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A Hölder-continuous Extended State Observer for Rigid Body Attitude Dynamics

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Abstract: Estimating rigid body attitude dynamics in the presence of unknown or uncertain torques, has applications to unmanned aerial, ground, (under)water and space vehicles. This work provides a new approach to estimating the attitude states and unknown, time-varying, (disturbance) torque vector acting on a rigid body, using an extended state observer. The observer design uses the concept of geometric homogeneity to obtain its stability. A Lyapunov stability analysis is carried out to prove its stability properties. The resulting observer for the attitude states and disturbance torque is smooth, Hölder-continuous, and exhibits almost globally finite-time stable (AG-FTS) with a constant disturbance torque in body frame. These properties are theoretically shown, and a numerical simulation is carried out to demonstrate how the observer works in a realistic attitude estimation scenario.

Keywords: Disturbance estimation, Nonlinear observers, Attitude Dynamics

1. INTRODUCTION

Extended state observers (ESO) are used to simultaneously estimate states of a system along with disturbance inputs acting on it. This is a topic of growing research interest in recent years in the areas of vehicle motion estimation and control. In particular, this is very important for estimation of disturbance torques acting on a vehicle's attitude dynamics.

Researches on this particular topic pursue three main approaches to ESO design. The first is to design a linear ESO that is designed to be asymptotically stable. Using this approach, Shao et al. (2018) design a linear ESO for a vehicle's translational and attitude dynamics to observe disturbance force and torque. The second approach utilizes the super-twisting algorithm Levant (1998) to design an ESO for attitude dynamics that is finite-time stable (FTS). An example of this approach is given in Liu et al. (2020). The third approach utilizes the geometric homogenity Bhat and Bernstein (2005); Rosier (1992); Guo and Zhao (2011). In recent years, this approach has been utilized for ESO design with FTS property for attitude dynamics Shao et al. (2019); Tian et al. (2018). In all of the prior literature on attitude dynamics ESO, the attitude kinematics and dynamics are either linearized locally or represented using local coordinates (like Euler angles) or quaternions. These representations can cause singularities for local coordinate representations (e.g., gimbal lock with Euler angles) and unwinding instability with quaternions Bhat and Bernstein (2000); Chaturvedi et al. (2011).

This work provides an ESO design to estimate the disturbance torque acting on a rotating rigid body. The proposed ESO can either be utilized to give disturbance torques estimations for disturbance rejection control, or as an attitude sensor, or actuator fault detector. The idea of geometric homogeneity is utilized in the provided ESO design. To avoid harmful chattering and oscillations, the signum function is not used in this ESO design. Unlike much of the existing research on attitude observer design that directly uses Euler angles or quaternion representations of attitude motion in vector spaces, this work represents attitude directly on the Lie group of rigid body rotations. Moreover, attitude information is obtained from vector measurements, which is the more common way to obtain attitude information from such on-board sensors as vision sensors, accelerometers and magnetometers. Based on Wahba's problem Wahba (1965), we use a Morse function Bullo and Lewis (2019) in the ESO design which is also directly utilized in its stability proof. This ESO is guaranteed to be singularity-free and proved to be almost globally finite-time stable (AG-FTS) with noise-free measurements and a constant disturbance torque. A simulation is carried out to show stable behavior of the ESO design with a disturbance torque that is varying with the attitude.

The remainder of this paper is organized as follows. Section 2 outlines the preliminaries on geometric homogeneity, and the framework used to represent rigid body attitude kinematics and dynamics. The static attitude determination problem from vector measurements is posed in Section 3. The framework for the ESO design for attitude dynamics is given in Section 4. Section 5 presents the observer law for the disturbance observer in details. A Lyapunov stability analysis for the ESO is also carried out

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in Section 5. Numerical simulation results of the proposed ESO are presented in Section 6. This numerical integration scheme preserves the geometry of the state space of rigid body rotational motion. Section 7 summarizes the results presented, and mentions related research directions to be pursued in the future.

2. PRELIMINARIES

2.1 Preliminaries on Homogeneous System

Definition 1. Guo and Zhao (2011) $V : \mathbb{R}^n \mapsto \mathbb{R}$ is called homogeneous of degree d with respect to weights $\{r_i{>}0\}_{i=1}^n$, if $\forall\lambda > 0$ and $\forall(x_1, x_2, ..., x_n)^{\mathrm{T}} \in \mathbb{R}^n$, there is : $V(\lambda^{r_1}x_1, \lambda^{r_2}x_2, ..., \lambda^{r_n}x_n) = \lambda^d V(x_1, x_2, ..., x_n)$.

Definition 2. A vector $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is called homogeneous of degree d with respect to weights $\{r_i{>}0\}_{i=1}^n$, if for all $\lambda > 0$ and for all $(x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, there is :

$$\begin{aligned} &F_i(\lambda^{r_1}x_1, \lambda^{r_2}x_2, ..., \lambda^{r_i}x_i, ..., \lambda^{r_n}x_n) \\ &\quad = \lambda^{d+r_i}F_i(x_1, x_2, ..., x_i, ..., x_n) \end{aligned} \tag{1}$$

If $V : \mathbb{R}^n \mapsto \mathbb{R}$ satisfies (1) and is differentiable with respect to x_i, then the partial derivative of V in x_i satisfies

$$\begin{aligned} &\lambda^{r_i}\frac{\partial}{\partial x_i}V(\lambda^{r_1}x_1, \lambda^{r_2}x_2, ..., \lambda^{r_i}x_i, ..., \lambda^{r_n}x_n) \\ &\quad = \lambda^d\frac{\partial}{\partial x_i}V(x_1, x_2, ..., x_i, ..., x_n) \end{aligned} \tag{2}$$

With the knowledge that V is homogeneous, (2) can be conveniently used to check the homogeneity of $\frac{\partial V}{\partial x_i}$.

Proposition 1. Bhat and Bernstein (2005) Suppose V_1 and V_2 are continuous real-valued functions on $\mathbb{R}^n$, homogeneous with respect to ν of degrees $l_1{>}0$ and $l_2{>}0$, respectively, and $V_1 > 0$. Then $\forall x \in \mathbb{R}^n$, there is:

$$\begin{aligned} &[\min_{z:V_1(z)=1} V_2(z)][V_1(x)]^{\frac{l_2}{l_1}} \le V_2(x) \\ &\quad \le [\max_{z:V_1(z)=1} V_2(z)][V_1(x)]^{\frac{l_2}{l_1}}. \end{aligned} \tag{3}$$

Proposition 2. Rosier (1992) Let f be a vector field satisfying $f \in C(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, f is homogeneous:

$$\begin{aligned} \forall\lambda > 0,\ &f_i(\lambda^{r_1}x_1, \lambda^{r_2}x_2, ..., \lambda^{r_i}x_i, ..., \lambda^{r_n}x_n) \\ &= \lambda^{\tau+r_i}f_i(x_1, x_2, ...x_i, ..., x_n) \end{aligned} \tag{4}$$

and the trivial solution $x = 0$ of system $\dot{x} = f(x)$ is locally asymptotically stable. Then let p be positive integer, and k be a real number larger than $p\cdot\max_{1\le i\le n} r_i$. There exists a function $V : \mathbb{R}^n \mapsto \mathbb{R}$ such that:

(i) $V \in C^p(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n\backslash\{0\}, \mathbb{R})$, and $V(0) = 0.$;
(ii) $\forall x \neq 0, V(x) > 0$ and $V(x) \mapsto +\infty$ as $||x|| \mapsto +\infty$;
(iii) V is homogeneous: $\forall\lambda > 0$,

$$V(\lambda^{r_1}x_1, \lambda^{r_2}x_2, ..., \lambda^{r_n}x_n) = V^k f_i(x_1, x_2, ..., x_n)$$

(iv) $\forall x \neq 0, \nabla V(x) \cdot f(x) < 0$.

2.2 Preliminaries on Special Orthogonal Group SO(3)

The set of possible attitudes of a rigid body is the special orthogonal group SO(3) Murray (1994), given by:

$$\mathrm{SO}(3) = \left\{R \in \mathbb{R}^{3\times3} | R^{\mathrm{T}}R = RR^{\mathrm{T}} = I,\ \det(R) = 1\right\}.$$

$\mathrm{SO}(3) \subset \mathbb{R}^{3\times3}$ is a matrix Lie group under matrix multiplication. The Lie algebra (tangent space at identity) of SO(3) is denoted $\mathfrak{so}(3)$ and defined as,

$$\mathfrak{so}(3) = \left\{S \in \mathbb{R}^{3\times3} \mid S = -S^T\right\},$$

which is identical to the set of 3×3 skew-symmetric matrices. Let $(.)^\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ denote the bijective map from three dimensional Euclidean space to $\mathfrak{so}(3)$. For a vector $s = [s_1\ s_2\ s_3]^{\mathrm{T}} \in \mathbb{R}^3$, the matrix $s^\times$ represents the vector cross product operator, that is $s \times r = s^\times r$, where $r \in \mathbb{R}^3$. The inverse of $(.)^\times$ is denoted $\mathrm{vex}(.) : \mathfrak{so}(3) \to \mathbb{R}^3$, such that $\mathrm{vex}(a^\times) = a$, for all $a^\times \in \mathfrak{so}(3)$. Define the trace inner product on $\mathbb{R}^{m\times n}$, $\langle\cdot,\cdot\rangle$, as $\langle A_1, A_2\rangle = \mathrm{tr}(A_1^{\mathrm{T}}A_2)$. Any square matrix $A \in \mathbb{R}^{n\times n}$ can be written as a sum of unique symmetric and skew-symmetric matrices as follows: $A = \mathrm{sym}(A) + \mathrm{skew}(A)$, where the symmetric and skew-symmetric components are defined as, $\mathrm{sym}(A) = \frac{1}{2}(A + A^{\mathrm{T}})$, and $\mathrm{skew}(A) = \frac{1}{2}(A - A^{\mathrm{T}})$. Additionally, the following property holds. Let $A_1 \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $A_2 \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix. Then, $\langle A_1, A_2\rangle = 0$.

2.3 System Kinematics and Dynamics

The dynamics model for rotational motion represented in the body-fixed frame, is based on eqs. (12)-(13) in Hamrah and Sanyal (2020). Let $\mathcal{I}$ denote an inertial frame that is spatially fixed. A body-fixed frame is fixed to the rigid body with its origin at the center of mass of the body, and is denoted $\mathcal{B}$. We denote the attitude of the rigid body by $R \in \mathrm{SO}(3)$, which transforms vectors in the body frame $\mathcal{B}$ to their counterparts in the inertial frame $\mathcal{I}$. The rigid body attitude kinematics and dynamics are given by:

$$\begin{cases} \dot{R} = R\Omega^\times \\ J\dot{\Omega} = J\Omega \times \Omega + \tau + \tau_d \end{cases}, \tag{5}$$

where $\Omega \in \mathbb{R}^3$ denotes the angular velocity vector of the vehicle in body-fixed frame $\mathcal{B}$, $J \in \mathbb{R}^{3\times3}$ represent the inertia tensor of the body, τ_d is the unmodeled and unknown (disturbance) dynamics and τ is the control torque acting on the rigid body.

For the convenience of disturbance observer design, the dynamics model is simplified from (5) as follows:

$$\begin{cases} \dot{R} = R\Omega^\times \\ \dot{\Omega} = J^{-1}(J\Omega \times \Omega + \tau) + \sigma \\ \dot{\sigma} = \delta. \end{cases} \tag{6}$$

The term $J^{-1}\tau_d$ is replaced by the term σ, which contains all of the unknown dynamics involved during the flight. The following assumption is made for this unknown dynamics acting on the vehicle.

Assumption 1. The rate of σ, $\dot{\sigma}$, is unknown but bounded, and satisfies the inequality: $\|\dot{\sigma}\| \le \bar{\delta}$. Further, $\bar{\delta}$ denotes the upper bound on $\|\delta\|$.

3. STATIC ATTITUDE DETERMINATION FROM VECTOR MEASUREMENTS

The aim of this section is to formulate the problem of attitude determination from vector measurements.

3.1 Vector Measurements

The rigid body attitude is determined from body-fixed measurements of k known inertial vectors. Let $e_1, e_2, \cdots e_k$, $k \in \mathbb{N}$ be the known inertial vectors and $u_1^m, u_2^m, \cdots u_k^m$ be the corresponding body-fixed measurements. The i^{th} vector measurement in the body-fixed frame $\mathcal{B}$ satisfies, $u_i^m = R^{\mathrm{T}}e_i + \sigma_i$, where $\sigma_i \in \mathbb{R}^3$ is the noise in the i^{th} vector measurement, for all $i \in 1, 2, \cdots k$. The attitude of the rigid body can be calculated from the vector measurements provided the following assumption is satisfied.

Assumption 2. There are at least three non-collinear vectors in the set $\{e_1, \cdots, e_k\}$ for attitude determination.

Define the matrix consisting of k known inertial vectors e_i as column vectors,

$$E = [e_1\ e_2\ \ldots e_k] \in \mathbb{R}^{3\times k}, \quad k > 2. \tag{7}$$

The assumption 2 can be alternatively specified as follows: matrix E should have rank equal to 3. The corresponding matrix composed of body-fixed measurements as column vectors can be defined as,

$$U^m = [u_1^m\ u_2^m\ \ldots\ u_k^m] \in \mathbb{R}^{3\times k}, \quad k > 2. \tag{8}$$

The matrix consisting of inertial vectors E and the matrix containing the body frame vectors U^m are related by:

$$U^m = R^{\mathrm{T}} E + \Xi \tag{9}$$

where the columns of matrix Ξ correspond to the measurement errors $\sigma_i \in \mathbb{R}^3$. Let the true vectors in body frame be denoted by $u_i = R^{\mathrm{T}} e_i$, then the matrix of the body vectors corresponding to the inertial vectors e_i is given by

$$U = R^{\mathrm{T}} E \tag{10}$$

in the absence of measurement errors.

3.2 Cost Function For Attitude Determination

The objective is to obtain an estimate of the attitude denoted by $\widehat{R} \in \mathrm{SO}(3)$ from k known inertial vectors $e_1, \ldots, e_k$ and corresponding measured vectors $u_1^m, \ldots, u_k^m$. The static attitude estimation can be formulated as an optimization problem as follows,

$$\text{Minimize}_{\widehat{R}}\, \mathcal{U} = \frac{1}{2}\sum_{i}^{k} w_i (e_i - \widehat{R}u_i^m)^{\mathrm{T}} (e_i - \widehat{R}u_i^m) \tag{11}$$

where $w_i > 0$ are weight factors. This is well-known in the relevant literature as Wahba's problem Wahba (1965). The cost function can be re-expressed as,

$$\mathcal{U} = \frac{1}{2}\left\langle E - \widehat{R}U^m, (E - \widehat{R}U^m)W \right\rangle \tag{12}$$

where $W = \mathrm{diag}([w_1,\ w_2,\ \ldots,\ w_k])$ and E and U^m are given by equations (7) and (8) respectively. The structure of the generalized cost function in the absence of measurement errors, is detailed in the following lemma.

Lemma 1. Izadi and Sanyal (2014) Define $Q = R\widehat{R}^{\mathrm{T}}$ as the attitude estimation error. Let $E \in \mathbb{R}^{3\times k}$ be as defined in (7) with $rank(E) = 3$. Let the gain matrix W of the generalized Wahba cost function be given by,

$$W = E^{\mathrm{T}}\left(EE^{\mathrm{T}}\right)^{-1} K \left(EE^{\mathrm{T}}\right)^{-1} E \tag{13}$$

where $K = \mathrm{diag}([k_1, k_2, k_3])$ and $k_1 > k_2 > k_3 \geq 1$. Then, in the absence of measurement errors,

$$\mathcal{U} = \frac{1}{2}\left\langle E - \widehat{R}U^m, (E - \widehat{R}U^m)W \right\rangle = \langle K, I - Q\rangle \tag{14}$$

is a Morse function on SO(3) whose critical points are given by the set,

$$\mathcal{C} = \{I, \mathrm{diag}([-1,-1,1]), \mathrm{diag}([1,-1,-1])\\ \mathrm{diag}([-1,1,-1])\} \tag{15}$$

In addition, $\mathcal{U}$ has a global minimum at $Q = I$.

4. PROBLEM FORMULATION FOR ESO ON SO(3)

4.1 Dynamic Attitude Estimation

With the kinematics and dynamics (6), let the measured angular velocity, denoted by Ω^m, be given by

$$\Omega^m = \Omega + \nu \tag{16}$$

where $\nu \in \mathbb{R}^3$ is the vector of additive noise in angular velocity components. Let $(\widehat{R}, \widehat{\Omega}) \in \mathrm{SO}(3) \times \mathbb{R}^3$ be the estimated attitude and angular velocity states provided by the estimation scheme, satisfying the following relation:

$$\dot{\widehat{R}} = \widehat{R}\widehat{\Omega}^{\times}, \quad \widehat{\Omega} = \Omega^m - \tilde{\Omega} \tag{17}$$

where $\tilde{\Omega} \in \mathbb{R}^3$ is the "excess" or error in estimating the angular velocity. In addition, define $\widehat{\sigma}$ to be the estimate of the disturbance torque, which is not directly measured.

The objective of the ESO is to obtain estimates of the attitude, angular velocity, and disturbance torque ($\widehat{R}$, $\widehat{\Omega}$ and $\widehat{\sigma}$) in real time, from the matrix of known inertial vectors E, the corresponding vector measurements made in the body-fixed frame U^m, and the biased angular velocity measurement Ω^m.

Lemma 2. Izadi and Sanyal (2014) Define $L = EW(U^m)^{\mathrm{T}}$. Let K be as defined in Lemma 1. Then, in the absence of measurement errors, the time derivative of $\mathcal{U}$ along the trajectories satisfying the kinematic equations (6) and (17), is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U} = \frac{\mathrm{d}}{\mathrm{d}t}\langle K, I - Q\rangle = s_K(Q)\cdot\left(\widehat{R}\widetilde{\Omega}\right) \tag{18}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}(K - L^{\mathrm{T}}\widehat{R}) = -s_L(\widehat{R})\cdot\widetilde{\Omega}, \tag{19}$$

where

$$\widetilde{\Omega} = \Omega - \widehat{\Omega} \tag{20}$$

$$s_K(Q) = \mathrm{vex}(KQ - Q^{\mathrm{T}}K) \tag{21}$$

$$s_L(\widehat{R}) = \mathrm{vex}(L^{\mathrm{T}}\widehat{R} - \widehat{R}^{\mathrm{T}}L). \tag{22}$$

With the above-defined $s_K(Q)$ and $s_L(\widehat{R})$, the following lemma is used to prove stability of the ESO in Section 5.

Lemma 3. Bohn and Sanyal (2015),Bullo and Lewis (2019) Let K be as defined in Lemma 1 and $s_K(Q)$ be as defined by equation (21). Let $\mathcal{S} \subset \mathrm{SO}(3)$ be a closed subset containing the identity in its interior, defined by

$$\mathcal{S} = \{Q \in \mathrm{SO}(3) : Q_{ii} \geq 0 \text{ and } Q_{ij}Q_{ji} \leq 0\\ \forall i,\ j \in \{1,2,3\},\ i \neq j\} \tag{23}$$

Then for $Q \in \mathcal{S}$, for $\theta \in]0,1[$ we have

$$s_K(Q)^{\mathrm{T}} s_K(Q) \geq \mathrm{tr}(K - KQ) \geq \theta s_K(Q)^T s_K(Q) \tag{24}$$

Lemma 4. Sanyal et al. (2019) Let $s_L(\widehat{R})$ and $s_K(Q)$ be as defined earlier. Then the following holds:

$$s_L(\widehat{R})^{\mathrm{T}} s_L(\widehat{R}) = s_K(Q)^{\mathrm{T}} s_K(Q) \tag{25}$$

Note that the attitude estimation error $Q = R\widehat{R}^{\mathrm{T}}$ is defined on the group of rigid body rotations, SO(3), not a vector space. The angular velocity estimation error, $\widetilde{\Omega}$, and disturbance error, $\widetilde{\sigma}$, are expressed on the vector space $\mathbb{R}^3$. Therefore, for Lyapunov stability analysis of the observer designed on $\mathrm{SO}(3)\times\mathbb{R}^3\times\mathbb{R}^3$, a Morse-Lyapunov function is required, where the Morse function $\mathcal{U} = \langle K, I - Q\rangle$ on SO(3) shall be used as the component of the Morse-Lyapunov function depending on the attitude component of the full state and disturbance. The Morse-Lyapunov function is subsequently used to guarantee convergence of state estimation errors $(Q, \widetilde{\Omega}, \widetilde{\sigma})$ to a small neighbourhood near $(I, 0, 0)$.

5. ATTITUDE ESO

The ESO design is described in details in this section, along with a stability analysis. The finite-time stability

(FTS) of the ESO is first shown for the case that the disturbance torque is constant in body frame and angular velocity measurement error is zero. Then finite-time input-to-state stability (FTISS) with time-varying disturbance torque and angular velocity measurement error is shown.

Following two definitions are provided to establish the backstepping error terms for the ESO design.

Definition 3. Define $z_L(\widehat{R}) : \mathrm{SO}(3) \to \mathbb{R}^3$ and $w_L(\widehat{R}, \widehat{\Omega}, \Omega^m) : \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, such that

$$z_L(\widehat{R}) = \big(s_L^{\mathrm{T}}(\widehat{R}) s_L(\widehat{R})\big)^{\frac{\alpha-1}{\gamma}} s_L(\widehat{R}) \tag{26}$$

$$\begin{aligned} w_L(\widehat{R}, \widehat{\Omega}, \Omega^m) &= \frac{\mathrm{d}}{\mathrm{d}t} s_L(\widehat{R}) \\ &= \mathrm{vex}\big(L^{\mathrm{T}} \widehat{R} \widehat{\Omega}^\times + \widehat{\Omega}^\times \widehat{R}^{\mathrm{T}} L\big) \\ &\quad - \mathrm{vex}\big(\widehat{R}^{\mathrm{T}} L (\Omega^m)^\times + (\Omega^m)^\times L^{\mathrm{T}} \widehat{R}\big). \end{aligned} \tag{27}$$

Definition 4. Define $H : \mathbb{R}^3 \to \mathrm{Sym}(3)$, the space of symmetric 3×3 matrix, as follows:

$$H(x) = I - 2\frac{(1-\alpha)}{\gamma} \frac{x x^{\mathrm{T}}}{x^{\mathrm{T}} x}. \tag{28}$$

γ and α in the Definition 3 and 4 are restricted as follows and the ESO is proposed.

Proposition 3. Let $\gamma \in \mathbb{Z}^+$ and $\frac{1}{2} < \alpha < 1$ satisfy $\gamma - 3\alpha + 1 \geq 0$ and let κ be sufficiently large. Define the auxiliary (backstepping) error:

$$\Psi = \tilde{\Omega} - \kappa z_L(\widehat{R}), \tag{29}$$

where z_L is provided in Definition 3.

Consider the rigid body rotational kinematics and dynamics given by (6), with known inertia J, control torque τ, attitude and angular velocity measurements. Then the following equations give an ESO for system (6):

$$\begin{aligned} \dot{\widehat{R}} &= \widehat{R} \widehat{\Omega}^\times = \widehat{R}(\Omega^m - \tilde{\Omega})^\times, \\ \dot{\widehat{\Omega}} &= J^{-1}(J\Omega^m \times \Omega^m + \tau) + \hat{\sigma} + (\Psi^{\mathrm{T}}\Psi)^{\frac{\alpha-1}{2}} k_\Psi \Psi \\ &\quad - \kappa (s_L(\widehat{R})^{\mathrm{T}} s_L(\widehat{R}))^{\frac{\alpha-1}{\gamma}} H(s_L(\widehat{R})) w_L(\widehat{R}, \widehat{\Omega}, \Omega^m), \\ \dot{\hat{\sigma}} &= k_\sigma (\Psi^{\mathrm{T}}\Psi)^{\alpha-1}\Psi, \end{aligned} \tag{30}$$

where s_L, H and w_L have been defined in Lemma 4, Definition 4 and 3 respectively.

With (6), (30) and the estimation errors defined earlier, the error dynamics of the ESO are given by:

$$\begin{aligned} \dot{Q} &= Q(\widehat{R}\tilde{\Omega})^\times, \\ \dot{\Psi} &= -k_\Psi (\Psi^{\mathrm{T}}\Psi)^{\frac{\alpha-1}{2}} \Psi + \tilde{\sigma}, \\ \dot{\tilde{\sigma}} &= -k_\sigma (\Psi^{\mathrm{T}}\Psi)^{\alpha-1}\Psi + \delta. \end{aligned} \tag{31}$$

Theorem 1 gives the stability of the error dynamics (31) without the terms δ and ν, which are the time derivative of disturbance torque σ and the angular velocity measurement error respectively. Lemma 5 is presented here to support the stability proof .

Lemma 5. Hardy et al. (1952) Let $x, y \geq 0$ and let $p \in]1, 2[$. Then $x^{(1/p)} + y^{(1/p)} \geq (x+y)^{(1/p)}$.

Theorem 1. Consider the auxiliary system given by (32) below, where α is as defined in Proposition 3:

$$\begin{aligned} \dot{Q} &= Q(\widehat{R}\tilde{\Omega})^\times \\ \dot{\Psi} &= -k_\Psi (\Psi^{\mathrm{T}}\Psi)^{\frac{\alpha-1}{2}} \Psi + \tilde{\sigma} \\ \dot{\tilde{\sigma}} &= -k_\sigma (\Psi^{\mathrm{T}}\Psi)^{\alpha-1}\Psi. \end{aligned} \tag{32}$$

In the absence of measurement errors, the attitude, angular velocity and disturbance estimation errors $(Q, \tilde{\Omega}, \tilde{\sigma})$ for this system converge to $(I, 0, 0) \in \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3$ in a finite time stable (FTS) manner, from almost all initial conditions except those in a set of measure zero.

proof 1. The proof of this result is based on the main results of Rosier (1992) (given by Proposition 2) and Bhat and Bernstein (2005) (given by Proposition 1), both stated in Section 2. Similar methodology is previously used in Wang and Sanyal (2021). Define the vectors $s = [\Psi^{\mathrm{T}}, \tilde{\sigma}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^6$ and $f(\Psi, \tilde{\sigma}) \in \mathbb{R}^6$, such that the stability proof of the last two equations in (32) is reduced to the stability proof of:

$$\dot{s} = [\dot{\Psi}^{\mathrm{T}}, \dot{\tilde{\sigma}}^{\mathrm{T}}]^{\mathrm{T}} = f(\Psi, \tilde{\sigma}), \tag{33}$$

where $f(\cdot, \cdot)$ is given by the RHS of the last two equations in (32). Define the Lyapunov candidate $V_a(\Psi, \tilde{\sigma})$ for the stability analysis of (33) as:

$$V_a(\Psi, \tilde{\sigma}) = \frac{1}{2}\Big[\frac{k_\sigma}{\alpha}(\Psi^{\mathrm{T}}\Psi)^\alpha + \tilde{\sigma}^{\mathrm{T}}\tilde{\sigma}\Big] \tag{34}$$

The Lie derivative of $V_a(\Psi, \tilde{\sigma})$ along the vector field $f(\Psi, \tilde{\sigma})$, $L_f V_a$, is obtained as follows:

$$L_f V_a = \dot{V}_a(\Psi, \tilde{\sigma}) = -\frac{k_\sigma}{\alpha} k_\Psi (\Psi^{\mathrm{T}}\Psi)^{\frac{\alpha+1}{2}} \leq 0. \tag{35}$$

From (35), (33) and the invariance principle, we see that the auxiliary system (33) is asymptotically stable. According to Definition 2, it can also be verified that $f(\Psi, \tilde{\sigma})$ is a homogeneous vector field of degree $\alpha - 1$ with respect to the weight vector $\{1, \alpha\}$. According to Proposition 2, there exists a homogeneous Lyapunov function $V_b(\Psi, \tilde{\sigma})$ of degree $\gamma \in \mathbb{N}$ such that $L_f V_b = \nabla V_b(\Psi, \tilde{\sigma}) \cdot f(\Psi, \tilde{\sigma}) < 0$. Although V_b is not given explicitly by this result, through Proposition 2 in Rosier (1992), V_b can be obtained from V_a. According to Definition 1, the real-valued function $L_f V_b = \nabla V_b(\Psi, \tilde{\sigma}) \cdot f(\Psi, \tilde{\sigma})$ is homogeneous of degree $\gamma + \alpha - 1$. Thus,

$$\begin{aligned} \dot{V}_b(\Psi, \tilde{\sigma}) &= \frac{\partial V_b}{\partial \Psi}[\tilde{\sigma} - k_\Psi (\Psi^{\mathrm{T}}\Psi)^{\frac{\alpha-1}{2}} \Psi] - k_\sigma \frac{\partial V_b}{\partial \tilde{\sigma}} (\Psi^{\mathrm{T}}\Psi)^{\alpha-1}\Psi \\ &< -c_1 V_b^{\frac{\gamma+\alpha-1}{\gamma}}. \end{aligned} \tag{36}$$

Further, according to Proposition 1, we get:

$$c_1 = -\max_{\{\Psi, \tilde{\sigma} : V_b(\Psi, \tilde{\sigma}) = 1\}} \nabla V_b(\Psi, \tilde{\sigma}) \cdot f(\Psi, \tilde{\sigma}). \tag{37}$$

Note that $\nabla V_b(\Psi, \tilde{\sigma}) \cdot f(\Psi, \tilde{\sigma})$ is negative definite, and therefore $c_1 > 0$. Since $\frac{1}{2} < \alpha < 1$, the auxiliary system (33) is proved to be FTS at the origin $(\Psi, \tilde{\sigma}) = (0, 0)$. Now define the Morse-Lyapunov function V_c:

$$V_c(Q, \Psi, \tilde{\sigma}) = V_c = V_b + \mathcal{U}. \tag{38}$$

With Ψ converging to the origin in finite time, the following equation is true after finite time: $\tilde{\Omega} = \kappa z_L(\widehat{R})$. Combining with Lemmas 2, 3, 4 and 5, the time-derivative of V_c can be obtained as follows:

$$\begin{aligned} \dot{V}_c &\leq -c_1 V_b^{\frac{\gamma+\alpha-1}{\gamma}} - \tilde{\Omega}^{\mathrm{T}} s_L(\widehat{R}) \\ &\leq -\min\{c_1, \kappa\}(V_b + \langle K, I - Q\rangle)^{\frac{\gamma+\alpha-1}{\gamma}} \end{aligned} \tag{39}$$

Based on (39) and Theorem 1 of Sanyal and Bohn (2015), the system (32) is proved to be AG-FTS. □

Now with the term δ, the reduced error dynamics (33) is augmented to the following equations.

$$\dot{s} = [\dot{\Psi}^{\mathrm{T}}, \dot{\tilde{\sigma}}^{\mathrm{T}}]^{\mathrm{T}} = f(\Psi, \tilde{\sigma}) + [0_{1\times 3}, \delta^{\mathrm{T}}]^{\mathrm{T}} \tag{40}$$

The following theorem is obtained for system (40).

Theorem 2. Consider the ESO design in Theorem 1, c_1 as defined in (37) and c_2 defined as:

$$c_2 = \max_{\{\psi,\tilde{\sigma}:V_b(\psi,\tilde{\sigma})=1\}} \|\frac{\partial V_b}{\partial \tilde{\sigma}}\|. \tag{41}$$

If the initial condition $V_b(\Psi_0, \tilde{\sigma}_0)$ satisfies:

$$V_b^{\frac{\gamma-3\alpha+1}{\gamma}}(\Psi_0, \tilde{\sigma}_0) < \frac{2c_1}{c_2} \tag{42}$$

system (40) is Finite-Time Input-to-State Stable (FTISS).

proof 2. Use the Lyapunov function V_b in Theorem 1 for the observer error dynamics. Note that V_b is homogenous of degree γ with respect to $\{1, \alpha\}$. According to Definitions 1 and 2, it can be further concluded that $|\frac{\partial V_b}{\partial \tilde{\sigma}}|$ is homogeneous of degree $\gamma - \alpha$ with repect to $\{1, \alpha\}$. Applying Proposition 1, the following inequality holds:

$$\|\frac{\partial V_b}{\partial \tilde{\sigma}}\| < c_2 V_b^{\frac{\gamma-\alpha}{\gamma}}, \quad c_2 > 0. \tag{43}$$

Now considering the term δ to be non-zero we obtain the time derivative of V_b for the system (40) as:

$$\begin{aligned}\dot{V}_b(\Psi, \tilde{\sigma}) &= \frac{\partial V_b}{\partial \Psi}[\tilde{\sigma} - k_\Psi(\Psi^{\mathrm{T}}\Psi)^{\frac{\alpha-1}{2}}\Psi] \\ &\quad + \frac{\partial V_b}{\partial \tilde{\sigma}}[-k_\sigma(\Psi^{\mathrm{T}}\Psi)^{\alpha-1}\Psi] + \frac{\partial V_b}{\partial \tilde{\sigma}}\delta \\ &\leq -c_1 V_b^{\frac{\gamma+\alpha-1}{\gamma}} + \|\frac{\partial V_b}{\partial \tilde{\sigma}}\|\|\delta\|.\end{aligned} \tag{44}$$

With γ and α as defined in Proposition 3, $\gamma - 3\alpha + 1 \geq 0$ and inequality (43), by applying the Cauchy-Schwarz inequality to (44), it can be deduced that:

$$\begin{aligned}\dot{V}_b(\Psi, \tilde{\sigma}) &\leq -c_1 V_b^{\frac{\gamma+\alpha-1}{\gamma}} + \frac{c_2}{2}(V_b^{\frac{2(\gamma-\alpha)}{\gamma}} + \|\delta\|^2) \\ &\leq (-c_1 + \frac{c_2}{2}V_b^{\frac{\gamma-3\alpha+1}{\gamma}})V_b^{\frac{\gamma+\alpha-1}{\gamma}} + \frac{c_2\|\delta\|^2}{2}\end{aligned} \tag{45}$$

From (45), when V_b at the initial time $V_b(\Psi_0, \tilde{\sigma}_0)$ satisfies (42), system (40) can be derived to be FTISS, as defined in Hong et al. (2010). This implies that s converges to a small neighbourhood around the origin in finite time.

Remark 1. Hong et al. (2010) Based on Theorem 2, for the solution of the auxiliary system (40), $\exists T > 0$ such that $\forall t > T$: $\|s(t)\| \leq \gamma(\sup_{0<\tau<t} \|\delta\|) = \gamma(\bar{\delta})$, where $\gamma(\cdot)$ is a class-$\mathcal{K}$ function.

Corollary 1. If the initial value of the Morse-Lyapunov function $V_c(Q_0, \Psi_0, \tilde{\sigma}_0) = V_b(\Psi_0, \tilde{\sigma}_0) + \langle K, I - Q_0\rangle$ satisfies:

$$V_b^{\frac{\gamma-3\alpha+1}{\gamma}}(\Psi_0, \tilde{\sigma}_0) < \frac{2c_1}{c_2}, \quad \langle K, I - Q_0\rangle^{\frac{1-\alpha}{\gamma}} < \frac{1}{2}\kappa, \tag{46}$$

then the error dynamics (31) converges to the origin $(I, 0, 0)$ in an almost global FTISS (AG-FTISS) manner.

proof 3. With $\|s(t)\| \leq \gamma(\bar{\delta})$ from Theorem 2 and as stated in Remark 1, it can be further concluded that $\|\Psi\| \leq \|s\| \leq \gamma(\bar{\delta})$. Now consider the Morse-Lyapunov function V_c. The time derivative of V_c satisfies the following inequality:

$$\begin{aligned}\dot{V}_c =& \dot{V}_b - \tilde{\Omega}^{\mathrm{T}} s_L(\widehat{R}) \\ \leq& (-c_1 + \frac{c_2}{2}V_b^{\frac{\gamma-3\alpha+1}{\gamma}})V_b^{\frac{\gamma+\alpha-1}{\gamma}} + \frac{c_2\|\delta\|^2}{2} \\ & - [\kappa z_L(\widehat{R}) + \Psi]^{\mathrm{T}} s_L(\widehat{R})\end{aligned} \tag{47}$$

By applying Lemmas 3 and 4 and Definition 3, it can be further derived that:

$$\begin{aligned}\dot{V}_c \leq& (-c_1 + \frac{c_2}{2}V_b^{\frac{\gamma-3\alpha+1}{\gamma}})V_b^{\frac{\gamma+\alpha-1}{\gamma}} - \kappa\langle K, I - Q\rangle^{\frac{\gamma+\alpha-1}{\gamma}} \\ & + \frac{c_2\|\delta\|^2}{2} + \|\Psi\|\|s_L(\widehat{R})\|\end{aligned} \tag{48}$$

Now applying the Cauchy-Schwarz inequality again for the term $\|\Psi\|\|s_L(\widehat{R})\|$, one obtains:

$$\begin{aligned}\dot{V}_c \leq& (-c_1 + \frac{c_2}{2}V_b^{\frac{\gamma-3\alpha+1}{\gamma}})V_b^{\frac{\gamma+\alpha-1}{\gamma}} - \kappa\langle K, I - Q\rangle^{\frac{\gamma+\alpha-1}{\gamma}} \\ & + \frac{c_2\bar{\delta}^2}{2} + \frac{1}{2}\Psi^{\mathrm{T}}\Psi + \frac{1}{2}s_L(\widehat{R})^{\mathrm{T}}s_L(\widehat{R}).\end{aligned} \tag{49}$$

By applying Lemma 3, one obtains

$$\begin{aligned}\dot{V}_c \leq& (-c_1 + \frac{c_2}{2}V_b^{\frac{\gamma-3\alpha+1}{\gamma}})V_b^{\frac{\gamma+\alpha-1}{\gamma}} - \kappa\langle K, I - Q\rangle^{\frac{\gamma+\alpha-1}{\gamma}} \\ & + \frac{c_2\bar{\delta}^2}{2} + \frac{1}{2}\gamma(\bar{\delta})^2 + \frac{1}{2}\langle K, I - Q\rangle.\end{aligned} \tag{50}$$

Applying Lemma 5, (50) leads to:

$$\begin{aligned}\dot{V}_c \leq& (-c_1 + \frac{c_2}{2}V_b^{\frac{\gamma-3\alpha+1}{\gamma}})V_b^{\frac{\gamma+\alpha-1}{\gamma}} + \frac{1}{2}c_2\bar{\delta}^2 + \frac{1}{2}\gamma(\bar{\delta})^2 \\ & + (-\kappa + \frac{1}{2}\langle K, I - Q\rangle^{\frac{1-\alpha}{\gamma}})\langle K, I - Q\rangle^{\frac{\gamma+\alpha-1}{\gamma}} \\ \leq& -c(V_b, Q)V_c^{\frac{\gamma+\alpha-1}{\gamma}} + \frac{1}{2}c_2\bar{\delta}^2 + \frac{1}{2}\gamma(\bar{\delta})^2,\end{aligned} \tag{51}$$

where $c(V_b, Q) = \min\{c_1 - \frac{c_2}{2}V_b^{\frac{\gamma-3\alpha+1}{\gamma}}, \kappa - \frac{1}{2}\langle K, I - Q\rangle^{\frac{1-\alpha}{\gamma}}\}$, which is a time-dependent coefficient. With the given initial condition (46) and Theorem 1, we can conclude that $\forall t > 0$, $c(V_b, Q) > 0$. Therefore, from (51), we conclude that the error dynamics (31) is AG-FTISS. □

6. NUMERICAL SIMULATION

The ESO is numerically implemented using a geometric numerical integration scheme. Unlike commonly used numerical integration methods like Runge-Kutta, geometric integration schemes Lee et al. (2005); Nordkvist and Sanyal (2010) preserve the geometry of the state-space without any projection or parameterization. Let $\mathcal{P}_i$ and $\mathcal{Q}_i$ denote the right-hand sides of the last two equations for $\widehat{\Omega}$ and $\widehat{\sigma}$ in (30) at time step t_i. Let $h = t_{i+1} - t_i$ be the time step size. The initial state estimates for this simulation are chosen as $\widehat{R}_0 = I, \widehat{\Omega}_0 = [0, 0, 1]^{\mathrm{T}}, \widehat{\sigma}_0 = [2, 3, 5]^{\mathrm{T}}$. The ESO parameters are selected as: $\alpha = 0.85$, $\gamma = 2$, $\kappa = 0.15$ and $k_\sigma = k_\Psi = 5$. The inertia matrix of the simulated rigid body is $J = \mathrm{diag}([4, 4, 10])$ kg-m^2. The attitude motion of this simulated rigid body is stabilized by a control torque τ given by: $\tau = -1/2\mathrm{vex}(K_c R - R^{\mathrm{T}} K_c)$, where $K_c = \mathrm{diag}([0.1, 0.2, 0.3]^{\mathrm{T}})$. The disturbance torque acting on the rigid body is selected as

$$\tau_d = [0.2\sin(0.2\pi t); 0.1\sin(0.4\pi t); 0.5\cos(0.1\pi t)](\mathrm{N \cdot m}).$$

The initial attitude and angular velocity of the rigid body for this simulation, are selected as follows: $R_0 = \exp(([\pi, 0, 0]^{\mathrm{T}})^\times), \Omega_0 = [1, 0.5, 0]^{\mathrm{T}}$rad/s. Four inertial vectors are considered to be measured at a constant rate by body-fixed sensors with an additive uniform random noise of $0.05 \times \mathrm{rand}(3, 4)$ m. The angular velocity measurement is assumed to have random noise of $0.05 \times \mathrm{rand}(3, 1)$ rad/s.

The simulation results are illustrated in Fig. 1, 2 and 3. As can be observed from these results, the stability and convergence of attitude state and disturbance torque estimation errors to a neighborhood of $(Q, \tilde{\Omega}, \tilde{\sigma}) = (I, 0, 0)$ agree with the analytically-shown properties of the ESO scheme designed in the previous section.

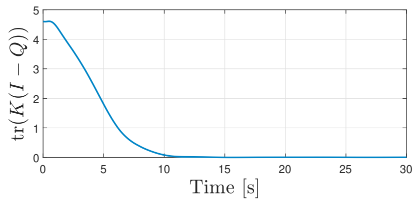

Fig. 1. Attitude estimation error

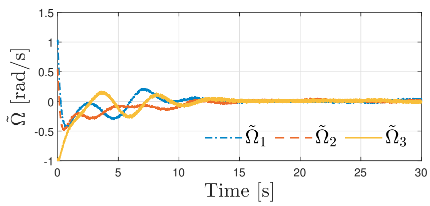

Fig. 2. Angular velocity estimation error

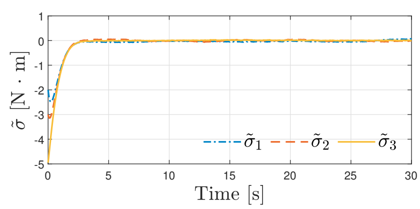

Fig. 3. Disturbance torque estimation error

7. CONCLUSION AND FUTURE WORK

This work provides a new design of an extended state observer (ESO) to estimate attitude states and an unknown torque affecting the attitude dynamics of a rigid body. The finite-time stability of this ESO is established using Propositions 1 and 2, which are based on the concept of geometric homogeneity. The ESO is proved to be AGFTS with noise-free measurements and a constant disturbance torque. Furthermore, when the disturbance torque changes with time or states, the ESO is proved to be AG-FTISS for certain ranges of observer design parameters. Numerical simulation results support the theoretical results and show the robustness of the proposed FTS-ESO as an estimation scheme for states and disturbance inputs. Continuing work will look into ESO design for coupled translational and rotational motion of rigid bodies.

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