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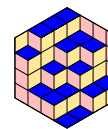


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Almost all wreath product character values are divisible by given primes

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ABSTRACT For a finite group G with integer-valued character table and a prime p , we show that almost every entry in the character table of $G \wr S_N$ is divisible by p as $N \rightarrow \infty$. This result generalizes the work of Peluse and Soundararajan on the character table of S_N .

1. INTRODUCTION

Let S_N be the symmetric group on N letters. The complex irreducible characters of S_N were calculated by Frobenius in 1900; in particular, Frobenius showed that the characters are integer-valued [1]. In 2019, Alex Miller investigated the distribution of the parity of entries of the character table of S_N . He made the remarkable conjecture that for any prime p and exponent $\ell \geq 1$, the proportion of entries of the character table of S_N divisible by p (and later p^ℓ for $\ell \geq 1$) tends to 1 as $N \rightarrow \infty$ [7, 6]. This conjecture was recently proved by Peluse and Soundararajan in the case $\ell = 1$ in [8].

This leaves the question of investigating the distribution of residues modulo p for more general finite groups with integer-valued character tables. For a fixed group G with integer-valued character table, a natural infinite family of such is the wreath product $G \wr S_N$ as $N \rightarrow \infty$. When G has integer-valued character table, it is known that the characters of $G \wr S_N$ are also integer-valued [4, Corollary 4.4.11]. These families of wreath products include the Weyl group of type B_N , when $G = \mathbb{Z}/2\mathbb{Z}$, and wreath products $S_M \wr S_N$ of symmetric groups.

Our main result is a generalization of Peluse and Soundararajan's theorem:

THEOREM (see Theorem 3.9 below). *Let G be a group with integer-valued character table and let $G \wr S_N$ be the wreath product of G with S_N . For all primes p , the proportion of entries in the character table of $G \wr S_N$ which are divisible by p tends to 1 as $N \rightarrow \infty$.*

The proof relies on the combinatorics of the representations of $G \wr S_N$. If G has k conjugacy classes, then conjugacy classes and representations of $G \wr S_N$ are both naturally labelled by k -multipartitions of N . One of the key inputs is characterizing when two elements of $G \wr S_N$ have columns in the character table congruent modulo p . In Lemma 3.2, we give a combinatorial characterization directly generalizing the corresponding criterion for S_N .

It is known that the character tables of all Weyl groups are integer-valued. The Weyl groups of type A are the symmetric groups, where our question was answered by Peluse and Soundararajan. The Weyl groups of type B_N and C_N are both equal to

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$\mathbb{Z}/2\mathbb{Z} \wr S_N$, handled by our main theorem. The only remaining infinite family of Weyl groups is that of type D . In Section 4, we also show that the proportion of character values of the Weyl group of type D_N divisible by a prime p tends to 1 as $N \rightarrow \infty$.

2. PRELIMINARIES

2.1. REPRESENTATION THEORY OF THE WREATH PRODUCT. Let G be a finite group with integer-valued character table and let S_N be the symmetric group on N letters.

DEFINITION 2.1. *The wreath product of G with S_N , denoted $G \wr S_N$, is the group of $N \times N$ permutation matrices with nonzero entries in G .*

We begin by recalling the representation theory of $G \wr S_N$. The representation theory of wreath products was first studied in Specht's dissertation [11], anticipated by Young's work on the case $G = \mathbb{Z}/2\mathbb{Z}$ [12]; see also [4, 13] for more modern treatments. If we take the representation theory of G as input data and let N vary, the representation theory has structural similarities to the representation theory of S_N , the case when $G = 1$. While representations and conjugacy classes of the symmetric group are labelled by partitions of N , representations and conjugacy classes of the wreath product are labelled by multipartitions:

DEFINITION 2.2. *A k -multipartition of an integer N is $\lambda = (\lambda_1, \dots, \lambda_k)$ where λ_i is a partition for all i such that $\sum_{i=1}^k |\lambda_i| = N$.*

We now describe how to label conjugacy classes of $G \wr S_N$. For an element of $G \wr S_N$, assign to each cycle in its projection to S_N a conjugacy class of G , called the *cycle product*, as follows: if $(i_1 i_2 \cdots i_m)$ is a cycle of σ , then the cycle product of $(g_1, g_2, \dots, g_N)\sigma \in G \wr S_N$ corresponding to $(i_1 i_2 \cdots i_m)$ is defined to be the conjugacy class of $g_{i_m} \cdots g_{i_2} g_{i_1}$ [4, (4.2.1)].

PROPOSITION 2.3 ([4, Theorem 4.2.8]). *If G has k conjugacy classes, then the conjugacy classes of $G \wr S_N$ are indexed by k -multipartitions of N . Given $x \in G \wr S_N$, the multipartition λ corresponding to x is formed as follows: for each cycle in x of length ℓ , if the corresponding cycle product is the i th conjugacy class of G , then add a part of size ℓ to λ_i .*

One can check the assignment of a conjugacy class to a multipartition is well-defined by checking under conjugation by S_N and by diagonal matrices $G^N \subseteq G \wr S_N$. Conjugating an element of $G \wr S_N$ by S_N does not change the set of cycle products at all. If $(g_1, \dots, g_N) \in G^N$ and $(12 \cdots N)$ is an N -cycle, the conjugate of $(12 \cdots N)(g_1, g_2, \dots, g_N)$ by $(g, 1, \dots, 1)$ is $(12 \cdots N)(g_1 g^{-1}, g_2, \dots, g_N)$; these two elements have conjugate cycle products $g_N \cdots g_2 g_1$ and $g(g_N \cdots g_2 g_1)g^{-1}$. The general case of conjugation by G^N reduces to the above case. We will not need to use the specific form of this bijection in this paper; it is used in the proofs of character formulas in Propositions 2.4 and 2.11, which we omit.

To find the complex irreducible representations of $G \wr S_N$, we need the complex irreducible representations of G as input; call the irreducible G -representations V_1, \dots, V_k .

PROPOSITION 2.4 ([4, Theorem 4.4.3]). *If G has k conjugacy classes, then the irreducible representations of $G \wr S_N$ are in bijection with k -multipartitions of N . For $\lambda = (\lambda_1, \dots, \lambda_k)$ a k -multipartition of N , let $a_i = |\lambda_i|$ and $G_a = G \wr S_a$. Then the irreducible representation of $G \wr S_N$ corresponding to λ is*

$$V^\lambda = \text{Ind}_{G_{a_1} \times \cdots \times G_{a_k}}^{G_N} \left(\boxtimes_{i=1}^k (S^{\lambda_i} \otimes V_i^{\otimes a_i}) \right)$$

where S^{λ_i} is the Specht module for S_N corresponding to λ_i .

Character values of wreath products can be calculated using a modified version of the Murnaghan–Nakayama rule for the symmetric group. Let χ^λ be the character of V^λ and χ_μ^λ be the value of χ^λ on the conjugacy class corresponding to μ . Then χ_μ^λ is calculated by decomposing the Young diagrams of the partitions λ_i for all i using rimhooks:

DEFINITION 2.5. A rimhook of a k -multipartition $\lambda = (\lambda_1, \dots, \lambda_k)$ consists of k adjacent boxes in the Young diagram of a single λ_i such that no other boxes are remaining south or east after the rimhook has been removed and no box in the rimhook has a southeast neighbor.

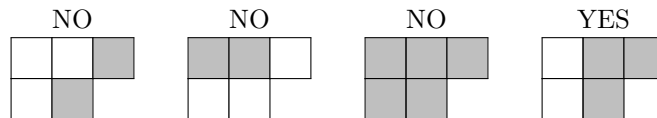


FIGURE 1. Examples of three invalid and one valid rimhooks in $\lambda = ((3^1 2^1))$.

DEFINITION 2.6. For k -multipartitions λ and μ , a rimhook decomposition of λ by μ is obtained by repeatedly removing rimhooks in λ according to a fixed ordering of the parts of μ . Further, we define $RHD(\lambda, \mu)$ to consist of all rimhook decompositions of λ by μ .

As in the Murnaghan–Nakayama rule for the usual symmetric group, formulas for irreducible characters involve the height of a rimhook decomposition:

DEFINITION 2.7. The height of a rimhook is one less than the number of rows included in that rimhook. The height of a rimhook decomposition ρ , denoted $ht(\rho)$, is the sum of the heights of all rimhooks in the decomposition.

The Murnaghan–Nakayama rule can be modified for wreath products as follows:

PROPOSITION 2.8 ([4, Theorem 4.4.10]). Let λ and μ be k -multipartitions of N . Let $\chi^1, \chi^2, \dots, \chi^k$ be the irreducible characters of G and c_1, c_2, \dots, c_k the conjugacy classes of G . For $\rho \in RHD(\lambda, \mu)$, let $\psi(\rho)$ be defined by

$$\psi(\rho) = \prod_{i=1}^k \prod_{j=1}^k \left(\chi^i(c_j)^{\#\{\text{rimhooks } h \in \rho \text{ from } \mu_j \text{ in } \lambda_i\}} \right).$$

Then

$$\chi_\mu^\lambda = \sum_{\rho \in RHD(\lambda, \mu)} (-1)^{ht(\rho)} \psi(\rho).$$

The permutation module characters of wreath products form another basis for the space of class functions of $G \wr S_N$ that is easier to work with.

DEFINITION 2.9. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a k -multipartition of N and let $a_i = |\lambda_i|$. For each λ_i , let S_{λ_i} be the Young subgroup of S_{a_i} corresponding to λ_i and let $G_{\lambda_i} = G \wr S_{\lambda_i}$. Then the permutation module M^λ for $G \wr S_N$ is defined by

$$M^\lambda = \text{Ind}_{G_{\lambda_1} \times \dots \times G_{\lambda_k}}^{G \wr S_N} \left(\boxtimes_{i=1}^k V_i^{\otimes a_i} \right).$$

There is a character formula for M^λ using row decompositions instead of rimhook decompositions. It is as follows:

DEFINITION 2.10. Let λ and μ be k -multipartitions of N . A row decomposition of λ by μ is a function $\rho : \{\text{rows of } \mu\} \rightarrow \{\text{rows of } \lambda\}$ such that if r is a row of λ , then the rows in $\rho^{-1}(r)$ have the same total length as r . The set of all row decompositions of λ by μ is denoted $RD(\lambda, \mu)$.

We will think of row decompositions of λ by μ as a tiling of the Young diagrams of λ by rows, where rows of μ are placed in a fixed ordering.



FIGURE 2. All valid row decompositions of $(\square\square\square, \square\square)$ by $(\color{red}\square, \color{blue}\square)$. The numbers in the boxes indicate the order in which parts of μ are placed into rows of λ , with fixed right-to-left placement.

PROPOSITION 2.11. Let λ and μ be k -multipartitions of N . Let $\chi^1, \chi^2, \dots, \chi^k$ be the irreducible characters of G and c_1, \dots, c_k the conjugacy classes of G . For $\rho \in RD(\lambda, \mu)$, let $\alpha(\rho)$ be defined by

$$\alpha(\rho) = \prod_{i=1}^k \prod_{j=1}^k \left(\chi^i(c_j)^{\#\{\text{rows from } \mu_j \text{ placed into } \lambda_i \text{ by } \rho\}} \right).$$

Then the character for permutation module M^λ at μ is

$$M_\mu^\lambda = \sum_{\rho \in RD(\lambda, \mu)} \alpha(\rho).$$

The proof follows from the character formula for induced representations.

We now describe the change-of-basis between irreducible and permutation characters.

DEFINITION 2.12. The dominance order on partitions is defined by $\lambda \succcurlyeq \eta$ if the row lengths $\lambda^1 \geq \lambda^2 \geq \dots$ and $\eta^1 \geq \eta^2 \geq \dots$ of λ and η satisfy $\sum_{i=1}^j \lambda^i \geq \sum_{i=1}^j \eta^i$ for all $j \geq 1$. The dominance order on k -multipartitions is defined by $\lambda \succcurlyeq \eta$ if and only if $\lambda_i \succcurlyeq \eta_i$ for all i .

LEMMA 2.13. The matrix of multiplicities $[M^\lambda : V^\eta]$ of the irreducible representations of $G \wr S_N$ in permutation modules is unimodular and lower-triangular with respect to dominance order.

Proof. Recall that the Kostka numbers $K^{\beta, \gamma}$ for β, γ partitions of N are defined by

$$M^\beta = \text{Ind}_{S_\beta}^{S_N} 1 = \bigoplus_{\gamma} (V^\gamma)^{\oplus K^{\gamma, \beta}},$$

where S_β is the Young subgroup corresponding to β and V^γ is the Specht module corresponding to γ . Note that our notation for M^β and V^γ agrees with that of wreath products $G \wr S_N$ when $G = 1$. The Kostka numbers satisfy $K^{\beta, \beta} = 1$ and $K^{\gamma, \beta} > 0$ if and only if $\gamma \succcurlyeq \beta$ in dominance order [5, I, (6.5)].

We claim that

$$(1) \quad M^\lambda = \bigoplus_{\eta} (V^\eta)^{\oplus c(\lambda, \eta)}, \quad c(\lambda, \eta) = \left(\prod_{i=1}^k K^{\eta_i, \lambda_i} \right).$$

By Definition 2.9, if $a_i = |\lambda_i|$ for all i and $H = G_{a_1} \times G_{a_2} \times \dots \times G_{a_k}$, then

$$M^\lambda = \text{Ind}_{G_\lambda}^{G_N} (\boxtimes_{i=1}^k V_i^{\otimes a_i}) = \text{Ind}_H^{G_N} (\boxtimes_{i=1}^k M^{\lambda_i} \otimes V_i^{\otimes a_i}),$$

where S_{a_i} acts diagonally on the tensor product $M^{\lambda_i} \otimes V_i^{\otimes a_i}$ and G^{a_i} acts naturally on $V_i^{\otimes a_i}$. Then (1) follows from multilinearity of the tensor product and linearity of induction.

Now since the matrix of Kostka numbers is unimodular and upper-triangular with respect to dominance order, the matrix $\{c(\lambda, \mu)\}_{\lambda, \mu}$ is unimodular and lower-triangular with respect to dominance order. \square

2.2. ASYMPTOTICS OF PARTITIONS. We recall a form of the Hardy–Ramanujan asymptotic for the number of partitions of N , denoted $p(N)$.

PROPOSITION 2.14 ([3, (1.36)]). *If $\delta > 0$, then*

$$\left(\frac{2\pi}{\sqrt{6}} - \delta\right) \sqrt{N} \leq \log p(N) \leq \left(\frac{2\pi}{\sqrt{6}} + \delta\right) \sqrt{N}$$

for sufficiently large N .

Let $p_k(N)$ denote the number of k -multipartitions of N .

CLAIM 2.15. *If $\delta > 0$, then*

$$\left(\frac{2\pi}{\sqrt{6}} - \delta\right) \sqrt{kN} \leq \log p_k(N) \leq \left(\frac{2\pi}{\sqrt{6}} + \delta\right) \sqrt{kN}$$

for sufficiently large N .

This formula also appears in [9]. We provide an elementary inductive proof.

Proof. We proceed by induction on k . The base case $k = 1$ is Proposition 2.14.

For $\delta > 0$, let $\delta' = \frac{4}{5}\delta$. By inductive hypothesis, there exists a constant B such that if $C \geq B$, then

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right) \left(\sqrt{(k-1)C}\right)\right) \leq p_{k-1}(C) \leq \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right) \left(\sqrt{(k-1)C}\right)\right)$$

and

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right) \left(\sqrt{C}\right)\right) \leq p(C) \leq \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right) \left(\sqrt{C}\right)\right).$$

By considering the size of the first partition in a k -multipartition, it follows that

$$p_k(N) = \sum_{a=0}^N p(a) p_{k-1}(N-a).$$

Now assume that $N \geq 2B - 2$; we can then break up the sum for $p_k(N)$ into the following distinct parts: let

$$\begin{aligned} D_1 &= \sum_{a=0}^{B-1} p(a) p_{k-1}(N-a), \\ D_2 &= \sum_{a=B}^{N-B} p(a) p_{k-1}(N-a), \\ D_3 &= \sum_{a=N-B+1}^N p(a) p_{k-1}(N-a). \end{aligned}$$

In D_2 , for $B \leq a \leq N - B$, we have

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right) \left(\sqrt{a} + \sqrt{(k-1)(N-a)}\right)\right) \leq p(a) p_{k-1}(N-a),$$

and the right hand side in turn satisfies

$$p(a)p_{k-1}(N-a) \leq \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\left(\sqrt{a} + \sqrt{(k-1)(N-a)}\right)\right).$$

Note that $\sqrt{a} + \sqrt{(k-1)(N-a)} \leq \sqrt{kN}$, with equality achieved at $a = \frac{N}{k}$. Summing over $a \in [B, N-B]$, we get

$$(2) \quad \exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta\right)\sqrt{kN}\right) \leq D_2 \leq (N-2B+1) \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right).$$

We now consider D_1 and D_3 . Note that for $a \in [0, B]$, we have $p(a)p_{k-1}(N-a) \leq p(B)p_{k-1}(N)$, and for $a \in (N-B, N]$, we have $p(a)p_{k-1}(N-a) \leq p(N)p_{k-1}(B)$. Hence

$$(3) \quad 0 \leq D_1 \leq Bp(B)p_{k-1}(N) \leq B \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right)$$

for sufficiently large N . Likewise,

$$(4) \quad 0 \leq D_3 \leq Bp(N)p_{k-1}(B) \leq B \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right).$$

Combining (2), (3), and (4), we have that

$$\begin{aligned} \exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta\right)\sqrt{kN}\right) &\leq p_k(N) \leq (N+1) \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right) \\ &\leq \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta\right)\sqrt{kN}\right) \end{aligned}$$

for sufficiently large N . □

The above estimate implies k -multipartitions concentrate around having close to equal-size parts:

COROLLARY 2.16. *For all $\delta > 0$, the proportion of k -multipartitions $\lambda = (\lambda_1, \dots, \lambda_k) \vdash N$ such that*

$$\frac{N}{k}(1-\delta) < |\lambda_i| < \frac{N}{k}(1+\delta)$$

for all $1 \leq i \leq k$ goes to 1 as $N \rightarrow \infty$.

Proof. Pick $1 \leq i \leq k$. By partitioning k -multipartitions according to the size of λ_i , the number of k -multipartitions of N where $|\lambda_i| \notin (\frac{N}{k}(1-\epsilon), \frac{N}{k}(1+\epsilon))$ is

$$(5) \quad \sum_{a \notin (\frac{N}{k}(1-\epsilon), \frac{N}{k}(1+\epsilon))} p(a)p_{k-1}(N-a).$$

Note that $\sqrt{a} + \sqrt{(k-1)(N-a)}$ increases for $a < \frac{N}{k}$ and decreases for $a > \frac{N}{k}$. Then by Claim 2.15, the rate of growth (5) is significantly slower than the rate of growth of $p_k(N)$. □

3. MAIN RESULTS

3.1. CHARACTER TABLE COLUMN CONGRUENCES. Corollary 3.3 below, which we call “the mashing rule,” gives a criterion for mod p congruence of two columns of the character table of $G \wr S_N$ in terms of k -multipartitions.

In this section, we must assume that G has integer-valued character table. By [10, §13.1], the group G has integer-valued character table if and only if $\sigma \in G$ is conjugate to σ^j whenever j is prime to the order of σ .

DEFINITION 3.1. Let \sim_p be the equivalence relation on k -multipartitions generated by the following: $\mu \sim_p \nu$ if there is j such that $\mu_i = \nu_i$ for $i \neq j$, and ν_j is formed by replacing one part of size mp in μ_j with p parts of size m in ν_j .

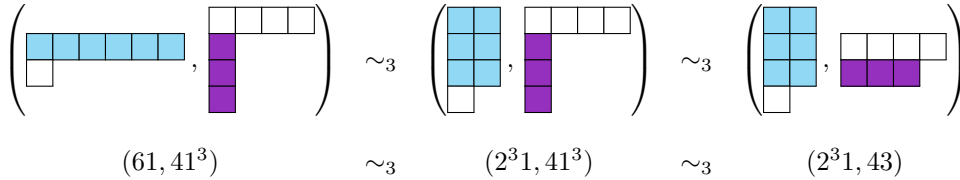


FIGURE 3. Example of three conjugacy classes which are congruent mod 3 in $\mathbb{Z}/2\mathbb{Z} \wr S_N$ (note that $m = 2$ in the first cycle type and $m = 1$ in the second).

LEMMA 3.2. Let p be a prime and G be a group with integer-valued character table. Let $\mu = (\mu_1, \dots, \mu_k)$ and $\nu = (\nu_1, \dots, \nu_k)$ be k -multipartitions of N , indexing conjugacy classes of $G \wr S_N$. If $\mu \sim_p \nu$, then $M_\mu^\lambda \equiv M_\nu^\lambda \pmod{p}$ for all k -multipartitions λ of N .

Proof. It suffices to show $M_\mu^\lambda \equiv M_\nu^\lambda \pmod{p}$ if there exists j such that $\mu_i = \nu_i$ for all $i \neq j$, $\mu_j = (\xi, mp)$ for some ξ , and $\nu_j = (\xi, m^p)$. We break $RD(\lambda, \nu)$ into two cases. In case one, we consider the row decompositions of ν where the p rows not in ξ are tiled into the same row of λ . In case two we consider the row decompositions where the p rows not in ξ are not tiled in the same row. Recalling our formula for characters of permutation modules in Proposition 2.11, let

$$(6) \quad \beta = \sum_{\substack{\rho \in RD(\lambda, \nu) \text{ s.t. } m^p \text{ is tiled} \\ \text{in the same row}}} \alpha(\rho)$$

and

$$(7) \quad \gamma = \sum_{\substack{\rho \in RD(\lambda, \nu) \text{ s.t. } m^p \text{ is not tiled} \\ \text{in the same row}}} \alpha(\rho),$$

so that $M_\nu^\lambda = \beta + \gamma$. In case one, we will show $\beta \equiv M_\mu^\lambda \pmod{p}$ and in case two that $\gamma \equiv 0 \pmod{p}$. Together, these two congruences imply $M_\mu^\lambda \equiv M_\nu^\lambda \pmod{p}$.

In both cases, we break into subcases based on the ways to tile μ_i for $i \neq j$ and ξ . In case one, we have compatible tilings for μ and ν , and in case two, we have additional tilings for ν .

In case one, assume we have tiled all rows of μ_i for all $i \neq j$ and we have tiled ξ . We now have one row remaining. There is only one way to tile the last row for both μ and ν : put the remaining pieces into the remaining row. Let these row decompositions be denoted ρ_μ and ρ_ν respectively.

For ρ_μ , say that we place the final row r of size mp in the partition λ_q . The associated cycle product is c_j because mp comes from μ_j . Then mp contributes $\chi_q(c_j)$ to the product $\alpha(\rho_\mu)$. Then for ρ_ν , the p rows of size m are placed into λ_q . The conjugacy class of G associated with the p rows of size m is again c_j , so m^p contributes a factor of $\chi_q(c_j)^p$ to $\alpha(\rho_\nu)$.

By assumption, the character values of G are integral, so by Fermat's little theorem, $\chi_q(c_j) \equiv \chi_q(c_j)^p \pmod{p}$. All other factors in $\alpha(\rho_\mu)$ contributed by μ_i for $i \neq j$ and ξ are identical to the corresponding factors in $\alpha(\rho_\nu)$. Hence, $\alpha(\rho_\mu) \equiv \alpha(\rho_\nu) \pmod{p}$.

Summing over all the tilings in case one, we find $M_\mu^\lambda \equiv \beta \pmod{p}$.

In case two, assume we have tiled all rows of μ_i for $i \neq j$ and ξ , after which there are $t > 1$ remaining unfilled rows of the Young diagrams of λ . If $T \subseteq RD(\lambda, \nu)$ is the set of row decompositions extending our given tiling by μ_i for $i \neq j$ and ξ , then we will show

$$\sum_{\rho \in T} \alpha(\rho) \equiv 0 \pmod{p}.$$

Then γ is the sum over all such T of $\sum_{\rho \in T} \alpha(\rho)$, from which it will follow $\gamma \equiv 0 \pmod{p}$.

Call the lengths of the remaining rows $(m\ell_1, m\ell_2, \dots, m\ell_t)$. Since the elements of T are in bijection with choices of placements of p cycles of length m into these rows,

$$(8) \quad |T| = \binom{p}{\ell_1, \ell_2, \dots, \ell_t}.$$

Let $\rho \in T$. Note that all pieces of m^p come from μ_j , and thus have cycle product c_j , while all other cycles in μ are in the same place in T . Thus $\alpha(\rho) = \alpha(\rho')$ for all $\rho, \rho' \in T$. Hence $\sum_{\rho \in T} \alpha(\rho)$ is a sum of $|T|$ identical terms. Then $\sum_{\rho \in T} \alpha(\rho) \equiv 0 \pmod{p}$ because $|T|$ is divisible by p .

Case one has shown that $M_\mu^\lambda \equiv \beta \pmod{p}$, and case two has shown that $\gamma \equiv 0 \pmod{p}$. Since $M_\nu^\lambda = \beta + \gamma$, we conclude $M_\mu^\lambda \equiv M_\nu^\lambda \pmod{p}$. \square

COROLLARY 3.3 (The mashing rule). *Let G have integer-valued character table and k conjugacy classes. Let μ and ν be k -multipartitions of N . If $\mu \sim_p \nu$, then $\chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p}$ for all irreducible characters χ^λ of $G \wr S_N$.*

Proof. The set of irreducible characters and the set of characters of permutation modules form bases for the space of class functions on $G \wr S_N$. Since the change of basis matrix between these two bases is unimodular and lower-triangular, as stated in Lemma 2.13, χ^λ can be expressed as an integral linear combination of M^η for all k -multipartitions λ . It follows from Lemma 3.2 that $\mu \sim_p \nu$ implies $\chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p}$. \square

REMARK 3.4. Corollary 3.3 may also be proved using the following criterion from modular representation theory: if G has integer-valued character table, $g \in G$, and g' is the p -prime part of g , then $\chi(g) \equiv \chi(g') \pmod{p}$ for all characters χ of G [10, 18.1(v)]. In this situation, it may be seen that the mashing preserves the conjugacy class of the p -prime part of a class in $G \wr S_N$. Nonetheless, we have chosen to give a combinatorial proof.

3.2. PROOF OF MAIN THEOREM. Using Corollary 3.3, the existence of one zero in the character table implies many more entries are divisible by p . We proceed, following Peluse and Soundararajan in [8], by using Proposition 2.8 to show sufficiently many entries of the character table are zero.

DEFINITION 3.5. *A partition is called a t -core if none of the hook lengths of its Young diagram are divisible by t where $t \in \mathbb{Z}$. For example, from Figure 4 one can see that $(4, 2, 1)$ is a 5-core.*

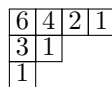


FIGURE 4. Hook-lengths for $\lambda_i = (4, 2, 1)$

In the course of proving [8, Proposition 1], Peluse and Soundararajan proved the following estimate of the number of t -cores when t is slightly larger than the typical longest cycle in a random conjugacy class:

PROPOSITION 3.6. *Let L be a positive integer, and let A be a real number with $1 \leq A \leq \log L / \log \log L$. Additionally suppose that t is a positive integer with*

$$(9) \quad t \geq \frac{\sqrt{6}}{2\pi} \sqrt{L} (\log L) \left(1 + \frac{1}{A}\right).$$

Then the number of partitions λ of L which are not t -cores is at most

$$O\left(p(L) \frac{\log L}{L^{\frac{1}{2A}}}\right),$$

independent of t satisfying (9).

Complementing the estimate in Proposition 3.6, Peluse and Soundararajan also estimated how many columns of the character table are congruent to a column corresponding to a partition with a large first part:

PROPOSITION 3.7 ([8, Proposition 2]). *Let $p \leq \frac{(\log L)}{(\log \log L)^2}$ be a prime. Starting with a partition μ of L , we repeatedly replace every occurrence of p parts of the same size m by one part of size mp until we arrive at a partition $\tilde{\mu}$ where no part appears more than $p-1$ times. Then the largest part of $\tilde{\mu}$ exceeds*

$$\frac{\sqrt{6}}{2\pi} \sqrt{L} (\log L) \left(1 + \frac{1}{5p}\right),$$

except for at most

$$O\left(p(L) \exp\left(-L^{\frac{1}{15p}}\right)\right)$$

partitions μ .

We now extend Peluse and Soundararajan's estimate in Proposition 3.7 to k -multipartitions.

PROPOSITION 3.8. *Let $p \ll N$ be a prime. Given a k -multipartition $\mu = (\mu_1, \dots, \mu_k)$ of N , for all μ_i with $1 \leq i \leq k$, we repeatedly replace every occurrence of p parts of the same size m by one part of size mp until we arrive at a k -multipartition $\tilde{\mu}$ where no part in any $\tilde{\mu}_i$ appears more than $p-1$ times.*

Then the largest part of $\tilde{\mu}$ is of size at least

$$(10) \quad \frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left(\log \frac{N}{k}\right) \left(1 + \frac{1}{5p}\right)$$

except for a number of multipartitions μ which is at most

$$O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) p_k(N)\right).$$

Proof. For a k -multipartition $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ of N , let $\tilde{\mu}$ be as above. We will bound above the number of k -multipartitions μ such that $\tilde{\mu}$ has largest part less than (10).

For any μ , we know that for some $1 \leq i \leq k$, $|\mu_i| \geq \frac{N}{k}$. Fix i such that μ_i has size $|\mu_i| = a \geq \frac{N}{k}$. Then Proposition 3.7 tells us that the largest part of $\tilde{\mu}_i$ exceeds

$$\frac{\sqrt{6}}{2\pi} \sqrt{a} (\log a) \left(1 + \frac{1}{5p}\right) \geq \frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left(\log \frac{N}{k}\right) \left(1 + \frac{1}{5p}\right)$$

except for at most

$$O\left(p(a) \exp\left(-a^{\frac{1}{15p}}\right)\right)$$

partitions μ_i of size a and therefore at most

$$O\left(p(a) \exp\left(-a^{\frac{1}{15p}}\right) p_{k-1}(N-a)\right)$$

total k -multipartitions μ with $|\mu_i| = a$. Furthermore, since $a \geq \frac{N}{k}$,

$$\exp\left(-a^{\frac{1}{15p}}\right) \leq \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right),$$

and therefore summing over all $a \geq \frac{N}{k}$ we have that the number of multipartitions μ such that $|\mu_i| \geq \frac{N}{k}$ with no part in $\tilde{\mu}_i$ exceeding (10) is at most an absolute constant times

$$\begin{aligned} \sum_{a=\frac{N}{k}}^N \exp\left(-a^{\frac{1}{15p}}\right) p(a) p_{k-1}(N-a) &\ll \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) \sum_{a=\frac{N}{k}}^N p(a) p_{k-1}(N-a) \\ &\ll \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) \sum_{a=0}^N p(a) p_{k-1}(N-a) \\ &= \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) p_k(N). \end{aligned}$$

Since this bound is identical for each i , the number of k -multipartitions μ such that $\tilde{\mu}$ does not have a part of size greater than (10) is at most a factor of k greater than the bound above, and therefore also at most

$$O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) p_k(N)\right). \quad \square$$

THEOREM 3.9. *Let G be a group with integer-valued character table, and let $G \wr S_N$ be the wreath product of G with the symmetric group S_N . For all primes p , the proportion of entries in the character table of $G \wr S_N$ divisible by p tends to 1 as $N \rightarrow \infty$.*

Proof. Let k be the number of conjugacy classes of G . Given a k -multipartition μ , let $\tilde{\mu}$ be the multipartition obtained by repeatedly replacing p parts of μ_i of size m with one part of size mp until no μ_i has a part appearing more than $p-1$ times. For $A = 5p$, Proposition 3.8 implies that the largest part of $\tilde{\mu}$ has size

$$(11) \quad t \geq \frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left(\log \frac{N}{k}\right) \left(1 + \frac{1}{A}\right)$$

for a proportion of μ tending to 1 as $N \rightarrow \infty$. Now pick $A' \geq 1$ and $\delta > 0$ such that

$$\left(\log \frac{N}{k}\right) \left(1 + \frac{1}{A}\right) \geq \sqrt{1+\delta} \left(\log \left(\frac{N}{k}(1+\delta)\right)\right) \left(1 + \frac{1}{A'}\right).$$

By Corollary 2.16, the proportion of k -multipartitions $\lambda = (\lambda_1, \dots, \lambda_k) \vdash N$ such that $|\lambda_i| \in \left(\frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta)\right)$ for all i tends to 1 as $N \rightarrow \infty$. Thus, consider only (λ, μ) satisfying the above conditions.

Our choice of δ and A' imply that

$$\frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left(\log \frac{N}{k}\right) \left(1 + \frac{1}{A}\right) \geq \frac{\sqrt{6}}{2\pi} \sqrt{\frac{N(1+\delta)}{k}} \left(\log \left(\frac{N}{k}(1+\delta)\right)\right) \left(1 + \frac{1}{A'}\right).$$

If (N_1, \dots, N_k) is a partition of sufficiently large N with $N_i \in (\frac{N}{k}(1 - \delta), \frac{N}{k}(1 + \delta))$, we can assume $A' \leq \frac{\log N_i}{\log \log N_i}$ for each i , as $N_i \geq \frac{N}{k(1 - \delta)} \rightarrow \infty$. Then by Proposition 3.6, the proportion of k -multipartitions λ with $|\lambda_i| = N_i$ such that some λ_i is not a t -core is

$$\sum_{i=1}^k O\left(\frac{\log N_i}{N_i^{\frac{1}{2A'}}}\right)$$

for all t satisfying (11), independent of t . Hence, over all k -multipartitions λ satisfying $|\lambda_i| \in (\frac{N}{k}(1 - \delta), \frac{N}{k}(1 + \delta))$, the proportion of λ such that some λ_i is not a t -core is

$$O\left(\frac{\log(\frac{N}{k}(1 + \delta))}{(\frac{N}{k}(1 - \delta))^{\frac{1}{2A'}}}\right).$$

It follows that most (λ, μ) satisfy that λ_i is a t -core for t the largest part of $\tilde{\mu}$. Thus, for a proportion of (λ, μ) tending to 1 as $N \rightarrow \infty$, we have $\chi_\mu^\lambda = 0$ by Proposition 2.8 and therefore $\chi_\mu^\lambda \equiv 0 \pmod{p}$ by Corollary 3.3. \square

4. WEYL GROUPS OF TYPE D

DEFINITION 4.1. *The Weyl group of type D_N is the group of $N \times N$ signed permutation matrices with an even number of entries equal to -1 .*

We will denote this group by D_N also (note that it is distinct from the dihedral group). The study of representations of D_N was taken up by Young in [12, §7-10]; see [2] for a more modern treatment.

Let $B_N = \{\pm 1\} \wr S_N$; then D_N is a subgroup of B_N of index two. Recall that conjugacy classes of B_N are labelled by 2-multipartitions (η, ν) of N , where we take ν to be those cycles with nontrivial cycle product.

PROPOSITION 4.2 ([2, Proposition 3.4.12]). *The conjugacy classes B_N which meet D_N correspond to 2-multipartitions (η, ν) of N where ν has an even number of parts. The conjugacy classes of B_N which split in D_N are exactly those (η, ν) where $\nu = \emptyset$ and η has only even parts.*

Since D_N is a subgroup of B_N of index two, Clifford theory determines its representations:

PROPOSITION 4.3 ([2, §5.6.1]). *The irreducible representations of the Weyl group of type D_N are as follows:*

- (1) *if (λ, μ) is a 2-multipartition of N such that $\lambda \neq \mu$, then*

$$\text{Res}_{D_N}^{B_N} V^{\lambda, \mu} = \text{Res}_{D_N}^{B_N} V^{\mu, \lambda}$$

is an irreducible representation of D_N ;

- (2) *if (λ, λ) is a 2-multipartition of N with equal parts, then*

$$\text{Res}_{D_N}^{B_N} V^{\lambda, \lambda}$$

is the sum of two irreducible representations of D_N .

- (3) *Each irreducible representation of D_N appears exactly once in (1) or (2).*

COROLLARY 4.4. *For all primes p , the proportion of entries in the character table of D_N which are divisible by p tends to 1 as $N \rightarrow \infty$.*

Proof. The number of irreducible representations of D_N of the form $\text{Res}_{D_N}^{B_N} V^{\lambda, \mu}$ for $\lambda \neq \mu$ equals $\frac{1}{2}(p_2(N) - p(N/2))$ when N is even, and $\frac{1}{2}p_2(N)$ when N is odd. The number of irreducible representations appearing as a summand of $\text{Res}_{D_N}^{B_N} V^{\lambda, \lambda}$

is $2p(N/2)$ when N is even and 0 when N is odd. By Claim 2.15, we have $p_2(N) \gg p(N/2)$ for large enough N , so the proportion of irreducibles of the form $\text{Res}_{D_N}^{B_N} V^{\lambda, \mu}$ goes to 1 as $N \rightarrow \infty$.

We must also analyze the splitting of conjugacy classes from B_N in D_N . By Proposition 4.2, the number of conjugacy classes of B_N which split in D_N is $p(N/2)$ when N is even and 0 when N is odd. We conclude just as above that the proportion of non-split conjugacy classes tends to 1 as $N \rightarrow \infty$.

We have shown that almost every entry of the character table of D_N is of the form $V_{\eta, \nu}^{\lambda, \mu}$ where $\lambda \neq \mu$ and (η, ν) is a non-split conjugacy class in D_N . Such entries occupy at least a constant fraction of the character table of B_N as $N \rightarrow \infty$. By Theorem 3.9, almost all of such entries are divisible by p as $N \rightarrow \infty$. Hence, the proportion of entries of the character table of D_N which are divisible by p goes to 1 as $N \rightarrow \infty$. \square

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