



Scale fragilities in localized consensus dynamics[☆]

Emma Tegling^{a,*}, Bassam Bamieh^b, Henrik Sandberg^c

^a Department of Automatic Control, Lund University, P.O. Box 118, SE 221 00, Lund, Sweden

^b Department of Mechanical Engineering at the University of California at Santa Barbara, Santa Barbara, CA 93106, USA

^c School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, SE 100 44, Stockholm, Sweden

ARTICLE INFO

Article history:

Received 14 March 2022

Received in revised form 2 February 2023

Accepted 16 March 2023

Available online 2 May 2023

Keywords:

Multi-agent networks

Large-scale systems

Fundamental limitations

ABSTRACT

We consider distributed consensus in networks where the agents have integrator dynamics of order two or higher ($n \geq 2$). We assume all feedback to be localized in the sense that each agent has a bounded number of neighbors and consider a scaling of the network through the addition of agents in a modular manner, i.e., without re-tuning controller gains upon addition. We show that standard consensus algorithms, which rely on relative state feedback, are subject to what we term scale fragilities, meaning that stability is lost as the network scales. For high-order agents ($n \geq 3$), we prove that no consensus algorithm with fixed gains can achieve consensus in networks of any size. That is, while a given algorithm may allow a small network to converge, it causes instability if the network grows beyond a certain finite size. This holds in families of network graphs whose algebraic connectivity, that is, the smallest non-zero Laplacian eigenvalue, is decreasing towards zero in network size (e.g. all planar graphs). For second-order consensus ($n = 2$) we prove that the same scale fragility applies to directed graphs that have a complex Laplacian eigenvalue approaching the origin (e.g. directed ring graphs). The proofs for both results rely on Routh–Hurwitz criteria for complex-valued polynomials and hold true for general directed network graphs. We survey classes of graphs subject to these scale fragilities, discuss their scaling constants, and finally prove that a sub-linear scaling of nodal neighborhoods can suffice to overcome the issue.

© 2023 Published by Elsevier Ltd.

1. Introduction

Characterizing the dynamic behaviors of networked or multi-agent systems has been an active research area for many years. In particular, since the works by Fax and Murray (2004), Jadbabaie, Lin, and Morse (2003), and Olfati-Saber and Murray (2004), the prototypical sub-problem of distributed consensus has been the subject of significant research efforts. While the particular modeling aspects vary, the consensus objective is to coordinate agents in a network to a common state of agreement. Applications range from distributed computing and sensing to power grid synchronization and coordination of unmanned vehicles (Olfati-Saber, Fax, & Murray, 2007).

The most traditional consensus problem is of first order, meaning that agents are modeled as single integrators with a state

that develops according to a weighted sum of differences between states of neighboring agents, that is, relative state feedback. Second-order consensus can model coordination of agents with mass and is used to study formation control in multi-vehicle networks. The corresponding higher-order problem, to which most results in this paper pertain, has also received significant attention, as in Jiang, Wang, and Jia (2009), Ni and Cheng (2010), Radmanesh, Naghash, and Mohamadifard (2017), Ren, Moore, and Chen (2007), Rezaee and Abdollahi (2015) and Zuo, Tian, Defoort, and Ding (2018). Here, each agent is modeled as an n th order integrator, and the control signal is a weighted sum of relative feedback terms. This can be viewed as an important theoretical generalization of the first- and second-order algorithms (Jiang et al., 2009), but also has practical relevance. For example, position, velocity, as well as acceleration feedback play a role in flocking behaviors, resulting in a model where $n = 3$ (Ren et al., 2007).

Existing literature has typically focused on deriving conditions for convergence of a given set of agents to consensus, and how such conditions depend on various properties of the network. For example, directed communication, a switching or random topology (Ni & Cheng, 2010), or a leader–follower structure (Zuo et al., 2018). This paper takes a different perspective and concerns the scalability of given consensus algorithms to large networks

[☆] The material in this paper was partially presented at the 2019 American Control Conference (ACC), July 10–12, 2019, Philadelphia, PA, USA. This paper was recommended for publication in revised form by Associate Editor Solmaz Kia under the direction of Editor Christos G. Cassandras.

* Corresponding author.

E-mail addresses: emma.tegling@control.lth.se (E. Tegling), bamieh@engineering.ucsb.edu (B. Bamieh), hsan@kth.se (H. Sandberg).

under a modular design principle. That is, we assume that the interaction rules between agents are *fixed*, (i.e., pre-designed) and *localized*, and grow the network through the addition of more and more agents. It has previously been observed that this type of modular scaling can lead to poor dynamic behaviors in first- and second-order consensus problems, such as a lack of network coherence (Bamieh, Jovanović, Mitra, & Patterson, 2012; Patterson & Bamieh, 2014; Siami & Motee, 2016; Tegling, Mitra, Sandberg, & Bamieh, 2019). These behaviors are a question of control *performance*. In this paper, we show that the question of scalability in high-order consensus is more fundamental: can *stability* be maintained as the network grows?

This paper shows that both second- and higher-order consensus ($n \geq 2$) are subject to *scale fragilities* in certain classes of network graphs. These imply that stability (and thereby convergence to consensus) is lost if the network grows beyond some finite size. For $n \geq 3$, our result is particularly clear-cut: the consensus algorithm treated in, for example, Ren et al. (2007) does not scale stably in any family of graphs whose algebraic connectivity decreases towards zero in network size.

The algebraic connectivity, that is, the smallest non-zero eigenvalue of the graph Laplacian, decreases towards zero in families of graphs where nodal neighborhoods are *localized* in the sense that they are bounded in size and reach (the formal definition is given through the graph's isoperimetric, or Cheeger, constant). Here, we review this property for graphs such as lattices, trees, and planar graphs, and derive the rates at which their respective algebraic connectivity decreases. In leader–follower consensus of order $n \geq 3$, the scale fragility arises in any undirected graph family where the neighborhood size is bounded. This latter result was observed in the context of vehicular strings by Barooah and Hespanha (2005) and Yadlapalli, Darbha, and Rajagopal (2006). Here, we generalize that result to leaderless consensus and general directed, weighted graphs.

For second-order consensus ($n = 2$), the scale fragility applies only to particular classes of directed graphs. These are characterized by a complex Laplacian eigenvalue that approaches the origin as the network size grows. This applies, for example, to directed ring graphs. The particular result for ring graphs has previously been reported in Cantos, Veerman, and Hammond (2016), Herman (2016) and Stüdli, Seron, and Middleton (2017), but our work provides a significant generalization. The result implies that ring-shaped vehicular formations, such as those where adaptive cruise control is used to regulate spacing and velocity to the preceding vehicle, see Gunter et al. (2021), are at risk of becoming unstable.

We remark that the phenomenon we describe in this paper is distinct from the issue of string stability in vehicular strings. String instability, see e.g. Seiler, Pant, and Hedrick (2004) and Yadlapalli et al. (2006), implies that disturbances are amplified along the string of vehicles, though the overall system dynamics can be stable. It is therefore a notion of performance rather than stability, see also Besselink and Knorn (2018). Here, we describe a loss of closed-loop stability, subject to a modular scaling of the network.

The fact that consensus may fail to scale stably to large networks has, to the best of our knowledge, not been observed in literature apart from the aforementioned works. While it is noted in Jiang et al. (2009) and Ren et al. (2007) that controller gains in high-order consensus must be chosen with care to ensure stability, we point out that no such choice can guarantee stability in a network that grows. For so-called *open* multi-agent systems (Franceschelli & Frasca, 2021; Hendrickx & Martin, 2017), where agents may come and leave while adhering to, e.g., a consensus protocol, our results imply limitations on the allowable size of the overall system (depending on the agent dynamics and the degree of locality).

The scale fragilities we describe can in principle be attributed to the relative state feedback upon which the consensus algorithm is based. It is known that a restriction to relative feedback imposes performance and design limitations; an issue that was analyzed formally in Jensen and Bamieh (2022). In this paper, we also discuss how the scalability can be achieved if the controller has access to absolute feedback.

The locality property, that is, bounded nodal neighborhoods, is also key for our results. A natural question is therefore how nodal neighborhoods would need to scale to alleviate the scale fragility. Interestingly, we prove using a ring graph topology that it can suffice to grow neighborhoods as $N^{2/3}$, where N is the network size. We note that this only holds for leaderless consensus; leader–follower consensus requires neighborhoods proportional to N .

The present paper extends our preliminary work in Tegling, Bamieh and Sandberg (2019), where the result on high-order ($n \geq 3$) consensus was first reported. The corresponding result herein is improved in its formalism and generalized to all directed graphs families. Our characterization of graphs with decreasing algebraic connectivity has been expanded with a general analytic criterion. All other results are new.

The remainder of this paper is organized as follows. We next introduce the n th order consensus algorithm along with important definitions and assumptions. In Section 3 we give the result for high-order consensus. We also discuss classes of graphs where the result applies and give numerical examples. Section 4 then presents corresponding results for second-order consensus. In Section 5 we discuss ways to retrieve scalable stability, e.g. by scaling nodal neighborhoods, and we conclude with a discussion in Section 6.

2. Problem setup

We now introduce the network model along with the n th order consensus algorithm. This algorithm is a straightforward extension to standard first- and second-order consensus and has previously been considered in Ni and Cheng (2010), Ren et al. (2007) and Rezaee and Abdollahi (2015).

2.1. Network model and definitions

Consider a network modeled by the graph $\mathcal{G}_N = \{\mathcal{V}_N, \mathcal{E}_N\}$ with $N = |\mathcal{V}_N|$ nodes. The set $\mathcal{E}_N \subset \mathcal{V}_N \times \mathcal{V}_N$ contains the edges, each of which has an associated nonnegative weight w_{ij} . We generally let the graph $\mathcal{G}_N$ be directed, so the edge $(i, j) \in \mathcal{E}_N$ points from node i (the tail) to node j (the head). The neighbor set $\mathcal{N}_i$ of node i is the set of nodes j to which there is an edge $(i, j) \in \mathcal{E}$. The outdegree of node i is defined as $d_i^+ = \sum_{j=1}^N w_{ij}$ and its indegree is $d_i^- = \sum_{j=1}^N w_{ji}$ ($w_{ij} = 0$ if $(i, j) \notin \mathcal{E}$). The graph $\mathcal{G}_N$ is *balanced* if $d_i^+ = d_i^-$ for all $i \in \mathcal{V}_N$ and *undirected* if $(i, j) \in \mathcal{E}_N \Rightarrow (j, i) \in \mathcal{E}$ for all $i, j \in \mathcal{V}_N$ and $w_{ij} = w_{ji}$. It has a *connected spanning tree* if there is a path from some node $i \in \mathcal{V}_N$ to any other node $j \in \mathcal{V}_N \setminus \{i\}$. The r -fuzz of a graph $\mathcal{G}_N$ is the graph obtained from $\mathcal{G}_N$ by adding an edge (u, v) for all v that are at most r steps away from u .

Going forward, we will model networks with an increasing numbers of agents. We therefore consider $\mathcal{G}_N$ as a member of a sequence, or a *family*, of graphs $\{\mathcal{G}_N\}$ in which the network size N is increasing. We remark that $\mathcal{G}_N$ need not be a subgraph of $\mathcal{G}_{N+1}$ for our results to hold.

The graph Laplacian L of $\mathcal{G}_N$ is defined as follows:

$$[L]_{ij} = \begin{cases} -w_{ij} & \text{if } j \neq i \text{ and } j \in \mathcal{N}_i \\ \sum_{k \in \mathcal{N}_i} w_{ik} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Denote by λ_l (or $\lambda_l(\mathcal{G}_N)$ where explicitness is needed) with $l = 1, \dots, N$ the eigenvalues of L . Zero is a simple eigenvalue of L if and only if the graph has a connected spanning tree, which will be the scenario of interest throughout. Remaining eigenvalues are in the complex right half plane (RHP), and numbered so that $0 = \lambda_1 < \text{Re}\{\lambda_2\} \leq \dots \leq \text{Re}\{\lambda_N\}$. The graph Laplacian L is called *normal* if $L^T L = L L^T$. If the graph is undirected, L is symmetric and thereby normal. For a directed graph, normality of L implies that $\mathcal{G}_N$ is balanced.

2.2. n th order consensus

The local dynamics of each agent $i \in \mathcal{V}_N$ is modeled as a chain of n integrators:

$$\begin{aligned} \frac{d}{dt} x_i^{(0)}(t) &= x_i^{(1)}(t) \\ &\vdots \\ \frac{d}{dt} x_i^{(n-2)}(t) &= x_i^{(n-1)}(t) \\ \frac{d}{dt} x_i^{(n-1)}(t) &= u_i(t) \end{aligned}$$

where we assume a scalar state $x_i(t) \in \mathbb{R}$ (see Remark 1), collected in the vector $x = [x_1, x_2, \dots, x_N]^T \in \mathbb{R}^N$. The notation for time derivatives is such that $x_i^{(0)}(t) = x_i(t)$, $x_i^{(1)}(t) = \frac{d}{dt} x_i(t) = \dot{x}_i(t)$ etc. until $x_i^{(n)}(t) = \frac{d^n}{dt^n} x_i(t)$. Going forward, we will often drop the time dependence in the notation.

We consider the following n th order consensus algorithm:

$$u_i = - \sum_{k=0}^{n-1} a_k \sum_{j \in \mathcal{N}_i} w_{ij} (x_i^{(k)} - x_j^{(k)}) \quad (2)$$

where the $a_k > 0$ are fixed gains. The feedback in (2) is termed *relative* as it is only based on differences between states of neighboring agents. The impact of absolute feedback, where the controllers have access to measurements of the absolute local state, is treated in Section 5.

Defining the full state vector $\xi = [x^{(0)}, x^{(1)}, \dots, x^{(n-1)}]^T \in \mathbb{R}^{Nn}$, we can write the system's closed-loop dynamics as

$$\frac{d}{dt} \xi = \underbrace{\begin{bmatrix} 0 & I_N & 0 & \cdots & 0 \\ 0 & 0 & I_N & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_N \\ -a_0 L & -a_1 L & -a_2 L & \cdots & -a_{n-1} L \end{bmatrix}}_{\mathcal{A}} \xi, \quad (3)$$

where the graph Laplacian L was defined in (1) and I_N denotes the $N \times N$ identity matrix.

Remark 1. We limit the analysis to a scalar information state, though an extension to $x_i(t) \in \mathbb{R}^m$ is straightforward if the same consensus algorithm is applied in all coordinate directions. In this case, the system dynamics can be written $\dot{\xi} = (\mathcal{A} \otimes I_m) \xi$, where $\otimes$ denotes the Kronecker product. Our results, which concern the stability of $\mathcal{A}$, would not change.

2.2.1. Leader-follower consensus

It will also be relevant to consider leader-follower consensus, where the state of one agent (the leader) is fixed at a desired setpoint and remaining agents converge to that same state (assuming there is a directed path to each of them from the leader node). Without loss of generality, take Agent 1 to be the leader

and set $x_1 = \dot{x}_1 = \dots = x_1^{(n)} \equiv 0$. The closed-loop dynamics for remaining agents can then be written

$$\frac{d}{dt} \bar{\xi} = \underbrace{\begin{bmatrix} 0 & I_{N-1} & 0 & \cdots & 0 \\ 0 & 0 & I_{N-1} & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{N-1} \\ -a_0 \bar{L} & -a_1 \bar{L} & -a_2 \bar{L} & \cdots & -a_{n-1} \bar{L} \end{bmatrix}}_{\bar{\mathcal{A}}} \bar{\xi}, \quad (4)$$

where $\bar{L}$ is the *grounded* graph Laplacian obtained by deleting the first row and column of L , and $\bar{\xi}$ is obtained by removing the states of Agent 1. Note that $\bar{L}$ unlike L has all of its eigenvalues in the right half plane (Xia & Cao, 2017).

2.3. Conditions for consensus and scalable stability

The network of agents is said to be achieving consensus if $x_i^{(k)} \rightarrow x_j^{(k)}$ for all $i, j \in \mathcal{V}_N$, all $k = 0, 1, \dots, n-1$, and for any initial state. It is known that the algorithm (2) achieves consensus if the eigenvalues of $\mathcal{A}$ are in the left half plane, apart from exactly n zero eigenvalues that are associated with the drift of the network average. This condition is in line with standard results for first- and second-order consensus, and is shown in Ren et al. (2007) for $n = 3$:

Lemma 2.1 (Ren et al., 2007, Theorem 3.1). *In the case of $n = 3$, the algorithm (2) achieves consensus exponentially if and only if $\mathcal{A}$ has exactly three zero eigenvalues and all of the other eigenvalues have negative real parts.*

We also require the following lemma:

Lemma 2.2 (Ren et al., 2007, Lemma 3.1). *In the case of $n = 3$, the matrix $\mathcal{A}$ has exactly three zero eigenvalues if and only if L has a simple zero eigenvalue.*

The proofs in Ren et al. (2007) extend straightforwardly to $n > 3$. This means that it suffices to verify that the $(N-1) \cdot n$ non-zero eigenvalues of $\mathcal{A}$ have negative real parts.

In this work, we describe systems where these conditions may hold for small network sizes N , but where one or more eigenvalues leaves the left half plane and causes instability when the network grows beyond some $\bar{N}$. In these cases, we say the control algorithm lacks *scalable stability*.

Definition 2.1 (Scalable Stability). A consensus control design is *scalably stable* if the resulting closed-loop system achieves consensus over any graph in the family $\{\mathcal{G}_N\}$.

2.4. Underlying assumptions: modularity and locality

The notion of scalable stability of a controller presumes a *modular* design principle. This means that new agents are added to the network with the pre-designed controller gains, which are not re-tuned as the network grows. This means that the following important assumptions will underlie our analysis of the control u in (2):

Assumption A1 (Fixed and Finite Gains). The gains a_k for all $k = 0, 1, \dots, n-1$ satisfy $a_k \leq a_{\max} < \infty$ and they do not change if the underlying graph changes. That is, the gains are fixed with respect to the graph family $\{\mathcal{G}_N\}$. In particular, they are independent of N .

When it comes to the network graph, our main result will rely on the property that the algebraic connectivity decreases in network size. When discussing families of graphs where this property holds, we will impose the following assumptions, unless otherwise stated:

Assumption A2 (Bounded Neighborhoods). All nodes in the graph family $\{\mathcal{G}_N\}$ have a neighborhood of size at most q , where q is fixed and independent of N . That is,

$$|\mathcal{N}_i| \leq q \quad \forall i \in \mathcal{V}_N. \quad (5)$$

Assumption A3 (Finite Weights). The edge weights in $\{\mathcal{G}_N\}$ are finite, that is, $w_{ij} \leq w_{\max} < \infty$ for all $(i, j) \in \mathcal{E}_N$, where $w_{\max}$ is fixed and independent of N .

Assumptions A2–A3 imply that we consider networks with bounded nodal degrees.

3. Scale fragility in high-order consensus

This section is devoted to our first important result. We prove that the high-order consensus algorithm ($n \geq 3$) lacks scalable stability in graph families with what we term a decreasing algebraic connectivity. This applies to all graphs where connections are, in a sense, localized.

3.1. Main result

This section's main result can be stated as follows.

Theorem 3.1. *If $n \geq 3$, no control on the form (2) subject to Assumption A1, is scalably stable in graph families where the sequence $\text{Re}\{\lambda_2(\mathcal{G}_N)\} \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. The first step of the proof is a (generalized) block-diagonalization of the system matrix $\mathcal{A}$. Let T be an invertible $N \times N$ matrix such that $\Lambda = T^{-1}LT$ is on Jordan normal form. That is, $\Lambda = \text{diag}\{\Lambda_1, \dots, \Lambda_k\}$, where Λ_l , $l = 1, \dots, k$ are $r_l \times r_l$ Jordan blocks, in which the Laplacian eigenvalue λ_l is repeated along the main diagonal and ones appear on the superdiagonal (see Horn and Johnson (1985, Chapter 3) for details). The number k of Jordan blocks is the number of linearly independent eigenvectors of L , which may be less than or equal to its number of distinct eigenvalues. If the graph $\mathcal{G}_N$ is undirected, then L is symmetric and thus diagonalizable. In this case, $r_i = 1$ for $i = 1, \dots, k = N$. Otherwise, we only impose that the eigenvalue $\lambda_1 = 0$ is simple, which is equivalent to the graph having a connected spanning tree. If this is not the case, the graph is disconnected, $\lambda_2(\mathcal{G}_N) = 0$, and the conditions in Section 2.3 do not hold. The system is then by definition not scalably stable. By pre- and post-multiplying $\mathcal{A}$ by the $(Nn \times Nn)$ matrix $\mathbf{T} = \text{diag}\{T, T, \dots, T\}$, we get

$$\mathbf{T}^{-1}\mathcal{A}\mathbf{T} = \underbrace{\begin{bmatrix} 0 & I_N & 0 & \cdots & 0 \\ 0 & 0 & I_N & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_N \\ -a_0\Lambda & -a_1\Lambda & -a_2\Lambda & \cdots & -a_{n-1}\Lambda \end{bmatrix}}_{\hat{\mathcal{A}}} \quad (6)$$

By pre- and post-multiplying by a suitable permutation matrix, the rows and columns of $\hat{\mathcal{A}}$ can be rearranged into the system

matrix $\text{diag}\{\hat{\mathcal{A}}_1, \dots, \hat{\mathcal{A}}_k\}$ with

$$\hat{\mathcal{A}}_l = \begin{bmatrix} 0 & I_{r_l} & 0 & \cdots & 0 \\ 0 & 0 & I_{r_l} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{r_l} \\ -a_0\Lambda_l & -a_1\Lambda_l & -a_2\Lambda_l & \cdots & -a_{n-1}\Lambda_l \end{bmatrix}$$

for $l = 1, \dots, k$. The eigenvalues of $\mathcal{A}$, equivalently $\hat{\mathcal{A}}$, are the union of the eigenvalues of all $\hat{\mathcal{A}}_l$ since these are decoupled from each other. Clearly, the n zero eigenvalues of $\mathcal{A}$ are obtained from $\hat{\mathcal{A}}_1$ since $\Lambda_1 = \lambda_1 = 0$. Therefore, to ensure scalable stability, we must require all eigenvalues of all $\hat{\mathcal{A}}_l$, $l = 2, \dots, k$ to have negative real parts for any N .

The characteristic polynomial of each $\hat{\mathcal{A}}_l$ is

$$p_l(s) = \underbrace{(s^n + a_{n-1}\lambda_l s^{n-1} + \cdots + a_1\lambda_l s + a_0\lambda_l)^{r_l}}_{p_l(s)}, \quad (7)$$

whose roots are given by the roots of $p_l(s)$. In general, the eigenvalue λ_l appearing in $p_l(s)$ is complex-valued. We therefore apply the Routh–Hurwitz criteria for polynomials with complex coefficients. As these criteria do not appear frequently in literature, we re-state them in Appendix.

The first Routh–Hurwitz criterion applied to $p_l(s)$ reads

$$a_{n-1}\text{Re}\{\lambda_l\} > 0. \quad (8)$$

Since $\text{Re}\{\lambda_l\} > 0$ for $l = 2, \dots, k$ this is always satisfied when $a_{n-1} > 0$. The second criterion, given in (A.4), can after some manipulation be written as

$$a_{n-1}(\text{Re}\{\lambda_l\})^2(a_{n-1}a_{n-2}\text{Re}\{\lambda_l\} - a_{n-3}) + a_{n-2}(\text{Im}\{\lambda_l\})^2(a_{n-1}^2\text{Re}\{\lambda_l\} - a_{n-2}) > 0, \quad (9)$$

which must hold for all $l = 2, \dots, k$. While the factors in front of the brackets remain positive for all λ_l (recall, $a_k > 0$), the brackets themselves are negative if $\text{Re}\{\lambda_l\}$ is sufficiently small. In particular, the condition (9) is violated if $\text{Re}\{\lambda_2\} = \min_l \text{Re}\{\lambda_l\}$ is sufficiently small.

This means that if the criterion (9) is evaluated for a graph family $\{\mathcal{G}_N\}$ in which $\text{Re}\{\lambda_2(\mathcal{G}_N)\} \rightarrow 0$ as $N \rightarrow \infty$, it will eventually (for a sufficiently large, but finite, N) be violated.

We can conclude that at least one root of the characteristic polynomial $p_2(s)$ will have a nonnegative real part for sufficiently large N . Lemma 2.1 is then not satisfied and the control is not scalably stable for $n \geq 3$. $\square$

Remark 2. If the graph is undirected, then the polynomial (7) has real-valued coefficients. The result can then be derived using the standard Routh–Hurwitz criteria. This gives the simpler condition

$$a_{n-1}a_{n-2}\lambda_2(\mathcal{G}_N) - a_{n-3} > 0, \quad (10)$$

which cannot remain satisfied if $\{\lambda_2(\mathcal{G}_N)\} \rightarrow 0$ as $N \rightarrow \infty$.

Theorem 3.1 implies that high-order consensus does not scale in certain graph families. Instability will occur at the smallest size N for which the Routh–Hurwitz criteria are violated, and at least one eigenvalue crosses to the right half plane. We will denote this critical network size $\bar{N}$. In Fig. 1 we display $\bar{N}$ for $n = 3, 4, 5$ in an unweighted path graph.

3.1.1. High-order leader–follower consensus

Leader–follower consensus (4) in undirected graphs lacks scalable stability under a weaker condition, namely, under bounded nodal degrees. This was also observed in Yadlapalli et al. (2006).

We first require the following Lemma:

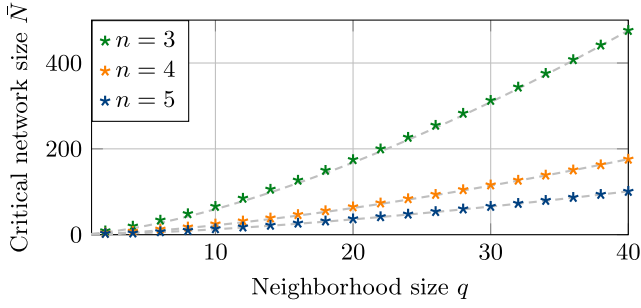


Fig. 1. Critical network size $\bar{N}$ at which the stability conditions are violated for an n th order consensus algorithm. The graph is an undirected path graph where each node is connected to its q nearest neighbors. Increasing the neighborhood size q here increases $\bar{N}$ faster than linearly – [Theorem 5.1](#) predicts $\bar{N} = \mathcal{O}(q^{3/2})$, indicated by dashed lines in the plot. Also note that for higher model order n , the stability conditions are violated at smaller $\bar{N}$.

Lemma 3.2. Consider the grounded Laplacian matrix $\bar{L}$ of an undirected graph $\mathcal{G}_N$ and let [Assumptions A2–A3](#) hold. The smallest eigenvalue $\bar{\lambda}_1$ of $\bar{L}$ then satisfies

$$\bar{\lambda}_1(\mathcal{G}_N) \leq \frac{q}{N-1} w_{\max}. \quad (11)$$

Proof. By the Rayleigh–Ritz theorem ([Horn & Johnson, 1985, Theorem 4.2.2](#)) it holds

$$\bar{\lambda}_1 \leq \frac{v^* \bar{L} v}{v^* v}, \quad \forall v \in \mathbb{C}^{N-1} \setminus \{0\}.$$

This implies in particular that

$$\bar{\lambda}_1 \leq \frac{\mathbf{1}_{N-1}^T \bar{L} \mathbf{1}_{N-1}}{\mathbf{1}_{N-1}^T \mathbf{1}_{N-1}} = \frac{\sum_{k \in \mathcal{N}_1} w_{1k}}{N-1} \leq \frac{q w_{\max}}{N-1},$$

where $\mathbf{1}_{N-1}^T \bar{L} \mathbf{1}_{N-1} = \sum_{k \in \mathcal{N}_1} w_{1k}$ is the weight sum of all edges leading to the leader node 1. The equality holds since each row k of the grounded Laplacian $\bar{L}$ sums to zero if the corresponding node k has no connection to the leader, and otherwise to $w_{1k} \leq w_{\max}$. $\square$

Clearly, $\bar{\lambda}_1(\mathcal{G}_N) \rightarrow 0$ as $N \rightarrow \infty$. The next theorem therefore follows.

Theorem 3.3. If $n \geq 3$, no leader–follower consensus algorithm on the form (4) is scalably stable in undirected graph families $\{\mathcal{G}_N\}$ under [Assumptions A1–A3](#).

Proof. The arguments in the proof of [Theorem 3.1](#) apply. In this case, $N-1$ real-valued characteristic polynomials $p_l(s)$ as in (7) are obtained. We can use the condition (10), which in this case reads $a_{n-1}a_{n-2}\bar{\lambda}_l - a_{n-3} > 0$ for $l = 1, \dots, N-1$. By [Lemma 3.2](#), that requires

$$a_{n-1}a_{n-2} > \frac{1}{q w_{\max}} a_{n-3}(N-1), \quad (12)$$

which will be violated for sufficiently large N , preventing scalable stability. $\square$

Remark 3. [Assumption A2](#) of bounded neighborhoods can be relaxed. As seen from (12), [Theorem 3.3](#) holds if $q/N \rightarrow 0$ as $N \rightarrow \infty$. That is, if nodal neighborhoods have sublinear growth in N .

3.2. Affected classes of graphs

We proved that high-order consensus lacks scalable stability in any network where the underlying graph family is such that

$\text{Re}\{\lambda_2(\mathcal{G}_N)\}$ is decreasing towards zero as N increases. For undirected graphs, the smallest non-zero Laplacian eigenvalue λ_2 is real-valued and known as the *algebraic connectivity* of the graph. For directed graphs, the notion of algebraic connectivity is not clear-cut, see e.g. [Chung \(2005\)](#). We can, however, make the following statement:

Lemma 3.4. If L is normal, then $\text{Re}\{\lambda_2\} = \lambda_2^s$, where λ_2^s is the smallest non-zero eigenvalue of $L^s = (L + L^T)/2$, that is, the symmetric part of L .

Proof. With v an eigenvector, $Lv = \lambda_2 v$, and since L is normal $L^T v = \lambda_2^* v$, where $*$ denotes complex conjugate. Then, $\frac{1}{2}(L + L^T)v = \frac{1}{2}(\lambda_2 + \lambda_2^*)v = \frac{1}{2}(2\text{Re}\{\lambda_2\})v$. $\square$

For any balanced graph, the matrix L^s is the graph Laplacian corresponding to the mirror graph $\hat{\mathcal{G}}_N$ of $\mathcal{G}_N$. The mirror graph (of any directed graph) is the undirected graph obtained as $\hat{\mathcal{G}}_N = \{\mathcal{V}_N, \mathcal{E}_N \cup \hat{\mathcal{E}}_N\}$, where $\hat{\mathcal{E}}_N$ is the set of all edges in $\mathcal{E}_N$, but reversed, and whose edge weights are $\hat{w}_{ij} = \hat{w}_{ji} = (w_{ij} + w_{ji})/2$ ([Olfati-Saber & Murray, 2004](#)). Clearly, the mirror graph of an undirected graph is the graph itself. [Lemma 3.4](#) implies that when L is normal, $\text{Re}\{\lambda_2(\mathcal{G}_N)\}$ is obtained as the algebraic connectivity of the mirror graph $\hat{\mathcal{G}}_N$.

We conclude that the result in [Theorem 3.1](#) will apply to graph families whose Laplacians are normal and where the corresponding mirror graph family has a decreasing algebraic connectivity. That is, where $\{\lambda_2(\hat{\mathcal{G}}_N)\} \rightarrow 0$ as $N \rightarrow \infty$. It is therefore meaningful to identify this property in undirected graph families, which is what the remainder of this section is devoted to. We first state a general condition, and then survey particular classes of graphs.

Remark 4. For directed graph families with non-normal Laplacians, a conclusion regarding the sequence $\text{Re}\{\lambda_2(\mathcal{G}_N)\}$ cannot in general be drawn from the mirror graphs. A notable counterexample is the directed path graph on N nodes with the edge set $\mathcal{E}_N = \{(i, i+1) \mid i = 1, \dots, N-1\}$. Here, $\text{Re}\{\lambda_2(\mathcal{G}_N)\} = 1$ for any N , while $\lambda_2(\hat{\mathcal{G}}_N) = 1 - \cos \frac{\pi}{N}$. For general directed graphs, the sequence $\text{Re}\{\lambda_2(\mathcal{G}_N)\}$ must therefore be checked case by case.

3.2.1. Condition on the Cheeger constant

In general, the algebraic connectivity decreases in N in any undirected graph family that is not an *expander family*. To define expander families, we require the Cheeger constant (also called isoperimetric constant), which for non-regular weighted graphs can be defined as ([Chung, 1997, Chapter 2](#)):

$$h(\mathcal{G}) = \inf_{X \subset \mathcal{V}} \frac{|\partial X|_d}{\min\{|X|_d, |\bar{X}|_d\}}. \quad (13)$$

Here, $\bar{X} = \mathcal{V} \setminus X$ and $\partial X = \{j \in \bar{X} \mid (i, j) \in \mathcal{E}, i \in X\}$ is called the boundary set of X . Sets of nodes are measured here as $|W|_d := \sum_{i \in W} d_i$, where the nodal degree $d_i = \sum_{j \in \mathcal{N}_i} w_{ij}$. Loosely speaking, a large Cheeger constant implies that any subset of nodes is well connected to the rest of the graph, and it is not possible to find a “bottleneck” that separates two graph partitions from each other as they grow. See [Tegling, Middleton and Seron \(2019\)](#) for an elaboration and an algebraic condition. Now, consider the following definition.

Definition 3.1 (Expander Family). Let $\{\mathcal{G}_N\}$ be a graph family in which $N \rightarrow \infty$. If the sequence $\{h(\mathcal{G}_N)\}$ is bounded away from zero, $\{\mathcal{G}_N\}$ is an expander family.

The following well-established result relates expander families to our problem:

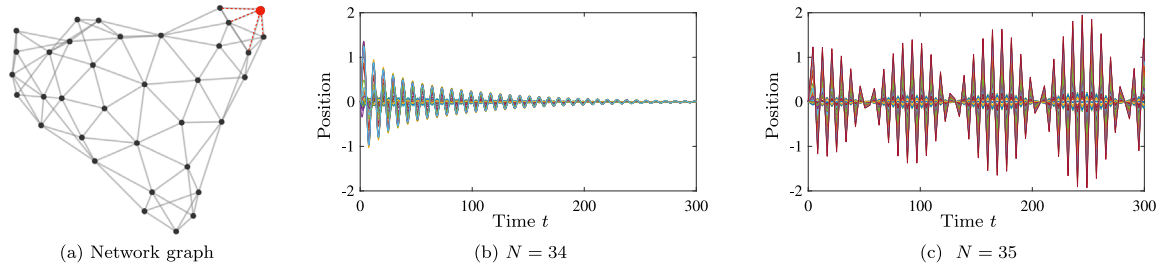


Fig. 2. Simulation of 3rd order consensus over graph depicted in (a) subject to random initial accelerations. In (b) the network's 34 agents converge to an equilibrium. In (c) a 35th node has been added, indicated by red color in the graph. This addition leads to instability. The plots (b) and (c) show position trajectories relative to Agent no. 1. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Lemma 3.5. *The sequence $\{\lambda_2(\mathcal{G}_N)\}$ is bounded away from zero as $N \rightarrow \infty$ if and only if $\{\mathcal{G}_N\}$ is an expander family.*

See e.g. Krebs and Shaheen (2011, Chapter 1) for a proof. Lemma 3.5 implies that a bounded-degree graph family can have an algebraic connectivity that does not decrease towards zero, if the same holds for the Cheeger constant. Eq. (13) reveals that this requires edges to connect across the entire network. In other words, that feedback is *non-localized*.

Expander graphs with bounded degrees are difficult to construct explicitly, but they may arise through random processes. For example, the regular random graph family constructed by assigning edges through equally likely permutations of the node set $\mathcal{V}_N$, will *almost surely* be an expander family (Friedman, 1991).

Next, we turn our attention to typical graph families that are non-expanding and thus have a decreasing algebraic connectivity.

Remark 5. It is noteworthy that Theorem 3.3 for leader–follower consensus applies even though $\{\mathcal{G}_N\}$ is an expander family. This means that leaderless consensus, despite being scalably stable in expander graphs, will be destabilized if one agent becomes a leader (“is grounded”). This fragility is described in detail in Tegling, Middleton et al. (2019).

3.2.2. Lattices, fuzzes and their embedded graphs

Consider a graph over the d -dimensional periodic lattice $\mathbb{Z}_M^d$ with $N = M^d$ nodes, and let each node be connected to its r neighbors in each lattice direction. We term this graph, which is the Cartesian product of d r -fuzzes of ring graphs, a d -dimensional r -fuzz lattice. This graph is regular and the neighborhood size is $q = 2rd$.

Lemma 3.6 (Algebraic Connectivity of r -Fuzz Lattices). *For undirected d -dimensional r -fuzz lattices*

$$\lambda_2(\mathcal{G}_N) = \mathcal{O}\left(\frac{1}{N^{2/d}}\right) \quad (14)$$

Proof. See Tegling et al. (2019). $\square$

The decay rate (14) also holds for any subgraph of the r -fuzz lattice, that is, any graph that is *embeddable* in it. In particular, lattices without periodic boundary conditions. This follows from the following important lemma:

Lemma 3.7. *Adding an edge to an undirected graph $\mathcal{G}_N$, or increasing the weight of an edge, can only increase (or leave unchanged) $\lambda_2(\mathcal{G}_N)$, and vice versa.*

Proof. Adding an edge: See Mohar (1991, Theorem 3.2). Increasing edge weight: If the weight of the edge (i', j') is increased by

Δw , the new graph Laplacian can be written $L' = L + \Delta L$, where ΔL is also a positive semidefinite graph Laplacian (of a disconnected graph). By Brouwer and Haemers (2012, Theorem 2.8.1) this implies that $\lambda'_l \geq \lambda_l$ for each $l = 1, \dots, N$, and in particular $\lambda'_2 \geq \lambda_2$. $\square$

3.2.3. Planar graphs

Planar graphs are embeddable in two-dimensional lattices, so Lemma 3.6 applies. For this important case, however, a more precise bound is available:

Lemma 3.8 (Algebraic Connectivity of Planar Graphs). *For undirected planar graphs,*

$$\lambda_2(\mathcal{G}_N) \leq \frac{8q w_{\max}}{N}. \quad (15)$$

Proof. See Spielman and Teng (2007, Theorem 6). $\square$

3.2.4. Tree graphs with growing diameter

The *diameter* $\text{diam}(\mathcal{G})$ of a graph $\mathcal{G}$ is defined as the longest distance between any two nodes in the graph. If we let $\mathcal{G}$ be a *tree graph*, then, by Grone, Merris, and Sunder (1990, Corollary 4.4) it holds $\lambda_2 \leq 2w_{\max} \left(1 - \cos\left(\frac{\pi}{\text{diam}(\mathcal{G})+1}\right)\right)$. This allows us to show the following lemma.

Lemma 3.9 (Algebraic Connectivity of Tree Graphs). *For undirected tree graphs,*

$$\lambda_2(\mathcal{G}_N) \leq \frac{\pi^2 w_{\max}}{(\text{diam}(\mathcal{G}_N) + 1)^2}. \quad (16)$$

Proof. Follows from the relation above and the fact that $1 - \cos x \leq \frac{x^2}{2}$ for any x . $\square$

In our case, the tree diameter will always increase in N as a consequence of Assumption A2. Therefore, $\{\lambda_2(\mathcal{G}_N)\} \rightarrow 0$ as $N \rightarrow \infty$.

3.3. Numerical examples

We next provide two numerical examples to illustrate the issue of scalable stability in high-order consensus.

3.3.1. Critical network size, locality and model order

Consider a family of undirected path graphs where each node is connected to its $q/2$ nearest neighbors in each direction (i.e., a $q/2$ -fuzz of a path graph). For any given N , the graph's connectivity is greater, the greater q is. Increasing q thus delays the violation of the stability criteria in Theorem 3.1.

In Fig. 1, we depict the critical network size $\bar{N}$ as a function of the neighborhood size q . Here, we have selected a consensus algorithm where $a_0 = 0.1$, $a_1 = 0.8$, $a_2, a_3, a_4 = 1$, and all edge

weights $w_{ij} = 1$. The plot shows that increasing q increases the critical network size, faster than linearly. In Section 5 we discuss the precise scaling of q in N required to defer instability completely.

We also note that as the model order n increases, the system becomes unstable at smaller $\bar{N}$. This is because the higher-order Routh–Hurwitz conditions in (A.2) are violated before the lower-order ones. It is also in line with common control-theoretic intuition.

3.3.2. Instability through node addition

Our second example illustrates the phase transition – from consensus to instability – that the system experiences as the critical network size is reached. Fig. 2(a) shows a planar graph that has been randomly generated by means of triangulation. Here, the maximum neighborhood size is $q = 8$ and the median is 5. All edge weights are set to 1.

We consider a third-order consensus algorithm:

$$\dot{x}_i^{(3)} = - \sum_{j \in \mathcal{N}_i} [0.5(x_i - x_j) + (\dot{x}_i - \dot{x}_j) + (\ddot{x}_i - \ddot{x}_j)],$$

which by Lemma 2.1 will achieve consensus if $\lambda_2 > 0.5$. With 34 nodes, the graph in Fig. 2(a) has $\lambda_2(\mathcal{G}_{34}) = 0.536$ and the system achieves consensus, as seen from the simulation in Fig. 2(b). We then add a 35th node along with 4 connecting edges, as indicated in red color in the graph in Fig. 2(a). Now, $\lambda_2(\mathcal{G}_{35}) = 0.493$ and the system becomes unstable.¹ Fig. 2(c) shows how the agents' positions x oscillate at an increasing amplitude.

4. Scale fragility in second-order consensus

Next, we turn our attention to consensus in second-order integrator networks ($n = 2$). This case is particularly relevant as this model is used in formation control problems (Olfati-Saber, 2006). Scalable stability is easily satisfied in second-order consensus if the underlying graph family is undirected² (though performance issues like string instability (Swaroop & Hedrick, 1996) and lack of coherence (Bamieh et al., 2012) may still be a concern). We show here, however, that it fails to scale stably in certain families of directed graphs with complex eigenvalues. More precisely, graph families where the real part of one or more Laplacian eigenvalues approaches zero as N grows and at least one of these eigenvalues has a relatively large imaginary part. The precise condition, which is illustrated in Fig. 3, is stated in Theorem 4.1. First, we remind the reader that the Laplacian eigenvalues are ordered as $0 = \lambda_1(\mathcal{G}_N) < \text{Re}\{\lambda_2(\mathcal{G}_N)\} \leq \dots \leq \text{Re}\{\lambda_N(\mathcal{G}_N)\}$.

Theorem 4.1. *If $n \geq 2$, no control on the form (2), subject to Assumption A1, is scalably stable in graph families where, for a fixed index $\bar{l} \in \{2, 3, \dots, N\}$,*

- (1) $\text{Re}\{\lambda_{\bar{l}}(\mathcal{G}_N)\} \rightarrow 0$ as $N \rightarrow \infty$, and
- (2) for each N and at least one $l \in \{2, 3, \dots, \bar{l}\}$ it holds $\arg\{\lambda_l(\mathcal{G}_N)\} > \psi$, where $\psi \in (0, \pi/2)$ is a constant angle independent of N .

Proof. For $n \geq 3$ the result follows immediately from Theorem 3.1 (note, $\text{Re}\{\lambda_{\bar{l}}(\mathcal{G}_N)\} \rightarrow 0 \Rightarrow \text{Re}\{\lambda_2(\mathcal{G}_N)\} \rightarrow 0$). For

¹ This particular value for $\lambda_2(\mathcal{G}_{35})$ depends on the placement of the 35th node. Other placements can allow the critical $\bar{N} > 35$, but instability occurs eventually.

² This is evident from the upcoming proof of Theorem 4.1. Consider (17) and note that if $\mathcal{G}_N$ is undirected, $\lambda_i(\mathcal{G}_N)$ are real-valued and positive. Since $a_1, a_0 > 0$, all roots of $p_l(s)$ are then in the left half plane for any N .

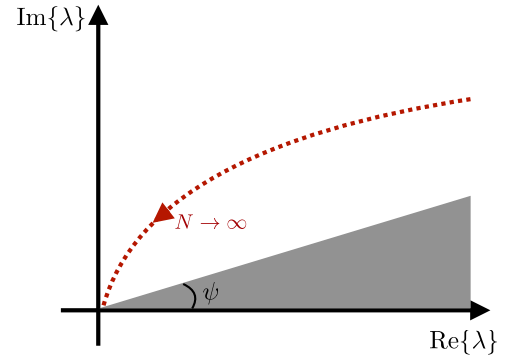


Fig. 3. Illustration of the conditions in Theorem 4.1. If a Laplacian eigenvalue approaches the origin at an angle greater than some ψ as the network grows, then second-order consensus lacks scalable stability. The example trajectory illustrates λ_2 of a family of directed ring graphs.

$n = 2$, we proceed as in the proof of Theorem 3.1 to obtain the characteristic polynomials

$$p_l(s) = s^2 + a_1 \lambda_l s + a_0 \lambda_l, \quad (17)$$

for $l = 2, \dots, N$. The Routh–Hurwitz criterion derived from $\Delta_4 > 0$ in (A.2) with $f_{n-3} = g_{n-3} = 0$ becomes

$$a_1^2 \text{Re}\{\lambda_l\} [(\text{Re}\{\lambda_l\})^2 + (\text{Im}\{\lambda_l\})^2] - a_0 (\text{Im}\{\lambda_l\})^2 > 0.$$

If $\text{Im}\{\lambda_l\} = 0$, this is clearly satisfied since $\text{Re}\{\lambda_l\} > 0$. For all $l \in \{2, \dots, N\}$ where $\text{Im}\{\lambda_l\} \neq 0$ we can re-formulate the condition as

$$a_1^2 \text{Re}\{\lambda_l\} \left[\left(\frac{\text{Re}\{\lambda_l\}}{\text{Im}\{\lambda_l\}} \right)^2 + 1 \right] - a_0 > 0. \quad (18)$$

If the expression in brackets is upper bounded by some constant, this condition will be violated whenever $\text{Re}\{\lambda_l\}$ is sufficiently small. Therefore, if the condition (18) is evaluated for a graph family $\{\mathcal{G}_N\}$ in which there are eigenvalues for which $\text{Re}\{\lambda_l(\mathcal{G}_N)\} \rightarrow 0$ as $N \rightarrow \infty$, and it holds $\left(\frac{\text{Re}\{\lambda_l(\mathcal{G}_N)\}}{\text{Im}\{\lambda_l(\mathcal{G}_N)\}} \right)^2 \leq \text{const.}$ for at least one of them, then the condition is eventually violated, and stability is lost. In our case, $\text{Re}\{\lambda_{\bar{l}}(\mathcal{G}_N)\} \rightarrow 0$ for some index $\bar{l}$ implies $\text{Re}\{\lambda_l(\mathcal{G}_N)\} \rightarrow 0$ for $2 \leq l \leq \bar{l}$, so we must check all eigenvalues $2 \leq l \leq \bar{l}$.

Now, $\left(\frac{\text{Re}\{\lambda_l(\mathcal{G}_N)\}}{\text{Im}\{\lambda_l(\mathcal{G}_N)\}} \right)^2 \leq \text{const.}$ is equivalent to having an upper bound on $\frac{\text{Re}\{\lambda_l(\mathcal{G}_N)\}}{\text{Im}\{\lambda_l(\mathcal{G}_N)\}}$ for an eigenvalue in the first quadrant (recall, the Laplacian eigenvalues appear in conjugate pairs in the RHP). This, in turn, is equivalent to having the argument $\arg\{\lambda_l(\mathcal{G}_N)\}$ bounded away from zero. In other words, $\arg\{\lambda_l(\mathcal{G}_N)\} > \psi$ for some fixed $\psi \in (0, \pi/2)$, and the theorem statement follows. $\square$

A simpler statement pertaining to the special case of λ_2 can be stated as follows:

Corollary 4.2. *If $n \geq 2$, no control on the form (2), subject to Assumption A1, is scalably stable in graph families where $\text{Re}\{\lambda_2(\mathcal{G}_N)\} \rightarrow 0$ as $N \rightarrow \infty$ while $\arg\{\lambda_2(\mathcal{G}_N)\} > \psi$ for some constant $\psi \in (0, \pi/2)$ that is independent of N .*

4.1. Affected classes of graphs

Theorem 4.1 states that if at least one Laplacian eigenvalue is complex valued and approaches the origin at a non-zero angle as $N \rightarrow \infty$, then second-order consensus fails to be scalably stable. See also Fig. 3. A particular graph family where this applies

is directed ring graphs³ with uniform edge weights, as already observed by Cantos et al. (2016), Herman (2016) and Stüdl et al. (2017). Here, we demonstrate that it applies to the more general family of directed lattices with periodic boundary conditions.

4.1.1. Directed periodic lattices

Consider again the d -dimensional r -fuzz lattice from . We impose location-invariant edge weights in the sense that, if $d = 1$, $w_{i,i+k} = w_k$ for all $i \in \mathbb{Z}_M$, $k = \pm\{1, \dots, r\}$. This means that the corresponding graph Laplacian for $d = 1$ is a $M \times M$ circulant matrix. In the higher-dimensional case, the Laplacian is the Kronecker sum of d such matrices (since the graph is the Cartesian product of d one-dimensional lattices) and $N = M^d$. Here, we assume the Laplacian is asymmetric:

Assumption A4. The edge weight $w_k \neq w_{-k}$ for at least one $k \in \pm\{1, \dots, r\}$.

Lemma 4.3. For the d -dimensional r -fuzz lattice under Assumption A4,

$$\operatorname{Re}\{\lambda_2(\mathcal{G}_N)\} = \mathcal{O}\left(\frac{1}{N^{2/d}}\right), \quad \operatorname{Im}\{\lambda_2(\mathcal{G}_N)\} = \mathcal{O}\left(\frac{1}{N^{1/d}}\right).$$

Proof. The smallest (in real part) non-zero eigenvalue of the r -fuzz lattice is given by

$$\lambda_2 = \sum_{\substack{k=-r \\ k \neq 0}}^r w_i \left(1 - \cos\left(\frac{2\pi k}{M}\right)\right) - j \sum_{\substack{k=-r \\ k \neq 0}}^r w_i \sin\left(\frac{2\pi k}{M}\right), \quad (19)$$

where $M = N^{1/d}$ is the lattice size (Tegling et al., 2019). The expression (19) is easily obtained from the case $d = 1$, since the Laplacian eigenvalues of a Cartesian product of any two graphs are given by every possible sum of their respective Laplacian eigenvalues (see e.g. Mohar, 1991), and one eigenvalue is zero in each. Next, note that since $\sin(-x) = -\sin(x)$, it is only under Assumption A4 that $\operatorname{Im}\{\lambda_2\} \neq 0$. Finally, recalling that r is bounded by Assumption A2, the lemma follows from Maclaurin series expansions of the real and imaginary parts. $\square$

Lemma 4.3 implies that $\arg\{\lambda_2(\mathcal{G}_N)\} \rightarrow \pi/2$ as $N \rightarrow \infty$, so the conditions in Theorem 4.1 clearly hold.

Remark 6. In fact, (19) will be an eigenvalue (though not necessarily λ_2) of a graph that results from a Cartesian product of any graph with a r -fuzz lattice. This follows from the proof of Lemma 4.3. Such product graphs would thus also be affected by Theorem 4.1.

4.1.2. General necessary condition – cyclicity

Characterizing the Laplacian spectra of general directed graph families is a difficult and largely unsolved problem. Even determining the properties of graphs that have a real-valued spectrum, and which are therefore certainly not affected by Theorem 4.1, is an open problem.

A necessary condition, however, for $\mathcal{G}_N$ having at least one complex eigenvalue is that $\mathcal{G}_N$ has a directed cycle. This is, however, not sufficient. The term *essentially cyclic graphs* has been proposed for graphs with non-real spectra, and properties of such graphs are examined in Agaev and Chebotarev (2010). To determine the eigenvalue behavior in N for families of such graphs, and thereby whether they are affected by Theorem 4.1, is a graph-theoretical endeavor that is outside the scope of the present paper.

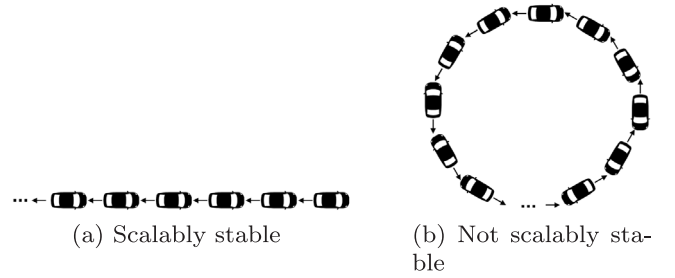


Fig. 4. Theorem 4.1 reveals a scale fragility in the vehicle formation dynamics $\ddot{x}_i = -a_0(x_i - x_{i-1}) - a_1(\dot{x}_i - \dot{x}_{i-1})$, where x_i is vehicle i 's displacement. These dynamics can model adaptive cruise control in commercial vehicles (Gunter et al., 2021). If the vehicles drive in a circle (let $x_{-1} = x_N$), the formation is destabilized at some size $\tilde{N}$. The same issue does not apply to the line formation.

4.2. Implications and numerical example

These results have interesting implications. First, that circular formations based on the consensus algorithm (2) are scale fragile. For example, vehicles driving with adaptive cruise controllers available in modern commercial vehicles can indeed be modeled as our second-order consensus with unidirectional nearest-neighbor connections, see Gunter et al. (2021, §II-A) (a constant reference spacing term can be eliminated by translating the state). If they drive in a circle, as on a ring road or as in many experimental set-ups (see e.g. Stern et al., 2018), our results show that the formation may be destabilized if too many vehicles join. See also Fig. 4. In such settings, however, it can be possible to recover scalable stability by including absolute feedback.

Second, we can note that even if the feedback in a ring formation is bidirectional, that is, if the graph is undirected, it can be destabilized if a slight change in the weights renders the graph directed. Therefore, formations on undirected ring graphs are also fragile. We note that the same issues do not apply to formations on a line. The two therefore have fundamentally different scalability and robustness properties.

Fig. 5 shows a simulation of a growing circular vehicle formation to illustrate this section's results. Here, each vehicle's displacement x_i is controlled with respect to the preceding vehicle so that $\ddot{x}_i = -w_{i,i-1}(x_i - x_{i-1}) - 3w_{i,i-1}(\dot{x}_i - \dot{x}_{i-1})$ for $i = 1, \dots, N$. Let $x_{-1} = x_N$. We relax the assumption of location-invariant edge weights w_{ij} used for Lemma 4.3. Instead, as vehicles are added, the edge weights take random values in the interval $(0, 1)$. In this example, the formation is destabilized at $\tilde{N} = 14$ and the vehicles collide.

5. Retrieving scalable stability

Having been presented with fundamental limitations to the scalability of modular, localized consensus, an obvious question is how to change the algorithm, or relax assumptions on the network topology, to retrieve scalability. We next address this question by pinning down on two key model assumptions.

5.1. Relaxing the locality assumption

Underlying our analysis was the assumption of locality in the sense of bounded nodal degrees, Assumption A2. Recall that under this assumption, all undirected graph families except expander families have decreasing algebraic connectivity and are thus affected by Theorem 3.1. If this assumption is relaxed, so that nodal neighborhoods are allowed to grow with N , the algebraic connectivity can remain bounded away from zero. Scalable

³ More precisely, a ring graph that is not undirected.

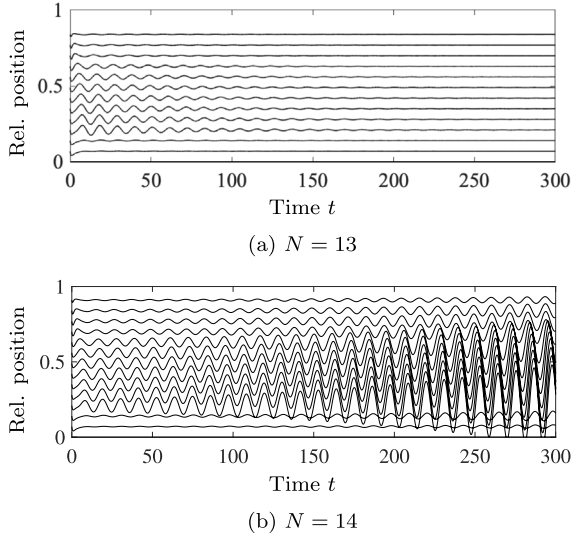


Fig. 5. Simulation of a circular vehicle formation with unidirectional feedback and random edge weights. Each line represents a vehicle's position relative to vehicle 1, which has a step change in its velocity at time $t = 0$. The formation is stable when $N = 13$, but the addition of a 14th vehicle destabilizes it.

stability can then be retrieved. Interestingly, even though weights are fixed, it can suffice to grow neighborhoods sub-linearly.

We show this for a ring graph topology, but note that the same result applies to any graph that is better connected due to [Lemma 3.7](#).

Theorem 5.1. *Let $\{\mathcal{G}_N\}$ be a family of undirected 1-dimensional $q/2$ -fuzz lattices (q even), that is, ring graphs with edges between each node and its q nearest neighbors. Then, if*

$$q \geq cN^{2/3},$$

with $c > 0$ a constant independent of N , the sequence $\{\lambda_2(\mathcal{G}_N)\}$ is bounded away from zero as $N \rightarrow \infty$.

Proof. The algebraic connectivity of $\mathcal{G}_N$ is $\lambda_2(\mathcal{G}_N) = \sum_{k=-q/2}^{q/2} w(1 - \cos \frac{2\pi k}{N})$ if edge weights are uniform, i.e., $w_{ij} = w$ for all $(i, j) \in \mathcal{E}_N$. The derivation of this expression is based on the Discrete Fourier Transform, see e.g. [Tegling et al. \(2019\)](#). Therefore, in a graph with non-uniform weights, but with $w_{ij} \geq w_{\min}$, we have

$$\begin{aligned} \lambda_2(\mathcal{G}_N) &\geq \sum_{k=-q/2}^{q/2} w_{\min} \left(1 - \cos \frac{2\pi k}{N}\right) \\ &= 2w_{\min} \left(1 - \cos \frac{2\pi}{N}\right) + \dots + 2w_{\min} \left(1 - \cos \frac{2\pi}{N}\right) \\ &\geq 2w_{\min} \frac{2}{\pi^2} \left(\left(\frac{2\pi}{N}\right)^2 + \left(\frac{2\pi \cdot 2}{N}\right)^2 + \dots + \left(\frac{2\pi \cdot q}{N}\right)^2 \right) \\ &= \frac{16w_{\min}}{N^2} (1^2 + 2^2 + \dots + q^2) \\ &= \frac{16w_{\min}}{N^2} \frac{q(q+1)(2q+1)}{6}, \end{aligned} \quad (20)$$

where the first inequality follows from [Lemma 3.7](#) and the second from the fact that $1 - \cos x \geq \frac{2}{\pi^2 x^2}$ for $x \in [-\pi, \pi]$. The last equality is a standard result for sums of sequences of squares. Now, if $q \geq cN^{2/3}$, where c is a positive constant, then (20) is lower bounded by $\frac{16w_{\min}}{N^2} \frac{2cN^2}{6} = \frac{16c^3 w_{\min}}{6}$, which is a positive constant independent of N . The theorem follows. $\square$

The sub-linear scaling in [Theorem 5.1](#) is surprising in light of well-known bounds on algebraic connectivity, which appear to require a linear scaling. One example is the bound based on the edge connectivity $e(\mathcal{G}_N)$: $\lambda_2(\mathcal{G}_N) \geq 2e(\mathcal{G}_N)(1 - \cos \frac{\pi}{N})$ ([Fiedler, 1973, §4.3](#)). Since the edge connectivity grows quadratically with the number of nearest-neighbor connections q and $(1 - \cos \frac{\pi}{N}) = \mathcal{O}(\frac{1}{N^2})$, this bound requires $q = \mathcal{O}(N)$.

It is also notable that leader–follower consensus indeed requires a linear scaling of q . This is evident from (12), which is a necessary stability condition. This again highlights an important difference in scalability between leaderless and leader–follower consensus.

Remark 7. [Theorem 5.1](#) is stated for a ring graph family that lets $\lambda_2(\mathcal{G}_N)$ be expressed as a fairly simple function of q and N . Numerical evaluations show, however, that the same result holds in path graphs, see also [Fig. 1](#). For more connected graph families, [Lemma 3.7](#) applies, making the result conservative.

5.2. Impact of absolute feedback

Scalable stability can be retrieved if the control includes *absolute* state feedback (equivalent to non-zero self-weights), if this feedback is carefully designed. To highlight this result while keeping the section brief, we consider the case of $n = 3$ and undirected graph families. In this case, the control algorithm becomes

$$u_i = - \sum_{k=0}^2 \left(a_k \sum_{j \in \mathcal{N}_i} w_{ij} (x_i^{(k)} - x_j^{(k)}) - a_k^{\text{abs}} x_i^{(k)} \right), \quad (21)$$

and we say that absolute feedback from the state $x^{(k)}$ is available if one can set $a_k^{\text{abs}} > 0$. The closed-loop system dynamics become

$$\frac{d}{dt} \xi = \begin{bmatrix} 0 & I_N & 0 \\ 0 & 0 & I_N \\ -a_0 L - a_0^{\text{abs}} I_N & -a_1 L - a_1^{\text{abs}} I_N & -a_2 L - a_2^{\text{abs}} I_N \end{bmatrix} \xi.$$

The following proposition lines out that absolute feedback from certain states is particularly important to retrieve scalable stability.

Proposition 5.2. *Let $\{\mathcal{G}_N\}$ be an undirected graph family in which $\{\lambda_2(\mathcal{G}_N)\} \rightarrow 0$ as $N \rightarrow \infty$. Then, a necessary condition for scalable stability of the controller (21), subject to [Assumption A1](#), is that at least one of $a_1^{\text{abs}}, a_2^{\text{abs}} > 0$.*

Proof. The proof follows that of [Theorem 3.1](#), with modifications lined out as follows. With absolute feedback terms, the characteristic polynomial corresponding to (7) becomes

$$p_l(s) = s^3 + (a_2 \lambda_l + a_2^{\text{abs}}) s^2 + (a_1 \lambda_l + a_1^{\text{abs}}) s + (a_0 \lambda_l + a_0^{\text{abs}}),$$

and the relevant stability condition is obtained from (9) by substituting $(a_k \text{Re}\{\lambda_l\} + a_k^{\text{abs}})$ for $a_k \text{Re}\{\lambda_l\}$. Since we let $\mathcal{G}_N$ be undirected, λ_l are real-valued, and the condition for $l = 2$ simplifies to

$$(a_1 \lambda_2 + a_1^{\text{abs}})(a_2 \lambda_2 + a_2^{\text{abs}}) - a_0 \lambda_2 - a_0^{\text{abs}} > 0 \quad (22)$$

(which compares to (10)). If both $a_1^{\text{abs}} = a_2^{\text{abs}} = 0$, (22) is eventually violated as $\lambda_2 \rightarrow 0$, regardless of a_0^{abs} . However if at least one of $a_1^{\text{abs}}, a_2^{\text{abs}} > 0$ the condition can stay satisfied, e.g. if $a_1^{\text{abs}} a_2 > a_0$ or $a_2^{\text{abs}} a_1 > a_0$ while $a_0^{\text{abs}} = 0$. If both $a_1^{\text{abs}}, a_2^{\text{abs}} > 0$, it is also allowed to set $a_0^{\text{abs}} > 0$. $\square$

This implies that absolute feedback from the high-order terms, that is, velocity or acceleration, is necessary to render the third-order consensus algorithm scalable. Reading the proof in more

detail also reveals the interesting observation that absolute feedback from positions cannot be included unless there is also absolute feedback from *both* velocity and acceleration (it will ruin scalable stability if included with only one of the two). This is somewhat counter-intuitive, as absolute feedback is usually beneficial for performance and stability, though often more difficult to implement (Jensen & Bamieh, 2022).

6. Discussion

This paper's results show that there is an important difference between the well-studied standard first-order consensus algorithm and the corresponding second- and higher-order algorithms, in that the latter are not always scalable in a modular manner to large networks. When subject to locality constraints, formally expressed through the network's Cheeger constant (13), high-order consensus will stop converging and become unstable at some finite network size. We remark that this result contradicts a statement made in Ren et al. (2007, §V), that convergence to consensus of a high-order multi-vehicle network "will not be impacted as the number of vehicles increases" (though the authors clearly note that controller gains must be chosen to ensure stability.) Second-order consensus is subject to the same scale fragility in certain families of directed networks, such as directed ring graphs.

An interesting consequence of both results is that, at some given network size, the addition of only one agent to a multi-agent network renders a previously converging system unstable. This can be thought of as a type of phase transition. For open multi-agent systems (Franceschelli & Frasca, 2021; Hendrickx & Martin, 2017) that obey a high-order consensus protocol, e.g. for flight formation, our results imply that special care must be taken to avoid this phase transition by limiting the network size or avoiding a localized network topology. We next discuss some further implications of our results.

6.1. Implications for distributed integral control

If distributed integral control is applied to a lower-order consensus network with relative feedback, the closed-loop dynamics can be formulated analogously to the high-order consensus algorithm. Our results can be used to reveal conditions on such integral control for scalable stability.

One example of such an integral controller is the distributed-averaging proportional-integral (DAPI) controller proposed for frequency control in electric power grids, see Andreasson, Dimarogonas, Sandberg, and Johansson (2014) and Simpson-Porco, Dörfler, and Bullo (2013). While in frequency control, absolute frequency feedback helps ensure scalable stability, the analogous control design based on relative feedback would lack scalable stability. In earlier work (Tegling et al., 2019, Theorem 5.4) we have stated a particular stability result for distributed integral control, but the topic is far from fully explored.

6.2. Asymptotic performance analysis

A further interesting consequence of our results is that an analysis of the asymptotic (in network size) performance of localized, consensus-like feedback control is only possible in first- and second-order integrator networks. This means that the analysis on coherence scaling in large-scale networks in Bamieh et al. (2012) cannot, as was conjectured there, be extended to chains of $n > 2$ integrators. We also note that the analysis for second-order networks in that work hinges on the assumption of symmetric feedback, since the scale fragility from Theorem 4.1 applies in directed tori.

6.3. Modular design vs. controller re-tuning

In order to be able to discuss a given controller's scalability in a network of increasing size, the assumption that it be fixed is necessary. This presumes a modular design, implying that the controller cannot be re-tuned as the network grows. By re-tuning the consensus algorithm from this paper, either by changing the gains a_k , weights w_{ij} , or by relaxing the locality assumption, consensus can be achieved also as the network grows.

Changing gains or weights would require adapting to the graph's changing algebraic connectivity. While this can indeed be estimated in a decentralized manner (Yang et al., 2010), dynamic weight re-tuning algorithms as in Kempton, Herrmann, and Bernardo (2018) require the entire network to participate in tuning to improve the connectivity. Still, the design of controller re-tuning protocols – which this paper shows to be necessary for scalable stability – is a highly interesting direction for future research.

Acknowledgments

We would like to thank Swaroop Darbha, Federica Garin, Rick Middleton, Ali Jadbabaie, and anonymous reviewers for their insightful comments relating to various aspects of this work.

This work was partially supported by the Wallenberg AI, Autonomous Systems and Software Program (WASP) funded by the Knut and Alice Wallenberg Foundation, the Swedish Research Council through grants 2016-00861 and 2019-00691, the Swedish Civil Contingencies Agency (MSB) through the project CERCES2, as well as by the National Science Foundation, USA through awards ECCS-1932777 and CMI- 1763064.

Appendix. Routh–Hurwitz criteria

We state the Routh–Hurwitz criteria for polynomials with complex coefficients as they appear in Tondl (1965, pp 21f).

Lemma A.1. Consider the polynomial

$$p(\mu) = \mu^n + (f_{n-1} + \mathbf{j}g_{n-1})\mu^{n-1} + \cdots (f_0 + \mathbf{j}g_0) = 0, \quad (\text{A.1})$$

where $\mathbf{j} = \sqrt{-1}$ denotes the imaginary unit. The roots μ will be such that $\text{Im}\{\mu\} > 0$ if and only if all inequalities

$$\begin{aligned} -\Delta_2 = -\begin{vmatrix} 1 & f_{n-1} \\ 0 & g_{n-1} \end{vmatrix} > 0, \quad \Delta_4 = \begin{vmatrix} 1 & f_{n-1} & f_{n-2} & f_{n-3} \\ 0 & g_{n-1} & g_{n-2} & g_{n-3} \\ 0 & 1 & f_{n-1} & f_{n-2} \\ 0 & 0 & g_{n-1} & g_{n-2} \end{vmatrix} > 0 \\ , \dots , \\ (-1)^n \Delta_{2n} = \begin{vmatrix} 1 & f_{n-1} & \cdots & f_0 & 0 & \cdots & \cdots & 0 \\ 0 & g_{n-1} & \cdots & g_0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & f_1 & f_0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & g_1 & g_0 & 0 & \cdots & 0 \\ & & & \vdots & & & & \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & f_1 & f_0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & g_1 & g_0 \end{vmatrix} > 0 \end{aligned} \quad (\text{A.2})$$

are satisfied.

Evaluating the determinants, the first two inequalities (which suffice to prove the theorems in this paper) read

$$g_{n-1} < 0, \quad (\text{A.3})$$

$$f_{n-1}g_{n-1}g_{n-2} - f_{n-2}g_{n-1}^2 + g_{n-3}g_{n-1} - g_{n-2}^2 > 0, \quad (\text{A.4})$$

for $n \geq 3$.

Note that Lemma A.1 gives a condition for all roots being in the upper half of the complex plane. To obtain a condition for poles in the left half plane ($\operatorname{Re}\{s\} < 0$), we substitute $\mu = -js$ in (A.1) and identify the coefficients with the polynomial

$$p(s) = s^n + b_{n-1}s^{n-1} + \dots + b_1s + b_0. \quad (\text{A.5})$$

Those coefficients that appear in (A.3)–(A.4) are then $f_{n-1} = \operatorname{Im}\{b_{n-1}\}$, $g_{n-1} = -\operatorname{Re}\{b_{n-1}\}$, $f_{n-2} = -\operatorname{Re}\{b_{n-2}\}$, $g_{n-2} = -\operatorname{Im}\{b_{n-2}\}$, $f_{n-3} = -\operatorname{Im}\{b_{n-3}\}$, $g_{n-3} = \operatorname{Re}\{b_{n-3}\}$. Note that these identifications hold regardless of n , as the coefficient of the highest order term is set to 1 in both (A.5) and (A.1).

References

- Agaev, R., & Chebotarev, P. (2010). Which digraphs with ring structure are essentially cyclic? *Advances in Applied Mathematics*, 45(2), 232–251.
- Andreasson, M., Dimarogonas, D., Sandberg, H., & Johansson, K. (2014). Distributed PI-control with applications to power systems frequency control. In *American control conf.* (pp. 3183–3188).
- Bamieh, B., Jovanović, M. R., Mitra, P., & Patterson, S. (2012). Coherence in large-scale networks: Dimension-dependent limitations of local feedback. *IEEE Transactions on Automatic Control*, 57(9), 2235–2249.
- Baroah, P., & Hespanha, J. P. (2005). Error amplification and disturbance propagation in vehicle strings with decentralized linear control. In *IEEE conf. on decision and control CDC*, (pp. 4964–4969).
- Besselink, B., & Knorn, S. (2018). Scalable input-to-state stability for performance analysis of large-scale networks. *IEEE Control Systems Letters*, 2(3), 507–512.
- Brouwer, A. E., & Haemers, W. H. (2012). *Spectra of graphs*. New York, NY.
- Cantos, C., Veerman, J., & Hammond, D. (2016). Signal velocity in oscillator arrays. *The European Physical Journal Special Topics*, 225(6), 1115–1126.
- Chung, F. (1997). *Spectral graph theory*. Providence, RI.
- Chung, F. (2005). Laplacians and the Cheeger inequality for directed graphs. *Annals of Combinatorics*, 9(1), 1–19.
- Fax, J. A., & Murray, R. M. (2004). Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9), 1465–1476.
- Fiedler, M. (1973). Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(2), 298–305.
- Franceschelli, M., & Frasca, P. (2021). Stability of open multiagent systems and applications to dynamic consensus. *IEEE Transactions on Automatic Control*, 66(5), 2326–2331.
- Friedman, J. (1991). On the second eigenvalue and random walks in random d -regular graphs. *Combinatorica*, 11, 331–362.
- Grone, R., Merris, R., & Sunder, V. (1990). The Laplacian spectrum of a graph. *SIAM Journal of Mathematical Analysis*, 11(2), 218–238.
- Gunter, G., Gloudemans, D., Stern, R. E., McQuade, S., Bhadani, R., Bunting, M., et al. (2021). Are commercially implemented adaptive cruise control systems string stable? *IEEE Transactions on Intelligent Transportation Systems*, 22(11), 6992–7003.
- Hendrickx, J. M., & Martin, S. (2017). Open multi-agent systems: Gossiping with random arrivals and departures. In *IEEE conf. on decision and control CDC*, (pp. 763–768).
- Herman, I. (2016). *Scaling in vehicle platoons* (Ph.D. thesis), Czech Technical University in Prague, URL: https://support.dce.felk.cvut.cz/mediawiki/images/d/d1/Diz_2017_herman_ivo.pdf.
- Horn, R. A., & Johnson, C. R. (1985). *Matrix analysis*. New York: Cambridge University Press.
- Jadbabaie, A., Lin, J., & Morse, A. S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6), 988–1001.
- Jensen, E., & Bamieh, B. (2022). On structured-closed-loop versus structured-controller design: the case of relative measurement feedback. *arXiv:2008.11291*.
- Jiang, F., Wang, L., & Jia, Y. (2009). Consensus in leaderless networks of high-order-integrator agents. In *American control conf.* (pp. 4458–4463).
- Kempton, L., Herrmann, G., & Bernardo, M. d. (2018). Self-organization of weighted networks for optimal synchronizability. *IEEE Transactions on Control of Network Systems*, 5(4), 1541–1550.
- Krebs, M., & Shaheen, A. (2011). *Expander families and cayley graphs, A beginner's guide*. Oxford: Oxford University Press.
- Mohar, B. (1991). The Laplacian spectrum of graphs. In *Graph theory, combinatorics, and applications* (pp. 871–898). Wiley.
- Ni, W., & Cheng, D. (2010). Leader-following consensus of multi-agent systems under fixed and switching topologies. *Systems & Control Letters*, 59(3), 209–217.
- Olfati-Saber, R. (2006). Flocking for multi-agent dynamic systems: algorithms and theory. *IEEE Transactions on Automatic Control*, 51(3), 401–420.
- Olfati-Saber, R., Fax, J. A., & Murray, R. M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1), 215–233.
- Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9), 1520–1533.
- Patterson, S., & Bamieh, B. (2014). Consensus and coherence in fractal networks. *IEEE Transactions on Control of Network Systems*, 1(4), 338–348.
- Radmanesh, A., Naghash, A., & Mohamadifard, A. (2017). Optimal distributed control of multi agents: Generalization of consensus algorithms for high-order state derivatives of SISO and MIMO systems. In *International conf. on control, automation and robotics* (pp. 606–611).
- Ren, W., Moore, K. L., & Chen, Y. (2007). High-order and model reference consensus algorithms in cooperative control of multi-vehicle systems. *Journal of Dynamic Systems, Measurement and Control*, 129(5), 678–688.
- Rezaee, H., & Abdollahi, F. (2015). Average consensus over high-order multiagent systems. *IEEE Transactions on Automatic Control*, 60(11), 3047–3052.
- Seiler, P., Pant, A., & Hedrick, K. (2004). Disturbance propagation in vehicle strings. *IEEE Transactions on Automatic Control*, 49(10), 1835–1842.
- Siami, M., & Motee, N. (2016). Fundamental limits and tradeoffs on disturbance propagation in large-scale dynamical networks. *IEEE Transactions on Automatic Control*, 61(12), 4055–4062.
- Simpson-Porco, J. W., Dörfler, F., & Bullo, F. (2013). Synchronization and power sharing for droop-controlled inverters in islanded microgrids. *Automatica*, 49(9), 2603–2611.
- Spielman, D. A., & Teng, S.-H. (2007). Spectral partitioning works: Planar graphs and finite element meshes. *Linear Algebra and its Applications*, 421(2), 284–305, Special Issue in honor of Miroslav Fiedler.
- Stern, R. E., et al. (2018). Dissipation of stop-and-go waves via control of autonomous vehicles: Field experiments. *Transportation Research Part C (Emerging Technologies)*, 89, 205–221.
- Städli, S., Seron, M. M., & Middleton, R. H. (2017). Vehicular platoons in cyclic interconnections with constant inter-vehicle spacing. In *20th IFAC world congress* (pp. 2511–2516).
- Swaroop, D., & Hedrick, J. K. (1996). String stability of interconnected systems. *IEEE Transactions on Automatic Control*, 41(3), 349–357.
- Tegling, E., Bamieh, B., & Sandberg, H. (2019). Localized high-order consensus destabilizes large-scale networks. In *American control conf. ACC*, (pp. 760–765).
- Tegling, E., Middleton, R. H., & Seron, M. M. (2019). Scalability and fragility in bounded-degree consensus networks. In *8th IFAC workshop on distributed estimation and control in networked systems (NecSys)*.
- Tegling, E., Mitra, P., Sandberg, H., & Bamieh, B. (2019). On fundamental limitations of dynamic feedback control in regular large-scale networks. *IEEE Transactions on Automatic Control*, 64(12), 4936–4951.
- Tondl, A. (1965). *Some problems of rotor dynamics*. Prague: Czechoslovak Academy of Sciences.
- Xia, W., & Cao, M. (2017). Analysis and applications of spectral properties of grounded Laplacian matrices for directed networks. *Automatica*, 80, 10–16.
- Yadlapalli, S. K., Darbha, S., & Rajagopal, K. R. (2006). Information flow and its relation to stability of the motion of vehicles in a rigid formation. *IEEE Transactions on Automatic Control*, 51(8), 1315–1319.
- Yang, P., Freeman, R., Gordon, G., Lynch, K., Srinivasa, S., & Sukthankar, R. (2010). Decentralized estimation and control of graph connectivity for mobile sensor networks. *Automatica*, 46(2), 390–396.
- Zuo, Z., Tian, B., Defoort, M., & Ding, Z. (2018). Fixed-time consensus tracking for multi-agent systems with high-order integrator dynamics. *IEEE Transactions on Automatic Control*, 63(2), 563–570.



Emma Tegling received her Ph.D. degree in Electrical Engineering from KTH Royal Institute of Technology, Sweden, in 2019, and her M.Sc. and B.Sc. degrees, both in Engineering Physics, from the same institute in 2013 and 2011, respectively. From 2019 to 2020 she was a Postdoctoral Research Fellow with the Institute of Data, Systems, and Society at the Massachusetts Institute of Technology (MIT), before joining the Department of Automatic Control at Lund University, Sweden, where she is currently a Senior Lecturer with a Wallenberg AI, Autonomous Systems and Software Program (WASP) professorship. Her research interests revolve around large-scale networked systems, fundamental limitations to their control, and applications including power networks and social networks. She is the recipient of the Swedish Research Council International Postdoc Grant and a Sweden-America Foundation Fellowship.



Bassam Bamieh received the B.Sc. degree in electrical engineering and physics from Valparaiso University, Valparaiso, IN, USA, in 1983, and the M.Sc. and Ph.D. degrees in electrical and computer engineering from Rice University, Houston, TX, USA, in 1986 and 1992, respectively. From 1991 to 1998 he was an Assistant Professor with the Department of Electrical and Computer Engineering, and the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, after which he joined the University of California at Santa Barbara (UCSB) where he is currently a Professor of Mechanical Engineering. His research interests include robust and optimal control, distributed and networked control and dynamical systems, shear flow transition and turbulence, and the use of feedback in thermoacoustic energy conversion devices. He is a past recipient of the IEEE Control Systems Society G. S. Axelby Outstanding Paper Award (twice), the AACC Hugo Schuck Best Paper Award, and the National Science Foundation CAREER Award. He is a Distinguished Lecturer of the IEEE Control Systems Society, a Fellow of the IEEE, and a Fellow of the International Federation of Automatic Control (IFAC).



Henrik Sandberg is Professor at the Division of Decision and Control Systems, KTH Royal Institute of Technology, Stockholm, Sweden. He received the M.Sc. degree in engineering physics and the Ph.D. degree in automatic control from Lund University, Lund, Sweden, in 1999 and 2004, respectively. From 2005 to 2007, he was a Post-Doctoral Scholar at the California Institute of Technology, Pasadena, USA. In 2013, he was a Visiting Scholar at the Laboratory for Information and Decision Systems (LIDS) at MIT, Cambridge, USA. He has also held visiting appointments at the Australian National University and the University of Melbourne, Australia. His current research interests include security of cyber-physical systems, power systems, model reduction, and fundamental limitations in control. Dr. Sandberg was a recipient of the Best Student Paper Award from the IEEE Conference on Decision and Control in 2004, an Ingvar Carlsson Award from the Swedish Foundation for Strategic Research in 2007, and a Consolidator Grant from the Swedish Research Council in 2016. He has served on the editorial boards of IEEE Transactions on Automatic Control and the IFAC Journal Automatica. He is Fellow of the IEEE.