IEEE Control Systems Letters paper presented at
2023 American Control Conference (ACC)
San Diego, CA, USA. May 31 - June 2, 2023

A Distributed Time-Varying Optimization Algorithm For Networked Lagrangian Agents Generating Continuous Control Torques

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Abstract— **In this paper, the distributed time-varying optimization problem is investigated for networked Lagrangian systems with parametric uncertainties. Due to the usage of the signum function in the control torque design, there might exist chattering while implementing the distributed time-varying optimization algorithms for networked Lagrangian agents in the existing works. To this end, we design a distributed optimization algorithm that is capable of generating continuous control torques and achieving exact optimum tracking. A simulation is presented to validate the effectiveness of the proposed algorithm.**

I. Introduction

In distributed optimization of networked systems, each member has a local cost function, and the objective is to cooperatively minimize the sum of all the local cost functions. There are a number of distributed optimization algorithms proposed in the literature (e.g., [1] and the references therein). Such results consider the cases where the agents have time-invariant local cost functions. However, the cost functions might be time-varying in practical applications (e.g., the economic dispatch problem [2]), and then the optimal point is changing over time and forming an optimal trajectory. Therefore, it is of great importance to investigate the distributed time-varying optimization problem.

In the literature, there are extensive distributed discrete-time algorithms (e.g. [3]–[5]) that solve the time-varying optimization problem. There usually exist bounded convergence errors to the optimal trajectory by using the discrete-time algorithms. There is another body of literature on distributed continuous-time optimization algorithms for time-varying cost functions. These distributed continuous-time optimization algorithms have various applications in practice. One important application lies in the coordination of a team of robots, where each robot's dynamics are described by differential equations. The distributed time-varying optimization problem is solved with exact optimum tracking for networked single-integrator agents [2], [6]–[8], double-integrator agents [9], and agents with nonlinear dynamics [10]. In reality, a broad class of robots can be modeled by Lagrangian dynamics, for example, the planar elbow manipulator and autonomous vehicles [11]. The Lagrangian dynamics are more complicated than single and double integrators, and are different from and cannot be included as special cases by the model considered in [10]. The complexity of the dynamics creates more challenges to solve the distributed optimization problem, especially the one with time-varying cost functions.

The works [12]–[14] focus on solving the distributed time-invariant optimization problem for networked Lagrangian agents. The idea behind these results is to introduce distributed observers at a higher level, where the agents communicate their observer states with their neighbors such that the observer states reach consensus on the desired trajectory. Then control algorithms are designed for the agents to track the observer states. The case considered in this paper is different from the ones in [12]–[14]. We consider the distributed optimization problem with time-varying cost functions in quadratic form, and design a distributed algorithm for Lagrangian agents to achieve exact optimum tracking. In the proposed algorithm, reference systems are constructed by using the mixture of physical and reference states, and the adaptive controllers are designed so that the physical velocities can track the reference states, and hence physical positions track the optimal trajectory.

The structure of the proposed algorithm is inspired by [15], [16], where the consensus and/or leader-following tracking of networked systems are addressed. However, the problem considered in this paper is more complex and challenging, and includes the consensus and leader-following tracking of networked agents as special cases. The proposed algorithm has a similar structure as in our previous work [17], and hence, it inherits some features from the one in [17]. It is worth noting that the algorithm in [17] generates control torque directly from the signum function and would cause chattering during implementation, which is undesirable for physical implementation. Although the proposed algorithm in the current paper also utilizes the signum function, the control torque is generated from a term that is essentially the integral of the signum function, and the resulting control torque is continuous. The continuity of the control torques is particularly beneficial for physical control systems for enhancing the reliability and safety (e.g., the motors of the system are subject to much mild actions when generating continuous torques while an abrupt action is often involved with chattering/discontinuous torques). In order to generate continuous control torques for the agents, one might replace

The work of Y. Ding and W. Ren was supported by National Science Foundation under Grant ECCS-2129949, and J. Mei's work was supported by the National Natural Science Foundation of China under Grant 62073098 and the Foundation Research Project of Shenzhen under Grants JCYJ20200109113210134 and JCYJ20210324132215038.

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the signum function with its continuous time-invariant or -varying approximation functions [18], [19], and such a method is referred to as the approximation method. However, the approximation method as an engineering practice might compromise between the alleviation of chattering and theoretical rigor. Our result ensures both the theoretical rigor and continuity of the control torques.

II. Preliminaries

A. Notations

Throughout this paper, let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the sets of all real numbers and all nonnegative real numbers, respectively. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$, and for a real number $x \in \mathbb{R}$, $|x|$ denotes the absolute value of x. The transpose of matrix A is denoted by A^T. For a given vector $x = [x_1, \ldots, x_p]^T \in \mathbb{R}^p$, define $\|x\|_1 = \sum_{i=1}^p |x_i|$, $\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_p|^2}$, and $\|x\|_\infty = \max_{i=1,\ldots,p} |x_i|$. For a symmetric matrix $A \in \mathbb{R}^{p\times p}$, let $\lambda_1(A), \cdots, \lambda_p(A)$ denote its eigenvalues. Let $\otimes$ and $\overline{\text{co}}$ denote the Kronecker product and the convex closure, respectively. Let $\text{diag}\{A_1, \ldots, A_p\}$, where $A_i \in \mathbb{R}^{n\times m}$, represent the block diagonal matrix with the i-th block in the main diagonal being A_i. For a vector $x \in \mathbb{R}^p$, define $\text{sgn}(x) = [\text{sgn}(x_1), \ldots, \text{sgn}(x_p)]^T$ where $\text{sgn}(x_i) = 1$ if $x_i > 0$, $\text{sgn}(x_i) = 0$ if $x_i = 0$, and $\text{sgn}(x_i) = -1$ if $x_i < 0$. Let $\mathbf{0}$ and $\mathbf{1}$ denote the zero and all-ones vectors/matrices with appropriate dimensions, respectively. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. For a time-varying signal x, let the k-th derivative of x be denoted by $x^{(k)}$, where k is a non-negative integer, and in particular, $x^{(0)} = x$. For a time-varying function $f : \mathbb{R}^p \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, its gradient, denoted by $\nabla f(q, t) \in \mathbb{R}^p$ with $q \in \mathbb{R}^p$ and $t \in \mathbb{R}_{\geq 0}$, is the partial derivative of $f(q, t)$ with respect to q, and its Hessian, denoted by $H(q, t) \in \mathbb{R}^{p\times p}$, is the partial derivative of the gradient $\nabla f(q, t)$ with respect to q. Define $\mathcal{L}_\infty = \{x \mid \sup_{t\geq 0} \|x(t)\|_\infty < \infty\}$ and $\mathcal{L}_2 = \{x \mid \sqrt{\int_0^\infty x^T(t)x(t)\text{d}t} < \infty\}$. A continuous function $\varpi : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\varpi(0) = 0$.

B. Graph Theory

For a multi-agent system consisting of N agents, the interaction topology can be modeled by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the node set and edge set, respectively. An edge denoted by $(i, j) \in \mathcal{E}$, means that agent i and j can obtain information from each other. In an undirected graph, the edges (i, j) and (j, i) are equivalent. It is assumed that $(i, i) \notin \mathcal{E}$. The neighbor set of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ of the graph G is defined such that $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. For an undirected graph, $a_{ij} = a_{ji}$. The Laplacian matrix $L = [L_{ij}] \in \mathbb{R}^{N\times N}$ associated with the adjacency matrix A is defined as $L_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$ and $L_{ij} = -a_{ij}$ for $i \neq j$. By arbitrarily assigning an orientation for every edge, let $B = [B_{ij}] \in \mathbb{R}^{N\times|\mathcal{E}|}$ denote the incidence matrix associated with graph $\mathcal{G}$, where $B_{ij} = -1$ if edge e_j leaves node i, $B_{ij} = 1$ if it enters node i, and $B_{ij} = 0$ otherwise. An undirected path between node i_1 and i_k is a sequence of edges of the form (i_1, i_2), (i_2, i_3), $\ldots$, (i_{k-1}, i_k), where $i_k \in \mathcal{V}$. A connected graph means that there exists an undirected path between any pair of nodes in $\mathcal{V}$.

Assumption 1: The graph $\mathcal{G}$ is connected.

C. Agents' Dynamics

In this paper, we consider N Lagrangian systems, whose dynamics are given by [11]

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i, \quad i \in \mathcal{V}, \tag{1}$$

where $q_i \in \mathbb{R}^p$ is the generalized position (or configuration), $M_i(q_i) \in \mathbb{R}^{p\times p}$ is the inertia matrix, $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{p\times p}$ is the Coriolis and centrifugal matrix, $g_i(q_i) \in \mathbb{R}^p$ is the gravitational torque, and $\tau_i \in \mathbb{R}^p$ is the exerted control torque. Three well-known properties associated with the dynamics (1) are listed as follows [11], [20].

Property 1: The inertial matrix $M_i(q_i)$ is symmetric and uniformly positive definite, and there exist positive constants $k_{\bar{C}}$ and $k_{\bar{g}}$ such that $\|C_i(q_i, \dot{q}_i)\|_2 \leq k_{\bar{C}} \|\dot{q}_i\|_2$ and $\|g_i(q_i)\|_2 \leq k_{\bar{g}}$, $\forall i \in \mathcal{V}$.

Property 2: The Coriolis and centrifugal matrix $C_i(q_i, \dot{q}_i)$ can be suitably chosen such that the matrix $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew-symmetric.

Property 3: The dynamics (1) depend linearly on an unknown constant parameter vector $\vartheta_i \in \mathbb{R}^m$, that is, $M_i(q_i)x + C_i(q_i, \dot{q}_i)y + g_i(q_i) = Y_i(q_i, \dot{q}_i, y, x)\vartheta_i$ holds for any $x, y \in \mathbb{R}^p$, where $Y_i(q_i, \dot{q}_i, y, x)$ is the regressor matrix.

III. Problem Statement

In the distributed time-varying optimization problem, each Lagrangian agent aims to cooperatively track the optimal trajectory $q^*(t) \in \mathbb{R}^p$ determined by the group objective function, which is defined as

$$q^*(t) = \underset{q(t)}{\arg\min} \left\{ \sum_{i=1}^N f_i[q(t), t] \right\}, \tag{2}$$

where $f_i[q(t), t] : \mathbb{R}^d \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the local cost function associated with agent i. It is assumed that $q^* \in \mathcal{L}_\infty$, which can be satisfied in most applications in practice. It is assumed that $f_i[q(t), t]$ is known only to agent i. Note that $\sum_{i=1}^N f_i[q(t), t] = \sum_{i=1}^N f_i[q_i(t), t]$ if $q_i(t) = q_j(t) = q(t)$ for all $i, j \in \mathcal{V}$, and hence to find $q^*(t)$ defined in (2) is equivalent to find the optimal solution

$$\{q_1^*(t), \ldots, q_N^*(t)\} = \underset{\{q_1(t),\ldots,q_N(t)\}}{\arg\min} \left\{ \sum_{i=1}^N f_i[q_i(t), t] \right\},$$
$$\text{Subject to } q_i(t) = q_j(t) \quad \forall i \neq j,$$

where $q_i^*(t) = q_j^*(t) = q^*(t)$ $\forall i \neq j$. In this paper, the goal is to design the control torques τ_i, $i \in \mathcal{V}$, for the agents (1) such that each agent's physical position $q_i(t)$ is capable of tracking $q^*(t)$, i.e., $\lim_{t\to\infty}[q_i(t) - q^*(t)] = \mathbf{0}_p$, $\forall i \in \mathcal{V}$. We make the following assumption on the cost functions.

Assumption 2: Each cost function $f_i(q_i, t)$, $i \in \mathcal{V}$, is twice continuously differentiable both in $q_i \in \mathbb{R}^p$ and t, and strongly convex in q_i and uniformly in t. That is, the Hessian

$H_i(q_i,t)$ is always positive definite and there exists a positive constant $\underline{m}$ such that $\lambda_j[H_i(q_i,t)] \geq \underline{m}$ $\forall j \in \{1,\dots,p\}$, $\forall i \in \mathcal{V}$ holds uniformly in t. In addition, each $H_i(q_i,t)$ is upper-bounded, i.e., $\|H_i(q_i,t)\|_2 \leq \bar{m}$ $\forall i \in \mathcal{V}$.

Lemma 1: [21] Let $f(x) : \mathbb{R}^p \to \mathbb{R}$ be a continuously differentiable convex function with respect to x. The function $f(x)$ is minimized at x^* if and only if $\nabla f(x^*) = \mathbf{0}_p$.

IV. Main Results

A. Algorithm

For each agent $i \in \mathcal{V}$, define

$$F_i(q_i,t) = H_i^{-1}(q_i,t)\Big[\frac{\partial}{\partial t}\nabla f_i(q_i,t) + \nabla f_i(q_i,t)\Big]. \quad (3)$$

Note that $\ddot{F}_i(q_i,t)$ is a function of q_i, $\dot{q}_i$, $\ddot{q}_i$ and t, denoted by $\widetilde{F}_i(q_i,\dot{q}_i,\ddot{q}_i,t)$. Construct a reference for each agent i as

$$\begin{aligned}\ddot{r}_i = &-\sum_{j\in\mathcal{N}_i}[\alpha(q_i-q_j)+\beta(\dot{q}_i-\dot{q}_j)+\alpha(\dot{r}_i-\dot{r}_j)] - \dot{q}_i \\ &- \dot{r}_i - \gamma\sum_{j\in\mathcal{N}_i}\mathrm{sgn}(q_i-q_j+\dot{q}_i-\dot{q}_j+\dot{r}_i-\dot{r}_j) + \varphi_i, \quad (4)\end{aligned}$$

where α, β, and γ are positive constants to be determined, and φ_i is defined as

$$\varphi_i = -F_i(q_i,t) - \dot{F}_i(q_i,t) - \widetilde{F}_i(q_i,\dot{q}_i,\dot{r}_i,t). \quad (5)$$

Define

$$s_i = \dot{q}_i - r_i. \quad (6)$$

The adaptive controller for the system (1) is given by

$$\tau_i = -K_i s_i + Y_i(q_i,\dot{q}_i,r_i,\dot{r}_i)\hat{\vartheta}_i, \quad (7)$$

$$\dot{\hat{\vartheta}}_i = -\Gamma_i Y_i^T(q_i,\dot{q}_i,r_i,\dot{r}_i)s_i, \quad (8)$$

where K_i and Γ_i are symmetric positive definite matrices, and $\hat{\vartheta}_i$ is the estimate of ϑ_i.

Remark 1: In brief, each agent has a virtual reference system (i.e., (4)) driven by both the physical states and the reference states to generate a reference signal tracking the optimal velocity, and the agents' control inputs (i.e., (7)) are designed to track their local reference signals and hence the optimal trajectory. Such an algorithmic structure results in strong coupling between the agents' dynamics and the reference systems. It is of great benefit since only one virtual variable is to be communicated, which is more efficient than the distributed observer method (see [10] for example). The structure of the proposed algorithm is inspired by [15], [16], where special cases of the distributed time-varying optimization problem, e.g., consensus and/or leader-following tracking, are investigated for networked systems. Similar structure is applied in [17] to solve the distributed time-varying optimization of networked Lagrangian systems. The constructed reference system in [17] is with first-order dynamics, for which the control torques are directly generated from of the signum function. Hence, there might exist the chattering during implementation, which is undesirable for physical implementation. While the signum function exists in (4), the control torque τ_i is generated from r_i and $\dot{r}_i$, which are obtained by integrating $\ddot{r}_i$ and hence continuous and differentiable. Compare with [17], the higher-order of the reference system (4) helps smooth out the chattering during the control torque generation. Hence, the proposed algorithm for Lagrangian agents is chattering-free.

B. Convergence Analysis

Assumption 3: For any $i \in \mathcal{V}$, the gradient of the cost function $f_i(q_i,t)$ can be written as $\nabla f_i(q_i,t) = H(t)q_i + g_i(t)$, where $H(t)$ is a matrix-valued function and $g_i(t)$ is a time-varying function. In addition, there exist positive constants $\bar{H}$ and $\bar{g}$ such that $\sup_{t\in[0,\infty)}\big\|H^{(l)}(t)\big\|_2 \leq \bar{H}$ and $\sup_{t\in[0,\infty)}\big\|g_i^{(k)}(t)\big\|_2 \leq \bar{g}$ $\forall l = 1,2$, $\forall k = 0,\dots,3$, $\forall i \in \mathcal{V}$.

Remark 2: In practice, Assumptions 2 and 3 can be satisfied in many applications. For instance, the distributed average tracking of networked systems, which has found several applications in region following formation control [22] and coordinated path planning [23], can be recasted as a time-varying optimization problem by constructing cost functions as $f_i(q_i,t) = \|q_i - g_i(t)\|_2^2$, $i \in \mathcal{V}$, where $g_i(t)$ is a smooth function denoting agent i's local reference signal. Assumptions 2 and 3 are satisfied under the boundedness assumption of $g_i(t)$, $\dot{g}_i(t)$, $\ddot{g}_i(t)$, and $g_i^{(3)}(t)$.

Define $\omega_i = \dot{r}_i$. By using (6), the reference system (4) can be rewritten as a perturbed third-order dynamics

$$\dot{q}_i = r_i + s_i \quad (9)$$

$$\dot{r}_i = \omega_i \quad (10)$$

$$\begin{aligned}\dot{\omega}_i = &-\sum_{j\in\mathcal{N}_i}[\alpha(q_i-q_j)+\beta(r_i-r_j+s_i-s_j) \\ &+\alpha(\omega_i-\omega_j)] - r_i - s_i - \omega_i - \gamma\sum_{j\in\mathcal{N}_i}\mathrm{sgn}(q_i-q_j \\ &+ r_i - r_j + s_i - s_j + \omega_i - \omega_j) + \varphi_i. \quad (11)\end{aligned}$$

Remark 3: From Assumption 3, it follows that $\varphi_i = -D_1 q_i - D_2\dot{q}_i - D\dot{r}_i - G_i$, where $D_1 = D + \dot{D} + \ddot{D}$, $D_2 = D + 2\dot{D}$, $G_i = \tilde{g}_i + \dot{\tilde{g}}_i + \ddot{\tilde{g}}_i$, $D = H^{-1}(t)\dot{H}(t) + I_p$ and $\tilde{g}_i = H^{-1}(t)[g_i(t) + \dot{g}_i(t)]$. From Assumption 2, it follows that $\|D\|_2 \leq 1 + \frac{\bar{H}}{\underline{m}} := d$, $\|\dot{D}\|_2 \leq \frac{\bar{H}}{\underline{m}}\big(1 + \frac{\bar{H}}{\underline{m}}\big)$, $\|\ddot{D}\|_2 \leq \frac{\bar{H}}{\underline{m}}\big(1 + 3\frac{\bar{H}}{\underline{m}} + \frac{\bar{H}^2}{\underline{m}^2}\big)$, and for any $i \in \mathcal{V}$, $\|G_i\|_2 \leq \frac{2\bar{g}}{\underline{m}}\big(3 + 3\frac{\bar{H}}{\underline{m}} + 2\frac{\bar{H}^2}{\underline{m}^2}\big) := \bar{G}$. In addition, it follows from the definitions of D_1 and D_2 that $\|I_N \otimes D_1\|_2 \leq d_1$ and $\|I_N \otimes D_2\|_2 \leq d_2$, where $d_1 := 1 + 3\frac{\bar{H}}{\underline{m}} + 4\frac{\bar{H}^2}{\underline{m}^2} + 2\frac{\bar{H}^3}{\underline{m}^3}$ and $d_2 := 1 + 3\frac{\bar{H}}{\underline{m}} + 2\frac{\bar{H}^2}{\underline{m}^2}$.

Proposition 1: Consider a group of N agents, and their interaction is described by the graph $\mathcal{G}$. Each agent's dynamics are given by (9)-(11). Suppose that Assumptions 1-3 hold, and let $\alpha > \frac{2d_1}{\lambda_2(L)}$, $\beta > \max\{\frac{2\lambda_N(L)}{\lambda_2(L)}\alpha, \frac{2d_1}{\lambda_2(L)} + 2\alpha\}$, and $\gamma > (N-1)\bar{G}$, where d_1 and $\bar{G}$ are given in Remark 3. The following statements hold.

1) It holds that $q_i - q^* \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$ if $s_i \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$.
2) It holds that $q_i(t) \to q^*(t)$ $\forall i \in \mathcal{V}$ as $t \to \infty$ if $s_i \to \mathbf{0}$ $\forall i \in \mathcal{V}$ as $t \to \infty$.

Proof: The proof is divided into three steps. In Step 1, it will be proved that $q_i - \frac{1}{N}\sum_{j=1}^N q_j \in \mathcal{L}_\infty$, $r_i - \frac{1}{N}\sum_{j=1}^N r_j \in$

$\mathcal{L}_\infty$ and $\omega_i - \frac{1}{N}\sum_{j=1}^N \omega_j \in \mathcal{L}_\infty$ if $s_i \in \mathcal{L}_\infty\ \forall i \in \mathcal{V}$, and that $q_i \to \frac{1}{N}\sum_{j=1}^N q_j$, $r_i \to \frac{1}{N}\sum_{j=1}^N r_j$ and $\omega_i \to \frac{1}{N}\sum_{j=1}^N \omega_j$ as $t \to 0$ if $s_i \to \mathbf{0}\ \forall i \in \mathcal{V}$ as $t \to 0$. Since the right hand of (11) is discontinuous, then the solution is investigated in terms of differential inclusions by using nonsmooth analysis. Since the signum function is measurable and locally essentially bounded, by [24], it is concluded that the Filippov solutions[1] [25] always exist and are abuslutely continuous. Let q, v, ω, s and φ be the column stack vector of q_i, r_i, ω_i, s_i and φ_i, $i \in \mathcal{V}$, respectively. Define $x = (M\otimes I_p)q$, $y = (M\otimes I_p)r$, and $z = (M\otimes I_p)\omega$, where $M = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T$. Define $X = [x^T, y^T, z^T]^T$. It then holds that $\dot{X} \in^{\text{a.e.}} \mathcal{K}[\mathcal{F}](X)$, where a.e. denotes "almost everywhere" and $\mathcal{F} = \left[\mathcal{F}_x^T, \mathcal{F}_y^T, \mathcal{F}_z^T\right]^T$ with $\mathcal{F}_x = y + (M\otimes I_p)s$, $\mathcal{F}_y = z$, and $\mathcal{F}_z = -(L\otimes I_p)\big(\alpha x + \beta y + \alpha z\big) - \left[(\beta L + M)\otimes I_p\right]s - y - z - \gamma(B\otimes I_p)\text{sgn}\left[(B^T\otimes I_p)\big(x+y+z+s\big)\right] + (M\otimes I_p)\varphi$.

Consider the Lyapunov function candidate $V = \frac{1}{2}X^T P X$, where $P = [P_{ij}]$, $i,j = 1,2,3$, $P_{11} = (\alpha+\beta)(L\otimes I_p) + I_{Np}$, $P_{12} = P_{21} = (\alpha+\kappa)(L\otimes I_p) + I_{Np}$, $P_{22} = (\beta+\kappa)(L\otimes I_p) + I_{Np}$, and $P_{13} = P_{31} = P_{23} = P_{32} = P_{33} = I_{Np}$. Note that V is positive definite if $\alpha > 0$ and $\beta > \frac{2\lambda_N(L)}{\lambda_2(L)}\alpha$. The set-valued Lie derivative [25] of V is

$$\dot{\tilde{V}} = \mathcal{K}\big[-\alpha x^T(L\otimes I_p)x - (\beta - 2\alpha)y^T(L\otimes I_p)y - \alpha z^T(L\otimes I_p)z + U_1 + U_2 + U_3\big],$$

where $U_1 = \alpha x^T(L\otimes I_p)s + (\beta-2\alpha)y^T(L\otimes I_p)s - \beta z^T(L\otimes I_p)s - (x+y+z)^T\big(M\otimes D_2\big)s$, $U_2 = -(x+y+z)^T\left[(I_N\otimes D_1)x + (I_N\otimes D_2)y + (I_N\otimes D)z\right]$, $U_3 = -\gamma(x+y+z)^T(B\otimes I_p)\text{sgn}\left[(B^T\otimes I_p)\big(x+y+z+s\big)\right] + (x+y+z)^T(M\otimes I_p)G$ and $G = \left[G_1^T, \ldots, G_N^T\right]^T$.

Note that, for any $\xi \in \{x,y,z\}$, it holds that $\xi^T(L\otimes I_p)s \le \lambda_N(L)\sqrt{Np}\,\|\xi\|_1\|s\|_\infty$, and that $\xi^T(M\otimes D_2)s \le d_2\sqrt{Np}\,\|\xi\|_1\|s\|_\infty$. Then, it holds that $-(x+y+z)^T\big(M\otimes D_2\big)s \le d_2\sqrt{Np}\big(\|x\|_1 + \|y\|_1 + \|z\|_1\big)\|s\|_\infty = d_2\sqrt{Np}\,\|X\|_1\|s\|_\infty$. Hence, for any $\widetilde{U}_1 \in \mathcal{K}[U_1]$, it holds that $\widetilde{U}_1 \le \lambda_N(L)\sqrt{Np}\,\|s\|_\infty\left[\alpha\|x\|_1 + (\beta - 2\alpha)\|y\|_1 + \beta\|z\|_1\right] + d_2\sqrt{Np}\,\|X\|_1\|s\|_\infty \le k_2\|X\|_2\|s\|_\infty$, where $k_1 = \max\{\alpha, \beta-2\alpha, \beta\}$ and $k_2 = \max\left\{k_1 Np\lambda_N(L)\sqrt{3}, d_2 Np\sqrt{3}\right\}$. For any $\widetilde{U}_2 \in \mathcal{K}[U_2]$, it holds that $\widetilde{U}_2 \le (\|x\|_2 + \|y\|_2 + \|z\|_2)\big(\|I_N\otimes D_1\|_2\|x\|_2 + \|I_N\otimes D_2\|_2\|y\|_2 + \|I_N\otimes D\|_2\|z\|_2\big) \le 2d_1\|X\|_2^2$. Note that $U_3 = -\gamma(x+y+z+s)^T(B\otimes I_p)\text{sgn}\left[(B^T\otimes I_p)\big(x+y+z+s\big)\right] + (x+y+z+s)^T(M\otimes I_p)G + \gamma s^T(B\otimes I_p)\text{sgn}\left[(B^T\otimes I_p)\big(x+y+z+s\big)\right] - s^T(M\otimes I_p)G$. It holds that $\mathcal{K}\left[-\gamma(x+y+z+s)^T(B\otimes I_p)\text{sgn}\left[(B^T\otimes I_p)\big(x+y+z+s\big)\right]\right] = \left\{-\gamma\left\|(B^T\otimes I_p)(x+y+z+s)\right\|_1\right\}$. It also holds that $\gamma s^T(B\otimes I_p)\text{sgn}\left[(B^T\otimes I_p)\big(x+y+z+s\big)\right] - s^T(M\otimes I_p)G \le \gamma\|s\|_1\left\|(B\otimes I_p)\text{sgn}\left[(B^T\otimes I_p)(x+y+z+s)\right]\right\|_\infty + \|s\|_1\|M\otimes I_p\|_\infty\|G\|_\infty \le \gamma N^2 p\|s\|_\infty + 2Np\bar{G}\|s\|_\infty = k_3\|s\|_\infty$, where $k_3 = \gamma N^2 p + 2Np\bar{G}$. Note that $(x+y+z+s)^T(M\otimes I_p)G \le (N-1)\bar{G}\left\|(B^T\otimes I_p)(x+y+z+s)\right\|_1$. Hence, for any $\widetilde{U}_3 \in \mathcal{K}[U_3]$, it holds that $\widetilde{U}_3 \le -[\gamma - (N-1)\bar{G}]\left\|(B^T\otimes I_p)(x+y+z+s)\right\|_1 + k_3\|s\|_\infty$, where $\bar{G}$ is given in Remark 3. Therefore, it follows that

$$\begin{aligned}\dot{V} &\le -[\alpha\lambda_2(L) - 2d_1]\|x\|_2^2 - [(\beta-2\alpha)\lambda_2(L) - 2d_1]\|y\|_2^2 \\ &\quad - [\alpha\lambda_2(L) - 2d_1]\|z\|_2^2 + k_2\|X\|_2\|s\|_\infty + k_3\|s\|_\infty \\ &\le -\lambda_m(1-2\eta)\|X\|_2^2 - 2\lambda_m\eta\|X\|_2^2 \\ &\quad + k_2\|X\|_2\|s\|_\infty + k_3\|s\|_\infty,\end{aligned}$$

where $\eta \in \left(0, \frac{1}{2}\right)$, $\lambda_m = \min\{\alpha\lambda_2(L) - 2d_1, (\beta - 2\alpha)\lambda_2(L) - 2d_1\}$. Note that the term $-2\lambda_m\eta\|X\|_2^2 + k_2\|X\|_2\|s\|_\infty + k_3\|s\|_\infty$ is non-positive if $\|X\|_2 \ge \max\{\frac{k_2}{\eta\lambda_m}\|s\|_\infty, \sqrt{\frac{k_3}{\eta\lambda_m}}\sqrt{\|s\|_\infty}\}$. Note that $\rho(r) = \max\{\frac{k_2}{\eta\lambda_m}r, \sqrt{\frac{k_3}{\eta\lambda_m}}\sqrt{r}\}$ is a class $\mathcal{K}$ function. It holds that

$$\dot{V} \le -\lambda_m(1-2\eta)\|X\|_2^2 \quad \forall \|X\|_2 \ge \rho(\|s\|_\infty).$$

It then follows from [26, Theorem 4.19] and the property of the input-to-state stability [26, p. 175] that $x \in \mathcal{L}_\infty$, $y \in \mathcal{L}_\infty$ and $z \in \mathcal{L}_\infty$ if $s \in \mathcal{L}_\infty$, and that $x(t) \to \mathbf{0}$, $y(t) \to \mathbf{0}$ and $z(t) \to \mathbf{0}$ as $t \to \infty$ if $s(t) \to \mathbf{0}$ as $t \to \infty$.

In Step 2, it will proved that $\sum_{i=1}^N \nabla f_i(q_i,t) \in \mathcal{L}_\infty$ if $s_i \in \mathcal{L}_\infty\ \forall i \in \mathcal{V}$, and that $\sum_{i=1}^N \nabla f_i(q_i,t) \to \mathbf{0}$ as $t \to \infty$ if $s_i \to \mathbf{0}\ \forall i \in \mathcal{V}$ as $t \to \infty$. Define $S_r = \sum_{i=1}^N [r_i + F_i(q_i,t)]$ and $S_\omega = \sum_{i=1}^N [\omega_i + \dot{F}_i(q_i,t)]$. From Assumption 1, the definition of φ_i in (5), and (9)-(11), it holds that $\dot{S}_r = S_\omega$ and $\dot{S}_\omega = -S_r - S_\omega - \sum_{i=1}^N s_i + \sum_{i=1}^N D\dot{s}_i$. Define $\widetilde{S}_\omega = S_\omega - \sum_{i=1}^N Ds_i$ and $S = \left[S_r^T, \widetilde{S}_\omega^T\right]^T$. It then holds that $\dot{S} = A_S S + \widetilde{D}\sum_{i=1}^N s_i$, where $A_S = [0,1;-1,-1]\otimes I_p$ and $\widetilde{D} = [D; I_p + D + \dot{D}]$. Note that $\dot{S} = A_S S$ is a standard exponentially stable linear time-invariant (LTI) system. Note that $\widetilde{D}$ is a bounded matrix. It then follows from the property of the input-to-state stability [26, p. 175] that $S \in \mathcal{L}_\infty$ if $s_i \in \mathcal{L}_\infty\ \forall i \in \mathcal{V}$ and $S \to \mathbf{0}$ as $t \to \infty$ if $s_i \to \mathbf{0}\ \forall i \in \mathcal{V}$ as $t \to \infty$. By the definitions of S and $\widetilde{S}_\omega$, it holds that $S_r \in \mathcal{L}_\infty$ and $S_\omega \in \mathcal{L}_\infty$ if $s_i \in \mathcal{L}_\infty\ \forall i \in \mathcal{V}$, and $S_r \to \mathbf{0}$ and $S_\omega \to \mathbf{0}$ as $t \to \infty$ if $s_i \to \mathbf{0}\ \forall i \in \mathcal{V}$ as $t \to \infty$.

Define $\chi = \sum_{i=1}^N \nabla f_i(q_i,t)$. It holds that $\dot{\chi} = -\chi + H(t)S_r + H(t)\sum_{i=1}^N s_i$. Note that $\dot{\chi} = -\chi$ is a standard exponentially stable LTI system. Hence, the Step 2 is proved by following from the property of the input-to-state stability [26, p. 175] and Assumptions 2 and 3.

In Step 3, the two statements will be proved by using the results obtained in Steps 1 and 2. Since the function $\sum_{j=1}^N f_j(q,t)$ is strongly convex in q by Assumption 2, it then follows from Assumption 3 that $N\underline{m}\|\bar{q} - q^*\|_2^2 \le [\sum_{j=1}^N \nabla f_j(\bar{q},t) - \sum_{j=1}^N \nabla f_j(q^*,t)]^T(\bar{q} - q^*) = [\sum_{j=1}^N \nabla f_j(\bar{q},t) - \sum_{j=1}^N \nabla f_j(q_j,t)]^T(\bar{q} - q^*) + [\sum_{j=1}^N \nabla f_j(q_j,t) - \sum_{j=1}^N \nabla f_j(q^*,t)]^T(\bar{q} - q^*)$, where $\bar{q} = \frac{1}{N}\sum_{j=1}^N q_j$. Then, it holds that $N\underline{m}\|\bar{q} - q^*\|_2 \le \left\|\sum_{j=1}^N \nabla f_j(\bar{q},t) - \sum_{j=1}^N \nabla f_j(q_j,t)\right\|_2 + \left\|\sum_{j=1}^N \nabla f_j(q_j,t) - \sum_{j=1}^N \nabla f_j(q^*,t)\right\|_2$. From Assumption 2 and Mean Value theorem, there exists a positive constant

[1]Consider the vector differential equation $\dot{x} = f(x,t)$, where $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is measurable and locally essentially bounded. A vector function $x(\cdot)$ is called a Filippov solution on $[t_0, t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$, $\dot{x} \in \mathcal{K}[f](x,t)$, where $\mathcal{K}[f](x,t) := \bigcap_{\Lambda>0}\bigcap_{\mu(\mathscr{N})=0}\overline{\text{co}} f(\mathcal{B}(x,\Lambda) - \mathscr{N}, t)$ is the Filippov set-valued map of $f(x,t)$ and $\bigcap_{\mu(\mathscr{N})=0}$ denotes the intersection over all sets $\mathscr{N}$ of Lebesgue measure zero.

M such that $\left\| \sum_{j=1}^{N} \nabla f_j(\bar{q},t) - \sum_{j=1}^{N} \nabla f_j(q_j,t) \right\|_2 \leq \sum_{j=1}^{N} M \|q_j - \bar{q}\|$. Since $\sum_{j=1}^{N} \nabla f_j(q^*,t) = \mathbf{0}$, then it can be shown that $\|\bar{q} - q^*\|_2 < \infty$, and hence $q_i - q^* \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. This completes the proof of the first statement. Recall from Steps 1 and 2 that if $s_i \to \mathbf{0}$ $\forall i \in \mathcal{V}$ as $t \to \infty$, it holds that $q_i - \frac{1}{N}\sum_{j=1}^{N} q_j \to \mathbf{0}$ $\forall i \in \mathcal{V}$ and $\sum_{j=1}^{N} \nabla f_i(q_i,t) \to \mathbf{0}$ as $t \to \infty$. Then, it holds that $q_i(t) \to q^*(t)$ $\forall i \in \mathcal{V}$ as $t \to \infty$. This completes the proof of the second statement. ∎

Theorem 1: Suppose that Assumptions 1-3 hold, and let $\alpha > \frac{2d_1}{\lambda_2(L)}$, $\beta > \max\{\frac{2\lambda_N(L)}{\lambda_2(L)}\alpha, \frac{2d_1}{\lambda_2(L)} + 2\alpha\}$, and $\gamma > (N-1)\bar{G}$, where d_1 and $\bar{G}$ are given in Remark 3. Using the controller (7)-(8) with r_i and $\dot{r}_i$ generated by the reference system (4) for the networked Lagrangian system (1) solves the distributed time-varying optimization problem, that is, $q_i(t) \to q^*(t)$ $\forall i \in \mathcal{V}$ as $t \to \infty$.

Proof: For any $i \in \mathcal{V}$, define Lyapunov function candidate $W_i = \frac{1}{2}s_i^T M_i(q_i)s_i + \frac{1}{2}\Delta\vartheta_i^T \Gamma_i^{-1}\Delta\vartheta_i$ with $\Delta\vartheta_i = \hat{\vartheta}_i - \vartheta_i$. By using Property 2, the derivative of W_i is given as $\dot{W}_i = -s_i^T K_i s_i \leq 0$. Then, it holds that $s_i \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $\hat{\vartheta}_i \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. Then, it follows from Proposition 1 that $q_i - q^* \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. Since $q^* \in \mathcal{L}_\infty$, then it holds that $q_i \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. It follows from the definition of $F_i(q_i,t)$ and Assumptions 2 and 3 that $F_i(q_i,t) \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. Recall from Step 2 in the proof of Proposition 1 that $\sum_{j=1}^{N}[r_j + F_j(q_j,t)] \in \mathcal{L}_\infty$. Then, $\sum_{j=1}^{N} r_j \in \mathcal{L}_\infty$. Since $s_i \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$, then it holds that $r_i - \frac{1}{N}\sum_{j=1}^{N} r_j \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. Then, it holds that $r_i \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. Then, by (6), it holds that $\dot{q}_i \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. From the definition of $\dot{F}_i(q_i,t)$ and Assumptions 2 and 3, it holds that $\dot{F}_i(q_i,t) \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$, and hence $\sum_{j=1}^{N} \omega_j \in \mathcal{L}_\infty$. Recall from Step 1 that $\omega_i - \frac{1}{N}\sum_{j=1}^{N} \omega_j \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$ if $s_i \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. Hence, $\omega_i \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. Then, it follows from (10) that $\dot{r}_i \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. Substituting (7) into (1) and using Property 3 yield that $M_i(q_i)\dot{s}_i + C_i(q_i,\dot{q}_i)s_i = -K_i s_i + Y_i(q_i,\dot{q}_i,r_i,\dot{r}_i)\Delta\vartheta_i$. Then by using Property 1, it holds that $\dot{s}_i \in \mathcal{L}_\infty$ $\forall i \in \mathcal{V}$. It can thus be shown that each s_i, $i \in \mathcal{V}$, is uniformly continuous. Using Barbalat's lemma [26, p. 175], we obtain that $s_i(t) \to \mathbf{0}$ $\forall i \in \mathcal{V}$ as $t \to \infty$. Then, from Proposition 1, it follows that $q_i - \frac{1}{N}\sum_{j=1}^{N} q_j \to \mathbf{0}$ $\forall i \in \mathcal{V}$ and $\sum_{j=1}^{N} \nabla f_j(q_j,t) \to \mathbf{0}$ as $t \to \infty$. Hence, it follows from Lemma 1 that $q_i(t) \to q^*(t)$ $\forall i \in \mathcal{V}$ as $t \to \infty$. ∎

Remark 4: In the literature, there is another approach used to generate continuous control inputs for the agents, which is replacing the signum function with its continuous time-invariant or -varying approximation [18], [19]. Such a method has been applied in [9] to address the time-varying optimization problem for networked single- and double-integrator agents. There usually exist approximation errors, and hence non-zero optimum tracking errors, while using time-invariant approximation function. Although the approximation errors, and hence the optimum tracking errors, converge to zero when using time-varying approximation functions, the effectiveness of alleviating the chattering issue might be weakened as time proceeds. In addition, the implementation of such a method requires exact values of the function arguments, while the proposed algorithm only needs the signs of the function arguments, which is with hight error allowance and computational efficiently.

V. An Illustrative Example

In this section, we provide an example to illustrate the results in this paper. Consider a group of $N = 10$ planar manipulators with two revolute joints modeled by (1) [11, pp. 259-262], and the interaction is characterized as a ring topology. Each agent $i \in \mathcal{V}$ has a local cost function $f_i(q_i,t) = [q_{i1} - 0.1i\sin(t)]^2 + [q_{i2} - 0.1i\cos(t)]^2$, and denote by $q^* = \left[q_1^*, q_2^*\right]^T$ the optimal trajectory that minimizes the sum of all the local cost functions $\sum_{i=1}^{10} f_i(q,t)$. In the simulation, we select $\Gamma_i = 0.8I_5$ and $K_i = 40I_2$ $\forall i \in \mathcal{V}$, $\alpha = 1$, $\beta = 4$ and $\gamma = 10$. The position trajectories and control torques are presented in Fig. 1 and Fig. 2, respectively. From Fig. 1, it shows that all the agents track the optimal trajectory, i.e., $q_i(t) \to q^*(t)$ $\forall i \in \mathcal{V}$. The distributed time-varying optimization algorithm proposed in [17] can also be used for the Lagrangian agents to track the optimal trajectory, and the agents' control torques are presented in Fig. 3, which shows the chattering exists. Compared with Fig. 3, one can see from Fig. 2 that the control torques are smooth and chattering-free.

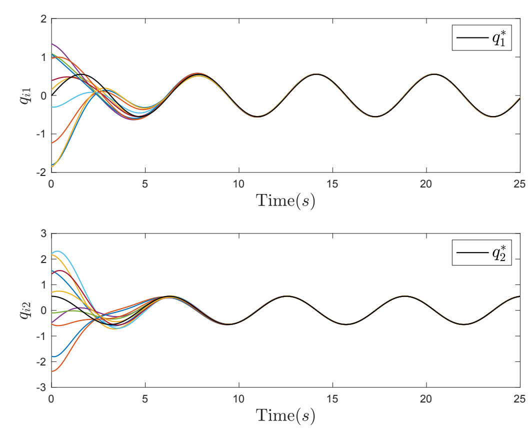

Fig. 1. The position trajectories of Lagrangian agents (1) by using the distributed time-varying optimization algorithm in Section IV-A.

VI. Conclusion

The distributed time-varying optimization problem has been solved for networked Lagrangian agents with parametric uncertainties. A algorithm has been proposed to solve the optimization problem, and the agents are able to generate continuous control torques and achieve exact optimum tracking. A example has been presented to illustrate the effectiveness of the proposed algorithm.

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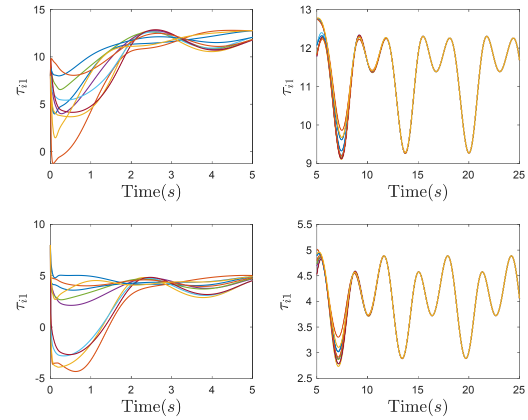

Fig. 2. The control torques of Lagrangian agents (1) by using the distributed time-varying optimization algorithm in Section IV-A.

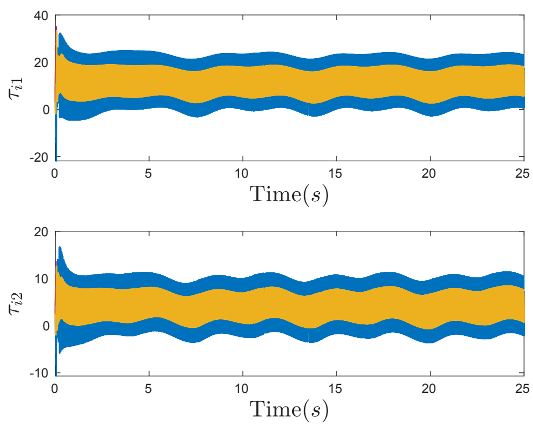

Fig. 3. The control torques of Lagrangian agents (1) by using the distributed time-varying optimization algorithm in [17].

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