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Continuous Patrolling Games

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We study a patrolling game played on a network Q, considered as a metric space. The Attacker chooses a point of Q (not necessarily a node) to attack during a chosen time interval of fixed duration. The Patroller chooses a unit speed path on Q and intercepts the attack (and wins) if she visits the attacked point during the attack time interval. This zero-sum game models the problem of protecting roads or pipelines from an adversarial attack. The payoff to the maximizing Patroller is the probability that the attack is intercepted. Our results include the following: (i) a solution to the game for any network Q, as long as the time required to carry out the attack is sufficiently short, (ii) a solution to the game for all tree networks that satisfy a certain condition on their extremities, and (iii) a solution to the game for any attack duration for stars with one long arc and the remaining arcs equal in length. We present a conjecture on the solution of the game for arbitrary trees and establish it in certain cases.

Key words: patrolling, zero-sum games, networks

1. Introduction

Patrolling games were introduced at the end of Alpern *et al.* (2011) to model the operational problem of how to optimally schedule patrols to intercept a terrorist attack, theft or infiltration. That paper, contrasting with earlier adversarial patrolling (Stackelberg) versions, modeled the problem as a zero-sum game between an Attacker and a Patroller, who wish to respectively maximize and minimize the probability of a successful attack. The domain on which the game was played out was taken to be a graph, with attacks restricted to the nodes and taking a given integer number of periods. A patrol is a walk on the graph, and intercepts the attack if it visits the attacked node during the attack period. This could model a guard in an art museum who enters a room while a thief is in the midst of removing a valuable painting from the wall. That paper was able to make some key observations about their game, giving bounds on the value, but was unable to find the value precisely or give optimal strategies except in some very limited cases. Papadaki *et al.* (2016) solved the game for line graphs, but the solution was very complicated even for this apparently simple graph. In the Conclusion section of the original paper Alpern *et al.* (2011), an extension of the problem to continuous space and time was suggested. The purpose of this paper is to carry out this suggestion.

We allow attacks that have a prescribed duration α to occur at any point of a continuous network Q. A unit speed patrol on Q is said to intercept the attack (and win for the Patroller) if it arrives at the attacked point at some time during the attack. The value of the game is the probability of interception, with best play on both sides. We find that optimal play for the Attacker typically involves mixing pure attacks that take place at different times.

After this type of continuous game was first proposed in 2011, it has been solved for some special networks. The circle network (or any Eulerian network) is easy to solve: a periodic traversal of the Eulerian tour, starting at a random point, is optimal for the Patroller; attacking starting at a fixed time at a uniformly random location is optimal for the Attacker (see Alpern *et al.* (2016) and Garrec (2019)). The line segment network was solved in Alpern *et al.* (2016). In Garrec (2019) a solution for some values of α is given for the network with two nodes connected by three unit length arcs, and a complete formulation of the general game is given, including a proof of the existence of the value. The present paper extends to some extent all three of these prior results to general classes of networks: Eulerian networks to networks without leaf arcs; the line segment network to trees; the three-arc network to networks with large girth - for small attack times. The Area Editor has observed that "in real-life the attacker has no incentive to hang out at the attack site - he would disappear as fast as he can following the attack. Therefore, small α is reasonable for many real-life situations."

Our main results and chapter organization are as follows. Section 3 presents several (mixed) strategies for the players that can be used or adapted to obtain solutions of the game for various classes of networks in later sections. We note that Eulerian networks have no leaves, and Section 4 generalizes the solution of the former to networks without leaves. In particular, as long as the attack time is sufficiently short, we show that the attack strategy that chooses a point uniformly at random is still optimal; an optimal strategy for the Patroller is to follow a double cover tour of the network which never traverses an arc consecutively in opposite directions (as described in Theorem 4). We also give a new algorithm for constructing such a tour in Theorem 3. In Section 5 we allow the network to have leaves, and modify the optimal strategies of the previous section to generate optimal strategies for arbitrary networks, as long as the attack time is sufficiently short (see Theorem 6).

Section 6 considers trees and in particular those that satisfy a condition we call the Leaf Condition. We give a precise definition of the condition, which requires some delicacy (Definition 8). In fact, any tree satisfies the Leaf Condition as long as the attack time is sufficiently short. Star networks (trees with only leaf arcs) also satisfy the Leaf Condition for sufficiently large attack times, and the only stars that do not satisfy the Leaf Condition are those that have an arc that is longer than half the total length of the network. In Theorem 7 we solve the game for all trees in the case that the Leaf Condition holds, giving a simple expression for the value of the game in terms of the length of the network, the attack time and another parameter. In Subsection 6.4 Conjecture 1 states that this expression is always equal to the value of the game on trees. We establish the conjecture for some stars that do not satisfy the Leaf Condition.

2. Literature Review

In addition to the papers discussed in the Introduction, which were the most relevant to continuous patrolling, there is a more extensive literature on adversarial patrolling. The problem of patrolling a perimeter has been analyzed by Zoroa *et al.* (2012) (where the attack location can move to adjacent locations) and Lin (2019), the latter in a continuous time context. Extensions of Alpern *et al.* (2016) where the costs of successful attacks are time and node dependent have been studied by Lin *et al.* (2013) (for random attack times), Lin *et al.* (2014) (with imperfect detection) and Yolmeh and Baykal-Gürsoy (2019) (which includes an application to an urban rail network).

Stackelberg approaches, with the Patroller as first mover, have been pioneered in an artificial intelligence context by Basilico *et al.* (2012) (which includes an algorithm for large cases) and Basilico *et al.* (2017) (where the optimal strategy in certain cases is for the Patroller to stay in place until the sensor reveals an attack an unknown location).

More applied approaches to patrolling are of practical importance. Applications to scheduling randomized security checks and canine patrols at Los Angeles Airport have been developed and deployed in Pita *et al.* (2008). The United States Coast Guard also uses a game-theoretic system to schedule patrols in the Port of Boston (An *et al.* 2013). Recently, a game theoretic approach to schedule patrols to guard against poachers has been explored in Fang *et al.* (2016) (where the novel algorithm PAWS was introduced) and Xu *et al.* (2019) (where the success of deploying PAWS in the field is described). Patrolling to detect radiation and consequently nuclear threats was modeled in the novel paper of Hochbaum *et al.* (2014).

The possibility that the Attacker could know when the Patroller is nearby (perhaps at the same node), raised in Alpern *et al.* (2011), has recently been studied in Alpern and Katsikas (2019), Alpern *et al.* (2021) and Lin (2019) in different contexts. In the former this knowledge helped the Attacker, in the latter, it did not. Multiple patrollers have been considered in the robotics and computer science literatures, where an important paper with a similar network structure to ours is Czyzowicz *et al.* (2017). A connection between patrols and inspection games is made in Baston and Bostock (1991) and between patrols and hide-seek games in Garrec (2019). Restricting the Patroller to periodic paths creates difficulties analyzed in Alpern *et al.* (2018).

3. Formal Definitions for Network and Game

In this section we define the continuous patrolling game and present definitions related to the connected network Q on which it is played. For Q, standard graph theoretic definitions must be modified for a network which is considered as a metric space and a measure space, not simply a combinatorial object.

To define Q, we begin with a graph G with edges and vertices, with the addition of a *length* $\lambda(e)$ assigned to each edge e. We can then identify an edge e with an open interval of length $\lambda(e)$, endowed with Lebesgue measure and Euclidean distance d, and consider λ as a measure on Q, called *length*. The total length of Q is denoted by $\mu = \lambda(Q)$. The topology on these intervals gives a topology on their union Q. A *path* in Q is a continuous function from a closed interval to Q. We take the metric $d(x, y)$ on Q as the minimum length of a path between x and y. A point x of Q is called a *regular* point if it has a neighborhood homeomorphic to an open interval. The remaining non-regular points are called *nodes*. The *degree* of a point y is defined as the number of connected components of a small neighborhood of y after y has been removed from it. Such a neighborhood is called a *punctured neighborhood* in the topology literature. A point of degree 2 is always by definition regular, and hence not a node. We say that two nodes of Q are *adjacent* if there is a path between them consisting only of regular points. Such a path is called an *arc*. A node of degree 1 is called a *leaf node*, and its incident arc is called a *leaf arc*. To ensure that every leaf arc has a single leaf node in its closure, we exclude the line segment network from consideration. In any case the continuous patrolling game has been solved for the line segment in Alpern *et al.* (2016).

A *circuit* in Q is a closed path (that is, with the same startpoint and endpoint) consisting of distinct adjacent arcs. A *tour* of Q is a closed path visiting all points of Q, and a tour of minimum length is called a *Chinese Postman Tour (CPT)*. The length of this path is denoted $\bar{\mu}$. It was shown by Edmonds and Johnson (1973) that a CPT can be found in polynomial time, with respect to the number of nodes. A closed path which is a circuit and a tour is called an Eulerian tour. As is well known, a connected network has an Eulerian tour if and only if it is Eulerian, defined as having nodes all of even degree. If we double every arc of a network Q, the resulting network is Eulerian with length 2μ, so Q has a tour of length 2μ and hence $\bar{\mu} \le 2\mu$.

The continuous patrolling game is played on Q as follows. The Attacker chooses a point x in Q to attack, and a closed time interval J of given length α during which to attack it. Since α is fixed, the *attack interval* $J = [\tau, \tau + \alpha]$ is determined by its starting time τ. The game and its value are determined by the pair (Q, α). The Patroller chooses a path $S(t)$, where $t \ge 0$, which we call a *patrol*, satisfying

$$d(S(t), S(t')) \le |t - t'|, \text{ for all } t, t' \ge 0. \tag{1}$$

For simplicity, we shall call a path satisfying the 1−Lipshitz condition (1) a *unit speed path*. We don't specify an upper bound on the starting time of the attack, but in every case we have studied there is an optimal mixed attack strategy in which all its (pure strategy) attacks are over by time 4μ. A patrol is said to *intercept* an attack if it visits the attacked point while it is being attacked. The game is very simply defined: the maximizing Patroller wins (payoff $P = 1$) if her patrol intercepts the attack. Otherwise, the Attacker wins (payoff $P = 0$ to the Patroller). The payoffs to the Attacker are reversed, so the game has constant sum 1. In other words, if the patrol is S and the attack is at point x during the interval $J = [\tau, \tau + \alpha]$, then the payoff P to the maximizing Patroller is given by

$$P(S, (x, J)) = \begin{cases} 1 \text{ if } x \in S(J), \\ 0 \quad \text{otherwise.} \end{cases}$$

For mixed strategies, the expected payoff can be interpreted as the probability that the attack is intercepted. The value of the game, denoted V, is the interception probability, with best play on both sides.

Garrec (2019) used the fact that P is lower semicontinuous to establish the existence of a value V for this infinite game. We note that if $\alpha = 0$ then the Attacker can win almost surely by attacking uniformly on Q (according to λ) at a fixed time; if $\alpha \ge \bar{\mu}$, the Patroller can ensure a win by adopting a Chinese Postman Tour, starting anywhere at time 0 and repeating the tour with period $\bar{\mu}$. So to avoid the trivial cases where one of the player can always win, we assume $0 < \alpha < \bar{\mu}$.

We follow Garrec (2019) in not imposing a finite time horizon. However, if we require that the attack ends by some time $T > \alpha$, this is only a restriction on the Attacker's strategy set. Hence, all Patroller estimates (lower bounds on the value) would remain valid. Attacker estimates (upper bounds on the value) also remain valid for sufficiently large T because all the optimal Attacker strategies presented in this paper end by a stated finite time. For example, the uniform attack strategy, discussed in the next subsection, ends by time $M + \alpha$, where M can be chosen arbitrarily.

Throughout the paper the complement $Q - Y$ of a set Y is denoted by Y^c.

3.1. The Uniform and the Independent Attack Strategies

Some networks, as we shall see in later sections, require Attacker strategies specifically suited to their structure, such as attacks on leaf nodes when the network is a tree. But there are also some general strategies that are available on any network. Here we define two of these and present the general bounds on the value that they give.

Definition 1 (**Uniform attack strategy**). A **uniform attack strategy** is a mixture of pure attacks that have a common attack time interval $J = [M, M + \alpha]$, where M can be chosen arbitrarily (for example $M = 0$). The attacked point is chosen uniformly at random. That is, the probability that the attacked point lies in a set Y is given by $\lambda(Y)/\mu$.

We restate a lemma from Alpern *et al.* (2016) for completeness (the proof is in the Online Appendix).

Lemma 1. *Against any patrol S, a uniform attack strategy is intercepted with probability not more than α/μ. Consequently $V \leq \alpha/\mu$ for any network.*

We now define independence for sets and strategies.

Definition 2 (**Independent set**). A subset I of Q is called **independent** if the distance between any two of its points is at least α. For any subset Y of Q, the set $W \equiv W(Y)$ is the subset of Q consisting of all points at distance at most $\alpha/2$ from Y.

Definition 3 (**Independent attack strategy**). Given an independent set I of cardinality l and the set $W \equiv W(I)$, the **independent attack strategy** is as follows for $p = \frac{l\alpha}{\lambda(W^c)+l\alpha}$.

1. With probability p attack at an element of I chosen equiprobably at a start time chosen uniformly at random in $J = [0, \alpha]$.

2. With probability $1-p$ attack uniformly on W^c at start time $\alpha/2$.

The independent attack strategy randomizes over both time and space, unlike the strategy of the same name defined in Alpern *et al.* (2011) for the discrete patrolling game, which randomizes only over space. The following result gives an upper bound on the strategy's interception probability.

Theorem 1. *Suppose I is an independent subset of Q of cardinality l. Then*

$$V \leq \frac{\alpha}{\lambda(W^c) + l\alpha},$$

which the Attacker can ensure by adopting the independent attack strategy. If $\lambda(W^c) = 0$ we have $V \leq 1/l$. Furthermore, if I is the set of leaf nodes, and leaf arcs have lengths exceeding $\alpha/2$, then

$$V \leq \frac{\alpha}{\mu + l\alpha/2}.$$

Proof. Let S denote any patrol and suppose the independent attack strategy is adopted. If S remains in W during J, it intercepts the attack with probability at most p/l, where l is the cardinality of I. Similarly, since S has unit speed, if it remains in W^c during time J, it intercepts an attack with probability at most $(1-p)(\alpha/\lambda(W^c))$. The chosen value of p is the one that makes these probabilities both equal to $\alpha/(\lambda(W^c) + l\alpha)$.

Finally, suppose the patrol S starts in W^c at time 0, reaches a point $x \in I$ at some time t, $\alpha \leq t \leq 2\alpha$, early enough to intercept some attacks on I and late enough to intercept some attacks on W^c. Since the latest such a patrol can leave W^c is at time $t - \alpha/2$, it can cover a set of length at most $(t-\alpha/2) - (\alpha/2) = t - \alpha$ in W^c after the attacks at time $\alpha/2$, intercepting a fraction $(t-\alpha)/\lambda(W^c)$ of the attacks there. In addition, the patrol can intercept the attacks at x starting between $t - \alpha$ and α, so a fraction $(2\alpha - t)/\alpha$ of the attacks at x, or $(2\alpha - t)/l\alpha$ of the attacks on

I. Thus the maximum probability that a patrol arriving at I at time t can intercept an attack is given by

$$(1-p)\,\frac{t-\alpha}{\lambda(W^c)}+p\frac{2\alpha-t}{l\alpha}=\frac{\alpha}{\lambda\,(W^c)+l\alpha.}$$

By time symmetry, the same bound holds if the patrol starts at a point of I and ends up in W^c.

If $\lambda\,(W^c)=0$ we have $V\leq 1/l$ trivially.

To prove the last assertion note that if I is the set of leaf nodes, and leaf arcs have lengths exceeding $\alpha/2$, then leaf nodes form an independent set I and $\lambda\,(W)=l\alpha/2$. □

3.2. A General Strategy Available to the Patroller

Some patrol strategies come from finding closed paths on the network with specific properties, and then have the Patroller go around them periodically starting at a random point. Normally the closed path will be a tour, but we give a more general definition in case it is not.

Definition 4 (Randomized periodic extension). If $S:[0,L]\rightarrow Q$ is a closed unit speed path, we can extend it to various patrols $S_\Delta:[0,\infty)\rightarrow Q$ of period L by the definition

$$S_\Delta\,(t)=S\,(\;(t+\Delta)\;\bmod\,L\;),\;\text{for all } t\geq 0.$$

Thus S_Δ is a periodic patrol that starts at the point $S\,(\Delta)$ at time 0. The **randomized periodic extension** $\tilde{S}$ of S is defined as the random mixture of the pure patrols S_Δ, with Δ chosen uniformly in the interval (or circle) $[0,L]$. In the special case that S is a Chinese Postman Tour, with $L=\bar{\mu}$, we call $\tilde{S}$ a Chinese Postman Tour strategy.

3.3. $k-$covering Tours and Identifying Points of Q

If a network Q has an Eulerian tour, its randomized periodic extension makes an effective patrolling strategy, because it visits all regular points equally often (once), so the Attacker is indifferent as to where to attack. If there is no Eulerian tour (the general case), we can still use this idea, if there is a tour which visits all regular points equally often. In Theorem 3 and Lemma 6, we will show that there is indeed such a tour which visits all regular points twice (a $2-$cover), with some additional properties. This idea is formalized in the following.

Theorem 2. *Suppose $S:[0,L]\to Q$ is a closed unit speed tour that visits every point of Q at k times which are separated by at least α (mod L). Suppose $\tilde{S}$ is the randomized periodic extension of S (from Definition 4). Then we have*

(i) $\tilde{S}$ intercepts any attack with probability at least $k\alpha/L$.

(ii) If $L=k\mu$, then the randomized periodic extension $\tilde{S}$ (for the Patroller) and a uniform attack strategy (for the Attacker) are optimal and the value of the game is given by α/μ.

Proof. For part (i), suppose the attack takes place at a point x in Q starting at some time τ. Let t_i, $i=1,\ldots,k$ be times, separated by at least α, such that $S(t_i)=x$. The attack will be intercepted by S_Δ if Δ is in the set $Y=\cup_i[t_i-\tau-\alpha,t_i-\tau]$ (modulo L), since in this case the Patroller will visit $x=S(t_i)$ at some time in $[\tau,\tau+\alpha]$. The separation assumption ensures that these intervals are disjoint, and since they all have length α, the length (Lebesgue measure) of Y is given by $|Y|=k\alpha$. By the definition of $\tilde{S}$, the probability that $\Delta\in Y$ is equal to $|Y|/L=k\alpha/L$, as claimed in (i), so we have $V\ge k\alpha/L=k\alpha/k\mu=\alpha/\mu$ under the assumption of part (ii). By Lemma 1, we also have that $V\le\alpha/\mu$, so the two inequalities give $V=\alpha/\mu$, with $\tilde{S}$ and the uniform attack strategy optimal. □

As suggested above in the introductory remarks of this subsection, taking $k=1$ in Theorem 2 gives another proof of the following elementary result of Alpern *et al.* (2016) and Garrec (2019).

Corollary 1. *If Q is Eulerian, with Eulerian tour S, then for $\alpha\le\mu$ we have $V=\alpha/\mu$. ($V=1$ if $\alpha\ge\mu$.) In this case the randomized periodic extension $\tilde{S}$ and the uniform attack strategy are optimal for the Patroller and Attacker, respectively. Furthermore, for a Chinese Postman Tour S of any network Q, taking $k=1$ and $L=\bar{\mu}$ gives $V\ge\alpha/\bar{\mu}$.*

It is useful to note for applications to patrolling by m robots, that if in Theorem 2 we require that S visits every point at k times separated by time intervals $m\alpha$, then m Patrollers can intercept any attack with probability at least $mk\alpha/L$ (or 1, if $mk\alpha/L\ge1$). To see this conclusion, pick Δ as above and let the path of the i'th Patroller (robot) be defined by $S_i(t)=S(\Delta+i(L/m)+t)$. The arrival times at any point of Q are then separated by at least α. This reasoning shows that in our later lower bounds for V, these can be multiplied by the number of Patrollers, with an upper bound of 1.

We conclude this section with an observation on the effect of identifying points of Q on the value. Alpern *et al.* (2011) considered the effect of identifying two nodes of a graph. Here, we identify two *points* of the network Q, using the well known quotient topology. In Figure 1 we identify the arc midpoints C and D of the network Q to produce a new network Q'.

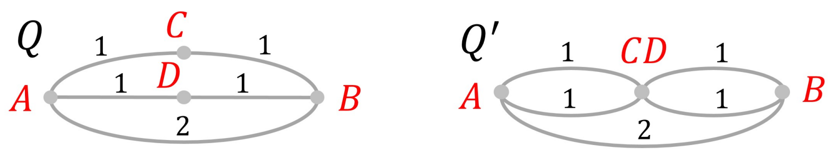

Figure 1 Identifying points C, D of Q to obtain Q'.

We may first look at two cases which have already been solved, the line segment $Q_{line} = [0,1]$ and the circle $Q_{circle} = [0,1] \mod 1$ (which is obtained from the line segment by identifying the endpoints), with say $\alpha = 1/2$. From Alpern *et al.* (2016), we have $V(Q_{line}) = \alpha/(\mu+\alpha) = 1/3$. However as the circle is Eulerian, we have $V(Q_{circle}) = \alpha/\mu = 1/2$, which is larger. It is easy to show that identifying points cannot decrease the value. Of course if we further identify points on the circle, we get new points of degree 4, so the resulting Eulerian network retains the value of 1/2.

Lemma 2. *Suppose Q', d' is the metric space obtained from Q, d by replacing the metric d with a smaller metric d', that is, with $0 \le d'(x,y) \le d(x,y)$ for all $x, y \in Q = Q'$. Then $V(Q', d') \ge V(Q, d)$. Furthermore, if Q' is obtained from Q by decreasing the length of an arc or simply identifying two points x and y, the same result holds.*

The proof of Lemma 2 is given in the Online Appendix. An application of it is given at the end of Section 4.

4. Networks Without Leaves

To extend Corollary 1 to general networks, we first note that Eulerian networks have no leaf arcs, so we attempt to find such a tour S satisfying the hypothesis of Theorem 2 for networks without leaf arcs. It turns out that taking $k = 2$ in Theorem 2 is high enough. We can find such a tour (see Theorem 4) if α is sufficiently small with respect to the *girth* g of Q, defined for networks as the minimum length of a circuit in Q, and if Q has no circuits then $g = \infty$. (For networks with unit length arcs, our definition of girth coincides with the usual integer definition of the girth of a graph.) Our first result is the following.

THEOREM 3. *For any network Q there is a tour S_2 which covers every arc twice and for which no arc is traversed consecutively in opposite directions, except for leaf arcs.*

Theorem 3 is not new; it was proved by Sabidussi (1977). See also Klavzar and Rus (2013) and Eggleton and Skilton (1984). We originally proved Theorem 3 independently and subsequently found it in the literature. Our proof, based on the new result, Lemma 3, is elementary.

The way we will prove Theorem 3 is to double every arc of Q to create an network $\hat{Q}$. Then $\hat{Q}$ is Eulerian and has an Eulerian tour. We note that in Euler's Theorem (finding an Eulerian tour in graphs of even degree), we can control to some extent the construction of the tour. The following refinement of Euler's Theorem (Lemma 3) is based on some simple modifications of the traditional proof and shows that we can control the pairing of entered and exited *passages* of the tour at every node. Formally, a *passage* at a node x is a pair (x, a), where a is an arc incident to x. So a node of degree d has d passages and every arc is part of two passages.

LEMMA 3. *Suppose Q is a connected Eulerian network such that at every node the passages are identified in pairs (they are "paired"). Then there is an Eulerian tour S of Q satisfying*

$$S \text{ never enters and leaves a node via paired passages.} \tag{2}$$

The proof of Lemma 3 can be found in the Online Appendix.

As mentioned at the beginning of Section 3 there are no nodes of degree 2. Thus, the minimum node degree in our Eulerian network is 4.

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Let $\hat{Q}$ be the Eulerian network obtained from Q by doubling every arc. (This action has the effect of replacing leaf arcs with loops of double the length.) At every node of $\hat{Q}$ we pair passages that correspond to the same passage of Q. Now apply Lemma 3 to $\hat{Q}$ to obtain an Eulerian circuit $\hat{S}$ of $\hat{Q}$ satisfying condition (2). The result is S_2, a *double cover of* Q (a tour of Q where every arc is traversed twice), in which consecutive arcs are distinct, except for leaf arcs. For loops, an arc may be repeated consecutively, but always in the same direction both times. □

The proof of Lemma 3 gives rise to an algorithm for constructing an Eulerian tour of $\hat{Q}$ satisfying condition (2), and hence a tour of Q of the form described in the statement of Theorem 3 (named S_2). Indeed, by following the rules listed in the proof of Lemma 3, we obtain a circuit C in $\hat{Q}$ satisfying (2); by recursively applying the rules to the connected components of $\hat{Q} - C$ and appending these circuits to C at appropriate points, we can obtain an Eulerian tour of $\hat{Q}$ satisfying (2).

We illustrate the creation of the $*$-circuit described above for the network K_4 depicted in Figure 2. Doubling each arc, we give the extra arc the same label as the original arc but with a prime. Applying the rules of the proof of Lemma 3, starting at the bottom left node, we obtain a circuit: $a, b, c, d, e, c', a', f, d'$. Removing this circuit leaves the network consisting of arcs b', e' and f', which is already a circuit. Adding this circuit at the first possible opportunity, we obtain the Eulerian tour $a, b', e', f', b, c, d, e, c', a', f, d'$.

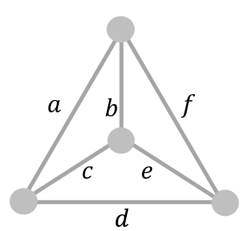

Figure 2 The network K_4.

Theorem 4. *Suppose Q is a network without leaf arcs. Then for $\alpha \leq g$, where g is the girth, we have the following:*

1. *The value of the game is $V = \alpha/\mu$.*

2. *For the Attacker, any uniform attack strategy is optimal.*

3. *For the Patroller, the randomized periodic extension $\tilde{S}_2$ is optimal, for any tour S_2 given by Theorem 3.*

Proof. Let S_2 be a tour of Q given by Theorem 3. Note that it has length $L = 2\mu$. Since there are no leaf arcs, any two consecutive arcs of S_2 are distinct. Suppose some point x of Q is reached by S_2 at consecutive times t and s with $t < s$. Let Z denote the restriction of S_2 to the interval $[t, s]$. Then Z is a circuit of length $s - t$ and hence $s - t \geq g$, by the definition of girth. Hence $V = \alpha/\mu$, by Theorem 2(ii) with $k = 2$ and since $\alpha \leq g$. □

For the network K_4 depicted in Figure 2, assuming all arcs have length 1, the girth g is 3. So for $\alpha \leq 3$, the uniform attack strategy is optimal and the Patroller strategy S_2 is optimal, where S_2 is the tour $a, b', e', f', b, c, d, e, c', a', f, d'$.

As a further example, consider Q to be a network with two nodes A and B connected by three arcs of lengths $a \leq b \leq c$. Then $g = a + b$ and $\mu = a + b + c$, so we have by Theorem 4 that the value is $V(\alpha) = \alpha/(a + b + c)$ for $\alpha \leq a + b$. This network, with $a = b = c = 1$ (and hence $g = 2$), was studied by Garrec (2019), who found (among other results) that $V(\alpha) = \alpha/3$ for $\alpha \leq 2$ and $V(\alpha) \leq f(\alpha) \equiv 1 - (1/3)(2 - \alpha/2)^2$ for $\alpha \in [2, 10/3]$. Since $f(\alpha) < \alpha/3$ for $\alpha \in (2, 10/3]$ ($f(\alpha) = \alpha/3$ for $a = 2$ and $f'(\alpha) = (4 - \alpha)/6 < 1/3$ for $\alpha > 2$), the Patroller cannot obtain an interception probability of $\alpha/3$ for α in this interval, so the bound $\alpha \leq g = 2$ in Theorem 4 is tight.

The condition $\alpha \leq g$ specified in Theorem 4 is a sufficient but not necessary condition. Consider a network Q_5 with two nodes connected by five arcs labeled as $1, 2, 3, 4, 5$, with arc i having length i. The girth is given by $g = g(Q_5) = 1 + 2 = 3$. However, suppose we obtain a double cover (with $k = 2$) S of Q described by the sequence $[1, 2', 3, 4', 5, 1', 2, 3', 4, 5']$, where unprimed arcs go from, say, node A to node B and primed arcs go from node B to node A. The shortest return time to a regular point is for a point x near node B on the arc of length 5. After leaving x, going to nearby B, the patrol traverses arcs of lengths $1 + 2 + 3 + 4 = 10$ before going back to x from B. Note that S returns to A after gaps of $3, 7, 6, 5$ and 9, so at two time points separated by 14 (at the start and after the gap of 6). Also B is visited twice separated by a gap of 14. So for the network Q_5 we have $V = \alpha/\mu$ for $\alpha \leq 10$ rather than just for $\alpha \leq 3$. This observation leads to combinatorial questions about the maximum shortest circuit in a k-cover of a network Q. As noted above based on Garrec's analysis of the three arc network, in certain cases $V = \alpha/\mu$ fails for all $\alpha > g$.

Now let Q be a network with two nodes connected by n arcs. If n is even, then Q is Eulerian and thus, by Corollary 1, $V = \alpha/\mu$ for all α. If n is odd then our example Q_5 generalizes easily to the following.

THEOREM 5. *Suppose Q is a network with two nodes connected by an odd number of arcs. Then $V = \alpha/\mu$ for $\alpha \leq \mu - D$, where D is the length of the longest arc.*

Proof. Label the arcs between the two nodes A and B as $a_1, \ldots, a_n$, in order of increasing length $b_1 \leq b_2 \leq \cdots \leq b_n$ where b_j is the length of arc a_j and $b_n = D$. We note that since the girth is given by $g = b_1 + b_2$, Theorem 4 says that $V = \alpha/\mu$ for $\alpha \leq g = b_1 + b_2$. We have to establish the stronger result that $V = \alpha/\mu$ for $\alpha \leq b_1 + b_2 + \cdots + b_{n-1} = \mu - D$. Following the construction of S for Q_5 given above, we define a double tour S of Q. Let j denote the traversal of arc a_j from A to B and j' denote the traversal of arc a_j from B to A. Let S be defined by the arc sequence $\left[1, 2', 3, 4', \ldots, (n-2), (n-1)', n, 1', 2, \ldots, n-1, n'\right]$. Returns to any regular point x of Q occur after traversing $n-1$ of the arcs once. So the shortest return occurs when the arc not traversed is the longest one, namely arc a_n of length $b_n = D$. So the shortest return time under S to any regular point is given by $b_1 + b_2 + \cdots + b_{n-1} = \mu - D$. So the double tour S reaches every regular point x twice at times separated by at least time $\mu - D$. So if $\alpha \leq \mu - D$ it reaches every regular point x twice at times separated by at least time α. Since the length of S is given by $L = 2\mu$, by Theorem 2, the value of the game is equal to α/μ.

We conclude this section with an application of our earlier result on identifying points.

EXAMPLE 1. *Consider the two networks Q and Q' drawn in Figure 1, with $\alpha = 3$. We would like to show that $V(Q') = \alpha/\mu = 3/6 = 1/2$. We know from Lemma 2 that $V(Q') \leq \alpha/\mu = 1/2$. So we only need $1/2$ as a lower bound on $V(Q')$. However we cannot apply Theorem 4 because it is not* *true that α is less than or equal to the girth of Q', which is 2. However we know either from Garrec (2019) or from Theorem 4 (which applies because $3=\alpha<g=4$) that $V(Q)=\alpha/\mu=1/2$. So by viewing Q' as coming from Q by identifying points C and D, Lemma 2 gives $V(Q')\geq V(Q)=1/2$.*

5. Brief Attacks on Arbitrary Networks

We now extend Theorem 4 to networks with leaves. We begin with a modified Patroller strategy based on the tour S_2 of Theorem 3.

Definition 5. Suppose S_2 is a tour given by Theorem 3. We denote by S_2^α the tour that follows the same trajectory as S_2 but stops for time α whenever it reaches a leaf node.

Lemma 4. *Suppose Q is a network with $l\geq 0$ leaf nodes and girth g. Then*

$$V\geq \frac{\alpha}{\mu+l\alpha/2},\ \textit{for } \alpha\leq g.$$

Proof. Tour S_2^α takes total time $2\mu+l\alpha$. Note that every point of Q is visited by S_2^α at two times differing by at least α. So by Theorem 2 part (i) with $k=2$, $L=2\mu+l\alpha$, we have $V\geq 2\alpha/(2\mu+l\alpha)$. (We observe that instead of stopping for time α, the tour S_2^α could do anything in this time interval, such as going away from the node a distance $\alpha/2$ and returning.) □

Definition 6 (**Generalized girth**). We define the **generalized girth** g^* of a network Q by considering a leaf arc of length L to be a circuit of length $2L$. So g^* is the smaller between (1) the shortest circuit length of Q and (2) twice the length of the shortest leaf arc.

In particular $g^*\leq g$, with equality if there are no leaf arcs or if all leaf arcs have length greater than $g/2$. Note that if $\alpha\leq g^*$ we know in particular that all leaf arcs have length at least $\alpha/2$ and hence Theorem 1 applies. Thus we have the following Attacker estimate (upper bound on V).

Lemma 5. *Suppose Q is a network with $l\geq 0$ leaf nodes and generalized girth g^*. Then by adopting the independent attack strategy on the set I of leaf nodes, the Attacker can ensure that the interception probability is less than $\frac{\alpha}{\mu+l\alpha/2}$ for $\alpha\leq g^*$. Hence,*

$$V\leq \frac{\alpha}{\mu+l\alpha/2},\ \textit{for } \alpha\leq g^*.$$

Proof. As noted above, the assumption on α ensures that all leaf arcs have length at least $\alpha/2$, so the result follows from Theorem 1. □

Since $g^* \leq g$, Lemmas 4 and 5 apply when $\alpha \leq g^*$ and hence we have the following extension of Theorem 4 to networks with leaf arcs.

THEOREM 6. *If Q is a network with $l \geq 0$ leaf nodes and generalized girth g^*, then*

$$V = \frac{\alpha}{\mu + l\alpha/2}, \text{ for } \alpha \leq g^*.$$

For the Patroller, an optimal strategy is S_2^α as defined above. For the Attacker, an optimal strategy is the independent attack strategy, taking I to be the independent set of leaf nodes.

Since g^* is always positive, Theorem 6 gives the solution of the game for some positive values of α on any network.

It is useful for later comparisons to specialize this result to trees.

COROLLARY 2. *If Q is a tree with l leaf arcs, then*

(i) $V \geq \frac{\alpha}{\mu + l\alpha/2}$,

(ii) with equality if all leaf arcs have length at least $\alpha/2$.

Proof. To establish (ii), note that trees have no circuits, so the generalized girth g^* is twice the length of its smallest leaf arc, so by assumption, $\alpha \leq g^*$. The result now follows from Theorem 6. For (i), consider the patrol S_2^α. Note that between any two visits by S_2^α to a point of Q, a leaf node is visited. Hence the return times exceed the time α that S_2^α stops at that node, and the result follows from Theorem 2(i) with $k = 2$ and $L = 2\mu + l\alpha$. □

For example, consider the tree Q depicted in Figure 3. The number of leaf arcs is $l = 5$, the generalized girth is $g^* = 2$ and total length is $\mu = 9$, so by Theorem 6, the value of the game is $\alpha/(9 + 5\alpha/2)$ for $\alpha \leq 2$. We will later solve the game for $\alpha \leq 4$, using Theorem 7.

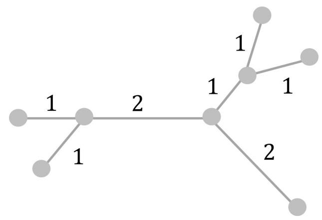

Figure 3 The tree Q.

6. Solving the Game for Trees

In Corollary 2 we gave some preliminary results for trees. Lemma 4 gave a lower bound on the value of the game based on the Patroller strategy S_2^α. Furthermore, for $\alpha \le g^*$, where g^* is the generalized girth, we showed in Theorem 6 that the independent attack strategy ensures that this lower bound is tight. Note that for a tree, g^* is twice the length of the shortest leaf arc. In this section, we extend these results and give optimal Patroller and Attacker strategies for some values of α which are greater than g^*. We start by defining the *extremity set* E, a subset of Q that is essential in describing optimal Patroller and Attacker strategies.

6.1. The Extremity Set E

The relationship between the network Q and the duration α of the attack interval determines the type of optimal player strategies. In this section we define the extremity set E that helps us explore this relationship for trees.

If B is a set of points then we denote by $\bar{B}$ the topological closure of B. If Q is a tree network, then its minimum tour time is 2μ, as every arc must be traversed twice. If x is a regular point of tree network Q, then $Q - \{x\}$ has two connected components $Q_1 = Q_1(x)$ and $Q_2 = Q_2(x)$, whose lengths satisfy $\lambda(Q_1) + \lambda(Q_2) = \lambda(Q) = \mu$. We introduce a subset E of Q called the *extremity set.*

Definition 7 (**The extremity set** E). Suppose Q is a tree. The extremity set $E \equiv E(Q, \alpha)$ is defined as the set of all regular points $x \in Q$ such that

$$\min_{i=1,2} \lambda(Q_i(x)) < \alpha/2. \tag{3}$$

Note that $\min_{i=1,2} \lambda(Q_i) \leq \mu/2$ and if additionally $\mu < \alpha$ then (3) holds for all regular points, which implies that $\bar{E} = Q$. The extremity set E consists of regular points whose minimum return time during a CPT is less than the attack duration α. It can be partitioned into maximal connected sets that we call *components* of E and we denote by E_j.

EXAMPLE 2. *We illustrate the extremity set E on the tree network of Figure 3 that has $\mu = 9$. Figure 4 shows how E changes for increasing values of α on this network. As α increases the components grow starting from points near the five leaf nodes of the tree. Initially there are five components (cases $\alpha = 1, 2, 3, 4$); but eventually points near non-leaf nodes become members of E and the number of components increase to seven (cases $\alpha = 5, 6, 7, 8$). Note that in case $\alpha = 8$ the closure $\bar{E}$ of E is equal to the whole network. The results from the previous sections (Theorem 6, Corollary 2) solve the game for cases $\alpha \leq g^* = 2$, but in this section we extend the results to cover all cases of $\alpha \leq 4$.*

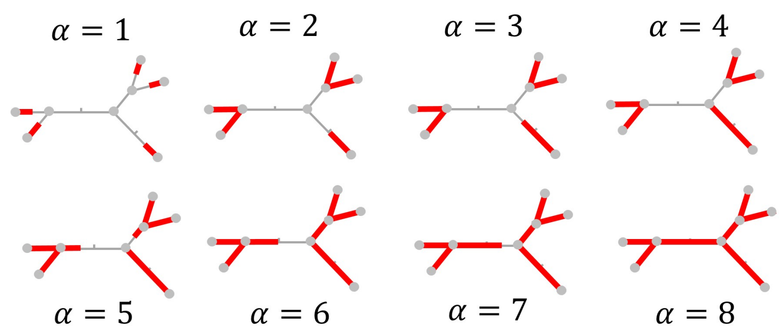

Figure 4 The extremity set $E(Q, \alpha)$, shown in thick (red) lines, for the tree Q of Figure 3 and $\alpha = 1, \ldots, 8$.

EXAMPLE 3. *Figure 5 depicts a star network. The extremity set E is depicted by red thick lines for attack time α, if the lengths of AD and AF are each greater than $\alpha/2$ and those of AB and AC are each less than or equal to $\alpha/2$. Note that the nodes indicated by the small disks are not part of E. Here, E decomposes into four components: (A, B), (A, C), (D, G), (F, H). We claim that $\lambda(DG) = \lambda(FH) = \alpha/2$; this is because on leaf arc AD (similarly for AF) if $\lambda(DG) < \alpha/2$ there would be a point X on the right of G whose distance from D would be $< \alpha/2$, implying $\lambda(DX) < \alpha/2$*

and thus contradicting $X \notin E$. Similarly, if $\lambda(DG) > \alpha/2$ there would be a point X on the left of G where $\lambda(DX) > \alpha/2$ contradicting $X \in E$. Thus, components E_j that are strict subsets of a leaf arc and whose closure contains the leaf node will have length $\alpha/2$. However, components E_j whose closure is the entire leaf arc (like AB and AC) must have length $\leq \alpha/2$; if they had length $> \alpha/2$ then there would be point X on the component AB near node A where $\lambda(BX) > \alpha/2$ contradicting $X \in E$.

Figure 5 A tree, with its extremity set E in thick red.

6.2. The E-patrolling Strategy S^E for Trees

We will see that for some trees, the uniform CPT strategy is still optimal for the Patroller, but its optimality depends on the size of the attack duration, α. As mentioned earlier, for a tree a CPT is simply any depth-first search which returns to its start point after completing its search, so that $\bar{\mu} = 2\mu$; every point of the tree except the leaf nodes is visited at least twice by a CPT. This means the leaf nodes and regular points near them are left "less protected" by a uniform CPT than the other points, and for sufficiently small values of α, there will be points in the tree whose two closest visit times (modulo $\bar{\mu}$) are at least time α apart, meaning that they are, in a sense "twice as protected" as the leaf nodes. (In all that follows, arithmetic on time will be performed modulo the length of the tour in question).

This observation motivates the introduction of a new Patroller strategy S^E for trees that we call the E-patrolling strategy. We construct it in such a way that each point is visited at least twice at times that differ by at least α, and then we use Theorem 2, part (i) to obtain a lower bound on the value. To describe the strategy, we use the extremity set $E \equiv E(Q, \alpha)$ that we defined earlier; in particular, we use the closure $\bar{E}$ of E and its components $\bar{E}^1, \ldots, \bar{E}^k$, each of which is a subtree of Q. We have $\lambda(\bar{E}) = \lambda(E)$ but by using the components of $\bar{E}$ rather than the components of E, we include the nodes and thereby unite adjacent components of E into a single component of $\bar{E}$. For example, in Figure 5 there are four components of E but only three components of $\bar{E}$, since in $\bar{E}$ the lines AB and AC join to form a single component BAC.

Let Q be a tree with $\bar{E} \neq Q$. We first construct a CPT S with the additional property that every component $\bar{E}^j$ is searched in a single CPT of $\bar{E}^j$, which we call C_j; note that some CPTs of Q might search different subsets of $\bar{E}^j$ during non-consecutive time intervals - we exclude this possibility by construction.

To obtain a CPT of Q with this property, we begin at any regular point not in $\bar{E}$ and go in either direction. When arriving at any node, we leave by a passage not already traversed, if there is such a passage. (This is the usual depth-first construction and ensures we obtain a CPT.) Furthermore, if the node belongs to some component $\bar{E}^j$ and there are untraversed passages staying in that component, we take one of these. For example, in Figure 5 if we start somewhere on GA going right, and tour the leaf arc to B from A, we must then take the passage to C (staying in component BAC) rather than the other untraversed passage out of A going to F. This rule ensures that the CPT say $ABAFACADA$ (in which the component BAC of $\bar{E}$ is not traversed in a single CPT of BAC) will not be constructed, but rather one like $[ABACA]FADA$, where the bracketed expression is a CPT of the component BAC.

Then we make two types of additions at every component. If $\lambda\left(\bar{E}^j\right) \geq \alpha/2$, we follow the CPT C_j of $\bar{E}^j$ in S by another identical one, before continuing with S. Note that this local CPT takes time $\geq \alpha$, so the time between the first and second CPT of $\bar{E}^j$ reaching any (regular) point is at least α.

If $\lambda\left(\bar{E}^j\right) < \alpha/2$ we wait until S comes back to $\bar{E}^j$ after the first occurrence of C_j in S, and then insert a second C_j. Let $[t_1, t_2]$ be the time interval during which S tours $\bar{E}^j$ so that $S(t_1) = S(t_2)$ and $t_2 - t_1 = 2\lambda(\bar{E}^j)$. We have $\alpha > t_2 - t_1$. In this case, we cannot simply tour $\bar{E}^j$ twice in succession, because some points in $\bar{E}^j$ will not be visited at two times that are at least time α apart. Let

$x = S(t_1) = S(t_2)$, and we claim that x is a (non-leaf) node of the network. For suppose not, so that x is a regular point, and let $x' \notin E$ be on the same arc with $\varepsilon := d(x,x') < \alpha/2 - \lambda(\bar{E}^j)$. Then the length of $S\left((t_1 - \varepsilon, t_2 + \varepsilon)\right)$, which is $\lambda(\bar{E}^j) + \varepsilon$, is less than $\alpha/2$. The set $S\left((t_1 - \varepsilon, t_2 + \varepsilon)\right)$ is a component of $Q - x'$ and by the definition of E, since the smaller component of $Q - x'$ has length less than $\alpha/2$, we have $x' \in E$, a contradiction. So x is a non-leaf node, and thus $Q - x$ has at least three components. If any component A of $Q - x$ has length less than $\alpha/2$, then its closure $\bar{A}$, which contains x, must be a subset of $\bar{E}$, and hence of $\bar{E}^j$ (since $x \in \bar{E}^j$). Hence, all components of $Q - x$ that are disjoint from $\bar{E}^j$ must have length at least $\alpha/2$. So the next time after t_2 that S arrives at x is $t_3 \geq t_2 + \alpha$, and the next time after t_3 that S arrives at x is at least $t_3 + \alpha$. Then S is updated by adding another tour of C_j at time t_3.

Observe that each additional local CPT of $\bar{E}^j$ takes time $2\lambda\left(\bar{E}^j\right)$, so the total length of the resulting tour S^E is $2\mu + 2\left(\sum_j \lambda\left(\bar{E}^j\right)\right) = 2\left(\mu + \lambda\left(\bar{E}\right)\right)$ and by construction it reaches every point of Q at two times separated by at least α (modulo the length of the tour). Note that if $\bar{E} = Q$, we simply take $S^E = S$. The optimal periodic strategy is thus S^E. For the network of Figure 5, taking S as $ABACADAFA$ we could have $S^E = ABACAGD\,[GDG]\,[ABACA]\,HF\,[HFH]\,A$, where the brackets indicate the three inserted local CPT's of the components of $\bar{E}$. Note that two of these are inserted right after their first occurrence, but the third one [ABACA] is inserted nonconsecutively. Our construction would not work directly on the CPT $ABAFACADA$.

Thus we have established the following result by explicit construction.

Lemma 6. *Suppose Q is a tree. Then there is a tour S^E, called an E-*patrolling strategy*, of length $2(\mu + \lambda(E))$ such that every point x of Q is visited at least twice at times that differ by at least α.*

We can obtain a lower bound on the value of the game obtained by using an E-patrolling strategy.

Lemma 7. *Suppose Q is a tree. Any E-patrolling strategy intercepts any attack with probability at least $v^* \equiv \alpha/(\mu + \lambda(E))$.*

Proof. Follows from Lemma 6 and Theorem 2 part (i) with $k = 2$, $S = S^E$, and $L = 2(\mu + \lambda(E))$. □

We conjecture the following on trees:

Conjecture 1 *If Q is a tree network, then for any α the E-patrolling strategy is optimal and the value of the game is $V = v^* \equiv \alpha/(\mu + \lambda(E))$.*

We later confirm the conjecture in some special cases.

Note that when $\alpha \leq g^*$ we have $\lambda(E) = l\alpha/2$, and the result of Lemma 7 becomes the same as the result of Corollary 2. In that case, the patrolling strategy S_2^α gives the same lower bound as an E-patrolling strategy.

6.3. The E-attack Strategy

In the previous section we showed that on a tree, any E-patrolling strategy intercepts any attack with probability at least v^*. Here, we define the *E-attack strategy*, whose attacks are intercepted with probability at most v^* on some trees. The condition that allows this strategy to be defined and to be optimal is given in Definition 8. It is useful to note that while for patrolling strategies we looked at the components of the closure $\bar{E}$ of E, for the attack strategy given here we look at the components of E itself.

Definition 8 (**Leaf Condition**). Suppose Q is a tree. We say that (Q, α) satisfies the **Leaf Condition** if the extremity set E consists of all points on every leaf arc within distance $\alpha/2$ of its leaf node.

For example, in Figure 4 the cases that satisfy the Leaf Condition are the first four ($\alpha = 1, 2, 3, 4$), where E consist of five components; all of these five components are subsets of leaf arcs and they are within $\alpha/2$ from the leaf node. Note that the Leaf Condition implies that every component E_j of E corresponds to a leaf node; this is easy to check in Figure 4. Cases $\alpha = 5, 6, 7, 8$ have seven components; five of these components are subsets of leaf arcs but two of them are subsets of non-leaf arcs and thus (Q, α) does not satisfy the Leaf Condition. (Recall that the extremity set does not contain nodes, thus the nodes separate the components.)

Definition 9 (E**-attack strategy**). Suppose (Q, α) satisfies the Leaf Condition, where Q is a tree. Let x_j denote the leaf node contained in the closure of the component E_j of E, and let $e_j = \lambda(E_j)$ and let $M = \max_j \lambda(E_j)$ be the maximum length of a component of E. We define the E**-attack strategy** as follows:

1. With probability $\lambda(E^c)/(\mu + \lambda(E))$, attack a uniformly random point of E^c at time M.
2. With probability $2e_j/(\mu + \lambda(E))$, attack at leaf node x_j at a start time chosen uniformly in the interval $[M - e_j, M + e_j]$.

Note that the Leaf Condition implies that $\sum_j e_j = \lambda(E)$, therefore the sum of the probabilities from 1. and 2. above sum to 1. Also, unlike the uniform attack strategy, the E-attack strategy is not synchronous. That is, the attack does not start at a fixed, deterministic time.

Example 4. *We revisit Figure 5, where the leaf arcs have lengths* $2, 1, 6, 6$ *and* $\alpha = 6$. *We illustrate the* E*-attack strategy on this star network in Figure 6. Here* $\mu = 15$*; the extremity set* E *is shown in thick red lines.* E *consists of four components that are subsets of leaf arcs and whose points are within* $\alpha/2$ *from the leaf node, thus the Leaf Condition is satisfied. Also, note that* $\lambda(E) = 9$ *and* $\mu + \lambda(E) = 24$. *The* E*-attack strategy then attacks as follows: with equal probabilities* $6/24$ *it attacks at nodes* D *and* F *with a starting time chosen uniformly on* $[0, 6]$*; with probabilities* $4/24$, $2/24$ *it attacks leaf nodes* B, C *with a starting time chosen uniformly on* $[1, 5]$, $[2, 4]$ *respectively; with probability* $6/24$ *it attacks uniformly on set* E^c *at time* $M = 3$.

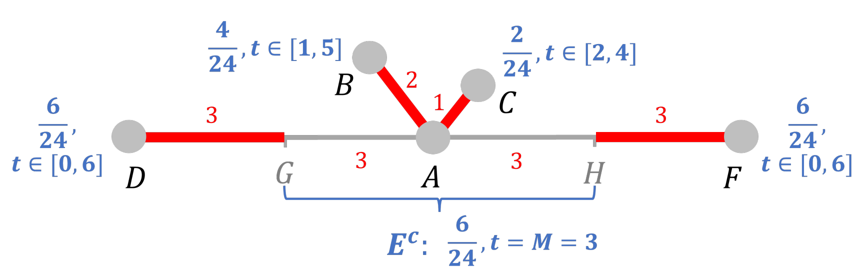

Figure 6 The E-attack strategy on an asymmetric star with arcs lengths 2,1,6,6 with $\alpha = 6$. The set E is shown in thick red lines.

We next prove that for trees Q, the E-attack strategy is optimal if (Q,α) satisfies the Leaf Condition.

LEMMA 8. *Suppose Q is a tree and (Q,α) satisfies the Leaf Condition. Then the E-attack strategy is intercepted by any patrol with probability at most $v^* = \alpha/(\mu+\lambda(E))$.*

The proof of Lemma 8 is in the Online Appendix. If we combine the results of Lemma 7 and Lemma 8 on patrolling and attack strategies for trees, we obtain the following exact result for the value of the game.

THEOREM 7. *Suppose Q is a tree and (Q,α) satisfies the Leaf Condition. Then any E-patrolling strategy is optimal, the E-attack strategy is optimal, and the value of the game is $V = v^*$.*

EXAMPLE 5. *We revisit the network Q from Figure 6 with $\alpha = 6$ and $\mu = 15$. We first consider patrolling strategies. The S_2^α patrolling strategy is $ADDABBACCAFFA$, where repeating a node means it stays there for duration α; this tour has length $2\mu + 4(6) = 54$. From Corollary 2 we have $V \geq \frac{\alpha}{\mu+l\alpha/2} = 6/27$. An E-patrolling strategy is $ADGDABACABACAFHFA$ with length $2\mu+2\lambda(E) = 48$; from Lemma 7 we have $V \geq v^* = \frac{\alpha}{\mu+\lambda(E)} = 6/24$. As we can see, an E-patrolling strategy, which is defined only for trees offers an improvement over the S_2^α patrolling strategy, which is a more general strategy.*

Now, we consider attacker strategies. Let I be the set of leaf nodes. The sets E and $W \equiv W(I)$ are shown in Figure 7 with solid thick red and dashed thick green lines respectively. Note that (Q,α) satisfies the Leaf Condition. The E-attack strategy is demonstrated in Figure 6 and it gives a lower bound, $v^* = \frac{\alpha}{\mu+\lambda(E)} = 6/24$, from Theorem 7, which is optimal. The bound given by Theorem 1 $\frac{\alpha}{\lambda(W^c)+l\alpha} = \frac{\alpha}{\mu+l\alpha/2} = 6/27$ does not hold in this case because I is not an independent set or, equivalently, leaf arcs do not have lengths exceeding $\alpha/2$.

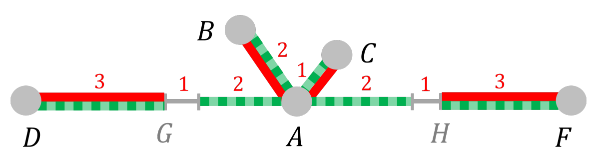

Figure 7 Star with arc lengths 6,6,2,1 and $\alpha = 6$. The solid thick red line is the set E and the thick dashed green line is the set $W \equiv W(I)$, where I is the set of leaf nodes; note that here I is not an independent set.

A star is a network consisting entirely of leaf arcs. We call a star *balanced* if no arc comprises more than half of its total length; otherwise we say that it is *skewed*. It is easy to check that balanced stars satisfy the Leaf Condition. All symmetric stars (whose arcs are all the same length) are balanced. An example of a skewed star is a star with n arcs of length 1 and one arc of length $x > n$, as shown in Figure 8; the long arc has length x, which is more than half of $\mu = n + x$.

It is also easy to see that if Q is a star (which may be balanced or skewed) whose longest arc has length at most $\alpha/2$, then $\bar{E} = Q$ and hence Q satisfies the Leaf Condition. So Theorem 7 gives the following.

Corollary 3. *Suppose Q is a star. Then the E-attack strategy and any E-patrolling strategy are optimal and the value of the game is $V = v^* = \alpha/(\mu + \lambda(E))$ if either*

(i) Q is balanced or

(ii) α is at least twice the length of the longest arc of Q.

Note that if Q is the line segment network, then by adding an artificial node in the center, we can apply Corollary 3, part (i), recovering the result for the value of this game, given previously in Alpern *et al.* (2016) (though the optimal strategies given here are different).

6.4. Stars Not Satisfying the Leaf Condition

In Lemma 7 we showed that the E-patrolling strategy intercepts any attack with probability at least $v^* = \alpha/(\mu + \lambda(E))$ and that (Lemma 8) for trees satisfying the Leaf Condition, the E-attack strategy avoids interception with probability at least v^*. Thus for trees we have $V = v^*$ if the Leaf Condition is satisfied, but what happens when it is not satisfied? In this subsection we present a class of trees Q for which the Leaf Condition fails for some values of α but nevertheless $V = v^*$ for all values of α. We do this by specifying particular attack strategies which are optimal on these trees.

We consider the class of skewed stars with n arcs of length 1 and one arc of length $x > n$, as shown in Figure 8. We refer to these skewed stars as *symmetric skewed stars*. The degree 1 nodes incident to the arcs of length 1 are denoted $a_1, \ldots, a_n$, the node of degree $n+1$ is denoted a_0 and the degree 1 node at the end of the arc of length x is denoted b. It is easy to see that symmetric skewed stars satisfy the Leaf Condition only for $\alpha \leq 2n$ and $\alpha \geq 2x$. In what follows we introduce attack strategies for these stars that guarantee v^* for the attacker for $2n \leq \alpha \leq 2x$, and thus show that Conjecture 1 holds for symmetric skewed stars for all values of α. Later, in Subsection 6.5 we give an attack strategy on a non-star tree that also guarantees the value v^* for the attacker and show that Conjecture 1 holds for this example.

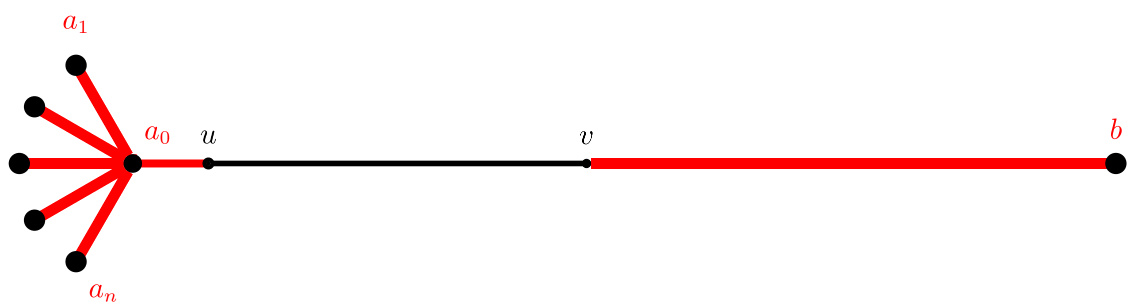

Figure 8 A symmetric skewed star. The extremity set E consists of the $n+2$ thick (red) lines. The black line is the set E^c.

We define an attack strategy that we will show is optimal for symmetric skewed stars for $2n \leq \alpha \leq 2x$. We note that for the a symmetric skewed star with $2n \leq \alpha \leq 2x$ it is easy to check that $\lambda(E) = \alpha$ if $2n \leq \alpha \leq \mu = x + n$ and $\lambda(E) = \mu$ (equivalently, $\lambda(E^c) = 0$) if $\alpha \geq \mu$. We denote the left and right boundary points of E^c with E by u and v respectively; since $2n \leq \alpha \leq 2x$, both of these points are on the long arc or on its boundary.

We note that the Leaf Condition for this star holds for $\alpha = 2n$ but not for $2n < \alpha \leq 2x$, thus the E-attack strategy is not defined for the latter set of values. Thus, we define a new attack strategy. For $\alpha = 2n$ either strategy can be used.

Definition 10 (**Symmetric-skewed attack**). The symmetric-skewed attack strategy is defined as follows:

Left attacks: With probability $(2\lambda(E)-\alpha)/(\mu+\lambda(E))$, attack equiprobably at nodes a_i, for $i=1,...,n$, starting uniformly at times in $[n-1,\alpha+n-1]$ if $2n\le\alpha\le x+n$ and at times in $[\alpha-x-1,x+2(n-1)+1]$ if $x+n\le\alpha\le 2x$.

Middle attacks: With probability $\lambda(E^c)/(\mu+\lambda(E))$, attack at a uniformly random point of E^c, starting equiprobably at times $\alpha/2+2j$ for $j=0,1,...,n-1$.

Right attacks: With probability $\alpha/(\mu+\lambda(E))$, attack node b, starting at a time in $[0,\alpha+2(n-1)]$ chosen as follows: conditional on the attack taking place here, the starting time is given by the following probability cumulative function. For $z=1,...,n-1$,

$$f(y)=\begin{cases}\frac{z(y-z+1)}{n\alpha} & \text{if } 2(z-1)\le y\le 2z,\\ \frac{n-1}{\alpha}+\frac{y-2(n-1)}{\alpha} & \text{if } 2(n-1)\le y\le\alpha,\\ \frac{\alpha-(n-1)}{\alpha}+\frac{z(z-1)}{n\alpha}+\frac{(y-\alpha)(n-z)}{n\alpha} & \text{if } \alpha+2(z-1)\le y\le\alpha+2z.\end{cases}$$

Note that when $\alpha\ge\mu$, we have $\lambda(E^c)=0$ so there are no middle attacks.

THEOREM 8. *Suppose Q is a symmetric skewed star. For any α the E-patrolling is optimal and the value of the game is $V=v^*=\alpha/(\mu+\lambda(E))$. If $x>n$ and $2n\le\alpha\le 2x$ then the symmetric-skewed attack strategy is optimal, otherwise the E-attack strategy is optimal.*

The proof of Theorem 8 can be found in the Online Appendix. Theorem 8 provides a counterexample to a conjecture in Alpern *et al.* (2016). The conjecture was that for trees, if α is at least the diameter of the network, the value of the game is $\alpha/\bar{\mu}=\alpha/(2\mu)$. For a symmetric skewed star, the diameter is $x+1$, and by Theorem 8, for $x+1\le\alpha<2x$, the value is $\alpha/(\mu+\lambda(E))$. This is not equal to $\alpha/(2\mu)$, since $\lambda(E)<\mu$ in that range of α, disproving the conjecture in Alpern *et al.* (2016).

6.5. A non-star tree with $\bar{E}=Q$ satisfying Conjecture 1.

We now consider the tree depicted in Figure 9 with unit length arcs and $\alpha=6$. This gives $\bar{E}=Q$ and thus $\lambda(E)=\mu$. Here $\mu=6$ and thus $v^*=\alpha/2\mu=1/2$.

We propose the following Attacker strategy for this specific tree with $\alpha=6$.

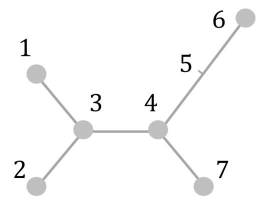

Figure 9 A tree with $\mu = 6$.

- At each leaf node 1 and 2 attack with probability 6/24 at a start time chosen uniformly in the interval $[0, 6]$ (total attack probability 12/24).
- At leaf node 6 attack with attack start time uniformly: in the interval $[0, 2]$ with probability 2/24, in the interval $[2, 4]$ with probability 4/24, in the interval $[4, 6]$ with probability 2/24 (total attack probability 8/24).
- At leaf node 7 attack takes place with probability 4/24 at a start time chosen uniformly in the interval $[1, 5]$ (total attack probability 4/24).

It is easy to verify that the probability of interception guaranteed by this strategy is $v^* = 1/2$, thus showing that the conjecture holds for this example; the proof is along the same lines as that of Theorem 8.

7. Conclusions

This paper models the problem of patrolling a pipeline or road system against attacks which can be made anywhere, not just at a discrete set of "targets". We do this by analyzing the continuous patrolling game on arbitrary metric networks Q, d, where d is the shortest path metric. The Attacker picks a point of Q to attack (not necessarily a node) during a chosen time interval of given length α. The Patroller chooses a unit speed path in the network and wins the game if the path crosses the attacked point during the attack; otherwise the Attacker wins. Mixed strategies are required for optimal play in this game, where the payoff to the maximizing Attacker is the probability that the attack is intercepted. Prior work of Alpern *et al.* (2016) and Garrec (2019) has solved the game for Eulerian networks, the line (or interval) network and a network consisting of two nodes connected by three arcs of certain lengths.

In this paper we show that for any network with total length μ and $l \geq 0$ leaf arcs, the value V of the game (probability that the attack is intercepted) is given by $V = \alpha / (\mu + l\alpha/2)$ when α is less than the minimum circuit length and also less than twice the length of any leaf arc. So the game is completely solved on *any* network for sufficiently small positive α. If there are no leaf arcs, the optimal patrol strategy reduces to a periodic cycle on the network which covers every arc exactly twice. (We give a new proof that such a cycle always exists.) Such a path is an efficient way of patrolling a network.

Of course many networks, for example museum corridors, have cul-de-sacs, which make them hard to patrol. Our general result, stated above, solves this problem for short attack durations α, but we also have results for larger durations. For networks which have a tree structure, we identify a useful technical property which implies that the value of the game is given by $V = v^*$, where $v^* = \alpha / (\mu + \Lambda)$ and Λ is the total length of certain points near the leaf nodes of the network. We conjecture that in fact $V = v^*$ for all trees. We show that our technical property (and hence $V = v^*$) holds for stars where no leaf arc has more than half the total length μ of the star. Finally, we show that for stars with a single arbitrarily long arc and the rest equal length short arcs, our conjecture $V = v^*$ holds. Star networks are important and often occur at airports where there is a central hub. The related problem of the "uniformed patroller" studied by Alpern and Katsikas (2019) and Alpern *et al.* (2021), where the presence of the Patroller at the node chosen for eventual attack can be detected by the Attacker, is studied in a spatial context that can be viewed as a star network.

The knowledge of our results would be useful in designing networks which are easier to patrol, as well as showing how to optimally patrol them. Even when the network is given, one might add additional links between some leaf nodes for the Patroller to use. A useful extension to this problem would be to make certain points of Q more valuable than others, so that successful attacks at such points are more costly to the Patroller and so would need to be patrolled more intensively.

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