

# ALGEBRAIC COMBINATORICS

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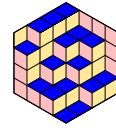
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# Almost all wreath product character values are divisible by given primes

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**ABSTRACT** For a finite group  $G$  with integer-valued character table and a prime  $p$ , we show that almost every entry in the character table of  $G \wr S_N$  is divisible by  $p$  as  $N \rightarrow \infty$ . This result generalizes the work of Peluse and Soundararajan on the character table of  $S_N$ .

## 1. INTRODUCTION

Let  $S_N$  be the symmetric group on  $N$  letters. The complex irreducible characters of  $S_N$  were calculated by Frobenius in 1900; in particular, Frobenius showed that the characters are integer-valued [1]. In 2019, Alex Miller investigated the distribution of the parity of entries of the character table of  $S_N$ . He made the remarkable conjecture that for any prime  $p$  and exponent  $\ell \geq 1$ , the proportion of entries of the character table of  $S_N$  divisible by  $p$  (and later  $p^\ell$  for  $\ell \geq 1$ ) tends to 1 as  $N \rightarrow \infty$  [7, 6]. This conjecture was recently proved by Peluse and Soundararajan in the case  $\ell = 1$  in [8].

This leaves the question of investigating the distribution of residues modulo  $p$  for more general finite groups with integer-valued character tables. For a fixed group  $G$  with integer-valued character table, a natural infinite family of such is the wreath product  $G \wr S_N$  as  $N \rightarrow \infty$ . When  $G$  has integer-valued character table, it is known that the characters of  $G \wr S_N$  are also integer-valued [4, Corollary 4.4.11]. These families of wreath products include the Weyl group of type  $B_N$ , when  $G = \mathbb{Z}/2\mathbb{Z}$ , and wreath products  $S_M \wr S_N$  of symmetric groups.

Our main result is a generalization of Peluse and Soundararajan's theorem:

**THEOREM** (see Theorem 3.9 below). *Let  $G$  be a group with integer-valued character table and let  $G \wr S_N$  be the wreath product of  $G$  with  $S_N$ . For all primes  $p$ , the proportion of entries in the character table of  $G \wr S_N$  which are divisible by  $p$  tends to 1 as  $N \rightarrow \infty$ .*

The proof relies on the combinatorics of the representations of  $G \wr S_N$ . If  $G$  has  $k$  conjugacy classes, then conjugacy classes and representations of  $G \wr S_N$  are both naturally labelled by  $k$ -multipartitions of  $N$ . One of the key inputs is characterizing when two elements of  $G \wr S_N$  have columns in the character table congruent modulo  $p$ . In Lemma 3.2, we give a combinatorial characterization directly generalizing the corresponding criterion for  $S_N$ .

It is known that the character tables of all Weyl groups are integer-valued. The Weyl groups of type  $A$  are the symmetric groups, where our question was answered by Peluse and Soundararajan. The Weyl groups of type  $B_N$  and  $C_N$  are both equal to

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$\mathbb{Z}/2\mathbb{Z} \wr S_N$ , handled by our main theorem. The only remaining infinite family of Weyl groups is that of type  $D$ . In Section 4, we also show that the proportion of character values of the Weyl group of type  $D_N$  divisible by a prime  $p$  tends to 1 as  $N \rightarrow \infty$ .

## 2. PRELIMINARIES

2.1. REPRESENTATION THEORY OF THE WREATH PRODUCT. Let  $G$  be a finite group with integer-valued character table and let  $S_N$  be the symmetric group on  $N$  letters.

DEFINITION 2.1. *The wreath product of  $G$  with  $S_N$ , denoted  $G \wr S_N$ , is the group of  $N \times N$  permutation matrices with nonzero entries in  $G$ .*

We begin by recalling the representation theory of  $G \wr S_N$ . The representation theory of wreath products was first studied in Specht's dissertation [11], anticipated by Young's work on the case  $G = \mathbb{Z}/2\mathbb{Z}$  [12]; see also [4, 13] for more modern treatments. If we take the representation theory of  $G$  as input data and let  $N$  vary, the representation theory has structural similarities to the representation theory of  $S_N$ , the case when  $G = 1$ . While representations and conjugacy classes of the symmetric group are labelled by partitions of  $N$ , representations and conjugacy classes of the wreath product are labelled by multipartitions:

DEFINITION 2.2. *A  $k$ -multipartition of an integer  $N$  is  $\lambda = (\lambda_1, \dots, \lambda_k)$  where  $\lambda_i$  is a partition for all  $i$  such that  $\sum_{i=1}^k |\lambda_i| = N$ .*

We now describe how to label conjugacy classes of  $G \wr S_N$ . For an element of  $G \wr S_N$ , assign to each cycle in its projection to  $S_N$  a conjugacy class of  $G$ , called the *cycle product*, as follows: if  $(i_1 i_2 \cdots i_m)$  is a cycle of  $\sigma$ , then the cycle product of  $(g_1, g_2, \dots, g_N) \sigma \in G \wr S_N$  corresponding to  $(i_1 i_2 \cdots i_m)$  is defined to be the conjugacy class of  $g_{i_m} \cdots g_{i_2} g_{i_1}$  [4, (4.2.1)].

PROPOSITION 2.3 ([4, Theorem 4.2.8]). *If  $G$  has  $k$  conjugacy classes, then the conjugacy classes of  $G \wr S_N$  are indexed by  $k$ -multipartitions of  $N$ . Given  $x \in G \wr S_N$ , the multipartition  $\lambda$  corresponding to  $x$  is formed as follows: for each cycle in  $x$  of length  $\ell$ , if the corresponding cycle product is the  $i$ th conjugacy class of  $G$ , then add a part of size  $\ell$  to  $\lambda_i$ .*

One can check the assignment of a conjugacy class to a multipartition is well-defined by checking under conjugation by  $S_N$  and by diagonal matrices  $G^N \subseteq G \wr S_N$ . Conjugating an element of  $G \wr S_N$  by  $S_N$  does not change the set of cycle products at all. If  $(g_1, \dots, g_N) \in G^N$  and  $(12 \cdots N)$  is an  $N$ -cycle, the conjugate of  $(12 \cdots N)(g_1, g_2, \dots, g_N)$  by  $(g, 1, \dots, 1)$  is  $(12 \cdots N)(g_1 g^{-1}, g_2, \dots, g_N)$ ; these two elements have conjugate cycle products  $g_N \cdots g_2 g_1$  and  $g(g_N \cdots g_2 g_1)g^{-1}$ . The general case of conjugation by  $G^N$  reduces to the above case. We will not need to use the specific form of this bijection in this paper; it is used in the proofs of character formulas in Propositions 2.4 and 2.11, which we omit.

To find the complex irreducible representations of  $G \wr S_N$ , we need the complex irreducible representations of  $G$  as input; call the irreducible  $G$ -representations  $V_1, \dots, V_k$ .

PROPOSITION 2.4 ([4, Theorem 4.4.3]). *If  $G$  has  $k$  conjugacy classes, then the irreducible representations of  $G \wr S_N$  are in bijection with  $k$ -multipartitions of  $N$ . For  $\lambda = (\lambda_1, \dots, \lambda_k)$  a  $k$ -multipartition of  $N$ , let  $a_i = |\lambda_i|$  and  $G_a = G \wr S_a$ . Then the irreducible representation of  $G \wr S_N$  corresponding to  $\lambda$  is*

$$V^\lambda = \text{Ind}_{G_{a_1} \times \cdots \times G_{a_k}}^{G_N} \left( \bigotimes_{i=1}^k (S^{\lambda_i} \otimes V_i^{\otimes a_i}) \right)$$

where  $S^{\lambda_i}$  is the Specht module for  $S_N$  corresponding to  $\lambda_i$ .

Character values of wreath products can be calculated using a modified version of the Murnaghan–Nakayama rule for the symmetric group. Let  $\chi^\lambda$  be the character of  $V^\lambda$  and  $\chi_\mu^\lambda$  be the value of  $\chi^\lambda$  on the conjugacy class corresponding to  $\mu$ . Then  $\chi_\mu^\lambda$  is calculated by decomposing the Young diagrams of the partitions  $\lambda_i$  for all  $i$  using rimhooks:

DEFINITION 2.5. A rimhook of a  $k$ -multipartition  $\lambda = (\lambda_1, \dots, \lambda_k)$  consists of  $k$  adjacent boxes in the Young diagram of a single  $\lambda_i$  such that no other boxes are remaining south or east after the rimhook has been removed and no box in the rimhook has a southeast neighbor.

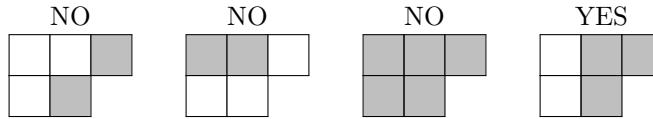


FIGURE 1. Examples of three invalid and one valid rimhooks in  $\lambda = ((3^1 2^1))$ .

DEFINITION 2.6. For  $k$ -multipartitions  $\lambda$  and  $\mu$ , a rimhook decomposition of  $\lambda$  by  $\mu$  is obtained by repeatedly removing rimhooks in  $\lambda$  according to a fixed ordering of the parts of  $\mu$ . Further, we define  $RHD(\lambda, \mu)$  to consist of all rimhook decompositions of  $\lambda$  by  $\mu$ .

As in the Murnaghan–Nakayama rule for the usual symmetric group, formulas for irreducible characters involve the height of a rimhook decomposition:

DEFINITION 2.7. The height of a rimhook is one less than the number of rows included in that rimhook. The height of a rimhook decomposition  $\rho$ , denoted  $ht(\rho)$ , is the sum of the heights of all rimhooks in the decomposition.

The Murnaghan–Nakayama rule can be modified for wreath products as follows:

PROPOSITION 2.8 ([4, Theorem 4.4.10]). Let  $\lambda$  and  $\mu$  be  $k$ -multipartitions of  $N$ . Let  $\chi^1, \chi^2, \dots, \chi^k$  be the irreducible characters of  $G$  and  $c_1, c_2, \dots, c_k$  the conjugacy classes of  $G$ . For  $\rho \in RHD(\lambda, \mu)$ , let  $\psi(\rho)$  be defined by

$$\psi(\rho) = \prod_{i=1}^k \prod_{j=1}^{ht(\rho)} \left( \chi^i(c_j)^{\#\{\text{rimhooks } h \in \rho \text{ from } \mu_j \text{ in } \lambda_i\}} \right).$$

Then

$$\chi_\mu^\lambda = \sum_{\rho \in RHD(\lambda, \mu)} (-1)^{ht(\rho)} \psi(\rho).$$

The permutation module characters of wreath products form another basis for the space of class functions of  $G \wr S_N$  that is easier to work with.

DEFINITION 2.9. Let  $\lambda = (\lambda_1, \dots, \lambda_k)$  be a  $k$ -multipartition of  $N$  and let  $a_i = |\lambda_i|$ . For each  $\lambda_i$ , let  $S_{\lambda_i}$  be the Young subgroup of  $S_{a_i}$  corresponding to  $\lambda_i$  and let  $G_{\lambda_i} = G \wr S_{\lambda_i}$ . Then the permutation module  $M^\lambda$  for  $G \wr S_N$  is defined by

$$M^\lambda = \text{Ind}_{G_{\lambda_1} \times \dots \times G_{\lambda_k}}^{G \wr S_N} \left( \bigotimes_{i=1}^k V_i^{\otimes a_i} \right).$$

There is a character formula for  $M^\lambda$  using row decompositions instead of rimhook decompositions. It is as follows:

DEFINITION 2.10. Let  $\lambda$  and  $\mu$  be  $k$ -multipartitions of  $N$ . A row decomposition of  $\lambda$  by  $\mu$  is a function  $\rho : \{\text{rows of } \mu\} \rightarrow \{\text{rows of } \lambda\}$  such that if  $r$  is a row of  $\lambda$ , then the rows in  $\rho^{-1}(r)$  have the same total length as  $r$ . The set of all row decompositions of  $\lambda$  by  $\mu$  is denoted  $RD(\lambda, \mu)$ .

We will think of row decompositions of  $\lambda$  by  $\mu$  as a tiling of the Young diagrams of  $\lambda$  by rows, where rows of  $\mu$  are placed in a fixed ordering.

$$\left( \begin{array}{c|c|c|c} 2 & 1 & 1 & 1 \\ \hline 4 & & & \end{array} \right), \left( \begin{array}{c|c} 3 & 3 \end{array} \right) \quad \left( \begin{array}{c|c|c|c} 4 & 1 & 1 & 1 \\ \hline 2 & & & \end{array} \right), \left( \begin{array}{c|c} 3 & 3 \end{array} \right)$$

FIGURE 2. All valid row decompositions of  $(\square\square\square\square, \square\square)$  by  $(31, 21)$ . The numbers in the boxes indicate the order in which parts of  $\mu$  are placed into rows of  $\lambda$ , with fixed right-to-left placement.

PROPOSITION 2.11. Let  $\lambda$  and  $\mu$  be  $k$ -multipartitions of  $N$ . Let  $\chi^1, \chi^2, \dots, \chi^k$  be the irreducible characters of  $G$  and  $c_1, \dots, c_k$  the conjugacy classes of  $G$ . For  $\rho \in RD(\lambda, \mu)$ , let  $\alpha(\rho)$  be defined by

$$\alpha(\rho) = \prod_{i=1}^k \prod_{j=1}^k \left( \chi^i(c_j)^{\#\{\text{rows from } \mu_j \text{ placed into } \lambda_i \text{ by } \rho\}} \right).$$

Then the character for permutation module  $M^\lambda$  at  $\mu$  is

$$M_\mu^\lambda = \sum_{\rho \in RD(\lambda, \mu)} \alpha(\rho).$$

The proof follows from the character formula for induced representations.

We now describe the change-of-basis between irreducible and permutation characters.

DEFINITION 2.12. The dominance order on partitions is defined by  $\lambda \succ \eta$  if the row lengths  $\lambda^1 \geq \lambda^2 \geq \dots$  and  $\eta^1 \geq \eta^2 \geq \dots$  of  $\lambda$  and  $\eta$  satisfy  $\sum_{i=1}^j \lambda^i \geq \sum_{i=1}^j \eta^i$  for all  $j \geq 1$ . The dominance order on  $k$ -multipartitions is defined by  $\lambda \succ \eta$  if and only if  $\lambda_i \succ \eta_i$  for all  $i$ .

LEMMA 2.13. The matrix of multiplicities  $[M^\lambda : V^\eta]$  of the irreducible representations of  $G \wr S_N$  in permutation modules is unimodular and lower-triangular with respect to dominance order.

*Proof.* Recall that the Kostka numbers  $K^{\beta, \gamma}$  for  $\beta, \gamma$  partitions of  $N$  are defined by

$$M^\beta = \text{Ind}_{S_\beta}^{S_N} 1 = \bigoplus_{\gamma} (V^\gamma)^{\oplus K^{\gamma, \beta}},$$

where  $S_\beta$  is the Young subgroup corresponding to  $\beta$  and  $V^\gamma$  is the Specht module corresponding to  $\gamma$ . Note that our notation for  $M^\beta$  and  $V^\gamma$  agrees with that of wreath products  $G \wr S_N$  when  $G = 1$ . The Kostka numbers satisfy  $K^{\beta, \beta} = 1$  and  $K^{\gamma, \beta} > 0$  if and only if  $\gamma \succ \beta$  in dominance order [5, I, (6.5)].

We claim that

$$(1) \quad M^\lambda = \bigoplus_{\eta} (V^\eta)^{\oplus c(\lambda, \eta)}, \quad c(\lambda, \eta) = \left( \prod_{i=1}^k K^{\eta_i, \lambda_i} \right).$$

By Definition 2.9, if  $a_i = |\lambda_i|$  for all  $i$  and  $H = G_{a_1} \times G_{a_2} \times \dots \times G_{a_k}$ , then

$$M^\lambda = \text{Ind}_{G_\lambda}^{G_N} (\boxtimes_{i=1}^k V_i^{\otimes a_i}) = \text{Ind}_H^{G_N} (\boxtimes_{i=1}^k M^{\lambda_i} \otimes V_i^{\otimes a_i}),$$

where  $S_{a_i}$  acts diagonally on the tensor product  $M^{\lambda_i} \otimes V_i^{\otimes a_i}$  and  $G^{a_i}$  acts naturally on  $V_i^{\otimes a_i}$ . Then (1) follows from multilinearity of the tensor product and linearity of induction.

Now since the matrix of Kostka numbers is unimodular and upper-triangular with respect to dominance order, the matrix  $\{c(\lambda, \mu)\}_{\lambda, \mu}$  is unimodular and lower-triangular with respect to dominance order.  $\square$

**2.2. ASYMPTOTICS OF PARTITIONS.** We recall a form of the Hardy–Ramanujan asymptotic for the number of partitions of  $N$ , denoted  $p(N)$ .

**PROPOSITION 2.14** ([3, (1.36)]). *If  $\delta > 0$ , then*

$$\left(\frac{2\pi}{\sqrt{6}} - \delta\right) \sqrt{N} \leq \log p(N) \leq \left(\frac{2\pi}{\sqrt{6}} + \delta\right) \sqrt{N}$$

for sufficiently large  $N$ .

Let  $p_k(N)$  denote the number of  $k$ -multipartitions of  $N$ .

**CLAIM 2.15.** *If  $\delta > 0$ , then*

$$\left(\frac{2\pi}{\sqrt{6}} - \delta\right) \sqrt{kN} \leq \log p_k(N) \leq \left(\frac{2\pi}{\sqrt{6}} + \delta\right) \sqrt{kN}$$

for sufficiently large  $N$ .

This formula also appears in [9]. We provide an elementary inductive proof.

*Proof.* We proceed by induction on  $k$ . The base case  $k = 1$  is Proposition 2.14.

For  $\delta > 0$ , let  $\delta' = \frac{4}{5}\delta$ . By inductive hypothesis, there exists a constant  $B$  such that if  $C \geq B$ , then

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right) \left(\sqrt{(k-1)C}\right)\right) \leq p_{k-1}(C) \leq \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right) \left(\sqrt{(k-1)C}\right)\right)$$

and

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right) \left(\sqrt{C}\right)\right) \leq p(C) \leq \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right) \left(\sqrt{C}\right)\right).$$

By considering the size of the first partition in a  $k$ -multipartition, it follows that

$$p_k(N) = \sum_{a=0}^N p(a)p_{k-1}(N-a).$$

Now assume that  $N \geq 2B - 2$ ; we can then break up the sum for  $p_k(N)$  into the following distinct parts: let

$$\begin{aligned} D_1 &= \sum_{a=0}^{B-1} p(a)p_{k-1}(N-a), \\ D_2 &= \sum_{a=B}^{N-B} p(a)p_{k-1}(N-a), \\ D_3 &= \sum_{a=N-B+1}^N p(a)p_{k-1}(N-a). \end{aligned}$$

In  $D_2$ , for  $B \leq a \leq N - B$ , we have

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right) \left(\sqrt{a} + \sqrt{(k-1)(N-a)}\right)\right) \leq p(a)p_{k-1}(N-a),$$

and the right hand side in turn satisfies

$$p(a)p_{k-1}(N-a) \leq \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\left(\sqrt{a} + \sqrt{(k-1)(N-a)}\right)\right).$$

Note that  $\sqrt{a} + \sqrt{(k-1)(N-a)} \leq \sqrt{kN}$ , with equality achieved at  $a = \frac{N}{k}$ . Summing over  $a \in [B, N-B]$ , we get

$$(2) \quad \exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta\right)\sqrt{kN}\right) \leq D_2 \leq (N-2B+1) \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right).$$

We now consider  $D_1$  and  $D_3$ . Note that for  $a \in [0, B]$ , we have  $p(a)p_{k-1}(N-a) \leq p(B)p_{k-1}(N)$ , and for  $a \in (N-B, N]$ , we have  $p(a)p_{k-1}(N-a) \leq p(N)p_{k-1}(B)$ . Hence

$$(3) \quad 0 \leq D_1 \leq Bp(B)p_{k-1}(N) \leq B \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right)$$

for sufficiently large  $N$ . Likewise,

$$(4) \quad 0 \leq D_3 \leq Bp(N)p_{k-1}(B) \leq B \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right).$$

Combining (2), (3), and (4), we have that

$$\begin{aligned} \exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta\right)\sqrt{kN}\right) &\leq p_k(N) \leq (N+1) \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right) \\ &\leq \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta\right)\sqrt{kN}\right) \end{aligned}$$

for sufficiently large  $N$ .  $\square$

The above estimate implies  $k$ -multipartitions concentrate around having close to equal-size parts:

**COROLLARY 2.16.** *For all  $\delta > 0$ , the proportion of  $k$ -multipartitions  $\lambda = (\lambda_1, \dots, \lambda_k) \vdash N$  such that*

$$\frac{N}{k}(1-\delta) < |\lambda_i| < \frac{N}{k}(1+\delta)$$

*for all  $1 \leq i \leq k$  goes to 1 as  $N \rightarrow \infty$ .*

*Proof.* Pick  $1 \leq i \leq k$ . By partitioning  $k$ -multipartitions according to the size of  $\lambda_i$ , the number of  $k$ -multipartitions of  $N$  where  $|\lambda_i| \notin (\frac{N}{k}(1-\epsilon), \frac{N}{k}(1+\epsilon))$  is

$$(5) \quad \sum_{a \notin (\frac{N}{k}(1-\epsilon), \frac{N}{k}(1+\epsilon))} p(a)p_{k-1}(N-a).$$

Note that  $\sqrt{a} + \sqrt{(k-1)(N-a)}$  increases for  $a < \frac{N}{k}$  and decreases for  $a > \frac{N}{k}$ . Then by Claim 2.15, the rate of growth (5) is significantly slower than the rate of growth of  $p_k(N)$ .  $\square$

### 3. MAIN RESULTS

**3.1. CHARACTER TABLE COLUMN CONGRUENCES.** Corollary 3.3 below, which we call “the mashing rule,” gives a criterion for mod  $p$  congruence of two columns of the character table of  $G \wr S_N$  in terms of  $k$ -multipartitions.

In this section, we must assume that  $G$  has integer-valued character table. By [10, §13.1], the group  $G$  has integer-valued character table if and only if  $\sigma \in G$  is conjugate to  $\sigma^j$  whenever  $j$  is prime to the order of  $\sigma$ .

DEFINITION 3.1. Let  $\sim_p$  be the equivalence relation on  $k$ -multipartitions generated by the following:  $\mu \sim_p \nu$  if there is  $j$  such that  $\mu_i = \nu_i$  for  $i \neq j$ , and  $\nu_j$  is formed by replacing one part of size  $mp$  in  $\mu_j$  with  $p$  parts of size  $m$  in  $\nu_j$ .

$$\left( \begin{array}{c|ccccc} \text{blue} & & & & & \\ \hline \text{white} & & & & & \end{array}, \begin{array}{c|ccccc} \text{white} & \text{white} & \text{white} & \text{white} & \text{white} \\ \hline \text{purple} & & & & \end{array} \right) \sim_3 \left( \begin{array}{c|ccccc} \text{blue} & \text{blue} & \text{blue} & \text{white} & \text{white} \\ \hline \text{white} & & & & \end{array}, \begin{array}{c|ccccc} \text{white} & \text{white} & \text{white} & \text{white} & \text{white} \\ \hline \text{purple} & & & & \end{array} \right) \sim_3 \left( \begin{array}{c|ccccc} \text{blue} & \text{blue} & \text{blue} & \text{white} & \text{white} \\ \hline \text{white} & & & & \end{array}, \begin{array}{c|ccccc} \text{white} & \text{white} & \text{white} & \text{purple} & \text{purple} \\ \hline \text{purple} & & & & \end{array} \right)$$

$$(61, 41^3) \sim_3 (2^3 1, 41^3) \sim_3 (2^3 1, 43)$$

FIGURE 3. Example of three conjugacy classes which are congruent mod 3 in  $\mathbb{Z}/2\mathbb{Z} \wr S_N$  (note that  $m = 2$  in the first cycle type and  $m = 1$  in the second).

LEMMA 3.2. Let  $p$  be a prime and  $G$  be a group with integer-valued character table. Let  $\mu = (\mu_1, \dots, \mu_k)$  and  $\nu = (\nu_1, \dots, \nu_k)$  be  $k$ -multipartitions of  $N$ , indexing conjugacy classes of  $G \wr S_N$ . If  $\mu \sim_p \nu$ , then  $M_\mu^\lambda \equiv M_\nu^\lambda \pmod{p}$  for all  $k$ -multipartitions  $\lambda$  of  $N$ .

*Proof.* It suffices to show  $M_\mu^\lambda \equiv M_\nu^\lambda \pmod{p}$  if there exists  $j$  such that  $\mu_i = \nu_i$  for all  $i \neq j$ ,  $\mu_j = (\xi, mp)$  for some  $\xi$ , and  $\nu_j = (\xi, m^p)$ . We break  $RD(\lambda, \nu)$  into two cases. In case one, we consider the row decompositions of  $\nu$  where the  $p$  rows not in  $\xi$  are tiled into the same row of  $\lambda$ . In case two we consider the row decompositions where the  $p$  rows not in  $\xi$  are not tiled in the same row. Recalling our formula for characters of permutation modules in Proposition 2.11, let

$$(6) \quad \beta = \sum_{\rho \in RD(\lambda, \nu) \text{ s.t. } m^p \text{ is tiled in the same row}} \alpha(\rho)$$

and

$$(7) \quad \gamma = \sum_{\rho \in RD(\lambda, \nu) \text{ s.t. } m^p \text{ is not tiled in the same row}} \alpha(\rho),$$

so that  $M_\nu^\lambda = \beta + \gamma$ . In case one, we will show  $\beta \equiv M_\mu^\lambda \pmod{p}$  and in case two that  $\gamma \equiv 0 \pmod{p}$ . Together, these two congruences imply  $M_\mu^\lambda \equiv M_\nu^\lambda \pmod{p}$ .

In both cases, we break into subcases based on the ways to tile  $\mu_i$  for  $i \neq j$  and  $\xi$ . In case one, we have compatible tilings for  $\mu$  and  $\nu$ , and in case two, we have additional tilings for  $\nu$ .

In case one, assume we have tiled all rows of  $\mu_i$  for all  $i \neq j$  and we have tiled  $\xi$ . We now have one row remaining. There is only one way to tile the last row for both  $\mu$  and  $\nu$ : put the remaining pieces into the remaining row. Let these row decompositions be denoted  $\rho_\mu$  and  $\rho_\nu$  respectively.

For  $\rho_\mu$ , say that we place the final row  $r$  of size  $mp$  in the partition  $\lambda_q$ . The associated cycle product is  $c_j$  because  $mp$  comes from  $\mu_j$ . Then  $mp$  contributes  $\chi_q(c_j)$  to the product  $\alpha(\rho_\mu)$ . Then for  $\rho_\nu$ , the  $p$  rows of size  $m$  are placed into  $\lambda_q$ . The conjugacy class of  $G$  associated with the  $p$  rows of size  $m$  is again  $c_j$ , so  $m^p$  contributes a factor of  $\chi_q(c_j)^p$  to  $\alpha(\rho_\nu)$ .

By assumption, the character values of  $G$  are integral, so by Fermat's little theorem,  $\chi_q(c_j) \equiv \chi_q(c_j)^p \pmod{p}$ . All other factors in  $\alpha(\rho_\mu)$  contributed by  $\mu_i$  for  $i \neq j$  and  $\xi$  are identical to the corresponding factors in  $\alpha(\rho_\nu)$ . Hence,  $\alpha(\rho_\mu) \equiv \alpha(\rho_\nu) \pmod{p}$ .

Summing over all the tilings in case one, we find  $M_\mu^\lambda \equiv \beta \pmod{p}$ .

In case two, assume we have tiled all rows of  $\mu_i$  for  $i \neq j$  and  $\xi$ , after which there are  $t > 1$  remaining unfilled rows of the Young diagrams of  $\lambda$ . If  $T \subseteq RD(\lambda, \nu)$  is the set of row decompositions extending our given tiling by  $\mu_i$  for  $i \neq j$  and  $\xi$ , then we will show

$$\sum_{\rho \in T} \alpha(\rho) \equiv 0 \pmod{p}.$$

Then  $\gamma$  is the sum over all such  $T$  of  $\sum_{\rho \in T} \alpha(\rho)$ , from which it will follow  $\gamma \equiv 0 \pmod{p}$ .

Call the lengths of the remaining rows  $(m\ell_1, m\ell_2, \dots, m\ell_t)$ . Since the elements of  $T$  are in bijection with choices of placements of  $p$  cycles of length  $m$  into these rows,

$$(8) \quad |T| = \binom{p}{\ell_1, \ell_2, \dots, \ell_t}.$$

Let  $\rho \in T$ . Note that all pieces of  $m^p$  come from  $\mu_j$ , and thus have cycle product  $c_j$ , while all other cycles in  $\mu$  are in the same place in  $T$ . Thus  $\alpha(\rho) = \alpha(\rho')$  for all  $\rho, \rho' \in T$ . Hence  $\sum_{\rho \in T} \alpha(\rho)$  is a sum of  $|T|$  identical terms. Then  $\sum_{\rho \in T} \alpha(\rho) \equiv 0 \pmod{p}$  because  $|T|$  is divisible by  $p$ .

Case one has shown that  $M_\mu^\lambda \equiv \beta \pmod{p}$ , and case two has shown that  $\gamma \equiv 0 \pmod{p}$ . Since  $M_\nu^\lambda = \beta + \gamma$ , we conclude  $M_\nu^\lambda \equiv M_\mu^\lambda \pmod{p}$ .  $\square$

**COROLLARY 3.3** (The mashing rule). *Let  $G$  have integer-valued character table and  $k$  conjugacy classes. Let  $\mu$  and  $\nu$  be  $k$ -multipartitions of  $N$ . If  $\mu \sim_p \nu$ , then  $\chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p}$  for all irreducible characters  $\chi^\lambda$  of  $G \wr S_N$ .*

*Proof.* The set of irreducible characters and the set of characters of permutation modules form bases for the space of class functions on  $G \wr S_N$ . Since the change of basis matrix between these two bases is unimodular and lower-triangular, as stated in Lemma 2.13,  $\chi^\lambda$  can be expressed as an integral linear combination of  $M^\eta$  for all  $k$ -multipartitions  $\lambda$ . It follows from Lemma 3.2 that  $\mu \sim_p \nu$  implies  $\chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p}$ .  $\square$

**REMARK 3.4.** Corollary 3.3 may also be proved using the following criterion from modular representation theory: if  $G$  has integer-valued character table,  $g \in G$ , and  $g'$  is the  $p$ -prime part of  $g$ , then  $\chi(g) \equiv \chi(g') \pmod{p}$  for all characters  $\chi$  of  $G$  [10, 18.1(v)]. In this situation, it may be seen that the mashing preserves the conjugacy class of the  $p$ -prime part of a class in  $G \wr S_N$ . Nonetheless, we have chosen to give a combinatorial proof.

**3.2. PROOF OF MAIN THEOREM.** Using Corollary 3.3, the existence of one zero in the character table implies many more entries are divisible by  $p$ . We proceed, following Peluse and Soundararajan in [8], by using Proposition 2.8 to show sufficiently many entries of the character table are zero.

**DEFINITION 3.5.** *A partition is called a  $t$ -core if none of the hook lengths of its Young diagram are divisible by  $t$  where  $t \in \mathbb{Z}$ . For example, from Figure 4 one can see that  $(4, 2, 1)$  is a 5-core.*

6	4	2	1
3	1		
1			

FIGURE 4. Hook-lengths for  $\lambda_i = (4, 2, 1)$

In the course of proving [8, Proposition 1], Peluse and Soundararajan proved the following estimate of the number of  $t$ -cores when  $t$  is slightly larger than the typical longest cycle in a random conjugacy class:

PROPOSITION 3.6. *Let  $L$  be a positive integer, and let  $A$  be a real number with  $1 \leq A \leq \log L / \log \log L$ . Additionally suppose that  $t$  is a positive integer with*

$$(9) \quad t \geq \frac{\sqrt{6}}{2\pi} \sqrt{L} (\log L) \left( 1 + \frac{1}{A} \right).$$

*Then the number of partitions  $\lambda$  of  $L$  which are not  $t$ -cores is at most*

$$O \left( p(L) \frac{\log L}{L^{\frac{1}{2A}}} \right),$$

*independent of  $t$  satisfying (9).*

Complementing the estimate in Proposition 3.6, Peluse and Soundararajan also estimated how many columns of the character table are congruent to a column corresponding to a partition with a large first part:

PROPOSITION 3.7 ([8, Proposition 2]). *Let  $p \leq \frac{(\log L)}{(\log \log L)^2}$  be a prime. Starting with a partition  $\mu$  of  $L$ , we repeatedly replace every occurrence of  $p$  parts of the same size  $m$  by one part of size  $mp$  until we arrive at a partition  $\tilde{\mu}$  where no part appears more than  $p-1$  times. Then the largest part of  $\tilde{\mu}$  exceeds*

$$\frac{\sqrt{6}}{2\pi} \sqrt{L} (\log L) \left( 1 + \frac{1}{5p} \right),$$

*except for at most*

$$O \left( p(L) \exp \left( -L^{\frac{1}{15p}} \right) \right)$$

*partitions  $\mu$ .*

We now extend Peluse and Soundararajan's estimate in Proposition 3.7 to  $k$ -multipartitions.

PROPOSITION 3.8. *Let  $p \ll N$  be a prime. Given a  $k$ -multipartition  $\mu = (\mu_1, \dots, \mu_k)$  of  $N$ , for all  $\mu_i$  with  $1 \leq i \leq k$ , we repeatedly replace every occurrence of  $p$  parts of the same size  $m$  by one part of size  $mp$  until we arrive at a  $k$ -multipartition  $\tilde{\mu}$  where no part in any  $\tilde{\mu}_i$  appears more than  $p-1$  times.*

*Then the largest part of  $\tilde{\mu}$  is of size at least*

$$(10) \quad \frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left( \log \frac{N}{k} \right) \left( 1 + \frac{1}{5p} \right)$$

*except for a number of multipartitions  $\mu$  which is at most*

$$O \left( \exp \left( - \left( \frac{N}{k} \right)^{\frac{1}{15p}} \right) p_k(N) \right).$$

*Proof.* For a  $k$ -multipartition  $\mu = (\mu_1, \mu_2, \dots, \mu_k)$  of  $N$ , let  $\tilde{\mu}$  be as above. We will bound above the number of  $k$ -multipartitions  $\mu$  such that  $\tilde{\mu}$  has largest part less than (10).

For any  $\mu$ , we know that for some  $1 \leq i \leq k$ ,  $|\mu_i| \geq \frac{N}{k}$ . Fix  $i$  such that  $\mu_i$  has size  $|\mu_i| = a \geq \frac{N}{k}$ . Then Proposition 3.7 tells us that the largest part of  $\tilde{\mu}_i$  exceeds

$$\frac{\sqrt{6}}{2\pi} \sqrt{a} (\log a) \left( 1 + \frac{1}{5p} \right) \geq \frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left( \log \frac{N}{k} \right) \left( 1 + \frac{1}{5p} \right)$$

except for at most

$$O\left(p(a) \exp\left(-a^{\frac{1}{15p}}\right)\right)$$

partitions  $\mu_i$  of size  $a$  and therefore at most

$$O\left(p(a) \exp\left(-a^{\frac{1}{15p}}\right) p_{k-1}(N-a)\right)$$

total  $k$ -multipartitions  $\mu$  with  $|\mu_i| = a$ . Furthermore, since  $a \geq \frac{N}{k}$ ,

$$\exp\left(-a^{\frac{1}{15p}}\right) \leq \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right),$$

and therefore summing over all  $a \geq \frac{N}{k}$  we have that the number of multipartitions  $\mu$  such that  $|\mu_i| \geq \frac{N}{k}$  with no part in  $\tilde{\mu}_i$  exceeding (10) is at most an absolute constant times

$$\begin{aligned} \sum_{a=\frac{N}{k}}^N \exp\left(-a^{\frac{1}{15p}}\right) p(a) p_{k-1}(N-a) &\ll \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) \sum_{a=\frac{N}{k}}^N p(a) p_{k-1}(N-a) \\ &\ll \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) \sum_{a=0}^N p(a) p_{k-1}(N-a) \\ &= \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) p_k(N). \end{aligned}$$

Since this bound is identical for each  $i$ , the number of  $k$ -multipartitions  $\mu$  such that  $\tilde{\mu}$  does not have a part of size greater than (10) is at most a factor of  $k$  greater than the bound above, and therefore also at most

$$O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) p_k(N)\right). \quad \square$$

**THEOREM 3.9.** *Let  $G$  be a group with integer-valued character table, and let  $G \wr S_N$  be the wreath product of  $G$  with the symmetric group  $S_N$ . For all primes  $p$ , the proportion of entries in the character table of  $G \wr S_N$  divisible by  $p$  tends to 1 as  $N \rightarrow \infty$ .*

*Proof.* Let  $k$  be the number of conjugacy classes of  $G$ . Given a  $k$ -multipartition  $\mu$ , let  $\tilde{\mu}$  be the multipartition obtained by repeatedly replacing  $p$  parts of  $\mu_i$  of size  $m$  with one part of size  $mp$  until no  $\mu_i$  has a part appearing more than  $p-1$  times. For  $A = 5p$ , Proposition 3.8 implies that the largest part of  $\tilde{\mu}$  has size

$$(11) \quad t \geq \frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left(\log \frac{N}{k}\right) \left(1 + \frac{1}{A}\right)$$

for a proportion of  $\mu$  tending to 1 as  $N \rightarrow \infty$ . Now pick  $A' \geq 1$  and  $\delta > 0$  such that

$$\left(\log \frac{N}{k}\right) \left(1 + \frac{1}{A}\right) \geq \sqrt{1+\delta} \left(\log \left(\frac{N}{k}(1+\delta)\right)\right) \left(1 + \frac{1}{A'}\right).$$

By Corollary 2.16, the proportion of  $k$ -multipartitions  $\lambda = (\lambda_1, \dots, \lambda_k) \vdash N$  such that  $|\lambda_i| \in (\frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta))$  for all  $i$  tends to 1 as  $N \rightarrow \infty$ . Thus, consider only  $(\lambda, \mu)$  satisfying the above conditions.

Our choice of  $\delta$  and  $A'$  imply that

$$\frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left(\log \frac{N}{k}\right) \left(1 + \frac{1}{A}\right) \geq \frac{\sqrt{6}}{2\pi} \sqrt{\frac{N(1+\delta)}{k}} \left(\log \left(\frac{N}{k}(1+\delta)\right)\right) \left(1 + \frac{1}{A'}\right).$$

If  $(N_1, \dots, N_k)$  is a partition of sufficiently large  $N$  with  $N_i \in (\frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta))$ , we can assume  $A' \leq \frac{\log N_i}{\log \log N_i}$  for each  $i$ , as  $N_i \geq \frac{N}{k(1-\delta)} \rightarrow \infty$ . Then by Proposition 3.6, the proportion of  $k$ -multipartitions  $\lambda$  with  $|\lambda_i| = N_i$  such that some  $\lambda_i$  is not a  $t$ -core is

$$\sum_{i=1}^k O\left(\frac{\log N_i}{N_i^{\frac{1}{2A'}}}\right)$$

for all  $t$  satisfying (11), independent of  $t$ . Hence, over all  $k$ -multipartitions  $\lambda$  satisfying  $|\lambda_i| \in (\frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta))$ , the proportion of  $\lambda$  such that some  $\lambda_i$  is not a  $t$ -core is

$$O\left(\frac{\log\left(\frac{N}{k}(1+\delta)\right)}{\left(\frac{N}{k}(1-\delta)\right)^{\frac{1}{2A'}}}\right).$$

It follows that most  $(\lambda, \mu)$  satisfy that  $\lambda_i$  is a  $t$ -core for  $t$  the largest part of  $\tilde{\mu}$ . Thus, for a proportion of  $(\lambda, \mu)$  tending to 1 as  $N \rightarrow \infty$ , we have  $\chi_{\mu}^{\lambda} = 0$  by Proposition 2.8 and therefore  $\chi_{\mu}^{\lambda} \equiv 0 \pmod{p}$  by Corollary 3.3.  $\square$

#### 4. WEYL GROUPS OF TYPE D

**DEFINITION 4.1.** *The Weyl group of type  $D_N$  is the group of  $N \times N$  signed permutation matrices with an even number of entries equal to  $-1$ .*

We will denote this group by  $D_N$  also (note that it is distinct from the dihedral group). The study of representations of  $D_N$  was taken up by Young in [12, §7-10]; see [2] for a more modern treatment.

Let  $B_N = \{\pm 1\} \wr S_N$ ; then  $D_N$  is a subgroup of  $B_N$  of index two. Recall that conjugacy classes of  $B_N$  are labelled by 2-multipartitions  $(\eta, \nu)$  of  $N$ , where we take  $\nu$  to be those cycles with nontrivial cycle product.

**PROPOSITION 4.2** ([2, Proposition 3.4.12]). *The conjugacy classes  $B_N$  which meet  $D_N$  correspond to 2-multipartitions  $(\eta, \nu)$  of  $N$  where  $\nu$  has an even number of parts. The conjugacy classes of  $B_N$  which split in  $D_N$  are exactly those  $(\eta, \nu)$  where  $\nu = \emptyset$  and  $\eta$  has only even parts.*

Since  $D_N$  is a subgroup of  $B_N$  of index two, Clifford theory determines its representations:

**PROPOSITION 4.3** ([2, §5.6.1]). *The irreducible representations of the Weyl group of type  $D_N$  are as follows:*

(1) *if  $(\lambda, \mu)$  is a 2-multipartition of  $N$  such that  $\lambda \neq \mu$ , then*

$$\text{Res}_{D_N}^{B_N} V^{\lambda, \mu} = \text{Res}_{D_N}^{B_N} V^{\mu, \lambda}$$

*is an irreducible representation of  $D_N$ ;*

(2) *if  $(\lambda, \lambda)$  is a 2-multipartition of  $N$  with equal parts, then*

$$\text{Res}_{D_N}^{B_N} V^{\lambda, \lambda}$$

*is the sum of two irreducible representations of  $D_N$ .*

(3) *Each irreducible representation of  $D_N$  appears exactly once in (1) or (2).*

**COROLLARY 4.4.** *For all primes  $p$ , the proportion of entries in the character table of  $D_N$  which are divisible by  $p$  tends to 1 as  $N \rightarrow \infty$ .*

*Proof.* The number of irreducible representations of  $D_N$  of the form  $\text{Res}_{D_N}^{B_N} V^{\lambda, \mu}$  for  $\lambda \neq \mu$  equals  $\frac{1}{2}(p_2(N) - p(N/2))$  when  $N$  is even, and  $\frac{1}{2}p_2(N)$  when  $N$  is odd. The number of irreducible representations appearing as a summand of  $\text{Res}_{D_N}^{B_N} V^{\lambda, \lambda}$

is  $2p(N/2)$  when  $N$  is even and 0 when  $N$  is odd. By Claim 2.15, we have  $p_2(N) \gg p(N/2)$  for large enough  $N$ , so the proportion of irreducibles of the form  $\text{Res}_{D_N}^{B_N} V^{\lambda, \mu}$  goes to 1 as  $N \rightarrow \infty$ .

We must also analyze the splitting of conjugacy classes from  $B_N$  in  $D_N$ . By Proposition 4.2, the number of conjugacy classes of  $B_N$  which split in  $D_N$  is  $p(N/2)$  when  $N$  is even and 0 when  $N$  is odd. We conclude just as above that the proportion of non-split conjugacy classes tends to 1 as  $N \rightarrow \infty$ .

We have shown that almost every entry of the character table of  $D_N$  is of the form  $V_{\eta, \nu}^{\lambda, \mu}$  where  $\lambda \neq \mu$  and  $(\eta, \nu)$  is a non-split conjugacy class in  $D_N$ . Such entries occupy at least a constant fraction of the character table of  $B_N$  as  $N \rightarrow \infty$ . By Theorem 3.9, almost all of such entries are divisible by  $p$  as  $N \rightarrow \infty$ . Hence, the proportion of entries of the character table of  $D_N$  which are divisible by  $p$  goes to 1 as  $N \rightarrow \infty$ .  $\square$

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