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Distributed Continuous-Time Resource Allocation Algorithm for Networked Double-Integrator Systems with Time-Varying Non-Identical Hessians and Resources

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Abstract— **This paper investigates the optimal resource allocation problem for networked double-integrator systems with time-varying cost functions and resources. Due to the coexistence of challenges caused by non-identical Hessians and more complicated agents' dynamics, the extension from existing related results on single-integrator agents is nontrivial. A distributed algorithm is proposed to address the time-varying resource allocation problem and achieve the exact optimum tracking. Finally, an example is provided to illustrate the effectiveness of the proposed algorithm.**

I. Introduction

In optimal resource allocation problem, a certain amount of resources must be distributed among a group of agents while minimizing the sum of all the agents' local cost functions. This problem can be found in various fields of research including power systems [1], [2], distributed computer systems [3], sensor networks [4], robot networks [5], and economic systems [6]. Recently, a number of distributed algorithms (see, for example, [2], [7]–[10]) have been established to address the optimal resource allocation problem by using local information and communication. These distributed approaches are addressing the problem with time-invariant cost functions and fixed amounts of resources to be distributed. In practical applications (e.g., the economic dispatch problem), however, the cost functions and/or the amount of resources to be distributed might be time varying, and hence the optimal solutions are trajectories changing over time instead of fixed points. It is meaningful and closer to the practical applications to investigate the optimal time-varying resource allocation problem.

In the literature, there are a few results on the distributed optimal time-varying resource allocation problem. The works [11] and [12] establish discrete-time distributed approaches to solve the constrained time-varying optimization problem, and there are usually nonzero tracking errors between the local decision variables and the optimal ones. There is another body of literature that devotes to derive continuous-time distributed algorithms to solve the resource allocation problem, and the established results can be used for robotic systems to accomplish certain tasks. The works [13] and [14] propose continuous-time algorithms to solve the resource allocation problem with time-invariant cost functions and time-varying resources. When implementing the results in [13], there exist non-zero tracking errors, and the results in [14] suit for quadratic cost functions. In [15]–[17], the optimal time-varying resource allocation problem is solved for the case where both the cost functions and the resource vectors are time varying. Specifically, in [15], [16], it is assumed that the cost functions have identical Hessians, and in [17], the case of non-identical time-varying diagonal Hessians is addressed.

Notice the fact that a broad class of vehicles can be modeled by double-integrator dynamics. Moreover, the results about the time-varying resource allocation problem mentioned above essentially assume single-integrator dynamics for the agents. These results cannot be directly applied to double-integrator agents. To this end, in this paper, the optimal time-varying resource allocation problem is investigated for networked double-integrator systems. First, a centralized approach is established, where a central virtual system is constructed to track the optimal Lagrange multiplier, and the central state information is used to design control inputs for each agent to track its own optimal decision trajectory. To remove the requirement of a central node, a distributed resource allocation algorithm only using local information and communication is proposed to achieve exact optimal-decision tracking. Specifically, each agent has a virtual system to track the optimal Lagrange multiplier, and the local virtual state is used in the controller design. Compared with the works in [15]–[17], this paper considers that the cost functions have non-identical time-varying Hessians, which is more general and includes them as special cases. Moreover, the agents' dynamics are double integrators, which is more complicated that the single-integrator systems considered in [15]–[17]. It is worth pointing out that the results obtained in this paper are not simple extensions from the existing results established for single-integrator systems.

II. Preliminaries

A. Notations

Throughout this paper, let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_+$ denote the sets of all real numbers, all nonnegative real numbers, and all positive real numbers, respectively. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$, and for a real number $x \in \mathbb{R}$, $|x|$ denotes the absolute value of x. The transpose of matrix A is denoted by $A^\top$. For a given vector $x = [x_1, \ldots, x_p]^T \in \mathbb{R}^p$,

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define $\|x\|_1 = \sum_{i=1}^{p} |x_i|$, $\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_p|^2}$, and $\|x\|_\infty = \max_{i=1,\ldots,p} |x_i|$. For a symmetric matrix $A \in \mathbb{R}^{p\times p}$, let $\lambda_1(A), \cdots, \lambda_p(A)$ denote its eigenvalues. Let $\otimes$ and $\overline{\text{co}}$ denote the Kronecker product and the convex closure, respectively. Let $\text{diag}\{A_1,\ldots,A_p\}$, where $A_i \in \mathbb{R}^{n\times m}$, represent the block diagonal matrix with the i-th block in the main diagonal being A_i. For a vector $x \in \mathbb{R}^p$, define $\text{sgn}(x) = [\text{sgn}(x_1),\ldots,\text{sgn}(x_p)]^\top$ where $\text{sgn}(x_i) = 1$ if $x_i > 0$, $\text{sgn}(x_i) = 0$ if $x_i = 0$, and $\text{sgn}(x_i) = -1$ if $x_i < 0$. Let $\mathbf{0}$ and $\mathbf{1}$ denote the zero and all-ones vectors/matrices with appropriate dimensions, respectively. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix.

B. Graph Theory

For a multi-agent system consisting of N agents, the interaction topology can be modeled by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1,\ldots,N\}$ and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ denote the node set and edge set, respectively. An edge denoted by $(i,j) \in \mathcal{E}$, means that agent i and j can obtain information from each other. In an undirected graph, the edges (i,j) and (j,i) are equivalent. It is assumed that $(i,i) \notin \mathcal{E}$. The neighbor set of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j,i) \in \mathcal{E}\}$. By arbitrarily assigning an orientation for every edge in $\mathcal{G}$, let $B = [B_{ij}] \in \mathbb{R}^{N\times|\mathcal{E}|}$ denote the incidence matrix associated with graph $\mathcal{G}$, where $B_{ij} = -1$ if edge e_j leaves node i, $B_{ij} = 1$ if it enters node i, and $B_{ij} = 0$ otherwise.

An undirected path between node i_1 and i_k is a sequence of edges of the form (i_1,i_2), (i_2,i_3), $\ldots$, (i_{k-1},i_k), where $i_k \in \mathcal{V}$. A connected graph means that there exists an undirected path between any pair of nodes in $\mathcal{V}$.

Assumption 1: The graph $\mathcal{G}$ is connected.

III. Problem Statement

Consider a multi-agent system consisting of N double-integrator agents, and agents' dynamics are described as

$$\dot{q}_i = v_i, \quad \dot{v}_i = u_i, \quad i \in \mathcal{V}, \tag{1}$$

where $q_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$, and $u_i \in \mathbb{R}^d$ denote, respectively, the position, velocity, and control input of agent i. In the distributed resource allocation problem, each agent aims to cooperatively track the optimal trajectory determined by the group objective function and the coupled equality constraint. Let $q^\star(t) = \left[q_1^{\star\top}(t),\ldots,q_N^{\star\top}(t)\right]^\top \in \mathbb{R}^{Nd}$ denote the optimal trajectories for the agents, and it is defined as

$$q^\star(t) = \underset{q(t)}{\arg\min} \left\{\sum_{i=1}^{N} f_i[q_i(t),t]\right\}, \tag{2}$$

$$\text{subject to } \sum_{i=1}^{N} q_i(t) = \sum_{i=1}^{N} c_i(t), \tag{3}$$

where $q(t) = \left[q_1^\top(t),\ldots,q_N^\top(t)\right]^\top$, $f_i[q(t),t] : \mathbb{R}^d\times\mathbb{R}_{\geq 0} \to \mathbb{R}$ is the local cost function associated with agent $i \in \mathcal{V}$, and $c_i(t) \in \mathbb{R}^d$ is agent i's time-varying resource vector. The objective is to design control inputs/torques for the agents such that the agents are capable of tracking the optimal trajectories defined in (2)-(3), i.e., $q_i(t) \to q_i^\star(t)\ \forall i \in \mathcal{V}$. We make the following assumptions on the cost functions and resource vectors.

Assumption 2: For any $i \in \mathcal{V}$, the cost function $f_i[q_i(t),t]$ is twice continuously differentiable and uniformly strongly convex with respect to q_i for all t. That is, there exists a positive constant $\underline{m}$ such that $\lambda_i\big(H_i[q_i(t),t]\big) > \underline{m}$, $i = 1,\ldots,d$, where $H_i[q_i(t),t]$ is the Hessian matrix of $f_i[q_i(t),t]$. In addition, each $H_i(q_i,t)$ is upper-bounded, i.e., $\|H_i(q_i,t)\|_2 \leq \overline{m}\ \forall i \in \mathcal{V}$, where $\overline{m}$ is a positive constant.

Assumption 3: There exists a positive constant $\bar{c}$ such that $\sup_{t\in[0,\infty)} \|c_i(t)\|_2 \leq \bar{c}$, $\sup_{t\in[0,\infty)} \|\dot{c}_i(t)\|_2 \leq \bar{c}$ and $\sup_{t\in[0,\infty)} \|\ddot{c}_i(t)\|_2 \leq \bar{c}\ \forall i \in \mathcal{V}$.

Assumption 4: For any $i \in \mathcal{V}$, the gradient of the cost function $f_i(q_i,t)$ can be written as $\nabla f_i(q_i,t) = H_i(t)q_i + g_i(t)$, where $H_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{d\times d}$ is a matrix-valued function, and $g_i(t)$ is a smooth time-varying function. In addition, there exist positive constants $\bar{H}$ and $\bar{g}$ such that $\sup_{t\in[0,\infty)} \left\|\dot{H}_i(t)\right\|_2 \leq \bar{H}$, $\sup_{t\in[0,\infty)} \left\|\ddot{H}_i(t)\right\|_2 \leq \bar{H}$, $\sup_{t\in[0,\infty)} \|g_i(t)\|_2 \leq \bar{g}$, $\sup_{t\in[0,\infty)} \|\dot{g}_i(t)\|_2 \leq \bar{g}$, and $\sup_{t\in[0,\infty)} \|\ddot{g}_i(t)\|_2 \leq \bar{g}$ hold for any $i \in \mathcal{V}$.

Remark 1: Note that Assumptions 2 and 4 can be satisfied in many real-world applications. For instance, by constructing the cost functions as $f_i(q_i,t) = \|q_i - g_i(t)\|_2^2$, the distributed average tracking problem, which has found several applications in region following formation control [18] and coordinated path planning [19], can be solved as a time-varying optimization problem. Similar assumptions have been applied in recent works on distributed time-varying optimization [20]–[22], especially on time-varying resource allocation [15]–[17], and Assumption 4 includes the case considered in all aforementioned results as special cases.

IV. Resource Allocation for Networked Double-Integrator Agents

Define the Lagrange function associated with the optimization problem in (2)-(3) as

$$\mathcal{L}(q,\mu,t) = \sum_{i=1}^{N} f_i[q_i(t),t] + \mu^\top(t)\sum_{i=1}^{N}[q_i(t) - c_i(t)], \tag{4}$$

where $\mu(t) \in \mathbb{R}^d$ is the Lagrange multiplier. From Assumption 2, it follows that the Lagrange function (4) is strongly convex in $q(t)$ and concave in $\mu(t)$. Then the optimal primal-dual pair $\{q^\star(t),\mu^\star(t)\}$ is unique at all time $t \geq 0$, and satisfy the following KKT condition:

$$\mathbf{0} = \nabla_q\mathcal{L}(q^\star,\mu^\star,t) = \nabla_q\left(\sum_{i=1}^{N} f_i[q_i^\star(t),t]\right) + \mathbf{1}\otimes\mu^\star(t),$$

$$\mathbf{0} = \nabla_\mu\mathcal{L}(q^\star,\mu^\star,t) = \sum_{i=1}^{N}\left[q_i^\star(t) - c_i(t)\right],$$

where $\nabla_\xi\mathcal{L}(q,\mu,t)$ denotes the partial derivative of the function $\mathcal{L}(q,\mu,t)$ with respect to $\xi \in \{q,\mu\}$. Define $z = [q^\top,\mu^\top]^\top$. The Lagrange function (4) can be rewritten as $\mathcal{L}(z,t)$, and the KKT condition can be rewritten as $\nabla_z\mathcal{L}(z^\star,t) = \mathbf{0}$, where $z^\star = \left[q^{\star\top},\mu^{\star\top}\right]^\top$. Then, we have the following lemma adapted from [23].

Lemma 1: If $\nabla_z \mathcal{L}(z,t) \to \mathbf{0}$ as $t \to \infty$, then $q_i(t) \to q_i^\star(t)$ $\forall i \in \mathcal{V}$ and $\mu(t) \to \mu^\star(t)$ as $t \to \infty$.

Proof: Let $H_z(z,t)$ denote the Hessian matrix associated with the Lagrange function $\mathcal{L}(z,t)$ and $H_z(z,t) = \begin{bmatrix} \boldsymbol{H}(q,t) & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}$, where $\boldsymbol{H}(q,t) = \operatorname{diag}\{H_1(q_1,t), \ldots, H_N(q_N,t)\}$. By Assumption 2, it holds that $H_z(z,t)$ is nonsingular [24, p. 523] and bounded. Then, there exists a positive constant m_H such that $\|H_z^{-1}(z,t)\|_2 \le m_H$. By mean-value theorem, it holds that $\|z(t) - z^\star(t)\|_2 = \|H_z^{-1}(z,t)\nabla_z \mathcal{L}(z,t)\|_2 \le m_H \|\nabla_z \mathcal{L}(z,t)\|_2$, where the fact that $\nabla_z \mathcal{L}(z^\star,t) = \mathbf{0}$ has been used to obtain the equality. Hence, it holds that $z(t) \to z^\star(t)$ as $t \to \infty$, which shows that the statement in Lemma 1 holds. ■

A. Centralized Algorithm

We establish a centralized algorithm for (1), and assume that there exists a central server that is connected with and can exchange information with all the agents. Construct a virtual system for the central server as

$$\ddot{\mu} = -\beta(\dot{\mu} - \widetilde{\mu}) + \dot{\widetilde{\mu}} \tag{5}$$

where

$$\widetilde{\mu} = -\alpha\mu - \left[\sum_{j=1}^N H_j^{-1}(q_j,t)\right]^{-1} \sum_{j=1}^N \Big\{ \dot{c}_j - \alpha(q_j - c_j) + H_j^{-1}(q_j,t)\left[\alpha \nabla f_j(q_j,t) \frac{\partial}{\partial t} \nabla f_j(q_j,t)\right]\Big\}, \tag{6}$$

and, β and α are positive constants. Design the control input for agent $i \in \mathcal{V}$ as

$$u_i = -v_i - F_i(q_i,\mu,\dot{\mu},t) - \dot{F}_i(q_i,\mu,\dot{\mu},t), \tag{7}$$

where the vector-valued function $F_i : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_{\ge 0} \to \mathbb{R}^d$ is defined as

$$F_i(q_i,\mu,\dot{\mu},t) = H_i^{-1}(q_i,t)\left[\frac{\partial}{\partial t}\nabla f_i(q_i,t) + \alpha \nabla f_i(q_i,t) + \alpha\mu + \dot{\mu}\right]. \tag{8}$$

By Assumption 2, it holds that the Hessians $H_i(q_i,t)$, $i \in \mathcal{V}$, are positive definite, which implies that all $H_i^{-1}(q_i,t)$, $i \in \mathcal{V}$, exist and are positive definite. Then, the matrix $\sum_{j=1}^N H_j^{-1}(q_i,t)$ is also positive definite, and hence invertible. Thus, the definition of $\widetilde{\mu}$ in (6) is justified. By Assumptions 2-4, $\dot{\widetilde{\mu}}$ exists. Then, $F_i(q_i,\mu,\dot{\mu},t)$ and $\dot{F}_i(q_i,\mu,\dot{\mu},t)$ exist, and hence, the controller (7) is well defined.

Proposition 1: Suppose that Assumptions 2-4 hold and let $\alpha, \beta \in \mathbb{R}_+$. Using (7) with μ, $\dot{\mu}$ and $\ddot{\mu}$ generated/given by (5) for the double-integrator agents (1) solves the distributed resource allocation problem, i.e., $q_i(t) \to q_i^\star(t)$ $\forall i \in \mathcal{V}$ as $t \to \infty$.

Proof: Define $\chi_i = v_i + F_i(q_i,\mu,\dot{\mu},t)$. By (7), it holds that $\dot{\chi}_i = -\chi_i$. Then, it holds that $\chi_i \to \mathbf{0}$ $\forall i \in \mathcal{V}$ as $t \to \infty$.

Define $\psi_i = \nabla f_i(q_i,t) + \mu$. By (1) and the definition of χ_i, it holds that $\dot{\psi}_i = H_i(q_i,t)\dot{q}_i + \frac{\partial}{\partial t}\nabla f_i(q_i,t) + \dot{\mu} = -\alpha\psi_i + H_i(q_i,t)\chi_i$, where the definition of $F_i(q_i,\mu,\dot{\mu},t)$ in (8) has been used to obtain the last equality. Note that $\dot{\psi}_i = -\alpha\psi_i$ is a standard exponentially stable linear time-invariant (LTI) system. Recall that $\chi_i \to \mathbf{0}$ as $t \to \infty$. Then, it follows from the property of the input-to-state stability [25, p. 175] and Assumption 2 that $\psi_i \to \mathbf{0}$ $\forall i \in \mathcal{V}$ as $t \to \infty$. That is, $\nabla f_i(q_i,t) + \mu \to \mathbf{0}$ $\forall i \in \mathcal{V}$ as $t \to \infty$. Hence, it holds that $\nabla_q \mathcal{L}(q,\mu,t) \to \mathbf{0}$ as $t \to \infty$.

Define $e_\mu = \dot{\mu} - \widetilde{\mu}$. By (5), it then holds that $\dot{e}_\mu = -\beta e_\mu$. It holds that $e_\mu \to \mathbf{0}$ as $t \to \infty$.

Define $\zeta = \sum_{i=1}^N (q_i - c_i)$. It holds that $\dot{\zeta} = -\alpha\zeta + \sum_{i=1}^N \chi_i - \sum_{i=1}^N H_i^{-1}(q_i,t)e_\mu$. Recall that $\chi_i, e_\mu \to \mathbf{0}$ $\forall i \in \mathcal{V}$ as $t \to \infty$, and note that $\dot{\zeta} = -\alpha\zeta$ is a standard exponentially stable LTI system. It then follows from the property of the input-to-state stability [25, p. 175] and Assumption 2 that $\zeta \to \mathbf{0}$ as $t \to \infty$. Hence, it holds that $\nabla_\mu \mathcal{L}(q,\mu,t) \to \mathbf{0}$ as $t \to \infty$.

From the analysis above, it holds that $\nabla_z \mathcal{L}(z,t) \to \mathbf{0}$ as $t \to \infty$, where $z = [q^\top, \mu^\top]^\top$. Therefore, the statement in Proposition 1 follows by Lemma 1. ■

From Proposition 1, it holds that $\sum_{i=1}^N [q_i(t) - c_i(t)] \to \mathbf{0}$ as $t \to \infty$, which implies that feasibility is achieved asymptotically. Moreover, if the initial condition is feasible, i.e., $\sum_{i=1}^N [q_i(0) - c_i(t)] = \mathbf{0}$, one can select appropriate values for $\mu(0)$, $\dot{\mu}(0)$ such that the solution $\sum_{i=1}^N q_i(t)$ will be feasible all the time, i.e., $\sum_{i=1}^N [q_i(t) - c_i(t)] = \mathbf{0}$ $\forall t \ge 0$.

Remark 2: Note that from the centralized algorithm, the optimal Lagrange multiplier is estimated by μ, and it requires global information, especially the Hessians $H_i(q_i,t)$, to calculate the terms G_1, G_2, $\dot{G}_1$ and $\dot{G}_2$, where $G_1 = \sum_{j=1}^N H_j^{-1}(q_j,t)$ and $G_2 = \sum_{j=1}^N \{H_j^{-1}(q_j,t)[\alpha\nabla f_j(q_j,t) + \frac{\partial}{\partial t}\nabla f_j(q_j,t)] + \dot{c}_j - \alpha(q_j - c_j)\}$. To derive the distributed counterpart, one can use distributed average tracking algorithms to estimate the those terms in a distributed manner, which is done in this way in [17] to address the time-varying resource allocation problem with diagonal Hessians. However, when it comes to non-diagonal Hessians, exchanging matrices (e.g., Hessians' inverses) among the agents is expensive and not practical. In the following, we propose a distributed algorithm for networked double-integrator agents (1) to cooperatively solve the time-varying resource allocation problem without exchanging matrices.

B. Distributed Algorithm

Each agent $i \in \mathcal{V}$ has a virtual system defined by

$$\ddot{\mu}_i = -\beta(\dot{\mu}_i - \widetilde{\mu}_i) + \dot{\widetilde{\mu}}_i + \dot{H}_i(q_i,t)H^{-1}(q_i,t)(\dot{\mu}_i - \widetilde{\mu}_i) - \gamma H_i(q_i,t)\sum_{j \in \mathcal{N}_i} \operatorname{sgn}\big[\alpha(\mu_i - \mu_j) + \dot{\mu}_i - \dot{\mu}_j\big], \tag{9}$$

where

$$\widetilde{\mu}_i = -\alpha\mu_i + \alpha H_i(q_i,t)(q_i - c_i) - H_i(q_i,t)\dot{c}_i - \frac{\partial}{\partial t}\nabla f_i(q_i,t) - \alpha\nabla f_i(q_i,t), \tag{10}$$

and α, β and γ are positive constants to be determined. Define the control input for agent $i \in \mathcal{V}$ as

$$u_i = -v_i - F_i(q_i, \mu_i, \dot{\mu}_i, t) - \dot{F}_i(q_i, \mu_i, \dot{\mu}_i, t), \qquad (11)$$

where the function $F_i(\cdot,\cdot,\cdot,\cdot)$ is given in (8).

Theorem 1: Suppose that Assumptions 1-4 hold and let positive constants α, β and γ be selected to satisfy that

$$\gamma \overline{m} \geq 2|\mathcal{E}| d\bar{\omega}\sqrt{Nd}, \qquad (12)$$

where

$$\begin{aligned}\bar{\omega} &= \big[(\alpha\beta + \alpha + 1)\bar{H} + (\alpha+\beta+1)\overline{m}\big]\bar{c} \\ &\quad + (\alpha+\beta+1)\bar{H}\bar{\zeta} + \left(\frac{\beta+1}{\alpha} + 2\right)\bar{H}\bar{\chi} \\ &\quad + \left(\frac{\beta+1}{\alpha} + 3\right)\bar{H}\bar{e} + (3\alpha+\beta+1)\bar{H}\frac{\gamma N\sqrt{d}}{\alpha\beta} \\ &\quad + (\alpha\beta+\alpha+\beta+1)\bar{g}, \qquad (13)\end{aligned}$$

$$\bar{\zeta} = \max_{i\in\mathcal{V}} \big\{ \|q_i(0) - c_i(0)\|_2 \big\}, \qquad (14)$$

$$\bar{\chi} = \max_{i\in\mathcal{V}} \big\{ \|v_i(0) + F_i[q_i(0), \mu_i(0), \dot{\mu}_i(0), 0]\|_2 \big\}, \qquad (15)$$

$$\bar{e} = \max_{i\in\mathcal{V}} \big\{ \big\|H_i^{-1}(0)[\mu_i(0) - \widetilde{\mu}_i(0)]\big\|_2 \big\}. \qquad (16)$$

Using (11) with μ_i, $\dot{\mu}_i$ and $\ddot{\mu}_i$ generated/given by (9) for the double-integrator agents (1) solves the distributed resource allocation problem, i.e., $q_i(t) \to q_i^\star(t)\ \forall i \in \mathcal{V}$ as $t \to \infty$.

Proof: The proof is divided into four steps.

In Step 1, it is proved that $\nabla f_i(q_i,t) + \mu_i \to \mathbf{0}\ \forall i \in \mathcal{V}$ as $t \to \infty$. Define $\chi_i = v_i + F_i(q_i, \mu_i, \dot{\mu}_i, t)$. By (11), it holds that $\dot{\chi}_i = -\chi_i$. It then holds that $\chi_i \to \mathbf{0}\ \forall i \in \mathcal{V}$ as $t \to \infty$. It also follows that $\chi_i(t) = e^{-t}\chi_i(0)\ \forall i \in \mathcal{V}$ for $t \in \mathbb{R}_{\geq 0}$. Then, $\|\chi_i(t)\|_2 \leq e^{-t}\|\chi_i(0)\|_2 \leq \|\chi_i(0)\|_2$. Define $\psi_i = \nabla f_i(q_i,t) + \mu_i$, and it holds that $\dot{\psi}_i = -\alpha\psi_i + H_i(q_i,t)\chi_i$. Then, it holds that $\psi_i \to \mathbf{0}\ \forall i \in \mathcal{V}$ as $t \to \infty$.

In Step 2, it is proved that $\sum_{i=1}^N [q_i - c_i(t)] \to \mathbf{0}$ as $t \to \infty$. Define $e_{\mu_i} = H_i^{-1}(q_i,t)(\dot{\mu}_i - \widetilde{\mu}_i)$. Then, it holds that

$$\dot{e}_{\mu_i} = -\beta e_{\mu_i} - \gamma \sum_{j\in\mathcal{N}_i} \rho_{i,j} \qquad (17)$$

where $\rho_{i,j} = \mathrm{sgn}\big[\alpha(\mu_i - \mu_j) + \dot{\mu}_i - \dot{\mu}_j\big]$, and the second equality follows from (9) and the fact that $\frac{\mathrm{d}}{\mathrm{d}t}H_i^{-1}(q_i,t) = -H^{-1}(q_i,t)\big[\frac{\mathrm{d}}{\mathrm{d}t}H_i(q_i,t)\big]H_i^{-1}(q_i,t)$.

Define $e_\mu = \sum_{i=1}^N e_{\mu_i}$. Note that $\sum_{i=1}^N \sum_{j\in\mathcal{N}_i} \rho_{i,j} = \mathbf{0}$ follows from Assumption 1. Then, it holds that $\dot{e}_\mu = -\beta e_\mu$. Hence, $e_\mu \to \mathbf{0}$ as $t \to \infty$.

Define $\zeta_i = q_i - c_i$. It then holds that

$$\dot{\zeta}_i = -\alpha\zeta_i + \chi_i - e_{\mu_i}, \qquad (18)$$

where the last equality is obtained by using the definition of $\widetilde{\mu}_i$ in (10). Define $\zeta = \sum_{i=1}^N \zeta_i$. It then follows that $\dot{\zeta} = -\alpha\zeta + \sum_{i=1}^N \chi_i - e_\mu$. Recall that $\chi_i, e_\mu \to \mathbf{0}$ as $t \to \infty$. It follows from the properties of the input-to-state stability that $\zeta \to \mathbf{0}$ as $t \to \infty$. That is, $\nabla_\mu \mathcal{L}(q,\mu,t) \to \mathbf{0}$ as $t \to \infty$.

In Step 3, it is proved that $\mu_i - \mu_j \to \mathbf{0}\ \forall i,j \in \mathcal{V}$ as $t \to \infty$. It follows from the definition of $\widetilde{\mu}_i$ in (10) that the system (9) can be rewritten as

$$\begin{aligned}\ddot{\mu}_i &= -\alpha\beta\mu_i - (\alpha+\beta)\dot{\mu}_i + \omega_i(q_i,v_i,t) \\ &\quad - \gamma H_i(q_i,t)\sum_{j\in\mathcal{N}_i} \mathrm{sgn}\big[\alpha(\mu_i-\mu_j) + \dot{\mu}_i - \dot{\mu}_j\big], \qquad (19)\end{aligned}$$

where $\omega_i(q_i,v_i,t) = \alpha\beta H_i(q_i,t)[q_i - c_i(t)] - \beta H_i(q_i,t)\dot{c}_i(t) - \beta\frac{\partial}{\partial t}\nabla f_i(q_i,t) - \alpha\beta\nabla f_i(q_i,t) + \alpha\big[\frac{\mathrm{d}}{\mathrm{d}t}H_i(q_i,t)\big][q_i - c_i(t)] + \alpha H_i(q_i,t)[v_i - \dot{c}_i(t)] - \big[\frac{\mathrm{d}}{\mathrm{d}t}H_i(q_i,t)\big]\dot{c}_i(t) - H_i(q_i,t)\ddot{c}_i - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial t}\nabla f_i(q_i,t) - \alpha\frac{\mathrm{d}}{\mathrm{d}t}\nabla f_i(q_i,t) + \big[\frac{\mathrm{d}}{\mathrm{d}t}H_i(q_i,t)\big]e_{\mu_i}$. Note from Assumption 4 that $\nabla f_i(q_i,t) = H_i(t)q_i + g_i(t)$, and then it follows from the definition of ζ_i that $\omega_i(q_i,v_i,t) = -\big[\alpha\beta H_i(t) + \alpha\dot{H}_i(t)\big]c_i(t) - \big[(\alpha+\beta)H_i(t) + \dot{H}_i(t)\big]\dot{c}_i(t) - H_i(t)\ddot{c}_(t) - \big[\beta\dot{H}_i(t) + \ddot{H}_i(t)\big][\zeta_i + c_i(t)] - \dot{H}_i(t)\big[\dot{\zeta}_i + \dot{c}_i(t)\big] - \alpha\beta g_i(t) - (\alpha+\beta)\dot{g}_i(t) - \ddot{g}_i + \dot{H}_i(t)e_{\mu_i}$.

Note that for any $i \in \mathcal{V}$, it follows from (17) that $\big\|\sum_{j\in\mathcal{N}_i}\rho_{i,j}\big\|_2 \leq N\sqrt{d}$. Then, it holds that $e_{\mu_i}(t) = e^{-\beta t}e_{\mu_i}(0) - \gamma\int_0^t e^{-\beta(t-\tau)}\sum_{j\in\mathcal{N}_i}\rho_{i,j}(\tau)\mathrm{d}\tau$. It follows that $\|e_{\mu_i}(t)\|_2 \leq \|e_{\mu_i}(0)\|_2 + \frac{\gamma N\sqrt{d}}{\beta}$. It follows from (18) that $\|\zeta_i(t)\|_2 \leq \|\zeta_i(0)\|_2 + \frac{1}{\alpha}[\|\chi_i(0)\|_2 + \|e_{\mu_i}(0)\|_2] + \frac{\gamma N\sqrt{d}}{\alpha\beta}$. It follows from (18) that $\big\|\dot{\zeta}_i\big\|_2 \leq \alpha\|\zeta_i(0)\|_2 + 2[\|\chi_i(0)\|_2 + \|e_{\mu_i}(0)\|_2] + \frac{2\gamma N\sqrt{d}}{\beta}$. By Assumption 4, it holds that $\|\omega_i(q_i,v_i,t)\|_\infty \leq \big[(\alpha\beta+\alpha+1)\bar{H} + (\alpha+\beta+1)\overline{m}\big]\bar{c} + (\alpha+\beta+1)\bar{H}\|\zeta_i(0)\|_2 + \big(\frac{\beta+1}{\alpha}+2\big)\bar{H}\|\chi_i(0)\|_2 + \big(\frac{\beta+1}{\alpha}+3\big)\bar{H}\|e_{\mu_i}(0)\|_2 + (3\alpha+\beta+1)\bar{H}\frac{\gamma N\sqrt{d}}{\alpha\beta} + (\alpha\beta+\alpha+\beta+1)\bar{g} \leq \bar{\omega}$, where $\bar{\omega}$ is given in (13).

Define $\delta_i = \alpha\mu_i + \dot{\mu}_i$ and $\delta = \big[\delta_1^\top, \ldots, \delta_N^\top\big]^\top$, and let $\widetilde{B} = B \otimes I_d$. Then, it holds that

$$\dot{\delta} = -\beta\delta - \gamma\boldsymbol{H}(t)\widetilde{B}\mathrm{sgn}\big(\widetilde{B}^\top\delta\big) + \omega(q,v,t), \qquad (20)$$

where $\boldsymbol{H}(t) = \mathrm{diag}\big\{H_1(t), \ldots, H_N(t)\big\}$ and $\omega(q,v,t) = \big[\omega_1^\top(q_1,v_1,t), \ldots, \omega_N^\top(q_N,v_N,t)\big]^\top$. Since the signum function is measurable and locally essentially bounded and $\omega(q,v,t)$ is bounded, by [26], the Filippov solutions [1] of (20) exist and are absolutely continuous, that is, δ is continuous. Hence, $\mathcal{K}[-\beta\delta] = \{-\beta\delta\}$ and $\mathcal{K}[\omega(q,v,t)] \subseteq [-\bar{\omega},\bar{\omega}]^{Nd}$. It holds that $\mathcal{K}\big[\dot{\delta}\big] \subseteq \mathcal{F}_\delta$, where $\mathcal{F}_\delta = \{-\beta\delta\} + \mathcal{K}[\omega(q,v,t)] - \gamma\boldsymbol{H}\widetilde{B}\mathcal{K}\Big[\mathrm{sgn}\big(\widetilde{B}^\top\delta\big)\Big]$. Note that, for any $r = [r_1, \ldots, r_p] \in \mathbb{R}^p$, it holds that $\mathcal{K}[\mathrm{sgn}(r)] = \mathcal{K}[\mathrm{sgn}(r_1)] \times \cdots \times \mathcal{K}[\mathrm{sgn}(r_p)]$ and $\mathcal{K}[\mathrm{sgn}(r_i)] = \{1\}$ if $r_i > 0$, $\mathcal{K}[\mathrm{sgn}(r_i)] = \{-1\}$ if $r_i < 0$, and $\mathcal{K}[\mathrm{sgn}(r_i)] = [-1,1]$ if $r_i = 0$. Consider the Lyapunov function candidate $V[\delta(t)] = \Big\|\widetilde{B}^\top\delta\Big\|_1$. Note that V is locally Lipschitz continuous but nonsmooth at some points. Then, by [27], it holds that $\frac{\mathrm{d}}{\mathrm{d}t}V[\delta(t)] \in \dot{\tilde{V}}$. The generalized gradient

[1]Consider the vector differential equation $\dot{x} = f(x,t)$, where $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is measurable and locally essentially bounded. A vector function $x(\cdot)$ is called a Filippov solution on $[t_0,t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0,t_1]$ and for almost all $t \in [t_0,t_1]$, $\dot{x} \in \mathcal{K}[f](x,t)$, where $\mathcal{K}[f](x,t) := \bigcap_{\Lambda>0}\bigcap_{\mu(\mathscr{N})=0}\overline{\mathrm{co}}f(\mathcal{B}(x,\Lambda) - \mathscr{N}, t)$ is the Filippov set-valued map of $f(x,t)$ and $\bigcap_{\mu(\mathscr{N})=0}$ denotes the intersection over all sets $\mathscr{N}$ of Lebesgue measure zero

of V is given by $\partial V = \widetilde{B}\mathcal{K}\left[\operatorname{sgn}\left(\widetilde{B}^\top\delta\right)\right]$. By [27], the set-valued Lie derivative of V is given by

$$\dot{\tilde{V}} \subseteq \bigcap_{\xi\in\mathcal{K}\left[\operatorname{sgn}\left(\widetilde{B}\delta\right)\right]} \xi^\top \widetilde{B}^\top \mathcal{F}_\delta. \tag{21}$$

By (21), it holds that if $\dot{\tilde{V}} \neq \emptyset$ and assume that $\tilde{a} \in \dot{\tilde{V}}$, then there exist $\tilde{\eta} \in \mathcal{K}\left[\operatorname{sgn}\left(\widetilde{B}^\top\delta\right)\right]$ and $\tilde{\omega} \in \mathcal{K}[\omega(q,v,t)]$ such that $\tilde{a} = \xi^\top\widetilde{B}\left[-\beta\delta - \gamma\boldsymbol{H}\widetilde{B}\tilde{\eta} + \tilde{\omega}\right]$ holds for any $\xi \in \mathcal{K}\left[\operatorname{sgn}\left(\widetilde{B}^\top\delta\right)\right]$. Define $\rho = \operatorname{sgn}\left(\widetilde{B}^\top\delta\right)$, and for such $\tilde{\eta}$ and $\tilde{\omega}$, one can choose $\xi = \tilde{\xi} \in \mathcal{K}\left[\operatorname{sgn}\left(\widetilde{B}^\top\delta\right)\right]$ such that $\tilde{\xi}_i = \rho_i$ if $\rho_i \neq 0$ and $\tilde{\xi}_i = \tilde{\eta}_i$ if $\rho_i = 0$, where $\tilde{\xi}_i$, ρ_i and $\tilde{\eta}_i$ denote the ith element of the vectors $\tilde{\xi}$, ρ and $\tilde{\eta}$, respectively. Note that $\rho_i = 0$ if and only if $X_i = 0$, where X_i is the ith element of the vector $X = \widetilde{B}^\top\delta$. Then, it holds that $-\beta\tilde{\xi}^\top\widetilde{B}^\top\delta = -\beta\left\|\widetilde{B}^\top\delta\right\|_1$. It also holds that $-\gamma\xi^\top\widetilde{B}^\top\boldsymbol{H}\widetilde{B}\tilde{\eta} \leq -\gamma\lambda_{\min}(\boldsymbol{H})\left\|\widetilde{B}\xi\right\|_2^2 \leq -\gamma\overline{m}\left\|\widetilde{B}\xi\right\|_2^2$ and $\xi^\top\widetilde{B}^\top\tilde{\omega} \leq \left\|\widetilde{B}\xi\right\|_2\|\tilde{\omega}\|_2 \leq \bar{\omega}\sqrt{Nd}\left\|\widetilde{B}\xi\right\|_1 \leq 2|\mathcal{E}|d\bar{\omega}\sqrt{Nd}\|\xi\|_\infty$. If there exists an edge $(i,j) \in \mathcal{E}$ such that $\delta_i \neq \delta_j$, then $\left\|\widetilde{B}\xi\right\|_2 \geq 1$. Then, it holds that $\tilde{a} \leq -\beta\left\|\widetilde{B}^\top\delta\right\|_1 - \gamma\overline{m} + 2|\mathcal{E}|d\bar{\omega}\sqrt{Nd}$. Hence, if α, β and γ are selected to satisfy (12), then it holds that $\tilde{a} \leq -\beta\left\|\widetilde{B}^\top\delta\right\|_1$. Therefore, for any $\tilde{a} \in \dot{\tilde{V}}$, if there exists an edge $(i,j) \in \mathcal{E}$ such that $\delta_i \neq \delta_j$, $\tilde{a} \leq -\beta\left\|\widetilde{B}^\top\delta\right\|_1$. Hence, it holds that $\widetilde{B}^\top\delta \to \mathbf{0}$ as $t \to \infty$. From the definition of δ, it follows that $(B^\top \otimes I_d)\mu \to \mathbf{0}$ as $t \to \infty$, which implies that Step 3 is complete.

In Step 4, the statement of Theorem 1 is finally proved. From Step 1, it holds that $\nabla f_i(q_i,t) + \mu_i \to \mathbf{0}\ \forall i \in \mathcal{V}$ as $t \to \infty$. From Step 3, it is can be derived that there exists a function $\mu(t)$ such that $\mu_i(t) - \mu(t) \to \mathbf{0}\ \forall i \in \mathcal{V}$ as $t \to \infty$. Note that $\nabla f_i(q_i,t) + \mu_i = \nabla f_i(q_i,t) + \mu(t) + \mu_i - \mu(t)$ $\forall i \in \mathcal{V}$. Then, it holds that $\nabla f_i(q_i,t) + \mu \to \mathbf{0}\ \forall i \in \mathcal{V}$ as $t \to \infty$, which implies that $\nabla_{q_i}\mathcal{L}(q,\mu,t) \to \mathbf{0}$ as $t \to \infty$. Combine with Step 2, it holds that $\nabla_z\mathcal{L}(z,t) \to \mathbf{0}$ as $t \to \infty$, where $z = [q^\top, \mu^\top]^\top$. Then, by Lemma 1, the statement in Theorem 1 holds. ∎

Remark 3: It is worth pointing out that one can always find positive constants α, β and γ satisfying (12). For instance, one can choose large enough values for α and β such that $\alpha\beta > (3\alpha + \beta + 1)\bar{H}N\sqrt{d}$, and then choose large enough value for γ. These parameters are constants, which can be determined off-line and then embedded into the agents.

V. Illustrative Examples

Consider a group of $N = 10$ double-integrator agents described by (1) where $d = 2$, and assume that each agent $i \in \mathcal{V}$ has a cost function $f_i(q_i,t) = \frac{1}{2}q_i^\top H_i(t)q_i + g_i(t)^\top q_i + h_i(t)$, where $H_i(t) = [10 + 0.1i, 10 + 0.1i; 10 + 0.1i, 13 + 0.1i\cos(t) + 0.1i]$, $g_i(t) = [i\cos(t), i\sin(t)]^\top$, and $h_i(t)$ is a time-varying function. The agents aim to cooperatively solve the resource allocation problem defined in (2)-(3), where $c_i(t) = [0.5i\cos(t) + i + 45, 0.5i\sin(t) + i + 20]^\top$. We select $\alpha = 0.5$, $\beta = 1$ and $\gamma = 9$. The trajectories of q_i and μ_i are presented in Fig. 1 and Fig. 2. In the two figures, the solid and dashed lines are the trajectories generated by using the distributed algorithm (in Section IV-B) and the centralized algorithm (in Section IV-A), respectively. The dash-dotted lines are the optimal solutions of $q_i^\star$ and $\mu^\star$, $i \in \mathcal{V}$. It can be seen that q_i and μ_i can track the optimal $q_i^\star$ and $\mu^\star$.

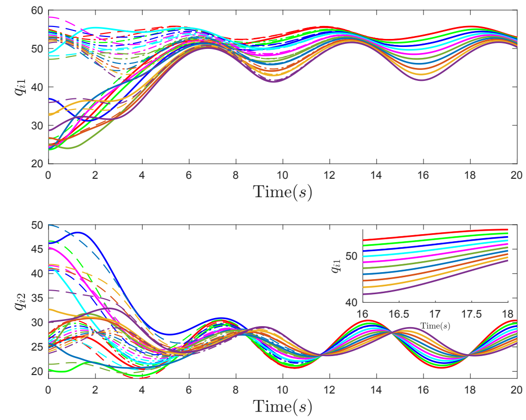

Fig. 1. The position trajectories of double-integrator agents (1) generated by using the algorithms in Sections IV-A and IV-B.

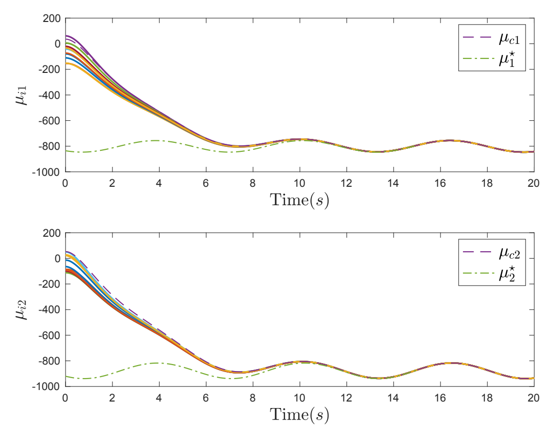

Fig. 2. The trajectories of μ_i generated by using the algorithm in Sections IV-A and IV-B.

VI. Conclusion

The distributed time-varying resource allocation problem has been investigated for networked double-integrator agents. A distributed algorithm has been proposed for the agents to track the optimal decision trajectories with zero errors. Finally, simulation results have been presented to validate the effectiveness of the proposed distributed algorithm.

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