. For intertwining rank $r = 1$, Theorem 1.1 recovers the special case of diagonal symbols considered in [ACP23], and the symbols of the form (1.4) considered in [EPV22]. For degree $k = 2$, by the spectral theorem, any quadratic leading form is diagonalizable under $\mathrm{GL}_n(\mathbb{R})$, which (after further renormalization) reduces the case of quadratic forms to the case of intertwining rank $r = 1$. For dimension $n = 2$, the only cases are intertwining rank $r = 1$, in which case Theorem 1.1 recovers a result of [EPV22], and $r = 2$, in

which case Theorem 1.1 states $s \geq 1/4$, which was previously known. It remains an interesting open question whether the regularity condition in Theorem 1.1 can be increased further.

In all dimensions, (1.3) holds for all $s > 1/2$, for a wide class of differentiable functions, including any real polynomial P of principal type of order α for $\alpha > 1$ (meaning $|\nabla P(\xi)| \gg (1+|\xi|)^{\alpha-1}$ for all sufficiently large $|\xi|$), by [BAD91, Thm. D] and [RVV06, Remark 2.2]. Positive results proving bounds related to (1.7) for $s \leq 1/2$, such as the celebrated work in the case $P(\xi) = |\xi|^2$ in [DZ19], must proceed by entirely different methods. For further notes on the vast literature on convergence results, maximal operators and connections to local smoothing, we refer to [ACP23, §1.2].

1.2 The role of Dwork-regular forms. Dwork-regular forms have been extensively developed by Dwork [Dwo62] and later Katz (e.g., [Kat08, Kat09]). To set the context for their definition, first recall that a form P_k is said to be nonsingular over $\mathbb{Q}$ if the polynomials $P_k, \partial P_k / \partial X_1, \dots, \partial P_k / \partial X_n$ have no common zeroes in $\mathbb{P}_{\mathbb{Q}}^{n-1}$ (correspondingly the projective hypersurface defined by $P_k = 0$ in $\mathbb{P}_{\mathbb{Q}}^{n-1}$ is nonsingular). (Here and throughout, $\bar{L}$ denotes a fixed algebraic closure of a given field L , and $\mathbb{P}_{\bar{L}}^{n-1}$ denotes the $(n-1)$ -dimensional projective space over the field $\bar{L}$.) In comparison, P_k is said to be Dwork-regular over $\mathbb{Q}$ in the variables $X_1, \dots, X_n$ if there are no simultaneous solutions in $\mathbb{P}_{\mathbb{Q}}^{n-1}$ to

$$(1.8) \quad P_k(X_1, \dots, X_n) = 0, \quad X_i \frac{\partial P_k}{\partial X_i}(X_1, \dots, X_n) = 0, \quad 1 \leq i \leq n.$$

A comparison of the definitions shows that any Dwork-regular form over $\mathbb{Q}$ is nonsingular over $\mathbb{Q}$. As mentioned before, any nonsingular form becomes Dwork-regular under an appropriate change of variables (see §3). Our interest in passing to Dwork-regular forms is that they are particularly amenable to applications of the Weil–Deligne bound (Lemma 4.2) even after fixing one or more variables (a consequence of Proposition 3.3). This allows us to make new progress on the convergence problem (1.3) despite a central difficulty that appears if each variable “interacts” with other variables in the leading form P_k . Intertwining rank captures the amount of such interaction. The novelty in our present work is that we can prove new results for forms of all intertwining ranks $1 < r < n$.

1.3 Genericity: an underlying motivating question. We are motivated by the question: what is the behavior of the initial value problem (1.2) when P_k is a “generic” form in $\mathbb{Q}[X_1, \dots, X_n]$? Technically, a class of forms is said to be generic if it corresponds to an open set (in the Zariski topology) in the moduli space of all degree k homogeneous forms in $\mathbb{Q}[X_1, \dots, X_n]$.

As one example, nonsingular forms are generic. It is equivalent to show that being singular is a condition on the coefficients of P_k that defines a closed set in the Zariski topology. Since P_k is singular if and only if $P_k, \partial P_k / \partial X_1, \dots, \partial P_k / \partial X_n$ have a common nonzero root, then P_k is singular if and only if the resultant of $P_k, \partial P_k / \partial X_1, \dots, \partial P_k / \partial X_n$ vanishes. This resultant is a (nonzero) polynomial in the coefficients of $P_k, \partial P_k / \partial X_1, \dots, \partial P_k / \partial X_n$, so that P_k being singular is characterized by its coefficients lying in the vanishing set of a polynomial, proving the claim.

As another example, indecomposable forms are generic; we describe this property thoroughly in §1.4 below. The union of two Zariski closed sets (e.g., the set of singular forms and the set of decomposable forms) is closed, and so the complement (e.g., the set of forms that are nonsingular and indecomposable) is open, and hence generic. Indeed, any generic condition will include nonsingular forms (generically), and indecomposable forms (generically). This fact cuts in two directions, one convenient and one inconvenient. First, on the one hand, even if we are interested in studying generic forms, it is reasonable only to consider nonsingular forms (which is advantageous for an application of Lemma 4.2). But on the other hand, it shows that to understand the generic situation, we must understand the case of indecomposable forms.

One strength of Theorem 1.1 is that it proves the first (nontrivial) counterexamples to (1.3) that apply to leading forms P_k that are indecomposable. Thus in the next section we describe decomposability/indecomposability in more detail. Nevertheless, Theorem 1.1 falls short of proving nontrivial results for a generic class of forms: it is only nontrivial for forms of rank strictly smaller than n , and these are not generic. In §8.4 we compute the codimension of Dwork-regular forms of intertwining rank $r < n$, among degree k forms in $\mathbb{Q}[X_1, \dots, X_n]$. This codimension quantifies that Theorem 1.1 proves nontrivial results for a class of forms that is not generic, but that nevertheless contains “many more” real symbols than were tractable in previous works.

1.4 Indecomposable forms of degree k : definition, remarks and examples. A form is called decomposable (or sometimes of Sebastiani–Thom type) over a field L if there is a $\mathrm{GL}_n(L)$ change of variables so that the form can be written as a sum of at least two forms in disjoint sets of variables, for example

$$(1.9) \quad P(X_1, \dots, X_n) = Q_1(X_1, \dots, X_m) + Q_2(X_{m+1}, \dots, X_n)$$

for some $1 \leq m < n$. Otherwise, a form is indecomposable over L . All of the forms considered in [ACP23, EPV22] were of the special shape (1.4), and thus have intertwining rank $r = 1$. All forms with intertwining rank $r = 1$ are

decomposable. Yet we are motivated to tackle indecomposable forms, since in the moduli space of degree $k \geq 3$ forms in $\mathbb{Q}[X_1, \dots, X_n]$, indecomposable forms are generic for $n \geq 2$, at least for $(n, k) \neq (2, 3)$. (This is related, e.g., to [Wan15, §6], [HLYZ22, Thm. 3.2], [OS03, p. 303]; see further details in §8.)

Note that the intertwining rank of any decomposable polynomial is at most $\lfloor n/2 \rfloor$; this can be seen by inspecting (1.9). We use this to deduce the following immediate corollary of Theorem 1.1, verified in §3.6.

Corollary 1.2. *Fix $n \geq 2$ and $k \geq 2$. Let $P \in \mathbb{R}[X_1, \dots, X_n]$ be a polynomial whose leading form $P_k \in \mathbb{Q}[X_1, \dots, X_n]$ is decomposable and nonsingular in $X_1, \dots, X_n$ over $\mathbb{Q}$. Suppose there is a constant C_s such that for all $f \in H^s(\mathbb{R}^n)$,*

$$\| \sup_{0 < t < 1} |T_t^P f| \|_{L^1(B_n(0,1))} \leq C_s \|f\|_{H^s(\mathbb{R}^n)}.$$

Then $s \geq \frac{1}{4} + \frac{n}{4((k-1)(n+2)+2)}$.

Since it has been remarked that it is challenging to exhibit indecomposable forms (see, e.g., [Pum06, p. 348], [Wan15, p. 576]), we provide explicit examples. For each degree $k \geq 3$ and rank $2 \leq r \leq n$, we exhibit indecomposable forms of degree k that are Dwork-regular over $\mathbb{Q}$ in $X_1, \dots, X_n$ with intertwining rank r , namely

$$\begin{aligned} P_k(X_1, \dots, X_n) &= X_1^k + \dots + X_n^k + \sum_{2 \leq j \leq r} X_1 X_j^{k-1} + \sum_{2 \leq i < j \leq n} X_i X_j^{k-1}, \quad k \geq 3 \text{ odd}; \\ P_k(X_1, \dots, X_n) &= X_1^k + \dots + X_n^k + \sum_{2 \leq j \leq r} X_1^2 X_j^{k-2} + \sum_{2 \leq i < j \leq n} X_i^2 X_j^{k-2}, \quad k \geq 4 \text{ even}. \end{aligned}$$

For example, in dimension $n = 3$ the examples with intertwining rank 2 are

$$\begin{aligned} P_k(X_1, X_2, X_3) &= X_1^k + X_2^k + X_3^k + X_1 X_2^{k-1} + X_2 X_3^{k-1}, \quad k \geq 3 \text{ odd}; \\ P_k(X_1, X_2, X_3) &= X_1^k + X_2^k + X_3^k + X_1^2 X_2^{k-2} + X_2^2 X_3^{k-2}, \quad k \geq 4 \text{ even}. \end{aligned}$$

In §8, we use a criterion of Harrison [Har75, HP88] to verify that these are indecomposable forms over $\mathbb{Q}$ (and hence in particular cannot be brought to have intertwining rank 1 by any $\mathrm{GL}_n(\mathbb{Q})$ change of variables).

1.5 Further directions. By the invariance of (1.6) under $\mathrm{GL}_n(\mathbb{Q})$ -action on P , the main result of Theorem 1.1 furthermore applies to any polynomial P with leading form $P_k \in \mathbb{Q}[X_1, \dots, X_n]$ lying in the $\mathrm{GL}_n(\mathbb{Q})$ -orbit of a Dwork-regular form of intertwining rank r . How big is such an orbit? This points to an interesting question, which is in fact typical when one encounters a notion of rank for a higher

degree form. Given a particular notion of rank, in an application one often wants to manipulate the original form (or class of forms) to make the (particular) rank more advantageous; the limits of this procedure may depend on the underlying field (and whether it is algebraically closed). For example, in the case of Schmidt rank, there is recent work on regularization and a new relative rank in [LZ21], and the relation to algebraic closure in [LZ22]. In the case of Waring rank, see the celebrated work of Alexander and Hirschowitz [AH95] over $\mathbb{C}$ (and a nice overview in [RS00]), or more recent work for monomials in [CCG12] and partial progress over $\mathbb{R}$ or $\mathbb{Q}$ in [HM22], with intriguing remarks on the dependence on the underlying field in [Rez13]. For our particular setting, this becomes the question: how does the intertwining rank behave under minimization via $\mathrm{GL}_n(\mathbb{Q})$? We pursue this question, which requires completely different methods, in other work.

1.6 Notation. In this paper we employ the convention $e(t) = e^{it}$. Correspondingly, $\hat{f}(\xi) = \int_{\mathbb{R}^m} f(x) e^{-ix \cdot \xi} dx$ and $f(x) = (2\pi)^{-m} \int_{\mathbb{R}^m} \hat{f}(\xi) e^{ix \cdot \xi} d\xi$, so that Plancherel's theorem takes the form $\|f\|_{L^2(\mathbb{R}^m)}^2 = (2\pi)^{-m} \|\hat{f}\|_{L^2(\mathbb{R}^m)}^2$. The Sobolev space $H^s(\mathbb{R}^m)$ is defined to be all $f \in \mathcal{S}'(\mathbb{R}^m)$ with finite Sobolev norm

$$\|f\|_{H^s(\mathbb{R}^m)}^2 = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi.$$

We use the convention that $B_m(c, r)$ is the Euclidean ball of radius r centered at c in $\mathbb{R}^m$. The notation $A \ll_{\kappa} B$ denotes that $|A| \leq C(\kappa)B$ for a constant $C(\kappa)$. It is harmless in our argument to allow constants to depend on the dimension n , the symbol P of degree k , the intertwining rank r , and a Schwartz function ϕ we will fix once and for all. Certain small constants, which we can choose freely, we will denote by $c_0, c_1, c_2, \dots$; we will demarcate these explicitly in inequalities when we are preparing to exploit their small size. For $v = (v_1, \dots, v_m)$, $w = (w_1, \dots, w_m) \in \mathbb{R}^m$, we define $v \circ w = (v_1 w_1, \dots, v_m w_m)$. For a multi-index α we set $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$; for two multi-indices α, β , $\alpha \geq \beta$ and $\alpha - \beta$ denote coordinate-wise relations.

2 Method of proof

To prove Theorem 1.1, we construct a family of test functions $\{f_j\}$ that are Fourier-supported in an annulus $\{(1/C)R_j \leq |\xi| \leq CR_j\}$ of radius R_j for a sequence of $R_j \rightarrow \infty$ as $j \rightarrow \infty$. By definition

$$R_j^s \|f_j\|_{L^2} \ll_{C,s} \|f_j\|_{H^s} \ll_{C,s} R_j^s \|f_j\|_{L^2}.$$

Hence Theorem 1.1 follows immediately from an explicit construction:

Theorem 2.1. *Let $n \geq 2$. Let $P \in \mathbb{R}[X_1, \dots, X_n]$ be a polynomial of degree $k \geq 2$ whose leading form $P_k \in \mathbb{Q}[X_1, \dots, X_n]$ is Dwork-regular over $\mathbb{Q}$ in the variables $X_1, \dots, X_n$, and has intertwining rank $r \leq n$. Fix any $s < \frac{1}{4} + \delta(n, k, r)$ with $\delta(n, k, r)$ as in Theorem 1.1. Then there exists an infinite sequence of $j \rightarrow \infty$ such that for $R_j = 2^j$, there exists a function $f_j \in L^2(\mathbb{R}^n)$, where $\|f_j\|_{L^2} = 1$ and $\hat{f}_j$ is supported in an annulus $\{(1/C)R_j \leq |\zeta| \leq CR_j\}$, and with the property that*

$$\lim_{j \rightarrow \infty} \frac{\| \sup_{0 < t < 1} |T_t^P f_j(x)| \|_{L^1(B_n(0,1))}}{R_j^s} = \infty.$$

To prove Theorem 2.1, we define each test function f so that $|T_t^P f(x)|$ can be approximated by an $(n - r)$ -dimensional exponential sum, which we show is “large” for many $x \in B_n(0, 1)$, after choosing t appropriately (depending on x). This strategy is motivated by ideas of Bourgain (for degree $k = 2$) as explained in [Pie20], and the flexible construction (applicable to degree $k \geq 3$) developed in our earlier work [ACP23], for the diagonal case $P(X_1, \dots, X_n) = X_1^k + \dots + X_n^k$. However, a difficulty arises if the leading form P_k of the real symbol has intertwining rank $r > 1$: in order to optimize the test functions f to violate the supposed upper bound (1.7) for s as large as possible, one is naturally led to dilate one variable within the exponential sum, say X_1 , by a large parameter. This large dilation contributes large error terms to certain approximation arguments. We overcome this by using the notion of intertwining rank. To assist the reader in tracking the main ideas of the method, we now present a series of heuristic computations that are not rigorous, but simply emphasize numerology. The remainder of the paper carries out each step rigorously.

2.1 Heuristic overview. Let $\Phi_n(x_1, \dots, x_n)$ be a non-negative Schwartz function with $\Phi_n(0) = 1$ and $\hat{\Phi}_n$ supported in $[-1, 1]^n$. As in Theorem 2.1, we think of R as a parameter that will go to infinity. Define a test function

$$(2.1) \quad f(x) = \Phi_n(S \circ x) \sum_{\substack{m \in \mathbb{Z}^n \\ m_j \approx R/\Lambda_j}} e((\Lambda \circ m) \cdot x)$$

for some parameters $S = (S_1, \dots, S_n)$ and $\Lambda = (\Lambda_1, \dots, \Lambda_n)$, with each S_i, Λ_i chosen later to be 1, R or a small power of R . Let $\|S\| = \prod S_j$ for the moment, and similarly for $\|\Lambda\|$. The Fourier transform $\hat{f}$ is supported in an annulus of radius $\approx R$ if each $S_j \ll R$, and $\|f\|_{H^s} \approx R^s \|S\|^{-1/2} R^{n/2} \|\Lambda\|^{-1/2}$, so that by normalizing appropriately, f fits the hypotheses of Theorem 2.1. For this test function,

$$T_t^P f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\Phi}_n(\zeta) e((S \circ \zeta) \cdot x) \sum_{\substack{m \in \mathbb{Z}^n \\ m_j \approx R/\Lambda_j}} e((\Lambda \circ m) \cdot x) e(P(S \circ \zeta + \Lambda \circ m)t) d\zeta.$$

For simplicity we temporarily assume P is a homogeneous form of degree $k \geq 2$, defined in terms of coefficients and multi-indices by

$$P(y_1, \dots, y_n) = \sum_{|\alpha|=k} c_\alpha y^\alpha.$$

Step 1. Use partial summation to remove all terms in the sum over m that depend on ζ (i.e., that are “not arithmetic”); this replaces the sum over m (up to an error term) by

$$(2.2) \quad \approx w(R/\Lambda_1, \dots, R/\Lambda_n) \sum_{\substack{m \in \mathbb{Z}^n \\ m_j \approx R/\Lambda_j}} e((\Lambda \circ m) \cdot x + P(\Lambda \circ m)t),$$

in which the new sum is the “arithmetic contribution” while

$$(2.3) \quad w(y_1, \dots, y_n) = e((P(S \circ \zeta + \Lambda \circ y) - P(\Lambda \circ y))t)$$

is the “weight” that has been removed by partial summation. The weight $w(R/\Lambda_1, \dots, R/\Lambda_n)$ contributes to the integral over ζ a factor with a linear phase in ζ , namely $\approx e(S \circ \zeta \cdot t \nabla P(\underline{R}))$, where $\underline{R} = (R, \dots, R)$.

Step 2. Use integration by parts to remove all terms in the phase of the integral over ζ that are order 2 or higher in ζ (up to an error term).

Step 3. After Step 2, one may immediately apply Fourier inversion to the remaining integral over ζ , so that the main contribution to $T_t^P f(x)$ is a product of the arithmetic contribution and

$$\approx \Phi_n(S \circ (x + t \nabla P(\underline{R}))).$$

Then place constraints on x and t so that $S \circ (x + t \nabla P(\underline{R})) \approx 0$, so that applying $\Phi_n(0) = 1$ (and continuity of Φ_n) implies that $\Phi_n(S \circ (x + t \nabla P(\underline{R}))) \approx 1$ and hence

$$(2.4) \quad |T_t^P f(x)| \approx \left| \sum_{\substack{m \in \mathbb{Z}^n \\ m_j \approx R/\Lambda_j}} e((\Lambda \circ m) \cdot x + P(\Lambda \circ m)t) \right|.$$

Step 4. Construct a set of $x \in \mathbb{R}^n$ that is a positive proportion of $B_n(0, 1)$ so that for each x in the set there exists $t \in (0, 1)$ for which the arithmetic contribution (2.4) is large. Up to some simple changes of variables, the set is a union of boxes centered at rationals $(a_1/q, a_2/q, \dots, a_n/q)$ for primes $q \approx Q$, where Q is a small power of R to be chosen later.

Step 5. Optimize the choices of $S = (S_1, \dots, S_n)$, $\Lambda = (\Lambda_1, \dots, \Lambda_n)$, and Q , subject to the constraints that the error terms in all previous steps are acceptable.

To unravel the chain of dependencies that make these steps efficient and compatible, first consider Step 3, which requires that for each $j = 1, \dots, n$

$$(2.5) \quad S_j(x_j + tR^{k-1}) \ll 1 \iff t \approx -\frac{x_j}{R^{k-1}} + O\left(\frac{1}{R^{k-1}S_j}\right).$$

(This sketch assumes in particular that $\partial_1 P(\underline{R}) \gg R^{k-1}$; to achieve this, we develop Lemma 3.5.) Given $(x_1, \dots, x_n)$, if we choose t to satisfy this for x_1 , then the only way it can simultaneously satisfy it for $x_2, \dots, x_n$ is for $x_2, \dots, x_n$ to all lie in $O(S_j^{-1})$ neighborhoods of x_1 . This is too limiting in Step 4 unless we set $S_2 = \dots = S_n = 1$, which we now do, so $S = (S_1, 1, \dots, 1)$ and $\|S\| = S_1$. From now on, because of the “large” rescaling factor S_1 , the first coordinate x_1 will play a special role.

Next consider Step 2, in which we use iterated integration by parts (coordinate by coordinate) to remove a “weight” from the integral that contains all terms in the phase that are order 2 or higher in ξ ; this weight takes the approximate form

$$\begin{aligned} W(\xi_1, \dots, \xi_n) &= e([P(S \circ \xi + \underline{R}) - L_0(\xi) - L_1(\xi)]t) \\ &= e\left(t \sum_{\substack{|\beta+\gamma|=k \\ |\beta|\geq 2}} c_{\beta+\gamma} C(\beta, \gamma) (S \circ \xi)^\beta \underline{R}^\gamma\right), \end{aligned}$$

in which L_0 (respectively L_1) represents terms in $P(S \circ \xi + \underline{R})$ that are order 0 (respectively order 1) in ξ , and $C(\beta, \gamma)$ are positive combinatorial constants. As usual, the error term when a weight is removed by integration by parts (or summation by parts) will be smaller if the weight is slowly-varying, and thus we must control the derivatives of W . The error accrued in Step 2 must be at most a small proportion of the main term (2.4). This will be achieved if for each nonzero multi-index $\kappa \in \{0, 1\}^n$, for all $\xi \in \text{supp } \hat{\Phi}_n \subseteq [-1, 1]^n$,

$$(2.6) \quad \left| \frac{\partial^{|\kappa|}}{\partial \xi^\kappa} W(\xi) \right| \ll 1 \iff t \sum_{\substack{|\beta+\gamma|=k \\ |\beta|\geq 2, \beta\geq \kappa}} c_{\beta+\gamma} C(\beta, \gamma) (S \circ \xi)^{\beta-\kappa} S^\kappa \frac{\beta!}{(\beta-\kappa)!} \underline{R}^\gamma \ll 1.$$

From Step 3 we know that $t \approx R^{-(k-1)}$, so we require the sum above to satisfy $\ll R^{k-1}$. Each term in the sum is roughly of size $S^\beta \underline{R}^\gamma = S_1^{\beta_1} R^{k-|\beta|}$ for some $|\beta| \geq 2$. There are two scenarios: if $|\beta| > \beta_1$, this term is $\ll R^{k-1}$ as long as $S_1 \ll R$. If $\beta_1 = |\beta|$, which can only occur if $\beta_1 \geq 2$, then this term is $\ll R^{k-1} (S_1^{\beta_1} / R^{\beta_1-1})$, which is $\ll R^{k-1}$ as long as $S_1 \ll R^{\frac{\beta_1-1}{\beta_1}}$, which is most restrictive when $\beta_1 = 2$. Thus we impose the condition $S_1 \ll R^{1/2}$.

Next consider Step 1, in which the error introduced by iterated partial summation must be at most a small proportion of the main term (2.4). This will be

achieved if for each nonzero multi-index $\kappa \in \{0, 1\}^n$, for all y with $y_j \approx R/\Lambda_j$,

$$(2.7) \quad \left| \frac{\partial^{|\kappa|}}{\partial y^\kappa} w(y) \right| \ll \Lambda^\kappa R^{-|\kappa|}$$

$$\iff t \sum_{\substack{|\beta+\gamma|=k \\ |\beta| \geq 1, \gamma \geq \kappa}} c_{\beta+\gamma} C(\beta, \gamma) (S \circ \zeta)^\beta (\Lambda \circ y)^{\gamma-\kappa} \Lambda^\kappa \frac{\gamma!}{(\gamma-\kappa)!} \ll \Lambda^\kappa R^{-|\kappa|}.$$

Note that in contrast to (2.6), in this case the phase in the weight $w(y)$ includes terms of order 1 in ζ , so that $|\beta| = 1$ is allowed in the sum immediately above. Each term in the sum immediately above is roughly of size

$$S^\beta \Lambda^\kappa (\Lambda \circ y)^{\gamma-\kappa} \approx S_1^{\beta_1} \Lambda^\kappa \underline{R}^{\gamma-\kappa} \approx S_1^{\beta_1} R^{k-|\beta|} \cdot \Lambda^\kappa R^{-|\kappa|}.$$

Thus the condition (2.7) will be met, recalling $t \approx R^{-(k-1)}$, as long as

$$S_1^{\beta_1} R^{k-|\beta|} \ll R^{k-1}$$

for all $|\beta| \geq 1$. There are again several scenarios: if $|\beta| > \beta_1$ or if $\beta_1 = |\beta| \geq 2$, this term is $\ll R^{k-1}$ by arguing as in Step 2, under our assumption $S_1 \ll R^{1/2}$. The problem is that there is now also a third case, with $\beta_1 = |\beta| = 1$ in which case the requirement is asking that $S_1^1 R^{k-1} \ll R^{k-1}$. These problematic terms can be seen as the contribution to the weight (2.3) that is varying the fastest with respect to y , namely the portion of the phase that is highest order in y (total degree $k-1$) and linear in ζ . (We also provide an explicit example of such terms in (5.6).) One way to achieve the requirement $S_1^1 R^{k-1} \ll R^{k-1}$ is to impose $S_1 \ll 1$, but this is inefficient in Step 5. The strategy we adopt is to modify the definition of the test function f so that such terms never appear.

Precisely, in the definition (2.1) of the test function f , we now restrict the sum over $m \in \mathbb{Z}^n$ to sum only over those coordinates m_j with the following property: for each multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, if $\alpha_j \geq 1$ and $\alpha_1 \geq 1$ then the coefficient $c_\alpha = 0$ in the original polynomial $P(y)$. Equivalently, we define the sum to be only over those coordinates m_j such that X_j never appears in a monomial with X_1 in the original polynomial $P(y)$. (Equivalently, set $\Lambda_j \approx R$ for each j such that X_j appears in a monomial with X_1 , and in all other coordinates take $\Lambda_j = L$, for L a small power of R to be chosen later.)

Because the exponential sum is a source of gain in Steps 4 and 5, we wish to sum over as many coordinates as possible, so depending on $P(y)$ we relabel coordinates in the beginning so that X_1 is the variable that appears in monomials with as few other coordinates as possible, say $X_2, \dots, X_r$ with $r < n$; this is the motivation for defining intertwining rank. We now let $m = (m_{r+1}, \dots, m_n)$, and

only sum over these coordinates. The conclusion is that in place of (2.4) we arrive at a main contribution of the form

$$(2.8) \quad |T_t^P f(x)| \approx \left| \sum_{\substack{(m_{r+1}, \dots, m_n) \in \mathbb{Z}^{n-r} \\ m_j \approx R/L}} e(m \cdot (Lx_{r+1}, \dots, Lx_n) + P(R/L, \dots, R/L, m)L^k t) \right|.$$

(Here we used homogeneity of P of degree k ; to make R/L integral, see Remark 6.1.)

In Step 4, to construct a set of x for which the above sum in (2.8) is “large,” imagine that each $Lx_j = a_j/q$ and $L^k t = a_1/q$ are rationals of prime denominator $q \approx Q$, so that the sum can be regarded as $\approx ((R/L)q^{-1})^{n-r}$ copies of a sum where each coordinate m_j runs over a complete set of residues modulo q . Since $P(y_1, \dots, y_n)$ is Dwork-regular, even after specializing the first r variables, the remaining polynomial is well-behaved (Proposition 3.3). A major feature of our argument shows that for a positive proportion ($\gg q^{n-r+1}$) of choices of $a_1, a_{r+1}, \dots, a_n$ the $(n-r)$ -dimensional sum mod q is of the optimal size $\approx q^{(n-r)/2}$ (Proposition 4.1). Hence at precisely such a point x and for such a t ,

$$(2.9) \quad |T_t^P f(x)| \approx \left(\frac{R}{Lq} \right)^{n-r} q^{(n-r)/2} \approx \left(\frac{R}{LQ^{1/2}} \right)^{n-r}.$$

To achieve a similar result for a positive measure of x , we need to show this continues to hold for Lx_j and $L^k t$ merely “close” to rationals with denominator q . Typically, deducing this from the case where they are precisely rationals would follow by applying partial summation to the sum in (2.8). The error incurred by partial summation will be too large if the “weight” removed involves terms of the highest order in m . Thus we must choose t so that (i): $L^k t = a_1/q$ precisely. Fortunately this is possible because of the wiggle room allowed in (ii): $L^k t \approx -L^k x_1/R^{k-1} + O(L^k/R^{k-1}S_1)$, from (2.5) in Step 3. Given x_1 with $-L^k x_1/R^{k-1}$ in an interval of length $O(1/q)$ centered at a_1/q we can always choose t meeting both requirements (i) and (ii) as long as $Q^{-1} \ll L^k/R^{k-1}S_1$, which we now assume.

In contrast, the coordinates $Lx_{r+1}, \dots, Lx_n$ appear as coefficients of the lowest-order (linear) terms in m so that partial summation will contribute reasonable errors when we allow Lx_j to vary in an interval around a_j/q . If the interval is of length V , say, the contributed error will be proportional to $(R/L)^{n-r}V^{n-r}$ times the size of the main term, so we require $(R/L)^{n-r}V^{n-r} \ll 1$. At the same time, we want V to be large in order for the boxes so constructed to cover a positive proportion of $(x_{r+1}, \dots, x_n) \in [0, 1]^{n-r}$. In this regard the principle of simultaneous Dirichlet approximation in $n-r$ dimensions motivates the choice $V = 1/qQ^{1/(n-r)} \approx Q^{-1-1/(n-r)}$ (see, e.g., [Pie20, Appendix B]). Taken together, these two requirements force the condition $Q^{-1-1/(n-r)} \ll L/R$.

Cumulatively, this construction yields boxes in the coordinates $x_1, x_{n-r}, \dots, x_n$, each of measure $\approx Q^{-1}V^{n-r}$, centered at around $\gg q^{n-r+1}$ rational tuples with denominator q for each prime $q \approx Q$. A naive calculation suggests the total measure of the union of the boxes could be

$$\approx (Q/\log Q)Q^{n-r+1}Q^{-1}(Q^{-1-1/(n-r)})^{n-r} \approx 1/\log Q.$$

Since the boxes can overlap significantly, a sophisticated justification is required although the conclusion agrees with the above (Proposition 6.2).

Upon reaching Step 5, these heuristics suggest that for the test function f so constructed, with $\|f\|_{H^s} \approx R^s S_1^{-1/2} (R/L)^{(n-r)/2}$, on a set $x \in B_n(0, 1)$ of measure $\gg 1/\log Q$,

$$|T_t^P f(x)| \gg (R/LQ^{1/2})^{n-r}.$$

This occurs under the constraints

$$S_1 \leq R^{1/2}, \quad Q^{-1-1/(n-r)} \ll L/R, \quad Q^{-1} \ll L^k/R^{k-1}S_1.$$

The claim of Theorem 2.1 consequently holds for each s such that

$$\frac{(\frac{R}{LQ^{1/2}})^{n-r}(\log Q)^{-1}}{S_1^{-1/2}(R/L)^{(n-r)/2}} \gg R^{s'}$$

for some $s' > s$. Upon setting $L = R^\lambda$, $Q = R^\kappa$, $S_1 = R^\sigma$ this is equivalent to a linear condition on λ, κ, σ and s subject to linear constraints, and it can be optimized, which we do in detail in §7.

2.2 Outline of the paper. In §3 we state and prove all the key properties of Dwork-regular polynomials we will use, including that upon fixing one or more variables, the resulting polynomial is a Deligne polynomial over $\mathbb{F}_q$ (for all but finitely many primes q). In §4 we prove upper and lower bounds on (complete and incomplete) exponential sums involving Deligne polynomials. In §5 we approximate $T_t^P f$, for appropriate test functions f , by an exponential sum. In §6 we define a set of $x \in B_n(0, 1)$ for which we can approximate this sum by complete exponential sums to which we can apply the arithmetic results of §4. In §7 we then optimize the choices of all parameters, thus proving Theorem 2.1. Finally, in §8 we provide details on examples of Dwork-regular, indecomposable forms of arbitrary intertwining rank and degree, and remark on the codimension of such forms.

3 Properties of Dwork-regular forms

In this section we first gather three algebraic properties of Dwork-regular forms that we will apply throughout the proof. Then in §3.4, we prove a lower bound on a partial derivative of a Dwork-regular form, in §3.5 we show that boundedness (or unboundedness) of the maximal operator is invariant under a $\mathrm{GL}_n(\mathbb{Q})$ change of variables, in §3.6 we verify Corollary 1.2, and finally in §3.7 we remark that the PDE's we consider are dispersive.

It is convenient to work temporarily in a more abstract setting, and simply fix a field L , which could for example be $\mathbb{Q}$, $\mathbb{R}$ or a finite field $\mathbb{F}_q$. We will later call upon the lemmas we prove both in the setting of infinite fields such as $\mathbb{Q}$ and $\mathbb{R}$ and finite fields $\mathbb{F}_q$ for q prime. Let L be a field and $H \in L[X_1, \dots, X_n]$ a homogeneous polynomial of degree $k \geq 2$. Then H is nonsingular over L if $H, \partial H / \partial X_1, \dots, \partial H / \partial X_n$ have no common zeroes in $\mathbb{P}_L^{n-1}$ (correspondingly the projective hypersurface defined by $H = 0$ in $\mathbb{P}_L^{n-1}$ is nonsingular). Recall that H is Dwork-regular in the variables $X_1, \dots, X_n$ over L if there are no solutions in $\mathbb{P}_L^{n-1}$ to the simultaneous equations

$$H(X_1, \dots, X_n) = 0, \quad X_i \frac{\partial H}{\partial X_i}(X_1, \dots, X_n) = 0, \quad 1 \leq i \leq n.$$

If H is Dwork-regular over L , then H is nonsingular over L . However, it can be that H is nonsingular but not Dwork-regular: for example, $X_1^k + \dots + X_{n-1}^k + X_n^{k-1}$ over $\mathbb{Q}$. However, if the field L is infinite, given any nonsingular form, there exists a $\mathrm{GL}_n(L)$ change of variables under which the form becomes Dwork-regular (see [Dwo62, pp. 67–68] and [Kat09, Lemma 3.1]). (In fact, for a given form, there are many such changes of variables: the proof of [Kat09, Lemma 3.1] can be adapted to show that the set of such elements is dense in $\mathrm{GL}_n(L)$.) We quote from Katz in [Kat08, p. 1252], that the archetypical Dwork-regular polynomial H would be of the form $H(X) = \sum_{i=1}^n X_i^k + \tilde{H}(X)$, where $\tilde{H}$ is any polynomial of degree at most $k - 1$. The antithesis to a Dwork-regular polynomial, when $n = 2m$ is even and L has odd characteristic, is something of the form $H(X) = \sum_{i=1}^m X_i X_{m+i}$. By Euler's identity, for a form H of degree k ,

$$kH = \sum_{i=1}^n X_i (\partial / \partial X_i) H,$$

so that when discussing Dwork-regular forms it is natural to assume that $\mathrm{char} L \nmid k$, if L is finite.

In this section, we first present an equivalent characterization of Dwork-regularity that is easier to work with. (This property has previously been remarked in the context of [Kat09, Lemma 3.1].)

Lemma 3.1. *Let L be a field and let $H(X_1, \dots, X_m) \in L[X_1, \dots, X_m]$ be a homogeneous polynomial of degree $d \geq 2$. For every nonempty $S \subseteq \{1, \dots, m\}$, define $H_S := H|_{X_j=0, j \notin S}$. Then H is Dwork-regular in $X_1, \dots, X_m$ over L if and only if*

- (1) *for all S with $|S| = 1$, H_S is not the zero polynomial, and*
- (2) *for all S with $|S| \geq 2$, the hypersurface defined by $H_S = 0$ is nonsingular in $\mathbb{P}_L^{|S|-1}$ in the variables $X_i, i \in S$.*

The first condition of Lemma 3.1 implies that a degree d form that is Dwork-regular in $X_1, \dots, X_m$ necessarily contains a nonzero multiple of each monomial $X_1^d, \dots, X_m^d$.

Second, we verify that if a form $H \in \mathbb{Z}[X_1, \dots, X_m]$ is Dwork-regular over $\mathbb{Q}$ in the variables $X_1, \dots, X_m$, then its reduction modulo q is Dwork-regular over $\mathbb{F}_q$ in the variables $X_1, \dots, X_m$ for all but finitely many primes q ; in particular this is true for all primes $q \geq K_1$ for a finite constant $K_1 = K_1(H)$. We can describe this abstractly over any field L as follows:

Lemma 3.2. *Let L be a field and let $H(X_1, \dots, X_m) \in L[X_1, \dots, X_m]$ be Dwork-regular over L in the variables $X_1, \dots, X_m$. Then there exists a finite set S of finite places of L such that for all finite places $\mathfrak{p} \notin S$, the reduced polynomial $H \pmod{\mathfrak{p}}$ is Dwork-regular over the residue field $\mathcal{O}_S/\mathfrak{p}$.*

We reserve the precise definitions of the notion of a finite place, a residue field, and the ring $\mathcal{O}_S$ to §3.2. In the case when $L = \mathbb{Q}$, a finite place corresponds to a prime number, and the conclusion of the lemma is that once a finite number of “bad” primes are excluded, then for all remaining prime numbers q , the reduction of H modulo q is Dwork-regular over the finite field $\mathbb{F}_q$.

We next recall that a polynomial $P \in \mathbb{Z}[X_1, \dots, X_m]$ of degree $d \geq 2$ is a **Deligne** polynomial over a finite field L of characteristic q if

- (1) $q \nmid d$, and
- (2) the hypersurface defined by the leading form $P_d(X_1, \dots, X_m) = 0$ is nonsingular in $\mathbb{P}_L^{m-1}$.

(In the case that $m = 1$, (2) is replaced by $P_d(X_1) \not\equiv 0$. Recall that the leading form of a polynomial is the homogeneous part of highest degree.) A crucial fact we apply later is that after specializing one or more coordinates of a Dwork-regular form, the remaining polynomial is Deligne:

Proposition 3.3. *Let L be a finite field and let $H(X_1, \dots, X_m) \in L[X_1, \dots, X_m]$ be a homogeneous polynomial of degree d that is Dwork-regular over L in the variables $X_1, \dots, X_m$, with $\text{char } L \nmid d$. Fix $1 \leq r \leq m - 1$. Then for any constants $c_1, \dots, c_r \in L$, $H|_{X_1=c_1, \dots, X_r=c_r}$ is a Deligne polynomial in $X_{r+1}, \dots, X_m$ over L .*

In particular, we remark that by Lemma 3.1, the constants c_i in Proposition 3.3 can be 0 in L . We now turn to the proof of these results.

3.1 Proof of Lemma 3.1. We suppose H is not Dwork-regular and then show either (1) or (2) is violated. For H not Dwork-regular, there exists an $a = [a_1 : \dots : a_m] \in \mathbb{P}_L^{m-1}$ such that

$$H(a) = 0, \quad \left(X_i \frac{\partial H}{\partial X_i} \right)(a) = 0 \quad \text{for } 1 \leq i \leq m.$$

In particular, $a_j \neq 0$ for some $1 \leq j \leq m$ and so for this j , $\frac{\partial H}{\partial X_j}(a) = 0$. Define the set $S = \{j : a_j \neq 0\} \subseteq \{1, \dots, m\}$. If $|S| = 1$ then H_S is either identically zero or a monomial in one variable, say in X_j . Moreover, when evaluated at the point $a = [0 : \dots : a_j : \dots : 0] \in \mathbb{P}_L^{m-1}$, the monomial in $a_j \neq 0$ satisfies $H_S(a) = H(a) = 0$. Thus the coefficient of the monomial is zero, and $H_S \equiv 0$ so that (1) is violated. If on the other hand $|S| \geq 2$, say $S = \{i_1, \dots, i_{|S|}\}$, then let $b = [a_{i_1} : \dots : a_{i_{|S|}}] \in \mathbb{P}_L^{|S|-1}$. Then upon regarding H_S as a polynomial in $X_{i_1}, \dots, X_{i_{|S|}}$, $H_S(b) = H(a) = 0$. For each $i \in S$, when we evaluate at the point b (or a respectively),

$$\frac{\partial H_S}{\partial X_i}(b) = \frac{\partial H}{\partial X_i}(a) = 0.$$

This produces a singular point on the projective hypersurface $H_S = 0$ in $\mathbb{P}_L^{|S|-1}$, violating (2).

Finally, suppose H is Dwork-regular; we will argue that (1) and (2) must hold, by contradiction. Indeed, suppose either H_S is identically zero for some S with $|S| = 1$ or $H_S = 0$ is singular for some S with $|S| \geq 2$. Write

$$S = \{i_1, \dots, i_{|S|}\} \subseteq \{1, \dots, m\}.$$

For the case $|S| = 1$, define $a = [a_1 : \dots : a_m] \in \mathbb{P}_L^{m-1}$ by $a_j = 1$ if $j \in S$ and 0 otherwise. Note that $H(a) = H_S(a) = 0$. For $i \notin S$, the coordinate $X_i(a) = 0$ while for $i \in S$, at the point a ,

$$\frac{\partial H}{\partial X_i}(a) = \frac{\partial H_S}{\partial X_i}(a) = 0$$

since $H_S \equiv 0$. Consequently,

$$(3.1) \quad H(a) = 0, \quad \left(X_i \frac{\partial H}{\partial X_i} \right)(a) = 0 \quad \text{for } 1 \leq i \leq m,$$

which violates Dwork-regularity of H , a contradiction. On the other hand, if $|S| \geq 2$, let $b = [b_{i_1} : \dots : b_{i_{|S|}}] \in \mathbb{P}_L^{|S|-1}$ be such that

$$H_S(b) = 0, \quad \frac{\partial H_S}{\partial X_i}(b) = 0 \quad \text{for } i \in S.$$

Define $a = [a_1 : \cdots : a_m] \in \mathbb{P}_L^{m-1}$ by $a_j = b_j$ if $j \in S$ and 0 otherwise. Then $H(a) = H_S(b) = 0$. Additionally, for $i \notin S$, the coordinate $X_i(a) = 0$ while for $i \in S$,

$$(\partial H / \partial X_i)(a) = (\partial H_S / \partial X_i)(b) = 0.$$

Thus a satisfies (3.1) and violates Dwork-regularity of H , a contradiction.

3.2 Proof of Lemma 3.2. Lemma 3.2 follows by a standard type of argument, which applies a version of Nullstellensatz; we provide the projective version we apply.

Lemma 3.4. *Let L be a field. Let $I \subseteq L[X_1, \dots, X_m]$ be a homogeneous ideal. Define $Z(I) = \{x \in \mathbb{P}_L^{m-1} : f(x) = 0 \text{ for all homogeneous } f \in I\} \subseteq \mathbb{P}_L^{m-1}$. Then $Z(I)$ is the empty set if and only if $(X_1^d, \dots, X_m^d) \subseteq I$ for some d .*

Proof. Suppose $Z(I) = \emptyset$ in $\mathbb{P}_L^{m-1}$. Define the affine set

$$Z_{\mathbb{A}}(I) = \{(a_1, \dots, a_m) \in \mathbb{A}_L^m : f(a) = 0 \text{ for all homogeneous } f \in I\}.$$

Then $Z_{\mathbb{A}}(I) = (0, \dots, 0)$ and so for each i the monomial X_i vanishes on $Z_{\mathbb{A}}(I) = (0, \dots, 0)$. Then by affine Nullstellensatz [Lan02, Theorem 1.5], $X_i^{d_i} \in I$ for some d_i . For the other direction, suppose $(X_1^d, \dots, X_m^d) \subseteq I$ for some d . Let $x \in Z(I)$ so that $f(x) = 0$ for all homogeneous $f \in I$. Then in particular the monomials X_i^d vanish on x and so $x_i = 0$ for all i . This is a contradiction to x belonging to $\mathbb{P}_L^{m-1}$, so $Z(I) = \emptyset$. $\square$

Now to prove Lemma 3.2, let us first recall some terminology. For the field L , we will denote a finite place by $\mathfrak{p}$ and its associated valuation by $v_{\mathfrak{p}}$. For example, in the case we will apply, $L = \mathbb{Q}$ so if we pick a finite place (prime number) p , then the associated valuation is $v_p(x) = \max\{a \in \mathbb{Z} : p^a | x\}$ for $x \in \mathbb{Q}$; for example, $v_p(x) \geq 0$ for all primes p precisely when x is an integer. When working with polynomials with rational coefficients, it can be convenient to multiply an identity of polynomials by a sufficiently large integer (say N) to “clear denominators;” alternatively, we could work in an enlarged set of “integers” that include rational numbers with denominators only divisible by primes $p | N$. For example, we could consider rational numbers with denominators only divisible by powers of 5 and 7; we call the set of all such rationals S -integers for the set $S = \{5, 7\}$. In general let S be a finite set of finite places of a field L . An associated ring $\mathcal{O}_S$ called the S -integers is defined by $\mathcal{O}_S = \{x \in L : v_{\mathfrak{p}}(x) \geq 0 \text{ for all finite places } \mathfrak{p} \notin S\}$. Finally, for any finite place $\mathfrak{p} \notin S$ we may consider the quotient $\mathcal{O}_S / \mathfrak{p}$, which is the residue field. In the case $L = \mathbb{Q}$ where we apply Lemma 3.2, given a particular prime $p \notin S$, $\mathcal{O}_S / \mathfrak{p} = \mathcal{O}_S / p\mathcal{O}_S$ is isomorphic to $\mathcal{O}_L / \mathfrak{p} = \mathbb{Z} / p\mathbb{Z} \cong \mathbb{F}_p$ since the map $\mathcal{O}_S \rightarrow \mathbb{Z} / p\mathbb{Z}$ given by $a/b \mapsto a \pmod{p}$ is surjective, and its kernel is $p\mathcal{O}_S$.

To prove the lemma, initially define S to be the set of places $\mathfrak{p}$ such that either $v_{\mathfrak{p}}(c) < 0$ for some coefficient c of H or $v_{\mathfrak{p}}(c) > 0$ for all coefficients of H . Then $H \in \mathcal{O}_S[X_1, \dots, X_m]$ and hence $X_i \frac{\partial H}{\partial X_i} \in \mathcal{O}_S[X_1, \dots, X_m]$. Define the ideal

$$I := \left(H, X_1 \frac{\partial H}{\partial X_1}, \dots, X_m \frac{\partial H}{\partial X_m} \right) \subseteq \mathcal{O}_S[X_1, \dots, X_m].$$

Since by assumption H is Dwork-regular over L , $Z(I) = \emptyset$ in $\mathbb{P}_L^{m-1}$ where we view I as an ideal in $L[X_1, \dots, X_m]$. Then by Lemma 3.4, $(X_1^d, \dots, X_m^d) \subseteq I$ for some d . In particular for each i , there exist $Q_i, Q_{i,1}, \dots, Q_{i,m} \in L[X_1, \dots, X_m]$ such that

$$X_i^d = HQ_i + X_1 \frac{\partial H}{\partial X_1} Q_{i,1} + \dots + X_m \frac{\partial H}{\partial X_m} Q_{i,m}.$$

Now we enlarge the set S so that it includes places $\mathfrak{p}$ such that $v_{\mathfrak{p}}(c) < 0$ for some coefficient of at least one of $Q_i, Q_{i,1}, \dots, Q_{i,m}$. Then

$$(X_1^d, \dots, X_m^d) \subseteq I \subseteq \mathcal{O}_S[X_1, \dots, X_m].$$

For $\mathfrak{p} \notin S$,

$$X_i^d \equiv HQ_i + X_1 \frac{\partial H}{\partial X_1} Q_{i,1} + \dots + X_m \frac{\partial H}{\partial X_m} Q_{i,m} \pmod{\mathfrak{p}}$$

and so $(X_1^d, \dots, X_m^d) \pmod{\mathfrak{p}} \subseteq (H, X_1 \frac{\partial H}{\partial X_1}, \dots, X_m \frac{\partial H}{\partial X_m}) \pmod{\mathfrak{p}}$; that is to say, working over the residue field $L' = \mathcal{O}_S/\mathfrak{p}$ and viewing the ideals now in $L'[X_1, \dots, X_m]$, the inclusion $(X_1^d, \dots, X_m^d) \subseteq (H, X_1 \frac{\partial H}{\partial X_1}, \dots, X_m \frac{\partial H}{\partial X_m})$ holds. We apply Lemma 3.4 again, now with $L' = \mathcal{O}_S/\mathfrak{p}$, to deduce that $H, X_1 \frac{\partial H}{\partial X_1}, \dots, X_m \frac{\partial H}{\partial X_m}$ have no common zeros in $\mathbb{P}_{\mathcal{O}_S/\mathfrak{p}}^{m-1}$ and hence by definition H is Dwork-regular over $\mathcal{O}_S/\mathfrak{p}$ in the variables $X_1, \dots, X_m$.

3.3 Proof of Proposition 3.3.

Let $c_1, \dots, c_r \in L$ be given, and let

$$G \in L[X_{r+1}, \dots, X_m]$$

denote the polynomial $H(c_1, \dots, c_r, X_{r+1}, \dots, X_m)$. Note that G has degree $d := \deg H$; by the remark following Lemma 3.1, H must contain a nonzero multiple of X_j^d for each j and in particular for $r+1 \leq j \leq m$, so $\deg G = d$. In particular, $\text{char } L \nmid \deg G$. Next, H can be written as $H = G_d F_0 + G_{d-1} F_1 + \dots + G_0 F_d$ where $G_i \in L[X_{r+1}, \dots, X_m]$ is homogeneous of degree i and $F_i \in L[X_1, \dots, X_r]$ is homogeneous of degree i (in particular $F_0 \equiv 1, G_0 \equiv 1$). The leading form of G is then precisely $G_d(X_{r+1}, \dots, X_m)$, and $G_d = H_S$ for $S = \{r+1, \dots, m\}$, in the notation of Lemma 3.1. Thus the leading form of G defines a nonsingular hypersurface in $\mathbb{P}_L^{m-r-1}$ (or is not the zero polynomial, if $m-r=1$), by Lemma 3.1, completing the proof.

3.4 A lower bound for a partial derivative. When constructing counterexamples, it will be important to find a lower bound for a partial derivative of the leading form P_k of the polynomial symbol. (Explicitly, this is used to guarantee that for each x in a small neighborhood of the origin in $\mathbb{R}^n$, a choice of t meets all the requirements of (5.4) simultaneously.) As a motivating example, in the special case where the symbol has diagonal leading form $P_k(X) = X_1^k + \cdots + X_n^k$, one immediately has $(\partial_1 P_k)(R, X_2, \dots, X_n) \gg_k R^{k-1}$ for all $(X_2, \dots, X_n) \in \mathbb{R}^{n-1}$, which suffices for our application; here and throughout the paper, R is a large parameter that will tend to infinity at the end of the argument. If P_k has higher intertwining rank, we must proceed differently; the following lemma provides an integral point where the partial derivative is sufficiently large.

Lemma 3.5. *Let $P_k \in \mathbb{Q}[X_1, \dots, X_n]$ be a Dwork-regular form of degree $k \geq 2$ with intertwining rank $r < n$; without loss of generality, say X_1 intertwines with $X_1, X_2, \dots, X_r$ but not with $X_{r+1}, \dots, X_n$. Then there exists a tuple*

$$(M_1, \dots, M_r) \in \mathbb{Z}^r,$$

with $M_i \geq 1$ for all i , such that for all $R \geq 1$,

$$|(\partial_1 P_k)(M_1 R, M_2 R, \dots, M_r R, X_{r+1}, \dots, X_n)| \gg R^{k-1},$$

uniformly in $(X_{r+1}, \dots, X_n) \in \mathbb{R}^{n-r}$.

Let $X' = (X_2, \dots, X_n)$ and rewrite P_k in terms of its coefficients c_β as

$$P_k(X_1, \dots, X_n) = \sum_{j=0}^k X_1^{k-j} \sum_{|\beta|=j} c_\beta X'^\beta$$

where β is a multi-index $(\beta_2, \dots, \beta_n)$ with order $|\beta| = \beta_2 + \cdots + \beta_n$, and

$$X'^\beta = X_2^{\beta_2} \cdots X_n^{\beta_n}.$$

By the hypothesis that X_1 does not intertwine with $X_{r+1}, \dots, X_n$, for each $|\beta| < k$, $c_\beta = 0$ for all β with $\beta_\ell > 0$ for some $\ell > r$. Consequently, the derivative $\partial_1 P_k$ is a function of $X_1, \dots, X_r$ and is independent of X_j for $j > r$:

$$(3.2) \quad (\partial_1 P_k)(X_1, \dots, X_n) = \sum_{j=0}^{k-1} (k-j) X_1^{k-j-1} \sum_{|\beta|=j} c_\beta X_2^{\beta_2} \cdots X_r^{\beta_r}.$$

Then for any value of a parameter $R \geq 1$, after plugging in any $(M_1, \dots, M_r) \in \mathbb{Z}^r$,

$$\begin{aligned} (\partial_1 P_k)(M_1 R, \dots, M_r R, X_{r+1}, \dots, X_n) \\ &= \sum_{j=0}^{k-1} (k-j)(M_1 R)^{k-j-1} \sum_{|\beta|=j} c_\beta (M_2 R)^{\beta_2} \dots (M_r R)^{\beta_r} \\ &= R^{k-1} \sum_{j=0}^{k-1} (k-j) M_1^{k-j-1} \sum_{|\beta|=j} c_\beta M_2^{\beta_2} \dots M_r^{\beta_r}. \end{aligned}$$

Thus we can conclude $|(\partial_1 P_k)(M_1 R, \dots, M_r R, X_{r+1}, \dots, X_n)| \gg R^{k-1}$ (uniformly in $X_{r+1}, \dots, X_n$) as long as

$$\sum_{j=0}^{k-1} (k-j) M_1^{k-j-1} \sum_{|\beta|=j} c_\beta M_2^{\beta_2} \dots M_r^{\beta_r} \neq 0.$$

From (3.2), we see that this condition is equivalent to $(\partial_1 P_k)(M_1, \dots, M_r) \neq 0$ where we view $\partial_1 P_k$ as an element of $\mathbb{Q}[X_1, \dots, X_r]$. Hence it remains to prove that there exists an integral point $(M_1, \dots, M_r) \in \mathbb{Z}^r$ with $M_i \geq 1$ for all i such that $(\partial_1 P_k)(M_1, \dots, M_r) \neq 0$.

First we note that $\partial_1 P_k$ is not the zero polynomial in $X_1, \dots, X_r$; this follows from Dwork-regularity, since by (1) of Lemma 3.1, P_k contains a term cX_1^k for some nonzero $c \in \mathbb{Q}$. If $r = 1$, $(\partial_1 P_k)(X_1)$ has at most $k - 1$ roots, so the claim holds. For $r \geq 2$, by a trivial upper bound (see, e.g., [BCLP22, Lemma 10.1] for a standard statement), for any $B \geq 1$, there are at most $\ll B^{r-1}$ integral solutions in $[1, B]^r$ to $(\partial_1 P_k)(X_1, \dots, X_r) = 0$. Since there are B^r integral points in that box, there exists a sufficiently large B such that there is an integral point, say $(M_1, \dots, M_r) \in [1, B]^r$ with $(\partial_1 P_k)(M_1, \dots, M_r) \neq 0$. The lemma is proved.

3.5 Invariance under $\mathrm{GL}_n(\mathbb{Q})$: nonsingular to Dwork-regular. Let

$$P \in \mathbb{R}[X_1, \dots, X_n]$$

be given, with leading form $P_k \in \mathbb{Q}[X_1, \dots, X_n]$. Now let $A \in \mathrm{GL}_n(\mathbb{Q})$ and let Q denote the polynomial after changing variables, i.e. $Q(\eta_1, \dots, \eta_n) = P(\xi_1, \dots, \xi_n)$ with $\xi_i = \sum_j a_{ij} \eta_j$ or equivalently $\eta = A^{-1} \xi$. (In particular, by [Kat09, Lemma 3.1], if P_k is nonsingular, there exists a choice of A so that $Q(\eta_1, \dots, \eta_n)$ is Dwork-regular over $\mathbb{Q}$ in $\eta_1, \dots, \eta_n$.) Then for any fixed $t > 0$,

$$T_t^P f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i(\xi \cdot x + P(\xi)t)} d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(A\eta) e^{i((A\eta) \cdot x + Q(\eta)t)} |A| d\eta.$$

Now define a function g such that $\hat{g}(\eta) = |A|\hat{f}(A\eta)$ where $|A|$ denotes the determinant of A ; equivalently $g(x) = f((A^{-1})^T x)$. Then

$$(T_t^P f)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(\eta) e^{i(\eta \cdot A^T x + Q(\eta)t)} d\eta = (T_t^Q g)(A^T x).$$

Now let $\tau : \mathbb{R}^n \rightarrow (0, 1)$ be a measurable stopping-time function, so that proving $f(x) \mapsto \sup_{0 < t < 1} |T_t^P f(x)|$ is bounded from H^s to L_{loc}^1 is equivalent to proving that $f(x) \mapsto |T_{\tau(x)}^P f(x)|$ is bounded from H^s to L_{loc}^1 , independent of the function τ . For a given stopping-time function τ , the computation above shows that

$$\|T_{\tau(x)}^P f(x)\|_{L^1(B_n(0,1))} = \int_{B_n(0,1)} |(T_{\tau(x)}^Q g)(A^T x)| dx = |A^T|^{-1} \int_{A^T B_n(0,1)} |(T_{\sigma(u)}^Q g)(u)| du,$$

for the stopping-time $\sigma(u) = \tau((A^T)^{-1}u)$. Thus

$$(3.3) \quad \|T_{\tau(x)}^P f(x)\|_{L^1(B_n(0,1))} \leq C_s \|f\|_{H^s}$$

is true for all $f \in H^s$, uniformly over all stopping-time functions, if and only if

$$(3.4) \quad \|T_{\sigma(x)}^Q g(x)\|_{L^1(A^T B_n(0,1))} \leq C_s C'_A \|g\|_{H^s}$$

is true for all $g \in H^s$, uniformly over all stopping-time functions. (Of course, $\|g\|_{H^s} \ll_A \|f\|_{H^s} \ll_A \|g\|_{H^s}$.) In particular, if we show for a given real $s > 0$ that there is no constant C_s such that

$$\left\| \sup_{0 < t < 1} |T_t^Q g(x)| \right\|_{L^1(B_n(0,1))} \leq C_s \|g\|_{H^s}$$

for all $g \in H^s$, then (3.4) fails for all constants C_s, C'_A and consequently (3.3) fails for all constants C_s . This invariance under $\text{GL}_n(\mathbb{Q})$ changes of variables shows that it is reasonable to study maximal operators for the class of Dwork-regular polynomial symbols in place of the class of nonsingular polynomial symbols, as we do. (The invariance demonstrated above holds for $\text{GL}_n(\mathbb{R})$ as well.)

3.6 Verification of Corollary 1.2. Let, as in the hypothesis, $P_k(X_1, \dots, X_n)$ be decomposable over $\mathbb{Q}$, so that after an appropriate $\text{GL}_n(\mathbb{Q})$ change of variables,

$$P_k(X_1, \dots, X_n) = Q_1(X_1, \dots, X_m) + Q_2(X_{m+1}, \dots, X_n)$$

holds for some $1 \leq m < n$. Note that P_k is nonsingular as a function of $X_1, \dots, X_n$ if and only if Q_1 is nonsingular as a function of $X_1, \dots, X_m$ and Q_2 is nonsingular as a function of $X_{m+1}, \dots, X_n$. By the disjointness of the variables in Q_1 and Q_2 , we can apply $\text{GL}_m(\mathbb{Q})$ (respectively $\text{GL}_{n-m}(\mathbb{Q})$) changes of variables separately

to Q_1 (respectively Q_2) so that both become Dwork-regular over $\mathbb{Q}$. This provides a block diagonal $\mathrm{GL}_n(\mathbb{Q})$ transformation that makes P_k Dwork-regular and still decomposed into a form in $X_1, \dots, X_m$ and a second form in $X_{m+1}, \dots, X_n$. In particular, its intertwining rank remains bounded above by $r \leq \lfloor n/2 \rfloor$. Since the parameter $\delta(n, k, r)$ in Theorem 1.1 is decreasing as a function of r , then $\delta(n, k, r) \geq \delta(n, k, n/2)$ for $r \leq \lfloor n/2 \rfloor$, and this verifies the corollary.

3.7 Dispersivity. A description of the broad principle of dispersion, in the context of an initial value problem like (1.2), can be found for example in [Pal97, §3.5] or [Tao06, Principle 2.1]. The following lemma verifies that for each real symbol considered in the main theorem, (1.2) is a dispersive PDE, in the sense of the criterion presented in [KPV91, Theorem 4.1].

Lemma 3.6. *Let $P(X_1, \dots, X_n) \in \mathbb{R}[X_1, \dots, X_n]$ be a polynomial of degree $k \geq 2$ such that its leading form $P_k(X_1, \dots, X_n) \in \mathbb{Q}[X_1, \dots, X_n]$ is Dwork-regular over $\mathbb{Q}$ in the variables $X_1, \dots, X_n$. Then $\nabla P_k(x_1, \dots, x_n) \neq 0$ for all $(x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$. Further, there exists a finite $M \in \mathbb{N}$ such that for each $i = 1, \dots, n$, for all $(c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n) \in \mathbb{R}^{n-1}$ and $c' \in \mathbb{R}$, the equation*

$$(3.5) \quad P(c_1, \dots, c_{i-1}, x, c_{i+1}, \dots, c_n) = c',$$

has at most M solutions.

Proof. Since P_k is Dwork-regular over $\mathbb{Q}$, then P_k is nonsingular over $\mathbb{Q}$, so that $\nabla P_k(X_1, \dots, X_n) = 0$ has no nontrivial solutions over $\overline{\mathbb{Q}}$; since P_k has rational coefficients, this implies that there are no nontrivial solutions over $\mathbb{R}$ either. Fix an $1 \leq i \leq n$. By the remark following Lemma 3.1, the leading form P_k contains a term that is a nonzero multiple of X_i^k . Hence $P(c_1, \dots, c_{i-1}, x, c_{i+1}, \dots, c_n) - c'$ contains a nonzero multiple of x^k , so that (3.5) has at most k solutions. $\square$

4 Upper and lower bounds for exponential sums

Now we prove three results about exponential sums involving a Deligne polynomial, which by Proposition 3.3 apply to Dwork-regular forms over $\mathbb{Q}$ with fixed variables, for all sufficiently large prime moduli. First, we prove in Proposition 4.1 that complete exponential sums with a Deligne polynomial in the phase are often “large”. Second, we prove in Proposition 4.5 that incomplete exponential sums are “not too large” (and in particular, only a logarithmic factor larger than complete exponential sums). Third, in Proposition 4.6 we approximate an exponential sum with real coefficients by complete exponential sums, up to an acceptable error.

4.1 A lower bound for many complete exponential sums.

Proposition 4.1. *Let $Q_k(X_1, \dots, X_m) \in \mathbb{Z}[X_1, \dots, X_m]$ be a polynomial of degree $k \geq 2$ and suppose that for all primes $q \geq K_1(Q_k)$, the reduction of Q_k modulo q is a Deligne polynomial over $\mathbb{F}_q$. Define for each prime q and $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q^m$,*

$$\mathbf{T}(a, b; q) := \sum_{x \pmod{q^m}} e\left(\frac{2\pi}{q}(aQ_k(x) + b \cdot x)\right).$$

Then there exist constants $0 < \alpha_1, \alpha_2 < 1$ with $\alpha_2 = \alpha_2(k, m)$, and a constant $K_2(k, m)$ such that for every prime $q > \max\{k, K_1(Q_k), K_2(k, m)\}$, there exist at least $\alpha_2 q^{m+1}$ choices of $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q^m$ such that

$$(4.1) \quad \alpha_1 q^{m/2} \leq |\mathbf{T}(a, b; q)| \leq (k-1)^m q^{m/2}.$$

In fact, we may take $\alpha_1 = 1/2$ and any $\alpha_2 \leq (1/8)(k-1)^{-2m}$. The Weil–Deligne bound is a key ingredient in the proof, which we now recall, following [Del74, Thm. 8.4] as stated in [IK04, Thm. 11.43].

Lemma 4.2. *Let $f(X_1, \dots, X_m) \in \mathbb{Z}[X_1, \dots, X_m]$ be a Deligne polynomial of degree $k \geq 2$ over $\mathbb{F}_q$ for q prime. Then*

$$\left| \sum_{\mathbf{x} \in \mathbb{F}_q^m} e\left(\frac{2\pi}{q}f(x_1, \dots, x_m)\right) \right| \leq (k-1)^m q^{m/2}.$$

The proof of the proposition is a mild variation on [ACP23, Prop. 2.2]. We first claim that

$$(4.2) \quad \sum_{\substack{a \pmod{q} \\ b \pmod{q^m}}} |\mathbf{T}(a, b; q)|^2 = q^{2m+1}.$$

The left-hand side may be expanded as

$$\sum_{x, \tilde{x}} \sum_{a \pmod{q}} e\left(\frac{2\pi a}{q}(Q_k(x) - Q_k(\tilde{x}))\right) \sum_{b \pmod{q^m}} e\left(\frac{2\pi}{q}b \cdot (x - \tilde{x})\right).$$

The innermost sum vanishes unless $x_i \equiv \tilde{x}_i \pmod{q}$ for all $1 \leq i \leq m$; in this case the left-hand side evaluates to q^{2m+1} , as claimed. Next we observe that since Q_k is Deligne over $\mathbb{F}_q$ of degree $k \geq 2$, so is $aQ_k(x) + b \cdot x$ for $a \neq 0 \in \mathbb{F}_q$, so we can apply Lemma 4.2 to bound $\mathbf{T}(a, b; q)$; for $a = 0$ the sum vanishes unless $b = 0$ and we address this below. Since (4.2) shows that $\mathbf{T}(a, b; q)$ is of size $q^{m/2}$ on average, and the Weil–Deligne bound shows it cannot ever be much larger, we can deduce that most of the time it is not much smaller either. Precisely, suppose for a given

pair $0 < \alpha_1, \alpha_2 < 1$ there are $< \alpha_2 q^{m+1}$ choices of $(a, b) \pmod{q^{m+1}}$ such that $|\mathbf{T}(a, b; q)| \geq \alpha_1 q^{m/2}$. Then

$$\sum_{\substack{a \pmod{q} \\ b \pmod{q^m}}} |\mathbf{T}(a, b; q)|^2 = q^{2m} + \sum_{\substack{a \neq 0 \\ |\mathbf{T}(a, b; q)| \geq \alpha_1 q^{m/2}}} |\mathbf{T}(a, b; q)|^2 + \sum_{\substack{a \neq 0 \\ |\mathbf{T}(a, b; q)| < \alpha_1 q^{m/2}}} |\mathbf{T}(a, b; q)|^2.$$

Here the first term is the contribution from $a \equiv 0$. The first sum on the right-hand side is $< \alpha_2 q^{m+1} ((k-1)^m q^{m/2})^2$; the second sum on the right-hand side is $< q^{m+1} (\alpha_1 q^{m/2})^2$. Hence

$$\sum_{\substack{a \pmod{q} \\ b \pmod{q^m}}} |\mathbf{T}(a, b; q)|^2 < (q^{-1} + \alpha_2 (k-1)^{2m} + \alpha_1^2) q^{2m+1} < (1/3 + \alpha_2 (k-1)^{2m} + \alpha_1^2) q^{2m+1}.$$

This is $< q^{2m+1}$, contradicting (4.2), for sufficiently small α_1, α_2 ; for example, we may take $\alpha_1 = 1/2$ and any $\alpha_2 \leq (1/4)(k-1)^{-2m}$. For such α_1, α_2 , there must then be $\geq \alpha_2 q^{m+1}$ choices of $(a, b) \pmod{q^{m+1}}$ for which the left-hand inequality in (4.1) holds.

Finally, the only case in which $|\mathbf{T}(a, b; q)| > (k-1)^m q^{m/2}$ is when $a = 0 \in \mathbb{F}_q$ (in which case $\mathbf{T}(a, b; q)$ is a linear exponential sum which vanishes unless $b = 0 \in \mathbb{F}_q^m$). Thus aside from $(a, b) = (0, 0)$, all those (a, b) satisfying the lower bound in (4.1) also satisfy the upper bound. We can summarize this by saying that at least $\alpha_2 q^{m+1}$ choices satisfy both bounds, with the modified choice $\alpha_2 = (1/8)(k-1)^{-2m}$, as long as we assume q is sufficiently large that $(1/4)(k-1)^{-2m} q^{m+1} \geq 2$, which is true for all $q \geq K_2$, for an appropriate choice of $K_2 = K_2(k, m)$. The proposition is proved.

4.2 An upper bound for incomplete exponential sums. To show the maximal operator associated to $T_t^P f$ is large, we will repeatedly approximate integrals and sums by complete exponential sums. We first state general formulas for partial summation and partial integration; these are proved simply by iteration one coordinate at a time, and we omit the proofs. Given a subset $J \subseteq \{1, \dots, n\}$, we define $I = \{1, \dots, n\} \setminus J$ and use the notation $(\mathbf{x}_{(J)}, \mathbf{x}_{(I)}) \in \mathbb{R}^n$ to indicate $\mathbf{x}_{(J)} \in \mathbb{R}^{|J|}$ is indexed by $j \in J$ and $\mathbf{x}_{(I)} \in \mathbb{R}^{n-|J|}$ is indexed by $j \in \{1, \dots, n\} \setminus J$.

Lemma 4.3. *Let $a(\mathbf{m})$ be a sequence of complex numbers indexed by $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$. Let $h(\mathbf{y})$ be a $C^{(n)}$ function on $\mathbb{R}^n$ such that for every $\kappa = (\kappa_1, \dots, \kappa_n) \in \{0, 1\}^n$, there exists a positive real number B_κ such that for*

$$\partial_\kappa h(\mathbf{y}) := \frac{\partial^{\kappa_1 + \dots + \kappa_n}}{\partial y_1^{\kappa_1} \dots \partial y_n^{\kappa_n}} h(\mathbf{y}),$$

we have $|\partial_\kappa h(\mathbf{y})| \leq B_\kappa$ uniformly in $\mathbf{y} \in [\mathbf{M}, \mathbf{M} + \mathbf{N}]$, where $\mathbf{M} = (M, M, \dots, M)$ and $\mathbf{N} = (N_1, N_2, \dots, N_n)$. Then

$$\sum_{\mathbf{M} \leq \mathbf{m} \leq \mathbf{M} + \mathbf{N}} a(\mathbf{m}) e(h(\mathbf{m})) = e(h(\mathbf{M} + \mathbf{N})) \sum_{\mathbf{M} \leq \mathbf{m} \leq \mathbf{M} + \mathbf{N}} a(\mathbf{m}) + E$$

where

$$|E| \ll_n \sup_{\substack{J \subseteq \{1, \dots, n\} \\ |J| \geq 1}} \left\{ \prod_{j \in J} N_j \cdot \sup_{\mathbf{u}_{(J)} \leq \mathbf{N}_{(J)}} |A(\mathbf{u}_{(J)}, \mathbf{N}_{(J)})| \cdot \sup_{1 \leq \ell \leq |J|} \sup_{\substack{\alpha_1, \dots, \alpha_\ell \in \{0, 1\}^n \\ \alpha_1 + \dots + \alpha_\ell = \mathbf{1}_J}} \prod_{i=1}^{\ell} B_{\alpha_i} \right\},$$

in which

$$A(\mathbf{u}_{(J)}, \mathbf{N}_{(J)}) = \sum_{\substack{\mathbf{M}_{(J)} \leq \mathbf{m}_{(J)} \leq \mathbf{M}_{(J)} + \mathbf{N}_{(J)} \\ \mathbf{M}_{(J)} \leq \mathbf{m}_{(J)} \leq \mathbf{M}_{(J)} + \mathbf{u}_{(J)}}} a(\mathbf{m}_{(J)}, \mathbf{m}_{(I)}).$$

Lemma 4.4. *Let $a < b$ be real numbers. Let $f(\mathbf{t})$ be an integrable function supported on $[a, b]^n$ and let $h(\mathbf{t})$ be a $C^{(n)}$ function on $\mathbb{R}^n$ such that for every $\kappa = (\kappa_1, \dots, \kappa_n) \in \{0, 1\}^n$, there exists a positive real number B_κ such that $|\partial_\kappa h(\mathbf{t})| \leq B_\kappa$, uniformly in $\mathbf{t} \in [a, b]^n$. Then*

$$\int_{[a, b]^n} f(\mathbf{t}) e(h(\mathbf{t})) d\mathbf{t} = e(h(\mathbf{b})) \int_{[a, b]^n} f(\mathbf{t}) d\mathbf{t} + E$$

where

$$|E| \ll_n \sup_{\substack{J \subseteq \{1, \dots, n\} \\ |J| \geq 1}} \left\{ (b - a)^{|J|} \sup_{\mathbf{u}_{(J)} \leq \mathbf{b}_{(J)}} |F(\mathbf{u}_{(J)}, \mathbf{b}_{(I)})| \cdot \sup_{1 \leq \ell \leq |J|} \sup_{\substack{\alpha_1, \dots, \alpha_\ell \in \{0, 1\}^n \\ \alpha_1 + \dots + \alpha_\ell = \mathbf{1}_J}} \prod_{i=1}^{\ell} B_{\alpha_i} \right\},$$

in which

$$F(\mathbf{u}_{(J)}, \mathbf{b}_{(I)}) = \int_{[a, b]^{|I|}} \int_{[a, u_j]} \dots \int_{[a, u_j]} f(\mathbf{t}_{(J)}, \mathbf{t}_{(I)}) d\mathbf{t}_{(J)} d\mathbf{t}_{(I)}.$$

We bound an incomplete exponential sum by completing the sum and applying the Weil bound:

Proposition 4.5. *Let $Q_k(X_1, \dots, X_n) \in \mathbb{Z}[X_1, \dots, X_n]$ be a Deligne polynomial of degree $k \geq 2$ over $\mathbb{F}_q$ for a prime q . Let $J \subseteq \{1, \dots, n\}$ and $I = \{1, \dots, n\} \setminus J$. Then for $\mathbf{H} = (H_1, \dots, H_n)$ with $1 \leq H_i \leq q$,*

$$\left| \sum_{\substack{\mathbf{1}_{(J)} \leq \mathbf{m}_{(J)} \leq \mathbf{q}_{(J)}, \\ \mathbf{1}_{(I)} \leq \mathbf{m}_{(I)} \leq \mathbf{H}_{(I)}}} e\left(\frac{2\pi}{q} Q_k(\mathbf{m}_{(J)}, \mathbf{m}_{(I)})\right) \right| \ll_{k,n} q^{n/2} (\log q)^{|J|}.$$

The sum on the left-hand side may be written as

$$\sum_{\mathbf{1}_{(J)} \leq \mathbf{m}_{(J)} \leq \mathbf{q}_{(J)}} \sum_{\mathbf{1}_{(I)} \leq \mathbf{a}_{(I)} \leq \mathbf{q}_{(I)}} e\left(\frac{2\pi}{q} Q_k(\mathbf{m}_{(J)}, \mathbf{a}_{(I)})\right) \sum_{\substack{\mathbf{1}_{(I)} \leq \mathbf{m}_{(I)} \leq \mathbf{H}_{(I)}, \\ \mathbf{m}_{(I)} \equiv \mathbf{a}_{(I)} \pmod{q}^{(I)}}} 1.$$

By the identity

$$\mathbf{1}_{m \equiv a \pmod{q}} = \frac{1}{q} \sum_{1 \leq h \leq q} e\left(\frac{2\pi}{q} h \cdot (m - a)\right)$$

we can expand the sum as

$$\frac{1}{q^{|I|}} \sum_{\mathbf{h}_{(I)} \leq \mathbf{q}_{(I)}} \left(\sum_{\substack{\mathbf{m}_{(J)} \leq \mathbf{q}_{(J)}, \\ \mathbf{a}_{(I)} \leq \mathbf{q}_{(I)}}} e\left(\frac{2\pi}{q} (Q_k(\mathbf{m}_{(J)}, \mathbf{a}_{(I)}) - \mathbf{h}_{(I)} \cdot \mathbf{a}_{(I)})\right) \sum_{\mathbf{m}_{(I)} \leq \mathbf{H}_{(I)}} e\left(\frac{2\pi}{q} \mathbf{h}_{(I)} \cdot \mathbf{m}_{(I)}\right) \right).$$

Since $Q_k(X_1, \dots, X_n)$ is a Deligne polynomial of degree $k \geq 2$, so is $Q_k(X_1, \dots, X_n) - \mathbf{h}_{(I)} \cdot \mathbf{X}_{(I)}$ so by Lemma 4.2, the complete sum over $\mathbf{m}_{(J)}, \mathbf{a}_{(I)}$ is bounded above by $(k-1)^n q^{n/2}$. For the remaining double sum over $\mathbf{h}_{(I)}$ and $\mathbf{m}_{(I)}$, we recall that for each $h, H \geq 1$,

$$\sum_{1 \leq m \leq H} e(2\pi h m / q) \ll \min\{H, \|h/q\|^{-1}\}.$$

(Here $\|t\|$ temporarily indicates the distance from t to the nearest integer.) By considering cases $1 \leq h \leq q/2$ and $q/2 < h \leq q$, note that

$$\sum_{1 \leq h \leq q} \min\{H, \|h/q\|^{-1}\} \ll q \log q.$$

Hence the double sum over $\mathbf{h}_{(I)}$ and $\mathbf{m}_{(I)}$ is bounded above by $(q \log q)^{|I|}$. In total, the sum considered in the lemma is thus bounded by $\ll (k-1)^n q^{n/2} (\log q)^{|I|}$, and the proof is complete.

4.3 Approximation of a sum with real coefficients by complete sums.

We next approximate an exponential sum with real coefficients in the linear term by complete exponential sums with prime moduli.

Proposition 4.6. *Let $Q_k(X_1, \dots, X_m) \in \mathbb{Z}[X_1, \dots, X_m]$ be a Deligne polynomial of degree $k \geq 2$ over $\mathbb{F}_q$ for a prime q . Let $1 \leq a < q$. Let $V \geq 0$ and $y \in \mathbb{R}^m$, and suppose that for each $1 \leq i \leq m$ there exists $1 \leq b_i \leq q$ such that*

$$|y_i - 2\pi b_i / q| < V.$$

Then for any $\mathbf{M} = (M_1, \dots, M_m) \in \mathbb{R}^m$ and $\mathbf{N} = (N_1, \dots, N_m) \in \mathbb{R}^m$ with $0 \leq N_i \leq N$,

$$\left| \sum_{\mathbf{M} \leq \mathbf{m} \leq \mathbf{M} + \mathbf{N}} e\left(\frac{2\pi a}{q} Q_k(\mathbf{m}) + y \cdot \mathbf{m}\right) \right| = \prod_{i=1}^m \left\lfloor \frac{N_i}{q} \right\rfloor \cdot \left| \sum_{\mathbf{m} \pmod{q^m}} e\left(\frac{2\pi}{q} (a Q_k(\mathbf{m}) + b \cdot \mathbf{m})\right) \right| + E$$

where, under the assumption $VN \leq 1$,

$$|E| \ll_{k,m} VN \cdot \prod_{i=1}^m \left\lfloor \frac{N_i}{q} \right\rfloor \cdot q^{m/2} + \sup_{|J| < m} \prod_{j \in J} \left\lfloor \frac{N_j}{q} \right\rfloor \cdot q^{m/2} (\log q)^{m-|J|}.$$

Remark 4.7. In our later application we will take $m = n - r$ and coordinates $(m_{r+1}, \dots, m_n) \in \mathbb{Z}^{n-r}$. Set $N = \max\{N_{r+1}, \dots, N_n\}$. The bound for $|E|$ is increasing as a function N_j so we can replace each N_j by N . Recalling the hypothesis $VN \leq 1$, the bound for $|E|$ can be crudely estimated by

$$|E| \ll_{k,n,r} \left\lfloor \frac{N}{q} \right\rfloor^{n-r} q^{(n-r)/2} \left(VN + \left\lfloor \frac{N}{q} \right\rfloor^{-1} (\log q)^{n-r} \right).$$

We need the error to be at most a small proportion (say half the size) of the main term. To achieve this, we will choose $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q^{n-r}$ so that the exponential sum in the main term is large (using Proposition 4.1) and we will impose conditions that force $V \leq d_0 N^{-1}$ for a constant $d_0 < 1$ as small as we like, and $N/q \gg q^{\Delta_0}$ for some $\Delta_0 > 0$ (see §6).

To prove the proposition, we apply partial summation with respect to $\mathbf{m}$, following Lemma 4.3, with the function $h(\mathbf{m}) = (y - \frac{2\pi}{q}b) \cdot \mathbf{m}$. Since this is linear, for a multi-index $\alpha \in \{0, 1\}^n$, the ∂_α partial derivative as a function of $\mathbf{m}$ vanishes unless $|\alpha| = 1$, say $\partial_\alpha = \partial_i$, in which case

$$\left| \partial_i \left(\left(y - \frac{2\pi}{q}b \right) \cdot \mathbf{m} \right) \right| = |y_i - 2\pi b_i/q| < V.$$

Thus using the notation of the lemma, for any nonempty subset $J \subseteq \{1, \dots, m\}$, we only need to consider the following expression in the case each $|\alpha_i| = 1$ (so that $\ell = |J|$):

$$\sup_{1 \leq \ell \leq |J|} \sup_{\substack{\alpha_1, \dots, \alpha_\ell \in \{0, 1\}^n \\ \alpha_1 + \dots + \alpha_\ell = \mathbf{1}_J}} \prod_{i=1}^{\ell} B_{\alpha_i} = \prod_{i=1}^{|J|} V = V^{|J|}.$$

Now apply partial summation to see that

$$\begin{aligned} (4.3) \quad & \sum_{\mathbf{M} \leq \mathbf{m} \leq \mathbf{M} + \mathbf{N}} e\left(\frac{2\pi a}{q} Q_k(\mathbf{m}) + y \cdot \mathbf{m}\right) \\ &= e\left(\left(y - \frac{2\pi}{q}b\right) \cdot (\mathbf{M} + \mathbf{N})\right) \sum_{\mathbf{M} \leq \mathbf{m} \leq \mathbf{M} + \mathbf{N}} e\left(\frac{2\pi}{q} (aQ_k(\mathbf{m}) + b \cdot \mathbf{m})\right) + E_1 \end{aligned}$$

where E_1 is dominated by

$$\sup_{|J| \geq 1} V^{|J|} \prod_{j \in J} N_j \cdot \sup_{\mathbf{u}_{(J)} \leq \mathbf{N}_{(J)}} \left| \sum_{\substack{\mathbf{M}_{(I)} \leq \mathbf{m}_{(I)} \leq \mathbf{M}_{(I)} + \mathbf{N}_{(I)}, \\ \mathbf{M}_{(J)} \leq \mathbf{m}_{(J)} \leq \mathbf{M}_{(J)} + \mathbf{u}_{(J)}}} e\left(\frac{2\pi}{q} (aQ_k(\mathbf{m}_{(J)}, \mathbf{m}_{(I)}) + b \cdot (\mathbf{m}_{(J)}, \mathbf{m}_{(I)}))\right) \right|$$

with nonempty $J \subseteq \{1, \dots, m\}$ and $I = \{1, \dots, m\} \setminus J$. The first factor is $\leq (VN)^{|J|} \leq VN \leq 1$. We may estimate the sum in the error E_1 by first breaking it into a main term of as many complete sums (complete in all m coordinates)

as possible, that is,

$$\prod_{i \in I} \left\lfloor \frac{N_i}{q} \right\rfloor \cdot \prod_{j \in J} \left\lfloor \frac{u_j}{q} \right\rfloor \cdot \sum_{\mathbf{m} \pmod{q^m}} e\left(\frac{2\pi}{q}(aQ_k(\mathbf{m}_{(J)}, \mathbf{m}_{(I)}) + b \cdot (\mathbf{m}_{(J)}, \mathbf{m}_{(I)}))\right),$$

plus an error term of incomplete sums, that is, of the form

$$\sum_{\substack{\bar{I} \subseteq I, \bar{J} \subseteq J, \\ |\bar{I} \cup \bar{J}| < m}} \prod_{i \in \bar{I}} \left\lfloor \frac{N_i}{q} \right\rfloor \cdot \prod_{j \in \bar{J}} \left\lfloor \frac{u_j}{q} \right\rfloor \cdot \sum^* e\left(\frac{2\pi}{q}(aQ_k(\mathbf{m}_{(J)}, \mathbf{m}_{(I)}) + b \cdot (\mathbf{m}_{(J)}, \mathbf{m}_{(I)}))\right)$$

in which the starred sum is over $\mathbf{1} \leq \mathbf{m}_{(\bar{I})} \leq \mathbf{q}_{(\bar{I})}$, $\mathbf{1} \leq \mathbf{m}_{(I \setminus \bar{I})} \leq \mathbf{H}_{(I \setminus \bar{I})}$, $\mathbf{1} \leq \mathbf{m}_{(\bar{J})} \leq \mathbf{q}_{(\bar{J})}$, $\mathbf{1} \leq \mathbf{m}_{(J \setminus \bar{J})} \leq \mathbf{H}_{(J \setminus \bar{J})}$, for some $\mathbf{H} = (H_1, \dots, H_m)$ with $1 \leq H_i < q$. Since Q_k is a Deligne polynomial of degree $k \geq 2$, we may apply the Weil–Deligne bound in Lemma 4.2 so that the contribution of the complete sums is at most

$$\ll_{k,m} \prod_{i \in I} \left\lfloor \frac{N_i}{q} \right\rfloor \cdot \prod_{j \in J} \left\lfloor \frac{u_j}{q} \right\rfloor \cdot q^{m/2} \ll \prod_{i=1}^m \left\lfloor \frac{N_i}{q} \right\rfloor \cdot q^{m/2}.$$

Again using that Q_k is a Deligne polynomial, we may apply Proposition 4.5 so the contribution of the incomplete sums is

$$\begin{aligned} &\ll \sup_{\substack{\bar{I} \subseteq I, \bar{J} \subseteq J, \\ |\bar{I} \cup \bar{J}| < m}} \prod_{i \in \bar{I}} \left\lfloor \frac{N_i}{q} \right\rfloor \cdot \prod_{j \in \bar{J}} \left\lfloor \frac{u_j}{q} \right\rfloor \cdot q^{m/2} (\log q)^{m - (|\bar{J}| + |\bar{I}|)} \\ &\ll \sup_{|J| < m} \prod_{j \in J} \left\lfloor \frac{N_j}{q} \right\rfloor \cdot q^{m/2} (\log q)^{m - |J|}. \end{aligned}$$

This is sufficient for bounding E_1 . Finally we can similarly separate the main term of the right-hand side of (4.3) into complete and incomplete sums, that is,

$$\prod_{i=1}^m \left\lfloor \frac{N_i}{q} \right\rfloor \cdot \left| \sum_{\mathbf{m} \pmod{q^m}} e\left(\frac{2\pi}{q}(aQ_k(\mathbf{m}) + b \cdot \mathbf{m})\right) \right| + E'_1,$$

where

$$E'_1 = \sum_{|J| < m} \prod_{j \in J} \left\lfloor \frac{N_j}{q} \right\rfloor \cdot \sum_{\substack{\mathbf{1}_{(J)} \leq \mathbf{m}_{(J)} \leq \mathbf{q}_{(J)}, \\ \mathbf{1}_{(I)} \leq \mathbf{m}_{(I)} \leq \mathbf{H}_{(I)}}} e\left(\frac{2\pi}{q}(aQ_k(\mathbf{m}_{(J)}, \mathbf{m}_{(I)}) + b \cdot (\mathbf{m}_{(J)}, \mathbf{m}_{(I)}))\right)$$

for some $H_i < q$. Another application of Proposition 4.5 shows that

$$|E'_1| \ll \sup_{|J| < m} \prod_{j \in J} \left\lfloor \frac{N_j}{q} \right\rfloor \cdot q^{m/2} (\log q)^{m - |J|}.$$

Combined with the bound for E_1 , this proves the proposition.

5 Reducing the maximal operator to an exponential sum

5.1 Initial definition of the test function. We now construct a collection of test functions to prove Theorem 2.1. The symbol $P \in \mathbb{R}[X_1, \dots, X_n]$ has leading form $P_k \in \mathbb{Q}[X_1, \dots, X_n]$ that is Dwork-regular in $X_1, \dots, X_n$ over $\mathbb{Q}$ and has intertwining rank r . By relabelling variables we may assume that X_1 intertwines with $X_1, X_2, \dots, X_r$ and does not intertwine with $X_{r+1}, \dots, X_n$. By a $\mathrm{GL}_n(\mathbb{Q})$ change of variables we may clear denominators and assume from now on that P_k has integral coefficients. Fix $R \geq 1$, which will later be a parameter that tends to infinity. Let

$$S_1 = R^\sigma, \quad L = R^\lambda$$

where $0 < \sigma < 1$, $0 < \lambda < 1$ are parameters we will choose optimally later. Once and for all, fix a Schwartz function ϕ on $\mathbb{R}$ with the properties (i) $\phi \geq 0$, (ii) $\phi(0) = (2\pi)^{-1} \int \hat{\phi}(\xi) d\xi = 1$, (iii) $\mathrm{supp}(\hat{\phi}) \subseteq [-1, 1]$; such a function may be constructed in a standard way, such as in [Pie20, §2.1]. Then define for each $m \geq 1$ and variables $u_1, \dots, u_m$ that $\Phi_m(u_1, \dots, u_m) = \phi(u_1) \cdots \phi(u_m)$.

We will define the test function f , tailored to the fact that X_1 does not intertwine with $X_{r+1}, \dots, X_n$. Fix an integral tuple $M = (M_1, M_2, \dots, M_r) \in \mathbb{Z}^r$ with $M_i \geq 1$ as provided by Lemma 3.5. Define the test function f to be

$$\begin{aligned} f(x) := & \Phi_r(S_1 x_1, x_2, \dots, x_r) e((M \circ R) \cdot (x_1, \dots, x_r)) \\ & \times \Phi_{n-r}(x_{r+1}, \dots, x_n) \sum_{\substack{m \in \mathbb{Z}^{n-r} \\ R/L \leq m_j < 2R/L}} e(Lm \cdot (x_{r+1}, \dots, x_n)) \end{aligned}$$

where we recall the notation $M \circ R = (M_1 R, \dots, M_r R)$. Let

$$M_* = \max\{2, M_1, M_2, \dots, M_r\}.$$

The Fourier transform of f is supported in

$$\begin{aligned} & [M_1 R - S_1, M_1 R + S_1] \times [M_2 R - 1, M_2 R + 1] \\ & \times \cdots \times [M_r R - 1, M_r R + 1] \times [R - 1, 2R + 1]^{n-r}, \end{aligned}$$

which is contained in $B_n(0, \sqrt{n}M_*R + \sqrt{n}S_1) \setminus B_n(0, \sqrt{n}R - \sqrt{n}S_1)$. Since $S_1 = R^\sigma$ with $\sigma < 1$ there exists some $R_1(\sigma)$ such that for $R \geq R_1(\sigma)$, $S_1 \leq (1/2)R$ so that this region is contained in the annulus $\{(1/C)R \leq |x| \leq CR\}$ with $C = 2M_*\sqrt{n}$, say, so that C depends only on the symbol P and the dimension n .

Finally, we note for later reference the size of the norm of f . Since $\hat{f}$ is supported in the annulus stated above, for all $R \geq 1$,

$$R^s \|f\|_{L^2(\mathbb{R}^n)} \ll_s \|f\|_{H^s(\mathbb{R}^n)} \ll_s R^s \|f\|_{L^2(\mathbb{R}^n)}.$$

Moreover,

$$(5.1) \quad S_1^{-1/2} \lfloor R/L \rfloor^{\frac{n-r}{2}} \|\phi\|_{L^2(\mathbb{R})}^n \leq \|f\|_{L^2(\mathbb{R}^n)} \leq S_1^{-1/2} \lceil R/L \rceil^{\frac{n-r}{2}} \|\phi\|_{L^2(\mathbb{R})}^n.$$

Explicitly, using the notation $\zeta = (\zeta_1, \dots, \zeta_r)$ and $\eta = (\zeta_{r+1}, \dots, \zeta_n)$,

$$\hat{f}(\zeta, \eta) = \sum_{\substack{m \in \mathbb{Z}^{n-r} \\ R/L \leq m_j < 2R/L}} g_m(\zeta, \eta),$$

in which

$$g_m(\zeta, \eta) = S_1^{-1} \hat{\Phi}_r(S^{-1} \circ (\zeta - M \circ R)) \hat{\Phi}_{n-r}(\eta - Lm),$$

where $S^{-1} = (S_1^{-1}, 1, \dots, 1)$. Thus g_m is supported in the set $\mathcal{B} + (M \circ R, Lm)$, where $\mathcal{B}$ is the box $[-S_1, S_1] \times [-1, 1]^{n-1}$. In particular, for $n \geq 2$, $n-r \geq 1$, and $L \geq 4$, the supports of g_m as m varies are distinct. Thus by Plancherel's theorem,

$$(2\pi)^n \|f\|_{L^2(\mathbb{R}^n)}^2 = \|\hat{f}\|_{L^2(\mathbb{R}^n)}^2 = \sum_{\substack{m \in \mathbb{Z}^{n-r} \\ R/L \leq m_j < 2R/L}} \|g_m\|_{L^2(\mathbb{R}^n)}^2,$$

and the claim (5.1) holds since by Plancherel's theorem,

$$\|g_m\|_{L^2(\mathbb{R}^n)}^2 = S_1^{-1} \|\hat{\phi}\|_{L^2(\mathbb{R}^n)}^{2n} = S_1^{-1} (2\pi)^n \|\phi\|_{L^2(\mathbb{R}^n)}^{2n}.$$

Remark 5.1. Note that if $n = r$ or $n = 1$ the above sum is empty; computing the $\|f\|_{H^s}$ norm as above and taking $\sigma = 1/2$ produces a counterexample to (1.7) for all $s < 1/4$, which is the claim of Theorem 1.1 in this case. Thus from now on we may assume $r < n$ and $n \geq 2$.

5.2 Approximation of the maximal operator by an exponential sum.

Since f treats the intertwined variables $x_1, \dots, x_r$ differently, it is convenient to define $v = (x_1, \dots, x_r)$ and $w = (x_{r+1}, \dots, x_n)$, and similarly use $\zeta = (\zeta_1, \dots, \zeta_r)$, $\eta = (\zeta_{r+1}, \dots, \zeta_n)$; finally, we will continue to denote $S \circ \zeta = (S_1 \zeta_1, \zeta_2, \dots, \zeta_r)$ and $M \circ R = (M_1 R, \dots, M_r R)$ for $(M_1, \dots, M_r)$ provided by Lemma 3.5. By definition, for the test function f ,

$$(5.2) \quad \begin{aligned} T_t^P f(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\Phi}_n(\zeta, \eta) e((S \circ \zeta + M \circ R) \cdot v + \eta \cdot w) \\ &\quad \times \sum_{\substack{m \in \mathbb{Z}^{n-r} \\ R/L \leq m_j < 2R/L}} e(Lm \cdot w) e(P(S \circ \zeta + M \circ R, \eta + Lm)t) d\zeta d\eta. \end{aligned}$$

The main result of this section is that $T_t^P f(x)$ is well-approximated by an exponential sum defined as follows: for any $u \in \mathbb{R}^{n-r}$ with each $u_j > R/L$, and

any $w \in \mathbb{R}^{n-r}$, $t \in \mathbb{R}$ set

$$\mathbf{S}(u; w, t) := \sum_{\substack{m \in \mathbb{Z}^{n-r} \\ R/L \leq m_j < u_j}} e(Lm \cdot w + P_k(M \circ R, Lm)t).$$

For simplicity, when $u = (2R/L, \dots, 2R/L)$, we denote this sum by $\mathbf{S}(2R/L; w, t)$.

Fix $0 < c_0 < 1/2$. Then since ϕ is smooth and $\phi(0) = 1$, there exists a small constant $\delta_0 = \delta_0(c_0, \phi)$ such that

$$(5.3) \quad \phi(y) \geq 1 - c_0/2 \quad \text{for all } |y| \leq \delta_0.$$

The main result of this section is:

Proposition 5.2. *Let $0 < c_0 < 1/2$ be a small constant and δ_0 be as in (5.3). There exist constants $0 < c_1(\delta_0, k, P)$, $c_2(k, P) < 1/2$ such that for all $c_1 < c_1(\delta_0, k, P)$, $c_2 < c_2(k, P)$ and $0 < c_3 < 1$ as small as we like, the following holds.*

Let $x \in (-c_1, -c_1/2] \times [-c_1, c_1]^{n-1}$, $\sigma \leq 1/2$ and $R \geq R_2(c_1, c_2, k, P, \sigma)$. Let $t \in (0, 1)$ satisfy

$$(5.4) \quad t = \frac{-x_1}{(\partial_1 P_k)(M \circ R, \tilde{R})} + \tau, \quad |\tau| \leq \frac{c_2 \delta_0}{S_1 R^{k-1}} \quad \text{and} \quad t \leq \frac{c_3}{R^{k-1}},$$

in which

$$\tilde{R} := (L(\lceil 2R/L \rceil - 1), \dots, L(\lceil 2R/L \rceil - 1)) \in \mathbb{R}^{n-r}.$$

Then with $w = (x_{r+1}, \dots, x_n)$,

$$|T_t^P f(x)| \geq (1 - c_0)^n |\mathbf{S}(2R/L; w, t)| + \mathbf{E}_1$$

where

$$|\mathbf{E}_1| \ll_{\phi, n, k, r, P} c_3 \sup_{u \in [R/L, 2R/L]^{n-r}} |\mathbf{S}(u; w, t)|.$$

Since we will bound $|\mathbf{S}(u; w, t)|$ by a function that is increasing as a function of u_j , we will obtain $|\mathbf{E}_1| \ll_{\phi, n, k, r, P} c_3 |\mathbf{S}(2R/L; w, t)|$, so that the error term is at most half the main term, by taking c_3 appropriately small, as we may. For the conditions in (5.4) to be compatible we need $(\partial_1 P_k)(M \circ R, \tilde{R})$ to be nonzero and moreover to be of size R^{k-1} . This is assured by Lemma 3.5.

To prove the proposition, we first use partial summation to remove all terms in the sum over m in (5.2) that do not appear in $\mathbf{S}(2R/L; w, t)$; to accrue only an acceptable error, we require the third constraint in (5.4), and intertwining rank plays a key role. The next step is to use integration by parts to remove all terms in the phase in (5.2) that are nonlinear in ξ, η , so that we may apply Fourier inversion

to the integral of $\hat{\Phi}_n$; this applies the third constraint in (5.4). Finally, after Fourier inversion, we arrive at a main term involving the evaluation of Φ_n , and to bound this from below we use the fact that $\Phi_n(0) = 1$ forces t to lie in a specified range around a specific point, which is encoded in the first and second constraints in (5.4).

5.3 Removing non-arithmetic terms in the exponential sum. We will use the notation

$$P = P_k + P_{k-1} + \cdots + P_0$$

where each P_i with $0 \leq i \leq k-1$ is a homogeneous polynomial of degree i with real coefficients. Then by Taylor expansion (multinomial expansion),

$$P(S \circ \zeta + M \circ R, \eta + Lm) = P_k(M \circ R, Lm) + Q(S \circ \zeta + M \circ R, \eta + Lm)$$

in which we define

$$Q(u + v) := \sum_{j=0}^{k-1} P_j(u) + \sum_{j=0}^k \sum_{1 \leq |\alpha| \leq j} \frac{(\partial_\alpha P_j)(u)}{\alpha!} v^\alpha, \quad \text{for any } u, v \in \mathbb{R}^n,$$

so that

$$(5.5) \quad Q(S \circ \zeta + M \circ R, \eta + Lm) = \sum_{j=0}^{k-1} P_j(M \circ R, Lm) + \sum_{j=0}^k \sum_{1 \leq |\alpha| \leq j} \frac{(\partial_\alpha P_j)(M \circ R, Lm)}{\alpha!} (S \circ \zeta, \eta)^\alpha.$$

Our goal is to extract the non-arithmetic weight $e(Q(S \circ \zeta + M \circ R, \eta + Lm)t)$ from the sum over m in (5.2) via partial summation, using the assumption on intertwining rank. To understand the role of intertwining rank, it is helpful to see a motivating example.

5.3.1 Example. Suppose the dimension $n = 2$ and $P(X_1, X_2) = X^\beta$ is simply a monomial, where β is a multi-index (β_1, β_2) with $\beta_1 + \beta_2 = k$. Then

$$(5.6) \quad \begin{aligned} & P(S_1 \zeta_1 + M_1 R, \zeta_2 + Lm_2) \\ &= \sum_{j=0}^{\beta_1} \binom{\beta_1}{j} (M_1 R)^{\beta_1-j} (S_1 \zeta_1)^j \cdot \sum_{\ell=0}^{\beta_2} \binom{\beta_2}{\ell} (Lm_2)^{\beta_2-\ell} \zeta_2^\ell. \end{aligned}$$

We isolate the arithmetic term $(M_1 R)^{\beta_1} (Lm_2)^{\beta_2}$ (the $j = \ell = 0$ term), leaving the non-arithmetic terms

$$H(m_2) := \sum_{\substack{\alpha \leq \beta \\ \alpha \neq (0,0)}} \frac{\beta!}{\alpha! (\beta - \alpha)!} (M_1 R, Lm_2)^{\beta - \alpha} (S_1 \zeta_1, \zeta_2)^\alpha.$$

In this example, we model the sum in (5.2) by summing over m_2 :

$$\sum_{R/L \leq m_2 < 2R/L} e(Lm_2x_2 + (M_1R)^{\beta_1}(Lm_2)^{\beta_2}t)e(h(m_2))$$

with

$$h(m_2) = H(m_2)t.$$

By partial summation with respect to m_2 , the weight $e(h(m_2))$ can be removed with an error term depending on the derivative $(\partial/\partial m_2)h(m_2)$. This derivative is

$$tL \sum_{\substack{\alpha \leq \beta - (0,1) \\ \alpha \neq (0,0)}} \frac{\beta!}{\alpha!(\beta - \alpha - (0,1))!} (M_1R, Lm_2)^{\beta - \alpha - (0,1)} (S_1\zeta_1, \zeta_2)^\alpha$$

$$\ll tL \sum_{\substack{\alpha \leq \beta - (0,1) \\ \alpha \neq (0,0)}} R^{k-|\alpha|-1} S_1^{\alpha_1},$$

where the upper bound holds uniformly for $(\zeta_1, \zeta_2) \in [-1, 1]^2$, $m_2 \in [R/L, 2R/L]$. In order for the resulting error term (after an application of Lemma 4.3) to be acceptable, we need

$$tL \sum_{\substack{\alpha \leq \beta - (0,1) \\ \alpha \neq (0,0)}} R^{k-|\alpha|-1} S_1^{\alpha_1} \ll L/R.$$

Since we will ultimately have $t \approx R^{-(k-1)}$, this requires $R^{k-|\alpha|} S_1^{\alpha_1} \ll R^{k-1}$ for each α . If the original monomial $P(X_1, X_2) = X_1^{\beta_1} X_2^{\beta_2}$ has $\beta_1 \geq 1$ and $\beta_2 \geq 1$, there will be a term with $|\alpha| = \alpha_1 = 1$, and this term will force the condition $S_1 \ll 1$. If however X_2 does not intertwine with X_1 in $P(X_1, X_2)$, $\beta_1 = 0$ so that $\alpha_1 = 0$, removing this difficulty. This concludes our example.

In general, we apply Lemma 4.3 to

$$(5.7) \quad \sum_{\substack{m \in \mathbb{Z}^{n-r} \\ R/L \leq m_j < 2R/L}} e(Lm \cdot w) e(P_k(M \circ R, Lm)t) e(h(m))$$

with $h(m) := Q(S \circ \zeta + M \circ R, \eta + Lm)t$. We claim that for each multi-index $\kappa = (\kappa_{r+1}, \dots, \kappa_n) \in \{0, 1\}^{n-r}$ with $|\kappa| \geq 1$,

$$(5.8) \quad \sup_{\substack{(\zeta, \eta) \in [-1, 1]^n \\ y \in [R/L, 2R/L]^{n-r}}} \left| \frac{\partial^{|\kappa|}}{\partial y_{r+1}^{\kappa_{r+1}} \dots \partial y_n^{\kappa_n}} Q((S \circ \zeta, \eta) + (M \circ R, Ly)) \right| \ll_{n,k,P} R^{k-1} (L/R)^{|\kappa|}.$$

Note that the derivatives controlled by κ apply only to the last $n - r$ coordinates. It is in proving this bound that we crucially use the definition of intertwining rank, and the construction of the test function f to accommodate the fact that X_1 does not intertwine with $X_{r+1}, \dots, X_n$ in P_k ; in particular, we will use the fact that for each $\ell > r$, the mixed partial $\partial_\ell \partial_1 P_k$ vanishes identically.

Assume (5.8) for the moment; then for all such κ ,

$$\sup_{\substack{(\zeta, \eta) \in [-1, 1]^n \\ y \in [R/L, 2R/L]^{n-r}}} \left| \frac{\partial^{|\kappa|}}{\partial y_{r+1}^{\kappa_{r+1}} \cdots \partial y_n^{\kappa_n}} h(y) \right| \ll R^{k-1} (L/R)^{|\kappa|} t =: B_\kappa,$$

in the notation of Lemma 4.3. (Here we also note that $B_\kappa \equiv 0$ if any entry in a multi-index κ exceeds the highest corresponding degree in the polynomial Q . However, unlike our previous application of partial summation in Proposition 4.6, in this case the total degree of Q is $k-1$ in m , so we could have $\ell = 1$ in the application below.) Consequently, for each nonempty $J \subseteq \{r+1, \dots, n\}$, and $\ell \leq |J|$,

$$\sup_{\substack{\alpha_1, \dots, \alpha_\ell \in \{0, 1\}^{n-r} \\ \alpha_1 + \dots + \alpha_\ell = \mathbf{1}_J}} \prod_{i=1}^{\ell} B_{\alpha_i} = \sup_{\substack{\alpha_1, \dots, \alpha_\ell \in \{0, 1\}^{n-r} \\ \alpha_1 + \dots + \alpha_\ell = \mathbf{1}_J}} \prod_{i=1}^{\ell} (R^{k-1} (L/R)^{|\alpha_i|} t) = R^{(k-1)\ell} (L/R)^{|J|} t^\ell.$$

This is $\ll R^{k-1} t (L/R)^{|J|}$ under the assumption $t \ll R^{-(k-1)}$, which we assume from now on.

Hence by Lemma 4.3 the sum in (5.7) is identical to

$$\mathbf{S}(2R/L; w, t) e(Q(S \circ \zeta + M \circ R, \eta + \tilde{R})t) + E_1$$

where we define $\tilde{R} := (L(\lceil 2R/L \rceil - 1), \dots, L(\lceil 2R/L \rceil - 1)) \in \mathbb{R}^{n-r}$ and we may take

$$|E_1| \ll_{n,k,r,P} \sup_{u \in [R/L, 2R/L]^{n-r}} |\mathbf{S}(u; w, t)| \cdot R^{k-1} t,$$

uniformly in $(\zeta, \eta) \in [-1, 1]^n$. Consequently, after integrating E_1 trivially in (ζ, η) by applying the compact support of $\hat{\Phi}_n$ in $[-1, 1]^n$, we conclude that $T_t^P f(x)$ is precisely

$$(5.9) \quad \frac{1}{(2\pi)^n} \mathbf{S}(2R/L; w, t) \int_{\mathbb{R}^n} \hat{\Phi}_n(\zeta, \eta) e((S \circ \zeta + M \circ R) \cdot v + \eta \cdot w) \\ \times e(Q(S \circ \zeta + M \circ R, \eta + \tilde{R})t) d\zeta d\eta + E_2$$

where

$$(5.10) \quad |E_2| \ll_{\phi,n,k,r,P} \sup_{u \in [R/L, 2R/L]^{n-r}} |\mathbf{S}(u; w, t)| \cdot R^{k-1} t.$$

It remains to verify (5.8). We recall the expansion (5.5), and bound the mixed partial of each term. First note that for $y = (y_{r+1}, \dots, y_n)$ and any $\ell > r$, for each degree j ,

$$|(\partial/\partial y_\ell)(P_j(M \circ R, Ly))| = L|(\partial_\ell P_j)(M \circ R, Ly)| \ll R^{j-1} L,$$

uniformly for $y \in [R/L, 2R/L]^{n-r}$. Thus for each fixed $j \leq k-1$, for each $\kappa = (\kappa_{r+1}, \dots, \kappa_n) \in \{0, 1\}^{n-r}$ with $|\kappa| \leq j$,

$$\sup_{y \in [R/L, 2R/L]^{n-r}} \left| \frac{\partial^{|\kappa|}}{\partial y_{r+1}^{\kappa_{r+1}} \dots \partial y_n^{\kappa_n}} P_j(M \circ R, Ly) \right| \ll_{n, P_j} L^{|\kappa|} R^{j-|\kappa|} \ll R^{k-1} (L/R)^{|\kappa|},$$

which suffices for (5.8). For the other terms in the expansion (5.5), fix some multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $1 \leq |\alpha| \leq k$, and recall the multi-index $\kappa = (\kappa_{r+1}, \dots, \kappa_n) \in \{0, 1\}^{n-r}$ with $|\kappa| \geq 1$ taking derivatives only with respect to coordinates of $y = (y_{r+1}, \dots, y_n)$. For each $0 \leq j \leq k$, for $|\kappa| \leq j - |\alpha|$,

$$\begin{aligned} \sup_{\substack{(\zeta, \eta) \in [-1, 1]^n \\ y \in [R/L, 2R/L]^{n-r}}} \left| \frac{\partial^{|\kappa|}}{\partial y_{r+1}^{\kappa_{r+1}} \dots \partial y_n^{\kappa_n}} (\partial_\alpha P_j)(M \circ R, Ly) \frac{(S \circ \zeta, \eta)^\alpha}{\alpha!} \right| \\ \ll_{n, k, P} L^{|\kappa|} R^{j-|\alpha|-|\kappa|} S_1^{\alpha_1} \ll R^j (L/R)^{|\kappa|} S_1^{\alpha_1} R^{-|\alpha|}, \end{aligned}$$

in which we recall that

$$S \circ \zeta = (S_1 \zeta_1, \zeta_2, \dots, \zeta_r).$$

If $j \leq k-1$, this already suffices for (5.8), since $S_1 \ll R$ shows that the right-hand side is visibly $\ll R^{k-1} (L/R)^{|\kappa|}$. (Effectively, for $j \leq k-1$ we are using that P_j is already of degree strictly smaller than k .)

Finally for the highest degree $j = k$ piece, we must be more careful, and use the fact that in the leading form P_k , X_1 does not intertwine with the last $n-r$ coordinates, over which we are carrying out the partial summation. Consider the expression $R^{k-1} (L/R)^{|\kappa|} S_1^{\alpha_1} R^{-(|\alpha|-1)}$, recalling $|\alpha| \geq 1$ in the cases we currently consider. As explained in the example in §5.3.1 and the heuristics in §2.1, the problematic case is when $|\alpha| = \alpha_1 = 1$. But since X_1 does not intertwine with $X_{r+1}, \dots, X_n$, $\partial_j \partial_1 P_k \equiv 0$ for each $j \geq r+1$. Consequently, for any α with $\alpha_1 \geq 1$, for every $\kappa = (\kappa_{r+1}, \dots, \kappa_n) \in \{0, 1\}^{n-r}$ with $|\kappa| \geq 1$,

$$\frac{\partial^{|\kappa|}}{\partial y_{r+1}^{\kappa_{r+1}} \dots \partial y_n^{\kappa_n}} (\partial_\alpha P_k)(M \circ R, Ly) \equiv 0.$$

Thus for the highest degree form P_k , we only need to verify that

$$R^{k-1} (L/R)^{|\kappa|} S_1^{\alpha_1} R^{-(|\alpha|-1)} \ll R^{k-1} (L/R)^{|\kappa|}$$

in cases where $|\alpha| \geq 1$ and $\alpha_1 = 0$, and this certainly holds. This completes the proof of (5.8).

5.4 Removing nonlinear terms in the phase, to apply Fourier inversion. We return our focus to (5.9), and now we remove all nonlinear terms in (ζ, η) by integration by parts, in order to apply Fourier inversion. First we Taylor expand, writing

$$\begin{aligned} Q(S \circ \zeta + M \circ R, \eta + \tilde{R}) &= \sum_{j=0}^{k-1} P_j(M \circ R, \tilde{R}) + \sum_{1 \leq |\alpha| \leq k} \frac{(\partial_\alpha P)(M \circ R, \tilde{R})}{\alpha!} (S \circ \zeta, \eta)^\alpha \\ &= (\nabla P_k)(M \circ R, \tilde{R}) \cdot (S \circ \zeta, \eta) + \tilde{Q}(S \circ \zeta + M \circ R, \eta + \tilde{R}) \end{aligned}$$

where we define $\tilde{Q}(S \circ \zeta + M \circ R, \eta + \tilde{R})$ to be

$$\sum_{j=0}^{k-1} P_j(M \circ R, \tilde{R}) + \sum_{j=1}^{k-1} (\nabla P_j)(M \circ R, \tilde{R}) \cdot (S \circ \zeta, \eta) + \sum_{2 \leq |\alpha| \leq k} \frac{(\partial_\alpha P)(M \circ R, \tilde{R})}{\alpha!} (S \circ \zeta, \eta)^\alpha.$$

Then the main term of (5.9) can be written as

$$\begin{aligned} &S(2R/L; w, t)e((M \circ R) \cdot v) \\ &\quad \times \frac{1}{(2\pi)^n} \int_{[-1,1]^n} \hat{\Phi}_n(\zeta, \eta) e((S \circ \zeta, \eta) \cdot ((v, w) + (\nabla P_k)(M \circ R, \tilde{R})t)) \\ &\quad \times e(\tilde{Q}(S \circ \zeta + M \circ R, \eta + \tilde{R})t) d\zeta d\eta. \end{aligned}$$

We will apply integration by parts as in Lemma 4.4 to remove the nonlinear terms in (ζ, η) , with $h(\zeta, \eta) := \tilde{Q}(S \circ \zeta + M \circ R, \eta + \tilde{R})t$. We claim that for every $\kappa = (\kappa_1, \dots, \kappa_n) \in \{0, 1\}^n$ with $|\kappa| \geq 1$,

$$(5.11) \quad \sup_{(\zeta, \eta) \in [-1,1]^n} \left| \frac{\partial^{|\kappa|}}{\partial \zeta_1^{\kappa_1} \dots \partial \zeta_r^{\kappa_r} \partial \eta_{r+1}^{\kappa_{r+1}} \dots \partial \eta_n^{\kappa_n}} h(\zeta, \eta) \right| \ll R^{k-1} t =: B_\kappa,$$

in the notation of Lemma 4.4. Then by integration by parts, the integral above becomes

$$\begin{aligned} &e(\tilde{Q}(S \circ 1 + M \circ R, 1 + \tilde{R})t) \\ &\quad \times \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\Phi}_n(\zeta, \eta) e((S \circ \zeta, \eta) \cdot ((v, w) + t(\nabla P_k)(M \circ R, \tilde{R}))) d\zeta d\eta + E_3 \end{aligned}$$

where $|E_3| \ll_{\phi, n} R^{k-1} t$. By Fourier inversion, the integral above evaluates precisely to

$$\Phi_n((S_1, 1, \dots, 1) \circ [(x_1, x_2, \dots, x_n) + t(\nabla P_k)(M \circ R, \tilde{R})]).$$

Consequently, in absolute value, (5.9) is

$$\begin{aligned} (5.12) \quad &|S(2R/L; w, t)| \\ &\times |\Phi_n((S_1, 1, \dots, 1) \circ [(x_1, x_2, \dots, x_n) + t(\nabla P_k)(M \circ R, \tilde{R})])| \\ &\quad + E_2 + E_4 \end{aligned}$$

with E_2 bounded as in (5.10) and

$$(5.13) \quad |E_4| \ll |\mathbf{S}(2R/L; w, t)| \cdot |E_3| \ll_{\phi, n} |\mathbf{S}(2R/L; w, t)| R^{k-1} t.$$

Finally, we verify (5.11). We bound each term in the expansion defining $\tilde{Q}$. The first sum is independent of (ζ, η) so it vanishes when we take partials ∂_κ with $|\kappa| \geq 1$. For each $1 \leq j \leq k-1$, recall that $(S \circ \zeta) = (S_1 \zeta_1, \zeta_2, \dots, \zeta_r)$ and $\tilde{R} \ll R$, so that at most

$$\sup_{(\zeta, \eta) \in [-1, 1]^n} \left| \frac{\partial^{|\kappa|}}{\partial \zeta_1^{\kappa_1} \cdots \partial \zeta_r^{\kappa_r} \partial \eta_{r+1}^{\kappa_{r+1}} \cdots \partial \eta_n^{\kappa_n}} (\nabla P_j)(M \circ R, \tilde{R}) \cdot (S \circ \zeta, \eta) \right| \ll R^{j-1} S_1 \ll R^{k-2} S_1.$$

This is $\ll R^{k-1}$ as desired, since $S_1 = R^\sigma < R$. Now fix a multi-index $2 \leq |\alpha| \leq k$. For each $0 \leq j \leq k$,

$$\sup_{(\zeta, \eta) \in [-1, 1]^n} \left| \frac{\partial^{|\kappa|}}{\partial \zeta_1^{\kappa_1} \cdots \partial \zeta_r^{\kappa_r} \partial \eta_{r+1}^{\kappa_{r+1}} \cdots \partial \eta_n^{\kappa_n}} \frac{(\partial_\alpha P_j)(M \circ R, \tilde{R})}{\alpha!} (S \circ \zeta, \eta)^\alpha \right| \ll R^{j-|\alpha|} S_1^{\alpha_1}.$$

For all the lower-degree terms with $j \leq k-1$, this is $\ll R^{k-1} S_1^{\alpha_1} R^{-|\alpha|} \ll R^{k-1}$. Finally, consider $R^{k-|\alpha|} S_1^{\alpha_1}$ in the case $2 \leq |\alpha| \leq k$ and $j = k$. (Note that unlike in the partial summation in the previous section, we do not need to consider the case $|\alpha| = 1$ and $j = k$, which is singled out as the main term here.) If $\alpha_1 = 0$, this is $\ll R^{k-1}$. If $|\alpha| > \alpha_1 \geq 1$, this is $\ll R^{k-1} (S_1/R)$. If $|\alpha| = \alpha_1 \geq 2$, then this is $\ll R^k (S_1/R)^2 = R^{k-1} (S_1^2/R)$. For this to be $\ll R^{k-1}$ we impose $S_1 = R^\sigma$ with

$$(5.14) \quad \sigma \leq 1/2.$$

This proves (5.11) and hence (5.12).

5.5 Restrictions on t to complete the proof of Proposition 5.2. Recall from (5.3) that $\phi(y) \geq 1 - c_0/2$ if $|y| \leq \delta_0$. Thus

$$\Phi_n((S_1, 1, \dots, 1) \circ [(x_1, x_2, \dots, x_n) + t(\nabla P_k)(M \circ R, \tilde{R})]) \geq (1 - c_0/2)^n,$$

which suffices for Proposition 5.2, if we choose

$$(5.15) \quad t = \frac{-x_1}{(\partial_1 P_k)(M \circ R, \tilde{R})} + \tau, \quad |\tau| \leq \frac{c_2 \delta_0}{S_1 R^{k-1}}$$

where $c_2 < 1$ is sufficiently small that $c_2 |(\partial_j P_k)(M \circ R, \tilde{R})|/R^{k-1} < 1/2$ for each $1 \leq j \leq n$, and furthermore restrict $x \in [-c_1, c_1]^n$ where $0 < c_1 < \delta_0/4$ and $c_1 |(\partial_j P_k)(M \circ R, \tilde{R})|/(\partial_1 P_k)(M \circ R, \tilde{R}) < \delta_0/4$ for each $2 \leq j \leq n$. (Recall

from Lemma 3.5 that M is chosen so that $|(\partial_1 P_k)(M \circ R, \tilde{R})| \gg R^{k-1} \neq 0$ for all R , and in the other direction it is always true that $|(\partial_j P_k)(M \circ R, \tilde{R})| \ll R^{k-1}$ for each $1 \leq j \leq n$.) For then, simultaneously

$$|S_1(x_1 + (\partial_1 P_k)(M \circ R, \tilde{R})t)| = |S_1 \cdot (\partial_1 P_k)(M \circ R, \tilde{R})\tau| < \delta_0/2,$$

and for $j = 2, \dots, n$,

$$|x_j + (\partial_j P_k)(M \circ R, \tilde{R})t| < \delta_0/4 + \delta_0/4 + \delta_0/2.$$

In fact if $x_1 \in [-c_1, -c_1/2]$ then we can ensure $t \in (0, 1)$ since for appropriate c_1, c_2 , for all sufficiently large $R \geq R_2 = R_2(c_1, c_2, \sigma, P, k)$, we have

$$c_1/(\partial_1 P_k)(M \circ R, \tilde{R}) + c_2/(S_1 R^{k-1}) < 1$$

and

$$c_1/(2(\partial_1 P_k)(M \circ R, \tilde{R})) - c_2/(S_1 R^{k-1}) > 0.$$

Finally, to bound E_2 and E_4 , we impose the condition that

$$(5.16) \quad t \leq \frac{c_3}{R^{k-1}}$$

for some small constant c_3 of our choice, and then $\mathbf{E}_1 \ll E_2 + E_4$ is bounded as stated in Proposition 5.2. For conditions (5.16) and (5.15) to be compatible, we verify that we can choose c_1 and c_2 so that $|R^{k-1}t| \leq c_1(R^{k-1}/(\partial_1 P_k)(M \circ R, \tilde{R})) + c_2(\delta_0/S_1)$ is as small as we want.

6 The arithmetic contribution

We now estimate the exponential sum $\mathbf{S}(2R/L; w, t)$ which we extracted from $T_t^P f(x)$ in Proposition 5.2. It is convenient to define the notation:

$$(6.1) \quad s := L^k \tau, \quad y_1 := -\frac{L^k}{(\partial_1 P_k)(M \circ R, \tilde{R})} x_1 \pmod{2\pi}, \quad y_j := Lx_j \pmod{2\pi},$$

for $r+1 \leq j \leq n$, and set

$$y = (y_{r+1}, \dots, y_n).$$

(Implicitly, we also take $y_j = x_j \pmod{2\pi}$ for $2 \leq j \leq r$, although these variables do not play a role here.) Then $y_1 + s = L^k t \pmod{2\pi}$ and so by recalling $w = (x_{r+1}, \dots, x_n)$ and using homogeneity of P_k ,

$$\begin{aligned} \mathbf{S}(2R/L; w, t) &= \sum_{\substack{m \in \mathbb{Z}^{n-r} \\ R/L \leq m_j < 2R/L}} e(Lm \cdot w + L^k P_k(M \circ (R/L), m)t) \\ &= \sum_{\substack{m \in \mathbb{Z}^{n-r} \\ R/L \leq m_j < 2R/L}} e(m \cdot y + P_k(M \circ (R/L), m)(y_1 + s)). \end{aligned}$$

Remark 6.1. Here we encounter a feature that has not arisen in previous special cases of P_k considered in [ACP23, EPV22], for which the first coordinate only appears in P_k in a diagonal term X_1^k , so that it can be pulled out of the exponential sum entirely. In our general setting, to evaluate the sum using number-theoretic methods, we require each $M_i R/L$ to be an integer. We will consider a sequence of $R = R_j \rightarrow \infty$ where each $R_j = 2^j$ for a sequence of integers j . Since $L = R^\lambda$ for some $0 < \lambda < 1$, we have $R/L = 2^{j(1-\lambda)}$ which is an integer if and only if $j\lambda$ is an integer. We will achieve this by choosing $\lambda = \lambda_1/\lambda_2$ to be rational, and then restricting to $j \rightarrow \infty$ along the arithmetic progression $j \equiv 0 \pmod{\lambda_2}$.

For now, we assume R and L are fixed, and R/L is an integer. We define for any prime q , and $a \in \mathbb{F}_q$, $b \in \mathbb{F}_q^{n-r}$, the complete exponential sum

$$\mathbf{T}(a, b; q) = \sum_{m \pmod{q}^{n-r}} e\left(\frac{2\pi}{q}(b \cdot m + aP_k(M \circ (R/L), m))\right).$$

Let $K_1(P_k)$ be the constant provided by Lemma 3.2, so that the reduction of $P_k(X_1, \dots, X_n)$ is Dwork-regular over $\mathbb{F}_q$ in $X_1, \dots, X_n$ for every prime $q \geq K_1(P_k)$. Then for such q , by Proposition 3.3, $P_k(M \circ (R/L), X_{r+1}, \dots, X_n)$ is a Deligne polynomial in $X_{r+1}, \dots, X_n$ over $\mathbb{F}_q$. Thus we will apply Proposition 4.1 to show that for many a, b the sum $\mathbf{T}(a, b; q)$ must be “large.” Consequently, we will construct a set Ω of $(y_1, \dots, y_n) \in [0, 2\pi]^n$ with (nearly) full measure on which $|\mathbf{S}(2R/L; w, t)|$, and hence $|T_t^P f(x)|$, is large. We state two propositions, first focused on the set Ω , and then focused on the operator $T_t^P f(x)$.

Proposition 6.2. *Let R, L be fixed with R/L an integer. There exists a parameter $K_3(P_k, k)$ such that the following holds for all $Q > K_3$. For every prime $q \in [Q/2, Q]$, let $\mathcal{G}(q)$ denote the set of all $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q^{n-r}$ for which*

$$(1/2)q^{(n-r)/2} \leq |\mathbf{T}(a, b; q)| \leq (k-1)^{n-r}q^{(n-r)/2}.$$

For any $0 < c_4, c_5 < 1$ sufficiently small of our choice, define a set $\Omega \subseteq [0, 2\pi]^n$ by

$$\Omega = \bigcup_{\substack{Q/2 \leq q \leq Q \\ q \text{ prime}}} \bigcup_{(a,b) \in \mathcal{G}(q)} B(a, b; q),$$

where each box $B(a, b; q)$ is defined to be

$$\left\{ |y_1 - 2\pi a/q| \leq c_4 q^{-1}, (y_2, \dots, y_r) \in [0, c_1]^{r-1}, \right. \\ \left. |y_j - 2\pi b_j/q| \leq c_5 q^{-1-1/(n-r)}, r+1 \leq j \leq n \right\}.$$

Then $|\Omega| \gg_{c_4, c_5, n, k} (\log Q)^{-1}$.

Next, recall that $L = R^\lambda$ for a parameter $0 < \lambda < 1$ we will choose later, and from now on we let $Q = R^\kappa$ for a parameter $0 < \kappa < 1$ we will choose later.

Proposition 6.3. *Let R, L be fixed with R/L an integer. Suppose*

$$(6.2) \quad \frac{1}{Q} \ll \frac{L^k}{S_1 R^{k-1}}, \quad \frac{1}{Q^{1+1/(n-r)}} \ll (R/L)^{-1}, \quad R/L \gg Q^{1+\Delta_0}$$

where $0 < \Delta_0 \leq 1/(n-r)$. Assume R, Q are sufficiently large, say

$$R \geq R_3(\delta_0, c_1, c_2, k, P, \sigma, \lambda, \kappa) \quad \text{and} \quad Q \geq K_4(n, k, r, P_k, \Delta_0).$$

Then there exists a set $\Omega^* \subseteq B_n(0, 1)$ with $|\Omega^*| \gg_{c_1, c_4, c_5, n, k, r} (\log Q)^{-1}$ and such that for every $x \in \Omega^*$, there exists a $t \in (0, 1)$ such that

$$|T_t^P f(x)| \geq (1 - c_0)^n 2^{-(n-r)-1} \left(\frac{R}{LQ^{1/2}} \right)^{n-r} - |\mathbf{E}_1| - |\mathbf{E}_2|$$

where

$$|\mathbf{E}_1| + |\mathbf{E}_2| \ll_{\phi, n, k, r, P} (c_3 + c_5 + Q^{-\Delta_0/2}) \left(\frac{R}{LQ^{1/2}} \right)^{n-r}.$$

By taking R sufficiently large and choosing the absolute constants c_3, c_5 sufficiently small, we will ultimately make the error terms less than half the size of the main term. We prove the propositions and then turn in §7 to the final choice of parameters, optimizing the counterexample and completing the proof of Theorem 2.1.

6.1 Proof of Proposition 6.2. Since $M \circ (R/L) \in \mathbb{Z}^r$ is fixed and P_k is Dwork-regular, by Proposition 3.3, $P_k(M \circ (R/L), X_{r+1}, \dots, X_n)$ is a Deligne polynomial in $X_{r+1}, \dots, X_n$ over $\mathbb{F}_q$ for all primes $q \geq K_1(P_k)$. By Proposition 4.1, for all primes $q > \max\{k, K_1, K_2\}$, $\alpha_2 q^{n-r+1} \leq |\mathcal{G}(q)| \leq q^{n-r+1}$ for some $\alpha_2 > 0$ depending only on n, k, r . We have defined the set Ω accordingly, so that for $(y_1, \dots, y_n) \in B(a, b; q)$ for some $(a, b) \in \mathcal{G}(q)$, a small s can be chosen so that $y_1 + s = 2\pi a/q$ precisely, and $(y_{r+1}, \dots, y_n)$ is well-approximated by $(2\pi b_{r+1}/q, \dots, 2\pi b_n/q)$. We will use this Diophantine behavior in Proposition 6.3 to show that for $(y_1, \dots, y_n) \in B(a, b; q)$, $\mathbf{S}(2R/L; w, t)$ is well-approximated by $\lfloor R/Lq \rfloor^{n-r}$ copies of $\mathbf{T}(a, b; q)$ and is hence $\gg \lfloor R/Lq \rfloor^{n-r} q^{(n-r)/2}$. For now we compute a lower bound on the measure of Ω . This is not immediate, since if $n-r > 1$, many of the boxes $B(a, b; q)$ can overlap as q varies; our construction of the set $\mathcal{G}(q)$ is also completely inexplicit about which a, b are chosen. However, we can apply [ACP23, Lemma 4.1], which shows that if a set of boxes is “well-distributed,” then the measure of their union is at least a positive proportion of the sum of their measures.

Precisely, choose K_0 sufficiently large that (by the prime number theorem) $\pi(x) \geq (1/4)x/\log x$ for all $x \geq K_0/2$. Let $\mathcal{J}$ denote the index set of $(a, b; q)$ over which the unions are taken in the definition of Ω . Then by [ACP23, Lemma 4.1(ii)],

$$\begin{aligned} |\Omega| &= \left| \bigcup_{\substack{Q/2 \leq q \leq Q \\ q \text{ prime}}} \bigcup_{(a,b) \in \mathcal{G}(q)} B(a, b; q) \right| \gg \sum_{\substack{Q/2 \leq q \leq Q \\ q \text{ prime}}} \sum_{(a,b) \in \mathcal{G}(q)} |B(a, b; q)| \\ &\gg (Q/\log Q)(\alpha_2 Q^{n-r+1})Q^{-1}(Q^{-1-1/(n-r)})^{n-r} \gg (\log Q)^{-1} \end{aligned}$$

(with implied constants depending on c_1, c_4, c_5, n, k, r) as long as all the boxes $B(a, b; q)$ have comparable size and

$$(6.3) \quad \#\{(a, b; q), (a', b'; q') \in \mathcal{J} : B(a, b; q) \cap B(a', b'; q') \neq \emptyset\} \ll |\mathcal{J}|,$$

with an acceptable implied constant. It is clearly true that

$$1 \ll |B(a, b; q)|/|B(a', b'; q')| \ll 1$$

for all pairs of indices $(a, b; q)$ and $(a', b'; q') \in \mathcal{J}$. The bound (6.3) is a statement that the boxes are well-distributed since a trivial upper bound for the cardinality would be $|\mathcal{J}|^2$; on the other hand, if all the boxes were pairwise disjoint, the cardinality would be precisely $|\mathcal{J}|$.

It remains to verify (6.3). Upon requiring $Q > 2 \max\{k, K_0, K_1, K_2\}$ and recalling the construction of the sets $\mathcal{G}(q)$,

$$(Q/\log Q)Q^{n-r+1} \ll |\mathcal{J}| \ll (Q/\log Q)Q^{n-r+1}.$$

The contribution to (6.3) when $(a, b; q) = (a', b'; q')$ as tuples is clearly $\ll \mathcal{J}$, so we consider instead the case when the tuples are distinct, and we suppose that $B(a, b; q) \cap B(a', b'; q') \neq \emptyset$, so that in particular,

$$\begin{aligned} |a/q - a'/q'| &\leq \frac{c_4}{2\pi}(1/q + 1/q'), \\ |b_j/q - b'_j/q'| &\leq \frac{c_5}{2\pi}(1/q^{1+1/(n-r)} + 1/(q')^{1+1/(n-r)}), \quad r+1 \leq j \leq n. \end{aligned}$$

If $q = q'$ then by taking $c_4, c_5 < 1$, the above relations impose $|a - a'| < 1$ and $|b_j - b'_j| < 1$ for all j , so that $(a, b; q) = (a', b'; q')$, which is a case we already considered. So we now assume $q \neq q'$ are distinct primes in $[Q/2, Q]$. Then the above relations show that

$$\begin{aligned} (6.4) \quad |aq' - a'q| &\leq \frac{c_4}{\pi}Q, \\ |b_jq' - b'_jq| &\leq \frac{c_5C}{\pi}Q^{1-1/(n-r)}, \quad r+1 \leq j \leq n \end{aligned}$$

where C is a constant depending on n, r . Recall that given an integer m , and distinct primes q, q' , there is a unique choice of a pair a, a' with $1 \leq a \leq q$ and $1 \leq a' \leq q'$ with $aq' - a'q = m$. Indeed, if there were another representation a_0, a'_0 then $aq' - a'q = a_0q' - a'_0q$ would imply that $(a - a_0)q' = (a' - a'_0)q$, so that $q'|(a' - a'_0)$ and $q|(a - a_0)$, implying $a = a_0$ and $a' = a'_0$, as claimed. Thus once a tuple $(m_1, m_{r+1}, \dots, m_n)$ of integers is chosen with $|m_1| \leq (c_4/\pi)Q$ and $|m_j| \leq (c_5C/\pi)Q^{1-1/(n-r)}$ for each $j = r+1, \dots, n$, there is at most one choice of a pair $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q^{n-r}$ and $(a', b') \in \mathbb{F}_{q'} \times \mathbb{F}_{q'}^{n-r}$ satisfying the $n-r+1$ relations in (6.4) above. Taking into account all possible values of such $(m_1, m_{r+1}, \dots, m_n)$, this shows that given $q \neq q'$, at most

$$\ll_{c_4, c_5, n, r} Q \cdot (Q^{1-1/(n-r)})^{n-r} \ll Q^{n-r}$$

pairs of index tuples $(a, b; q)$ and $(a', b'; q')$ can have $B(a, b; q) \cap B(a', b'; q') \neq \emptyset$. Taking a union over all pairs of primes $q \neq q' \in [Q/2, Q]$ bounds the left-hand side of (6.3) by $\ll (Q/\log Q)^2 Q^{n-r} \ll |J|$. This proves (6.3) and completes the proof of Proposition 6.2.

6.2 Proof of Proposition 6.3. The existence and measure of the set $\Omega^* \subseteq B_n(0, 1)$ follows directly from the construction of the set Ω in Proposition 6.2. Indeed, Ω^* is defined to be the set of those $x \in B_n(0, 1)$ such that the change of variables defining the y -coordinates in (6.1) map x to a point $(y_1, y_2, \dots, y_n) \in \Omega$. To bound the measure of Ω^* from below, one only needs to compute the measure of the pre-image of Ω under the change of variables (6.1). Under the assumption that $\lambda > (k-1)/k$ (which will hold for our final choice of λ), this simple rescaling argument follows precisely the argument given in [ACP23, §4.5], and we omit it.

For every $x \in \Omega^*$ there is thus a corresponding $y = (y_1, \dots, y_n) \in \Omega$ such that $y \in B(a, b; q)$ for some tuple a, b, q for which

$$(1/2)q^{(n-r)/2} \leq |\mathbf{T}(a, b; q)| \ll_k q^{(n-r)/2}.$$

By the construction of the box $B(a, b; q)$, $|y_1 - 2\pi a/q| \leq c_4 q^{-1}$. Thus we may choose a value of s with $|s| \leq c_4 q^{-1}$, such that $y_1 + s = 2\pi a/q$ exactly. We make this choice for s (which corresponds precisely via (6.1) to a choice of the time parameter t). Again by the construction of the box $B(a, b; q)$,

$$|y_j - 2\pi b_j/q| \leq c_5 q^{-1-1/(n-r)}$$

for each $r+1 \leq j \leq n$, so an application of Proposition 4.6 with $N_j = R/L$ for all $r+1 \leq j \leq n$ and $V = c_5 q^{-1-1/(n-r)}$, yields

$$\mathbf{S}(2R/L; w, t) = \left\lfloor \frac{R}{Lq} \right\rfloor^{n-r} \mathbf{T}(a, b; q) + \mathbf{E}_2$$

with $\mathbf{E}_2 = E$ as in the proposition. For every $q \in [Q/2, Q]$, in the notation of Proposition 4.6 we have $V \ll c_5 N^{-1}$ by the second constraint in (6.2), and $N \gg q^{1+\Delta_0}$ by the third constraint in (6.2). Hence we may apply the simplified upper bound for the error term given in Remark 4.7, which in the present setting yields

$$(6.5) \quad \begin{aligned} |\mathbf{E}_2| &\ll \left\lfloor \frac{R}{Lq} \right\rfloor^{n-r} q^{(n-r)/2} (c_5 + \left\lfloor \frac{R}{Lq} \right\rfloor^{-1} (\log q)^{n-r}) \\ &\ll (c_5 + Q^{-\Delta_0/2}) \left\lfloor \frac{R}{LQ^{1/2}} \right\rfloor^{n-r}. \end{aligned}$$

Here we have applied the third condition in (6.2) to see that for all Q sufficiently large,

$$\left(\frac{R}{LQ} \right)^{-1} (\log Q)^{n-r} \ll Q^{-\Delta_0} (\log Q)^{n-r} \ll Q^{-\Delta_0/2}.$$

Finally, for all R sufficiently large (with respect to λ, κ), $\lfloor \frac{R}{Lq} \rfloor \geq (1/2)R/Lq$, so that the main term of $|\mathbf{S}(2R/L; w, t)|$ satisfies

$$\left\lfloor \frac{R}{Lq} \right\rfloor^{n-r} |\mathbf{T}(a, b; q)| \geq 2^{-(n-r)-1} \left(\frac{R}{Lq} \right)^{n-r} q^{(n-r)/2} \geq 2^{-(n-r)-1} \left(\frac{R}{LQ^{1/2}} \right)^{n-r}.$$

The last step of proving Proposition 6.3 is to control the error term $\mathbf{E}_1$ from Proposition 5.2. By that proposition,

$$|\mathbf{E}_1| \ll \sup_{R/L \leq u_j \leq 2R/L} c_3 |\mathbf{S}(u; w, t)|,$$

and thus it will suffice to prove that uniformly in $R/L \leq u_j \leq 2R/L$, the sum obeys the upper bound $|\mathbf{S}(u; w, t)| \ll (R/LQ^{1/2})^{n-r}$. For this we can again apply Proposition 4.6 with $N_j = u_j$ and $V = c_5 q^{-1-1/(n-r)}$, so that

$$|\mathbf{S}(u; w, t)| = \prod_{j=r+1}^n \left\lfloor \frac{u_j}{q} \right\rfloor \cdot |\mathbf{T}(a, b; q)| + E_5$$

with $E_5 = E$ as in the proposition. We apply the bound $|\mathbf{T}(a, b; q)| \ll_k q^{(n-r)/2}$, valid for each pair $(a, b) \in \mathcal{G}(q)$. Upon noting that the expressions for both the main term and E given in Proposition 4.6 are increasing as each range N_j increases, we bound both from above by taking $u_j = 2R/L$ in each case. Hence we may in fact apply the upper bound (6.5) also to E_5 . In conclusion,

$$|\mathbf{E}_1| \ll c_3 \left\lfloor \frac{R}{Lq} \right\rfloor^{n-r} q^{(n-r)/2} + c_3 (c_5 + Q^{-\Delta_0/2}) \left(\frac{R}{LQ^{1/2}} \right)^{n-r} \ll c_3 \left(\frac{R}{LQ^{1/2}} \right)^{n-r}.$$

7 Choosing parameters and conclusion of the proof

From Proposition 6.3, by taking R sufficiently large (relative to $\phi, \delta_0, c_1, c_2, n, k, r, P, \sigma, \lambda, \kappa$) and choosing c_3, c_5 sufficiently small (relative to ϕ, c_0, n, k, r, P) we may conclude that under the hypotheses of the proposition,

$$\sup_{0 < t < 1} |T_t^P f(x)| \geq \frac{1}{2} (1 - c_0)^n 2^{-(n-r)-1} \left(\frac{R}{LQ^{1/2}} \right)^{n-r}.$$

It then follows from the measure of Ω^* and the computation (5.1) of $\|f\|_{L^2}$ that

$$\frac{\| \sup_{0 < t < 1} |T_t^P f(x)| \|_{L^1(B_n(0,1))}}{\|f\|_{L^2}} \gg \left(\frac{R}{LQ^{1/2}} \right)^{n-r} S_1^{1/2} (R/L)^{-(n-r)/2} (\log Q)^{-1}.$$

Set

$$\delta(n, k, r) = \frac{n-r}{4((k-1)(n-(r-1))+1)}.$$

To finish the proof of Theorem 2.1, it suffices to show that we can define the parameters $S_1 = R^\sigma$, $L = R^\lambda$ and $Q = R^\kappa$ such that R/L is an integer, (5.14) and (6.2) are satisfied, and for every $s < \frac{1}{4} + \delta(n, k, r)$,

$$(7.1) \quad \left(\frac{R}{LQ^{1/2}} \right)^{n-r} S_1^{1/2} (R/L)^{-(n-r)/2} (\log Q)^{-1} \geq A_s R^{s'}$$

for some $s' > s$ (and some nonzero constant A_s). Note that verifying (7.1) is equivalent to choosing σ, λ, κ such that

$$(7.2) \quad s < \frac{\sigma}{2} + \frac{n-r}{2} - (\kappa + \lambda) \frac{n-r}{2},$$

while (5.14) imposes $\sigma \leq 1/2$ and (6.2) imposes

$$(7.3) \quad \kappa + k\lambda \geq k-1 + \sigma, \quad \frac{n-(r-1)}{n-r} \kappa + \lambda \geq 1, \quad \lambda \leq 1 - \kappa(1 + \Delta_0).$$

By taking a linear combination of the first two relations in the line above (namely $1/(k-1)$ times the first relation plus $n-r$ times the second relation), we obtain

$$(7.4) \quad \kappa + \lambda \geq \frac{n-(r-1) + \sigma/(k-1)}{n-(r-1) + 1/(k-1)}.$$

To maximize the right-hand side of (7.2), we choose κ, λ so that equality holds in this relation, and substitute the resulting value for $\kappa + \lambda$ into (7.2). For all $k \geq 2$ the coefficient of σ on the right-hand side of (7.2) is then positive, so in order to enlarge the region in (7.2) as much as possible, we take $\sigma = 1/2$. Now solving for κ and λ that obey the first two relations in (7.3) and satisfy equality in (7.4) reveals

$$\kappa = \frac{n-r}{2((k-1)(n-(r-1))+1)}, \quad \lambda = 1 - \frac{n-(r-1)}{2((k-1)(n-(r-1))+1)}.$$

It is then true that $0 < \kappa$, $\lambda < 1$. Moreover, $\lambda = 1 - \kappa(1 + \Delta_0)$ with $\Delta_0 = 1/(n - r)$, so that the third relation in (7.3) holds. Additionally, $\lambda > (k - 1)/k$, as was required in §6.2. Finally, $\lambda = \lambda_1/\lambda_2$ is a rational number and hence we take a sequence of integers $j \rightarrow \infty$ with $j \equiv 0 \pmod{2((k - 1)(n - (r - 1)) + 1)}$. Then for each $R = R_j = 2^j$ and $L = L_j = R_j^\lambda$ as $j \rightarrow \infty$ in this sequence, we have $R_j/L_j = 2^{j(1-\lambda)}$ is an integer, as required in Remark 6.1. Finally, we conclude that with these choices, (7.1) holds for all $s < 1/4 + \delta(n, k, r)$, which ends the proof of Theorem 2.1.

8 Forms and intertwining rank: examples and remarks

8.1 Examples of Dwork-regular forms of any intertwining rank.

For each $k \geq 3$ and $2 \leq r \leq n$, we now prove the following forms of degree k are Dwork-regular over $\mathbb{Q}$ in $X_1, \dots, X_n$ with intertwining rank r , namely

$$\begin{aligned} P_k(X_1, \dots, X_n) &= X_1^k + \dots + X_n^k + \sum_{2 \leq j \leq r} X_1 X_j^{k-1} + \sum_{2 \leq i < j \leq n} X_i X_j^{k-1}, \quad k \geq 3 \text{ odd}; \\ P_k(X_1, \dots, X_n) &= X_1^k + \dots + X_n^k + \sum_{2 \leq j \leq r} X_1^2 X_j^{k-2} + \sum_{2 \leq i < j \leq n} X_i^2 X_j^{k-2}, \quad k \geq 4 \text{ even}. \end{aligned}$$

These visibly have intertwining rank r . In the next sections we additionally prove these forms are indecomposable, and we compute the codimension of all Dwork-regular forms of intertwining rank $r < n$, thus quantifying the set of forms for which Theorem 1.1 proves a new result.

First we prove that each example P_k defined above is Dwork-regular. We provide a full proof in the case $k \geq 3$ is odd; this relies on the fact that $k - 1$ is then even. In the case that $k \geq 4$ is even, the proof is analogous, and relies on the fact that $k - 2$ is even. By Lemma 3.1, it suffices to check that for all nonempty $S \subseteq \{1, \dots, n\}$, $P_S := P|_{X_i=0, i \notin S}$ is nonzero if $|S| = 1$ and P_S is nonsingular if $|S| \geq 2$. It is clear that P_S is nonzero for $|S| = 1$, so henceforth we assume $|S| \geq 2$. Suppose $S = \{\ell_1, \dots, \ell_m\}$ with $\ell_1 < \dots < \ell_m$. If $1 \notin S$ then P_S is of the form

$$(8.1) \quad X_{\ell_1}^k + \dots + X_{\ell_m}^k + \sum_{\substack{2 \leq \ell_i < \ell_j \leq n, \\ \ell_i, \ell_j \in S}} X_{\ell_i} X_{\ell_j}^{k-1}.$$

If $1 \in S$ and $S \setminus \{1\} \subseteq \{r + 1, \dots, n\}$ then P_S is of the form

$$X_1^k + X_{\ell_2}^k + \dots + X_{\ell_m}^k + \sum_{\substack{2 \leq \ell_i < \ell_j \leq n, \\ \ell_i, \ell_j \in S}} X_{\ell_i} X_{\ell_j}^{k-1}.$$

This is the sum of X_1^k and a polynomial $\tilde{P}_S$ of the form (8.1), and so it is nonsingular if and only if $\tilde{P}_S$ is nonsingular. (Note that when $|S| = 2$ this is diagonal and hence

nonsingular.) Lastly if $1 \in S$ and $S \setminus \{1\} \not\subseteq \{r+1, \dots, n\}$ then P_S is of the form

$$(8.2) \quad X_1^k + X_{\ell_2}^k + \dots + X_{\ell_m}^k + \sum_{\substack{2 \leq \ell_j \leq r, \\ \ell_j \in S}} X_1 X_{\ell_j}^{k-1} + \sum_{\substack{2 \leq \ell_i < \ell_j \leq n, \\ \ell_i, \ell_j \in S}} X_{\ell_i} X_{\ell_j}^{k-1}.$$

In particular, if $|S| = 2$ then P_S is of the form

$$(8.3) \quad X_1^k + X_{\ell_2}^k + X_1 X_{\ell_2}^{k-1}.$$

Consequently, in all cases (after relabelling variables) it suffices to check that for each $m \geq m' \geq 2$ and $\alpha \in \{0, 1\}$,

$$Q(Y_1, \dots, Y_m) := Y_1^k + \dots + Y_m^k + \sum_{2 \leq j \leq m'} Y_1 Y_j^{k-1} + \alpha \sum_{2 \leq i < j \leq m} Y_i Y_j^{k-1}$$

is nonsingular. (The case $m' = m, \alpha = 1$ corresponds to (8.1) with $Y_i = X_{\ell_i}$, the case $m' < m, \alpha = 1$ corresponds to (8.2), and the case $m' = m = 2, \alpha = 0$ corresponds to (8.3).)

Suppose there exists $a = [a_1 : \dots : a_m] \in \mathbb{P}^{m-1}$ that is a simultaneous solution to the system

$$\frac{\partial Q}{\partial Y_1} = kY_1^{k-1} + \sum_{1 < j \leq m'} Y_j^{k-1} = 0,$$

$$\frac{\partial Q}{\partial Y_\ell} = kY_\ell^{k-1} + (k-1)Y_1 Y_\ell^{k-2} + \alpha(k-1) \sum_{2 \leq i < \ell} Y_i Y_\ell^{k-2} + \alpha \sum_{\ell < j \leq m} Y_j^{k-1} = 0,$$

$$2 \leq \ell \leq m',$$

$$\frac{\partial Q}{\partial Y_\ell} = kY_\ell^{k-1} + \alpha(k-1) \sum_{2 \leq i < \ell} Y_i Y_\ell^{k-2} + \alpha \sum_{\ell < j \leq m} Y_j^{k-1} = 0, \quad m' + 1 \leq \ell \leq m.$$

Since k is odd, $k-1$ is even, and so the vanishing of $\partial Q / \partial Y_1$ forces $a_\ell = 0$ for $1 \leq \ell \leq m'$. If $\alpha = 0$, then the vanishing of the partials $\partial Q / \partial Y_{m'+1}, \dots, \partial Q / \partial Y_m$ forces $a_\ell = 0$ for $m' + 1 \leq \ell \leq m$. Otherwise if $\alpha = 1$, then the vanishing of $\partial Q / \partial Y_{m'}$ forces $a_\ell = 0$ for $m' + 1 \leq \ell \leq m$ (by recalling $a_\ell = 0$ for $1 \leq \ell \leq m'$). So $[a_1 : \dots : a_m]$ cannot represent a point in $\mathbb{P}^{m-1}$. Thus Q is nonsingular, as needed.

8.2 A criterion to check if a form is indecomposable. We will next prove that the examples given above are indecomposable, and in particular, there is no $\mathrm{GL}_n(\mathbb{Q})$ change of variables that bring them to the shape $X_1^k + Q_k(X_2, \dots, X_n)$ (e.g., the shape required in previous work [ACP23, EPV22]). We refer to the results of Harrison [Har75] and Harrison–Pareigis [HP88], who studied the theory

of higher-degree forms using the analogous theory for symmetric spaces. (Being decomposable is also referred to as being of Sebastiani–Thom type in algebraic geometry literature, for example in the literature on Carlson and Griffiths’ result [CG80] that the generic polynomial can be reconstructed (up to a constant multiplicative factor) from its Jacobian ideal; see the explicit relation to ST-type in [Wan15].)

Let L be a field, in our case with characteristic zero. A symmetric space of degree k over L is a pair (V, θ) where V is a vector space over L of dimension n and $\theta : V^k \rightarrow L$ is a symmetric multilinear map. This is equivalent to $\text{Sym}^k(V^*)$, the k th symmetric power of V^* , which is naturally identified with the space of homogeneous polynomials of n variables and degree k (see standard texts such as [DF04],[Har92]). To describe explicitly the identification between forms and symmetric spaces, write a homogeneous polynomial $F \in L[X_1, \dots, X_n]$ in the following symmetric form,

$$F(X_1, \dots, X_n) = \sum_{1 \leq i_1, \dots, i_k \leq n} c_{i_1 \dots i_k} X_{i_1} \cdots X_{i_k},$$

where $c_{i_1 \dots i_k} = c_{\sigma(i_1) \dots \sigma(i_k)}$ for all $\sigma \in S_k$ and S_k is the symmetric group on $\{1, \dots, k\}$. Let V be an n -dimensional vector space over the field L (we may view V as L^n), and let $v_1, \dots, v_n$ be a basis of V . Define $\theta(v_{i_1}, \dots, v_{i_k}) = c_{i_1 \dots i_k}$. Then for all $x_1, \dots, x_n \in L$, we have the relation

$$F(x_1, \dots, x_n) = \theta\left(\sum_{i=1}^n v_i x_i, \dots, \sum_{i=1}^n v_i x_i\right).$$

By definition, a symmetric space (V, θ) is nondegenerate if $\theta(v, v_2, \dots, v_k) = 0$ for all $v_2, \dots, v_k \in V$ implies $v = 0$. Further, a symmetric space is decomposable if there exist nonzero symmetric spaces (U, ϕ) and (W, ψ) such that

$$(V, \theta) = (U, \phi) \oplus (W, \psi).$$

Harrison showed that the decomposability of a symmetric space (V, θ) is characterized by its center $Z(V, \theta)$ which is defined as

$$Z(V, \theta) = \{f \in \text{End}_L(V) : \theta(f(v_1), v_2, \dots, v_k) = \theta(v_1, f(v_2), v_3, \dots, v_k)\}.$$

Note that this also implies

$$\theta(v_1, \dots, f(v_i), \dots, v_j, \dots, v_k) = \theta(v_1, \dots, v_i, \dots, f(v_j), \dots, v_k)$$

for any i, j since θ is symmetric. Precisely, Harrison proves in [Har75, Proposition 4.1]: let (V, θ) be a nondegenerate symmetric space of degree $k \geq 3$ over a

field L of characteristic zero. Then (V, θ) is indecomposable if and only if $Z(V, \theta)$ has no nontrivial idempotents. (An idempotent in a ring is an element a such that $a^2 = a$. The trivial idempotent elements are the 0 and 1, respectively the additive identity and the multiplicative identity in the ring.)

To translate this into the language of homogeneous polynomials, we define the center of a form F , following [HP88], as

$$Z(F) = \{A \in M_n(L) \mid A^T H_F = H_F A\}$$

where H_F denotes the Hessian matrix $(\partial^2 F / \partial X_i \partial X_j)_{1 \leq i, j \leq n}$. Then the center, and the decomposability, of a form coincide with those of its associated symmetric space. Precisely, let $F \in L[X_1, \dots, X_n]$ be a form of degree $k \geq 3$ and let (V, θ) be the symmetric space associated to F . Then it can be shown that $Z(F) \cong Z(V, \theta)$, and F is indecomposable as a form if and only if (V, θ) is indecomposable as a symmetric space. Thus to show a form is indecomposable, it is equivalent to show that its center has no nontrivial idempotents. This is the criterion we will exploit.

It is convenient to note that over a field L (with $\text{char} L \nmid k$), a form is called central if $Z(F) \simeq L$. Harrison showed that if a form is central (over L), then it is absolutely indecomposable (that is, indecomposable over any field extension of L). In our case, to show F is indecomposable over $\mathbb{Q}$ it suffices to show that $Z(F) \simeq \mathbb{Q}$, so the form has the even stronger property of being central. We remarked earlier that indecomposable forms are generic. This is implied for $(n, k) \neq (2, 3)$ over $\mathbb{C}$ by [HLYZ22, Thm. 3.2] (which shows the set of central forms is open and dense in the moduli space over $\mathbb{C}$; this proof can be adapted to hold over $\mathbb{Q}$). It is also shown directly for $n \geq 3, k \geq 3$ over $\mathbb{C}$ by [Wan15, Ex. 4.3, 4.4, Cor. 6.1]. The case $(n, k) = (2, 3)$ is more complicated, and we defer its study to a different work.

8.3 The examples are indecomposable. For $k \geq 3, n \geq 2$ and $2 \leq r \leq n$, the example form P_k defined above is indecomposable; we will prove this next by showing $Z(P_k)$ has no nontrivial idempotents. Precisely, when $(n, k) \neq (2, 3)$, we show that $Z(P_k) \simeq \mathbb{Q}$; for $(n, k) = (2, 3)$, the example P_3 is also indecomposable, but with a different center. (Note that if $r = 1$, the form is decomposable since X_1 only appears in a diagonal term. Thus we need only consider $2 \leq r \leq n$.)

We present the full proof for all odd $k \geq 5, n \geq 2$ and $2 \leq r \leq n$; the proof for $k = 3$ and for all even $k \geq 4$ is fundamentally analogous. Fix $P = P_k$ to be the example form defined above. Let H_P denote the Hessian of P . Then $H_P/(k-1)$ is

the $n \times n$ matrix

$$\begin{pmatrix} kX_1^{k-2} & X_2^{k-2} & \cdots & X_r^{k-2} & 0 & \cdots & 0 \\ X_2^{k-2} & kX_2^{k-2} + Q_2 & \cdots & X_r^{k-2} & X_{r+1}^{k-2} & \cdots & X_n^{k-2} \\ \vdots & & \ddots & & & & \vdots \\ X_r^{k-2} & X_r^{k-2} & \cdots & kX_r^{k-2} + Q_r & X_{r+1}^{k-2} & \cdots & X_n^{k-2} \\ 0 & X_{r+1}^{k-2} & \cdots & X_{r+1}^{k-2} & kX_{r+1}^{k-2} + Q_{r+1} & \cdots & X_n^{k-2} \\ \vdots & & & & & \ddots & \vdots \\ 0 & X_n^{k-2} & \cdots & X_n^{k-2} & X_n^{k-2} & \cdots & kX_n^{k-2} + Q_n \end{pmatrix}$$

where $Q_\ell = (k-2) \sum_{1 \leq i < \ell} X_i X_\ell^{k-3}$ for $\ell \leq r$ and $Q_\ell = (k-2) \sum_{2 \leq i < \ell} X_i X_\ell^{k-3}$ for $\ell > r$. Note that each $Q_\ell \not\equiv 0$. For $k \geq 5$, each Q_ℓ consists of monomials that differ from each other and from the entries of $H_P/(k-1)$ that are off the diagonal, and so the vanishing of any linear combination $c_{i_1} Q_{i_1} + \cdots + c_{i_m} Q_{i_m} \equiv 0$ would imply $c_{i_j} = 0$ for all $1 \leq j \leq m$.

Let $A = (a_{ij}) \in M_n(\mathbb{Q})$ and write

$$H_P/(k-1) = (h_{ij}).$$

Let B_A denote $(A^T H_P - H_P A)/(k-1)$. Then $A \in Z(P)$ if and only if $B_A = 0$. Note that a priori we have $\{cI_n : c \in \mathbb{Q}\} \subseteq Z(P)$. The assumption that all entries of B_A are 0 implies constraints on the entries of A that show the reverse inclusion

$$Z(P) \subseteq \{cI_n : c \in \mathbb{Q}\},$$

from which we deduce the equality $Z(P) = \{cI_n : c \in \mathbb{Q}\} \simeq \mathbb{Q}$.

Write $B_A = (b_{ij})$ so that $b_{ij} = \sum_{\ell=1}^n (a_{\ell i} h_{\ell j} - h_{i \ell} a_{\ell j})$. Note that since H_P is symmetric, $b_{ii} = 0$ and $B_A^T = -B_A$. Hence it suffices to consider the $(n^2 - n)/2$ entries above the diagonal, i.e., b_{ij} with $i < j$. Each entry b_{ij} is a polynomial, so it is $\equiv 0$ if and only if the coefficient of each term (after regrouping) is zero. We split the $(n^2 - n)/2$ entries of b_{ij} with $i < j$ into the following five cases of (i, j) , based on the shapes of the rows and columns:

- (1) $(1, j)$ with $2 \leq j \leq r$,
- (2) (i, j) with $2 \leq i < j \leq r$,
- (3) $(1, j)$ with $r+1 \leq j \leq n$,
- (4) (i, j) with $r+1 \leq j \leq n$ and $2 \leq i \leq r$, and
- (5) (i, j) with $r+1 \leq i < j \leq n$.

It suffices to learn from the assumption that $b_{ij} \equiv 0$ in the cases (1), (3), (4).

For case (1) with $(1, j)$ with $2 \leq j \leq r$, we compute

$$\begin{aligned} b_{1j} = & -a_{1j}kX_1^{k-2} - \sum_{\ell=2}^{j-1} a_{\ell j}X_\ell^{k-2} + \left(\sum_{\ell=1}^{j-1} a_{\ell 1} + a_{j1}k - a_{jj} \right) X_j^{k-2} \\ & + \sum_{\ell=j+1}^r (a_{\ell 1} - a_{\ell j})X_\ell^{k-2} + \sum_{\ell=r+1}^n a_{\ell 1}X_\ell^{k-2} + a_{j1}Q_j. \end{aligned}$$

The assumption $b_{1j} \equiv 0$ for all $2 \leq j \leq r$ can be seen to imply that

$$(8.4) \quad a_{ii} = a_{11}, \quad 2 \leq i \leq r,$$

$$(8.5) \quad a_{i1} = 0, \quad 2 \leq i \leq n,$$

$$a_{ij} = 0, \quad 1 \leq i \neq j \leq r.$$

These give the desired conditions on the top left $r \times r$ block and the first column of A . If $r = n$, the proof is complete; otherwise for $r < n$ we continue, as cases (3), (4) are non-vacuous.

For case (3), $(1, j)$ with $r+1 \leq j \leq n$, we compute

$$b_{1j} = -a_{1j}kX_1^{k-2} - \sum_{\ell=2}^r a_{\ell j}X_\ell^{k-2} + \left(\sum_{\ell=2}^{j-1} a_{\ell 1} + a_{j1}k \right) X_j^{k-2} + \sum_{\ell=j+1}^n a_{\ell 1}X_\ell^{k-2} + a_{j1}Q_j.$$

Thus the assumption $b_{1j} \equiv 0$ gives in particular the new condition

$$a_{ij} = 0, \quad 1 \leq i \leq r, \quad r+1 \leq j \leq n.$$

This is the desired result for the top right $r \times (n-r)$ block of A .

For case (4), (i, j) with $r+1 \leq j \leq n$ and $2 \leq i \leq r$, we compute

$$\begin{aligned} b_{ij} = & - \left(\sum_{\ell=1}^{i-1} a_{\ell j} + a_{ij}k \right) X_i^{k-2} - \sum_{\ell=i+1}^{j-1} a_{\ell j}X_\ell^{k-2} + \left(\sum_{\ell=2}^{j-1} a_{\ell i} + a_{ji}k - a_{jj} \right) X_j^{k-2} \\ & + \sum_{\ell=j+1}^n (a_{\ell i} - a_{\ell j})X_\ell^{k-2} - a_{ij}Q_i + a_{ji}Q_j. \end{aligned}$$

The assumption $b_{ij} \equiv 0$ for all $r+1 \leq j \leq n$ and $2 \leq i \leq r$ implies that

$$a_{ii} = a_{22}, \quad 2 \leq i \leq n,$$

$$a_{ij} = 0, \quad r+1 \leq i \leq n, \quad 2 \leq j \leq r,$$

$$a_{ij} = 0, \quad 2 \leq i \leq n, \quad r+1 \leq j \leq n, \quad i \neq j.$$

The second condition combined with (8.5) confirms that the entries in the lower-left $(n-r) \times r$ block of A are zeroes, while the third condition finalizes that the

off-diagonal entries of the lower-right $(n - r) \times (n - r)$ block of A are zeroes. Finally, combined with (8.4) the first condition shows that all diagonal entries in the lower-right block are also equal to a_{11} . Thus $A = a_{11}I_n$, and this completes the proof that $Z(P) \subseteq \{cI_n : c \in \mathbb{Q}\}$.

The above computations focused on the case of $k \geq 5$ odd. If $k \geq 4$ is even, the argument follows exactly the same structure; the assumption that $b_{ij} \equiv 0$ for indices in case (1) proves the top left $r \times r$ block has the desired structure, and (if $r < n$) indices from cases (3) and (4) complete the information about the remaining matrix. If $k = 3$, the proof is more complicated, because the polynomials Q_ℓ defined above now must be grouped with various other terms (the monomials appearing are no longer all distinct). Nevertheless, if $n \geq 3$, the assumption $b_{ij} \equiv 0$ in the cases (1)–(4) shows that $Z(P_3) \simeq \mathbb{Q}$. If $(n, k) = (2, 3)$, we need only consider rank $r = 2$, and only the index case (1) is non-vacuous. From $b_{ij} \equiv 0$ in case (1) we conclude $3a_{12} - a_{21} = 0$ and $a_{11} - a_{22} + 3a_{21} = 0$ so that

$$Z(P_3) = \left\{ \begin{pmatrix} \alpha - 9\beta & \beta \\ 3\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{Q} \right\}.$$

Any idempotent $A \in Z(P_3)$ must satisfy $A^2 = A$, by definition of being an idempotent. If $\beta = 0$, this forces $\alpha = 0$ or 1 , corresponding to the A being either of the trivial idempotents (the zero matrix or the identity matrix). If $\beta \neq 0$, the identity $A^2 = A$ produces three independent equations in α, β , and in particular, inspection of these equations shows that α must satisfy a quadratic equation with no rational roots. Thus $Z(P_3)$ contains no nontrivial idempotents over $\mathbb{Q}$. In conclusion, P_3 is not central over $\mathbb{Q}$ but it is indecomposable over $\mathbb{Q}$. (It is incidentally decomposable over $\overline{\mathbb{Q}}$, since its center contains nontrivial idempotents over $\overline{\mathbb{Q}}$.)

8.4 Codimension of the class of forms. Let $\mathcal{M}$ denote the moduli space of forms $P \in \mathbb{Q}[X_1, \dots, X_n]$ of degree $k \geq 2$. Then $\dim \mathcal{M} = \binom{n+k-1}{n-1}$. To see this by a “stars and bars” argument, note that $\dim \mathcal{M}$ is the number of monomials of degree k in n variables. Each such monomial can be represented as a configuration of k stars and $n - 1$ bars (e.g., $X_1X_2^2$ when $k = 3$ and $n = 4$ would be represented by the configuration $*|*|*|$.) The number of such configurations is equal to the number of ways to choose the location of the $n - 1$ bars (among $k + n - 1$ possible places), and this is the binomial coefficient $\binom{n+k-1}{n-1}$.

Let $\mathcal{D} \subseteq \mathcal{M}$ denote the set of Dwork-regular forms, and let $\mathcal{P}_r$ denote the set of forms of intertwining rank $\leq r$, for $1 \leq r \leq n$, so that $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \dots \subseteq \mathcal{P}_n = \mathcal{M}$. (We remark that the set of forms that have intertwining rank precisely r , that is, the set $\tilde{\mathcal{P}}_r = \mathcal{P}_r \setminus \mathcal{P}_{r-1}$, has dimension equal to that of $\mathcal{P}_r$, since it can be shown

that $\dim \mathcal{P}_{r-1} < \dim \mathcal{P}_r$.) For each $1 \leq r < n$, Theorem 1.1 proves a nontrivial result for all real symbols P with leading form $P_k \in \mathcal{D} \cap \mathcal{P}_r$. Here we compare the codimension of $\mathcal{D} \cap \mathcal{P}_{n-1}$ in $\mathcal{M}$ (the largest class to which our theorem applies) to the codimension of $\mathcal{D} \cap \mathcal{P}_1$ in $\mathcal{M}$ (equivalent to the largest class to which the previous works [ACP23, EPV22] applied); we focus on a brief summary, since a more complete study of such forms (and their $\mathrm{GL}_n(\mathbb{Q})$ -orbits) will be given in other work.

Fix $1 \leq r \leq n$. First, $\mathcal{D}$ is open in $\mathcal{M}$, and it can be shown that $\mathcal{P}_r$ is a finite union of (irreducible) affine varieties in $\mathcal{M}$, say $P_{r,i}$ for $i = 1, \dots, N$, so that $\mathcal{P}_r = \bigcup_{i=1}^N P_{r,i}$. Then $\mathcal{D} \cap \mathcal{P}_r$ is a quasi-affine variety, and consequently by [Har77, Prop 1.10], $\dim \mathcal{D} \cap \mathcal{P}_r = \dim \overline{\mathcal{D} \cap \mathcal{P}_r}$. In general, if $U \subseteq \mathbb{A}^m$ is an open set and $X = \bigcup_{i=1}^N X_i \subseteq \mathbb{A}^m$ is a union of irreducible affine varieties X_i , then as long as $U \cap X_i \neq \emptyset$ for every i , it follows that $\overline{U \cap X} = X$. For each i , it can be shown that $\mathcal{D} \cap P_{r,i}$ is nonempty, by precisely the examples stated above (up to re-ordering coordinates). Thus we conclude that $\overline{\mathcal{D} \cap \mathcal{P}_r} = \mathcal{P}_r$, so that it suffices to compute $\dim \mathcal{P}_r$ in $\mathcal{M}$.

We focus on the cases of $\mathcal{P}_1$ and $\mathcal{P}_{n-1}$; it is easier to count the codimension. By symmetry considerations, the dimension of $\mathcal{P}_{n-1}$ is the dimension of the class of forms for which X_1 does not intertwine with X_n , or equivalently all those forms that do not contain monomials with the factor $X_1 X_n$. Equivalently, all coefficients of terms of the form $X_1 X_n Q_{k-2}(X_1, \dots, X_n)$ with Q_{k-2} of degree $k-2$, must be zero. This constrains the coefficients of $\binom{n+(k-2)-1}{n-1}$ monomials, so that

$$\mathrm{codim}(\mathcal{P}_{n-1}) = \binom{n+k-3}{n-1}.$$

On the other hand, by symmetry considerations, the dimension of $\mathcal{P}_1$ is the dimension of the class of forms $cX_1^k + Q_k(X_2, \dots, X_n)$, where Q_k has degree k ; the dimension of such polynomials is $\binom{n+k-2}{n-2} + 1$. Equivalently,

$$\mathrm{codim}(\mathcal{P}_1) = \binom{n+k-1}{n-1} - \binom{n+k-2}{n-2} - 1 = \mathrm{codim}(\mathcal{P}_{n-1}) + \binom{n+k-3}{n-2} - 1.$$

Thus $\mathcal{P}_{n-1}$ is a larger class of forms, with $\dim(\mathcal{P}_{n-1}) - \dim(\mathcal{P}_1) = \binom{n+k-3}{n-2} - 1$ behaving asymptotically like $\sim c_n k^{n-2}$ if n is fixed and $k \rightarrow \infty$, or like $\sim c_k n^{k-1}$ if k is fixed and $n \rightarrow \infty$.

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Rena Chu and Lillian B. Pierce

DEPARTMENT OF MATHEMATICS

DUKE UNIVERSITY

120 SCIENCE DRIVE

DURHAM, NC 27708, USA

email: rena.chu@duke.edu, pierce@math.duke.edu

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