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Students' Thinking about the Structure of Constructive Existence Proofs

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Abstract
Undergraduate students are expected to produce and comprehend constructive existence proofs; yet, these proofs are notoriously difficult for students. This study investigates students' thinking about these proofs by asking students to validate two arguments for the existence of a mathematical object. The first argument featured a common structural error while the second was a valid argument of the claim. We found that the students often considered the logical structures of the arguments when validating them. They provided reasons for their evaluations, including why they thought the structure of the first argument functioned to prove the claim and why they thought the structure of the second argument did not function to prove the claim. We discuss how these reasons provide insights into why constructive existence proofs might be challenging for students. We end the paper with implications for the teaching and learning of constructive existence proofs and their proof frameworks.

Keywords Constructive existence proofs · Existence claims · Proof framework · Explanatory proof · Proof validation · Proof

Introduction

Proofs and proof-activity are central components of mathematics students' undergraduate courses and as such, have received wide attention from mathematics education researchers. Scholars often differentiate students' activity when working on the formal-rhetorical part of a proof from the problem-centered part (e.g.,

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Fukawa-Connelly, 2012; McKee et al., 2010; Selden & Selden, 2010, 2013). Creating the formal-rhetorical part includes unpacking and using the logical structure of the associated claim to create or make sense of the logical structure of an argument, the result of this process is a proof framework (see Fig. 1 for an example of a proof with its framework in bold). Selden and Selden (2013) explain that while proof frameworks can be difficult for students with little experience with proof, they generally do not require a deep understanding of the concepts involved. Yet, that is not to say that important mathematical work is absent from creating the formal-rhetorical part of proof. We see proof frameworks as one of the many conventional systems that the larger mathematics community uses to communicate mathematical meanings, thus providing a rich context for discussions about linguistic choices that are used to discern these meanings (Kontorovich & Zazkis, 2017; Larsen et al., 2022; Vroom, 2022). This study investigates students' thinking about the structure of proofs, specifically for proofs that argue the existence of a mathematical object.

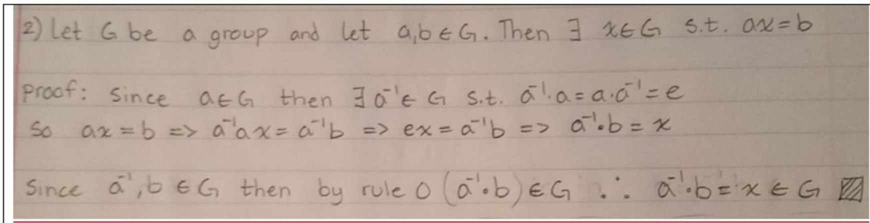
Undergraduate students are expected to produce and comprehend existence proofs, or proofs that argue the existence of a mathematical object. Claiming the existence of a mathematical object is prevalent in undergraduates' proof-activity as it is common in both theorems (e.g., Mean Value Theorem, Cauchy's Theorem in group theory) and defining properties (e.g., $\varepsilon - \delta$ definition of continuity, the identity element of a group). Regardless of its prevalence, scholars have recognized that existence proofs are notoriously difficult for undergraduate students, especially since their prior experiences in secondary mathematics rarely necessitate arguing the existence of an object (De Guzmán et al., 1998).

In our classroom experiences, as well as discussions with other experienced instructors of proof-based courses, we have noticed that when attempting to prove the existence of a mathematical object, students often produce an argument with a flawed framework. Their argument starts with a desired property and then continues solving for the object. For instance, in Fig. 2 a student produced such an argument to show that given a group G and two elements$a, b \in G$, there exists $x \in G$ such that $ax = b$. The argument deduced from the property $ax = b$ that $x = a^{-1} \cdot b$. Rather than proving the existence of the desired mathematical object, the argument shows that *if* $ax = b$, *then* $x = a^{-1} \cdot b$. Given the perceived prevalence of this framework error and our view that students can and will engage in meaning making, we wondered what mathematical thinking students engaged in as they produced or endorsed an argument of this nature.

In this study, we explored students' thinking about an argument for the existence of a mathematical object that featured an error in the framework, as well as students' thinking about a valid argument of the same existence claim. In doing

Fig. 1 Example of valid existence proof with framework in bold

Let G be a group and let $a, b \in G$. Then there exists an x in G such that $a * x = b$.
Proof:
Let G be a group with operation $*$.
Let a and b be in G.
Choose $\boldsymbol{x}$ $= a^{-1} * b$.
Then, x is in G since a^{-1} is in G, b is in G, and G is closed.
So, $\boldsymbol{a * x}$ $= a * (a^{-1} * b) = (a * a^{-1}) * b = e * b =$ **b.**
Therefore, there exists $\boldsymbol{x \in G}$ such that $\boldsymbol{a * x = b}$.



Let G be a group and let $a, b \in G$. Then $\exists\ x \in G$ s.t. $ax = b$.

proof: Since $a \in G$ then $\exists\ a^{-1} \in G$ s.t. $a^{-1} \cdot a = a \cdot a^{-1} = e$
So $ax = b \Rightarrow a^{-1}ax = a^{-1}b \Rightarrow ex = a^{-1}b \Rightarrow a^{-1} \cdot b = x$

Since $a^{-1}, b \in G$ then by rule 0, $(a^{-1} \cdot b) \in G \therefore a^{-1} \cdot b = x \in G$

Fig. 2 Example of student argument with logical error with typed proof below

so, we gain a better understanding of the mathematical work that students engage in when working on the formal-rhetorical part of a proof.

Theoretical Grounding

In what follows we first describe what we mean by proof, proof-activity, and constructive existence proofs. Then, we situate our work in relevant research literature.

Proof and Proof Activity

We draw on Stylianides' (2007) characterization of proof in school mathematics:

"*Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community" (p. 291).

We emphasize two points about this definition. First, what constitutes a proof is dependent on the audience's accepted statements, modes of argumentation, and representation. For example, an argument that is viewed as a proof by a mathematician may be rejected by a student because it uses a form of reasoning unfamiliar to the student. Throughout the paper, we specify when we consider the students' evaluation of an argument. When we classify an argument as valid (or invalid), we are referring to our view of whether the larger mathematics community would accept (or reject) the mathematical argument as a proof of a given statement based on formal logic.

Second, Stylianides' (2007) definition indirectly suggests that an argument that meets the three characteristics establishes the truth of the statement. To make this explicit, we further clarify that the associated mathematical claim plays a role in the second characteristic: when evaluating whether a mathematical argument is a proof, one should consider whether the argument uses acceptable modes of argumentation that are for (or against) a known mathematical claim. We see proof frameworks as a way to determine this. For instance, in the student argument above (Fig. 2), the students' proof framework assumes $ax = b$ to deduce $x = a^{-1} \cdot b$, which is inconsistent with the associated claim and thus does not establish its truth. While we acknowledge that a given argument could prove a different claim than what is given, when we say that an argument was (or was not) viewed as a proof, we mean that the argument was (or was not) viewed as a proof of the given claim.

Scholars have distinguished between proofs that only convince and proofs that simultaneously convince and explain (i.e., *explanatory proofs*) (e.g., Bartlo, 2013; Lockwood et al., 2020; Weber, 2010). In this study, we leverage two aspects of explanatory proofs that are consistent with Lockwood et al. (2020) characterization. First, a proof that is explanatory to a mathematician may not give the same insight to a student, and thus, explanatory proofs can be audience-dependent. Because of this, a prover who wanted to provide insight into why a statement is true would consider their audience. Second, the explanatory nature of a proof is closely tied to the activity of constructing the proof; a reader often finds a proof to be explanatory if it gives them insight into the informal reasoning that was used to create it. Thus, a prover who wanted to produce an explanatory proof for a particular reader might include an appropriate amount of their problem-solving that led them to determine why the statement was true.

The research literature on students' proof-activity has historically focused on construction, comprehension, or validation. *Constructing* a proof is the mental and physical actions of producing a proof, *comprehending* a proof is making meaning of the given text as one reads a proof, and *validating* a proof is checking whether a mathematical argument proves a given mathematical claim. As with many scholars (e.g., Melhuish et al., 2021; Selden & Selden, 2003, 2017), we see these activities as inherently connected. We see comprehension and validation as mutually dependent activities; one cannot validate a proof without comprehending it nor can one comprehend a proof without understanding its validity. Selden and Selden (2003, 2017) explain that students' proof construction and validation are necessarily interrelated: "…one constructs a proof with an eye toward ultimately validating it and may often validate parts of it during the construction process. In fact, the final part of a proof construction is likely to be a validation of that proof" (Selden & Selden, 2003, p. 6). Because of this, we argue that we can gain insight into why students construct an argument like the one in Fig. 2 by investigating their validation of such an argument.

One common aspect among these proof activities that is particularly relevant to this study is attention to the proof framework. While comprehending proofs, students can coordinate the statement being proven, the proof technique, and the necessary assumptions and conclusions (Mejía-Ramos et al., 2012; Weber, 2015). While validating proofs, students can evaluate whether the structure works to prove the claim (Selden & Selden, 2003). When constructing a proof, Selden et al. (2018) recommend students create a proof framework to organize the logical structure of their argument. The student introduces the claim's assumptions by introducing the appropriate variables, leaves space to later fill in the middle of the argument, and ends with the statement's conclusion. The prover can add another layer of their framework by "unpacking" the conclusion. Selden and Selden (2013) note that with enough practice, identifying an appropriate proof framework can be a relatively procedural task involving little to no problem-solving.

Once their framework is complete, they can begin problem solving how to fill in the middle of the argument. This problem-solving can sometimes include starting by fictitiously assuming the conclusion to deduce what is known to be true based on the givens of the statement. Then, the prover can reverse the argument to start with the givens and end with the sought-after conclusion. This strategy is consistent with what is called the analysis-synthesis method (Lakatos, 1978).

In this study, we asked students to engage in proof validation of two arguments: one is a valid argument while the other is invalid because it uses an invalid proof framework (similar to the argument in Fig. 2). We investigate the students' validation activity to not only understand students' reasons for the evaluations of the arguments, but also to better understand how students might think about the arguments and why students might construct a proof with this structural error.

Proofs of Existence Claims

Brown (2017) explained that existence proofs take two forms: constructive or nonconstructive. A *constructive existence proof* "either explicitly produces the desired result or provides an algorithm for its production" (p. 468–469). Whereas a *nonconstructive proof* "establishes the theorem as a consequence of previous theorems or as a logical necessity" (p. 469). Brown provided two proofs of the claim that there exist irrational numbers a and b such that a^b is rational. The constructive proof introduced the irrational numbers $a = \sqrt{2}$ and $b = log_2 9$ and then demonstrated that the desired property, a^b is rational, was met. The nonconstructive existence proof relied on the logical necessity that either (1) $a = b = \sqrt{2}$ or (2) $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ must satisfy the desired property since $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.

In Brown's (2017) study, she investigated students' conviction of the two proofs. Despite prior literature supporting the conjecture that students would be more convinced by the constructive argument (Harel & Sowder, 1998; Leron, 1985), Brown found that the students in her study tended to deem the constructive proof as *less* convincing, providing some evidence for students' skepticism for constructive existence proofs. Further, the students often gave unexpected reasons for their selection. For instance, more than half of the students that viewed the constructive proof as more convincing did not mention its constructive nature and some students seemed to interpret the non-constructive proof as a constructive one. Thus, Brown highlighted that students' interpretations of a proof might differ from the standard mathematical interpretation. This suggests that there are potential methodological issues in work that draws on preconceived notions about what is communicated with a given proof to understand how the students interpret it. Research that aims to understand students' interpretations should use approaches that bring to light the arguments from the students' point of view.

We argue that another aspect of constructive existence proofs may impact how students comprehend and validate them. Published existence proofs, and likely those provided by instructors, vary in the extent to which the process of identifying the desired object (or a procedure for generating it) are communicated. On one end of this spectrum, a prover offers an object as a candidate early in the proof without details of how the prover found it and then shows that the object fits the desired property (e.g., the proof in Fig. 1). On the other end, the prover details the construction process. The proof in Fig. 3 shows that given any $\varepsilon > 0$, there exists a mathematical object, denoted δ, that fits a desired property (i.e., if $0 < |x - 2| < \delta$ then $|g(x) - 4| < \varepsilon$). This proof shares with the reader more information about how the prover went about selecting $\delta = \min\{1, \varepsilon/5\}$.

The prover must strategically decide an adequate amount of information to share with the reader in such a way that the argument is logically connected with the associated claim. For example, presumably the prover of the argument in Fig. 1 selected $x = a^{-1} * b$ by first solving the desired equation $a * x = b$ for x. However, the prover tactically left this information out since proving if $a * x = b$, then $x = a^{-1} * b$ was not the goal of the argument. In contrast, the prover of the proof in Fig. 3 was quite strategic by including their problem-solving work without a logical flaw. The amount of detail that the prover shares with a reader might also impact how students comprehend and validate the proof. The framework of the proof in Fig. 1 is straightforward in that one could match the framework to the statement; however, such a concise framework contains a great deal of precise meanings that students may not yet be able to unpack. The proof in Fig. 3 describes more informal reasoning for how the prover found the proposed value for δ; however, a consequence of the lengthier description is that the framework is more hidden.

Fig. 3 An example of a more explanatory constructive existence proof (borrowed from Abbot, 2015, p. 117)

Let's show

$$\lim_{x \to 2} g(x) = 4,$$

where $g(x) = x^2$. Given an arbitrary $\epsilon > 0$, our goal this time is to make $|g(x) - 4| < \epsilon$ by restricting $|x - 2|$ to be smaller than some carefully chosen δ. As in the previous problem, a little algebra reveals

$$|g(x) - 4| = |x^2 - 4| = |x + 2||x - 2|.$$

We can make $|x - 2|$ as small as we like, but we need an upper bound on $|x+2|$ in order to know how small to choose δ. The presence of the variable x causes some initial confusion, but keep in mind that we are discussing the limit as x approaches 2. If we agree that our δ-neighborhood around $c = 2$ must have radius no bigger than $\delta = 1$, then we get the upper bound $|x + 2| \leq |3 + 2| = 5$ for all $x \in V_\delta(c)$.

Now, choose $\delta = \min\{1, \epsilon/5\}$. If $0 < |x - 2| < \delta$, then it follows that

$$|x^2 - 4| = |x + 2||x - 2| < (5)\frac{\epsilon}{5} = \epsilon,$$

and the limit is proved.

Insights from the Literature

While the research literature on students' thinking about proof and proving in general is mounting, there are only a handful of studies that explicitly discuss students' thinking about existence proofs (Brown, 2017; Samper et al., 2016; Schaub, 2021). In what follows, we review the literature on students' proving activity to gain insight into how students might think about constructive existence proofs.

There are several studies that have investigated students' attention (or lack thereof) to the structure of arguments in general. These studies show that students might view an invalid argument with a flaw in the framework as a proof because they ignore the logical flaw or doubt the structure of the argument matters (Selden & Selden, 2003; Weber, 2009, 2010). Weber (2009, 2010) interviewed 28 mathematics majors asking them to read various arguments presented as potential proofs, or purported proofs. One of which featured a logical flaw by assuming "n is divisible by 3" to prove that "If n^2 is divisible by 3, then n is divisible by 3" (i.e., assumed the conclusion). Similar to Selden and Selden's (2003) findings which also used this argument, Weber found that more than half of the students viewed the argument as a proof. Twelve of these students gave the argument the highest marks in terms of their personal conviction and its rigor despite the flaw in the proof framework. Weber explained that many of the students did not check whether the argument used appropriate assumptions and conclusions. This is consistent with other findings that students tend to focus on "surface features" such as computational aspects of an argument rather than the logical structure (Inglis & Alcock, 2012; Selden & Selden, 2003). Interestingly, Weber (2009) also found that some of the students who viewed the argument as a proof attended to the structure by identifying that the argument started with what they wanted to prove but, as one student put it, "I don't know if the formatting matters" (p. 15). Thus, when students are considering proofs of existence claims, the structure of the argument may not impact their validation.

Studies have also identified that students have difficulties with the mathematical language used in proofs (Moore, 1994; Selden & Selden, 1995), and specifically, that students may not understand the nuanced ways that mathematical objects are introduced and used (Lew & Mejía-Ramos, 2019). In the context of existence proofs, Samper et al. (2016) argue that teachers need to deliberately support the students in understanding the strategy for proving existence claims in geometry. This included supporting the students to understand what object to introduce, how they could introduce the object, and why it was guaranteed. This is because in their experience, students tended to introduce random objects and attempt to force the desired properties on them. Given that constructive existence proofs often introduce a particular object as a candidate early in the argument, these studies suggest that students might not fully understand the subtle way that such an argument establishes the existence of the desired object and may need intentional support to gain this understanding.

Of particular relevance to our study, Schaub (2021) presented a classroom episode of an inquiry-oriented Real Analysis class constructing a proof of an existence claim. During the episode, the class discussed leaving out the algebraic work that led the students to find a desired mathematical object from their final proof while proving a particular sequence converged by employing the $\varepsilon - N$ definition. In the case that a sequence converges, the definition claims the existence of a mathematical object, denoted N: for every $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$, $|a_n - L| < \varepsilon$. During the episode, the instructor transformed the class board work for finding an N into a proof, which included "Let $N > \lceil 3/\varepsilon - 1/2 \rceil$" immediately after introducing an arbitrary positive epsilon. After the instructor wrote the full proof on the board, she acknowledged that it seemed off putting because it was unclear why the prover chose the particular$N$, explaining:

"…[The proof] has this weird like 'where did you get that?' and the answer is we got that by working out the problem backwards. Assuming what we wanted to prove, finding what N should be, and then sticking it in and proving that it works. And this step is super important because this is a proof in the right order. Start with epsilon, find big N, and show that beyond that point, we are always close, and close means within epsilon, of our limit" (Schaub, 2021, p. 168).

A student then expressed some discomfort with this strategy, explaining, "I've always had a problem with this way of writing a proof" adding, "because we do things, we find the answer the only way we can, and then we pretend like we didn't do it that way" (Schaub, 2021, p. 168). This student's comment suggests that students might have issues with constructive existence proofs that are not transparent in how the prover found the mathematical object.

The reviewed literature suggests that constructive existence proofs are non-trivial for students. Students' difficulties may be related to the proof framework, how the object is introduced and used in the proof, and lack of transparency for how the prover went about finding the mathematical object. In the current study, we investigate: how do students think about arguments for the existence of a mathematical object? To answer this question, we created models of the students' thinking as they evaluated arguments for the existence of a mathematical object. That is, we used observable data (students' utterances, gestures) to describe our understanding of the students' thinking. This understanding adds to the current research literature by identifying why constructive existence proofs might be challenging for students. In doing so, we view our study as important steps towards better supporting students to engage in constructing, comprehending, and validating constructive existence proofs.

Methods

Interviews

This study is part of a larger design research project that is developing inquiry-oriented Introduction to Proof curriculum and instructor support materials (*Advancing Students' Proof Practices in Mathematics through Inquiry, Reinvention, and Engagement* project, NSF DUE #1916490). The course is designed so that the instructors can use materials in the context of real analysis (via exploring the Intermediate Value Theorem, IVT) and/or group theory (via exploring Symmetry groups). The course was intended for students who had not yet taken any undergraduate proof-based courses. During the course, students reinvented key concepts (e.g., sequence convergence or the definition of group) and engaged in various related proof construction tasks. Of particular relevance, the students were sometimes tasked to prove various existence claims, during which the instructors often mentioned the problem-solving needed to find the desired object prior to writing the proof. At this point in the design project, the curriculum did not include tasks designed to explicitly support students in learning about how constructive existence proofs can be a valid proof of an existence claim, or why an argument that starts with the desired property is not a valid proof.

Students exiting the courses were invited to complete a survey which asked them to interpret a collection of mathematical statements as well as describe their experiences in the course, views about mathematics, and demographic information. At the end of the survey, students were invited to participate in an additional interview. The data for this study comes from 16 semi-structured interviews (not including pilot interviews) from students at two universities enrolled in three different classes with different instructors. At one university two instructors implemented both group theory and real analysis materials during a 10-week term whereas the instructor at the other university implemented real analysis materials during a 15-week semester. The interview tasks were either in the context of groups or functions depending on the course context. See Table 1 for an overview of the data.

There were two interview components that corresponded to two distinct research goals. The first component followed-up with the students regarding their experiences in the course. The goal for this component was to better understand these experiences in order to inform later course implementations. The second component is the focus of this report and was designed to engage students in proof validation tasks. The goal was to understand how the students thought about the purported proofs. We then used our interpretations of their thinking to design additional instructional tasks for the course that would target students' proof comprehension, validation, and construction of existence claims. The full interview typically lasted an hour with the second component lasting between 30 and 40 min.

We were both present for each of the interviews. The interviews were facilitated remotely via Zoom and a shared Google Doc and were video-recorded capturing the students' gestures and typed work. When we refer to individual students, we use a code to indicate their proof validation for each argument and the context of the interview tasks. For instance, the label "NPG-7" represented that the participant did *not* view the first argument as a proof (represented by "N"), viewed the second argument as a proof (represented by "P"), the arguments were in the group context (represented by "G"), and they were the seventh interviewee (represented by "7"). We used "F" to represent the tasks in the function context and "U" to represent the one instance in which a student was ultimately undecided about the validity.

Interview Tasks

There were three main parts of the second component of the interview: we asked students to (1) (re)interpret a given existence statement, (2) describe their sense making and validation of the Invalid Argument, and then (3) describe their sense

Table 1 Overview of Data

Term/Semester	School	Course Content	Interview Context	Number of Participants
Summer 2020 Term	Community College A	Group Theory	Group	3 (Pilot Data)
Fall 2020 Term	University A	Group Theory + Real Analysis	Group	6
Winter 2021 Term	University A	Group Theory + Real Analysis	Group	5
Spring 2021 Semester	University B	Real Analysis	Function	5

making and validation of the Valid Argument. See Figs. 4 and 5 for the statements and the Invalid and Valid Arguments. The students did not see the labels "Invalid Argument" or "Valid Argument" during the interview. Additionally, the students did not initially see the underlined text in the arguments (we will elaborate on this methodological choice later). The students interviewed with the group context had previously discussed the statement in detail and approaches for proving it during the class. In the function context, the students had previously conjectured various function properties that would guarantee the existence of a real root (e.g., IVT). Prior to the interviews, the students had not engaged with tasks that asked them to evaluate the two arguments given in Figs. 4 or 5.

We constructed the Invalid Arguments in Figs. 4 and 5 based on our experiences with students. It was relatively common during the interviews for students to comment that the argument was how they would construct a proof of the given statement. For instance, before PPG-11 saw the Invalid Argument, he explained that he would start proving the given statement by "left multiplying both sides of this equation [$a * x = b$] by a^{-1} and then it would just be $x = a^{-1} * b$". When he was given the Invalid Argument, he confirmed that this was the approach that he explained earlier, saying "That's what I was getting at, yeah." While we acknowledge that in

Group Context
Statment: Consider the group G with operation $*$. Let b be in G. For every a in G, there exists an x in G such that $a * x = b$.

Invalid Argument:	*Valid Argument:*
Let G be a group with operation $*$. Let b be in G. Let a be in G. Then, $a * x = b$ Implies that $a^{-1} * (a * x) = a^{-1} * b$ Which implies that $(a^{-1} * a) * x = a^{-1} * b$ Which implies that $e * x = a^{-1} * b$ Which implies that $x = a^{-1} * b$. By closure x is in G since a^{-1} is in G and b is in G.	Let G be a group with operation $*$. Let b be in G. Let a be in G. Choose $x = a^{-1} * b$. By closure, x is in G since a^{-1} is in G and b is in G. Then, $a * x = a * (a^{-1} * b) = (a * a^{-1}) * b = e * b = b$.

Fig. 4 Group version of the statements and purported proofs

Function Context

Statement: Let f be a real-valued function defined by $f(x) = mx + b$. For all real numbers m and b with $m \neq 0$, there exists a real number t such that $f(t) = 0$.

Invalid Argument:	*Valid Argument:*
Let f be a real-valued function defined by $f(x) = mx + b$. Let m, b be in $\mathbb{R}$ such that $m \neq 0$. Then, $f(t) = 0$ Implies that $mt + b = 0$ Which implies that $mt = -b$ Which implies that $t = -b/m$. Since $m \neq 0$, t is a real number.	Let f be a real-valued function defined by $f(x) = mx + b$. Let m, b be in $\mathbb{R}$ such that $m \neq 0$. Choose $t = -b/m$. Since $m \neq 0$, t is a real number. Then, $f(t) = f(-b/m) = m(-b/m) + b = -b + b = 0$.

Fig. 5 Function version of the statements and purported proofs

both contexts the solved-for object must fit the desired criteria, we view this argument as invalid since this is not always the case for existence claims of this form. We constructed the Valid Arguments to show two parts of the desired property: 1) the mathematical object is a solution to the desired equation (e.g., $a * x = b$) and 2) the mathematical object is a member of the desired set (e.g., $x \in G$). The first part is accomplished through the string of equalities at the end of the proof while the second is accomplished by the underlined text after the mathematical object is introduced.

We conducted three pilot interviews which supported refinements to the interview protocol. We found one strategy to be particularly effective in eliciting students' thinking about the arguments: during the second and third part of the interview we periodically added, deleted, or adapted lines of the arguments, asking students whether the change altered the meaning of the argument and if the change was necessary. We found this to be a useful interviewing technique in two ways. First, by asking about the necessity of specific lines, we focused students' attention beyond the surface features and on the inner-workings of the argument in relation to the statement. Second, it supported us to see the arguments from the students' point of view: the students' responses included their view of how each line functioned to convey the prover's argument, and thus, centering the student's view of the prover's argument. We added this interviewing technique to the protocol in two ways. First, we added the underlined text in Figs. 4 and 5 after the students had discussed their initial thinking about the purported proof. We decided to focus on these lines because they were part of the proof frameworks. We decided to add the lines after their initial thinking to allow the students opportunities to identify potential disconnects between the proofs and the statements without our prompting. Second, we added individualized follow-up questions of this nature depending on the discussion. This methodological choice is consistent with Vroom's (2022) technique that was rooted in social semiotic theory (Halliday & Matthiessen, 2013) that surfaced students' views of meaningful grammatical choices. Additionally, we view this approach as following Brown's (2017) recommendation that research that engages students in comparing proofs should seek to understand student activity from the eyes of the student as opposed to interpreting it through an observer's predetermined criteria.

Part 1: Interpret the Statement

The task-based interview began with the student and researchers opening a shared Google Doc that displayed the statement. The students that were interviewed using the group context were previously asked to determine if the statement was true or false and explain why on the exit survey. The students were prompted to re-read the statement and their response and then asked whether they agreed with their earlier response. Students interviewed using the function context were not surveyed about their interpretation of the statement. In this case, the students were given the statement and then asked about the meaning, whether the statement was true or false, and why. In either case, we continued to ask follow-up questions until we were satisfied with our understanding of the students' interpretations. At this point in the interviews, all of the interviewees indicated that the given statement was true.

Part 2: Make Sense of the Invalid Argument

Next, we pasted the Invalid Argument (without the underlined text) into the document and presented it as a student's proof of the statement. After we gave the participant time to read the argument, we asked them to discuss their thinking about the purported proof following up with whether or not they thought it was a valid proof and why. We then added the underlined text to the argument and prompted the student to explain whether or not the new text was needed and whether it changed the meaning of the argument. We continued to follow up with the student regarding what they saw as the prover's argument.

Part 3: Make Sense of the Valid Argument

Then, we pasted the Valid Argument into the document explaining that it was a different student's proof of the statement. Again, we allowed participants time to read the argument and asked the same follow-up questions as we did with the Invalid Argument. We then added the two underlined texts one at a time to allow discussion for each. Throughout this discussion we elicited how the participant considered the first prover's argument as different or similar to the second prover's argument. Additionally, we asked how the students saw the statement as connected or disconnected to each of the purported proofs.

Data Analysis

Our data analysis process was consistent with a thematic analysis (Braun & Clarke, 2006). Together we engaged in a cyclic process examining each interview transcript and corresponding video data for evidence of the students' thinking about the existence statement and two arguments. To do so, we focused on the following guiding questions:

- How did the student describe their interpretation of the statement?
  - Did the student's description of the statement change over the course of the interview? (If yes, how so, and why?)
- How did the student describe their initial thinking about the Valid/Invalid Arguments?
  - Did the student indicate it was a proof or not? Why?
  - How did the student describe the logical argument?
  - Did the student change the way they described their thinking about the Valid/Invalid Argument over the course of the interview? (If yes, how so, and why?)
- How did the student discuss the two arguments in comparison to each other?

For each student, we discussed our answers to these guiding questions until we could come to an agreement on how we would interpret their thinking. We documented our answers to the questions with relevant quotes and our shared-interpretation of the quotes in an analytic memo. By answering these questions, we developed initial descriptions of our understanding of each of the student's thinking about the two arguments and why they did or did not view them as proofs. After creating each analytic memo, we compared our understanding of the student's thinking to the previous. During this comparison, we generated and refined a list of the ways in which we understood how the participants thought about the arguments and how this related to their views of the arguments as proofs or not.

Results

In this section, we share our interpretations of the students' ways of thinking about the mathematical arguments, and how, if at all, this could explain the students' view of the arguments as proofs. We summarize our reasons for why the students in our study viewed the arguments as proofs (or not) at the end of each subsection. Table 2 offers an overview of the students' assessments of the arguments by providing the frequency of each of the proof validation combinations.

The Invalid Argument Shows *How* to Find the Mathematical Object

The majority of the participants (N = 12) indicated that the Invalid Argument showed how to find the mathematical object. For instance, PNG-3 explained that it described "the process of finding that element [x] for which that [$a * x = b$] is true". The students whose thinking did not fit in this category either did not elaborate on the prover's logical argument after they pointed to an error (NPG-8, NPG-9,

Table 2 Overview of students' validations of the two arguments

Student Validations of Invalid Argument	Student Validations of Valid Argument	Frequency[a]
Not a proof	Proof	7
Proof	Proof	4
Proof	Not a proof	2
Not a proof	Not a proof	2

[a]This count does not include PUG-4 was undecided about the validity of the Valid Argument

NPG-10) or else explained that the prover showed that if there exists an x in G such that $a * x = b$, then x must be in the form $a^{-1} * b$ (NPG-1).

Some of the students who saw the Invalid Argument as showing how to find the mathematical object viewed it as a proof (N = 7), while others did not (N = 5). One salient reason that these students claimed it was a proof was that it elaborated why the existence claim was true by showing the algebraic work to construct the claimed object. For instance, when PPG-11 compared the two arguments, he explained that he had a preference for the Invalid Argument since it showed how to find the object, saying: "I prefer to see the steps, right. Like, I think I prefer to see like the thought process happen in real time…" It was common for these students to mention that the Invalid Argument was an appropriate proof for more novice readers. For instance, PPF-16 explained, "I could see where somebody at a lower math level would appreciate having that". We see these students as thinking the Invalid Argument was an explanatory proof for less experienced students as it provided insights about how to construct the claimed object.

Some of the students who viewed the Invalid Argument as a proof since it showed how to find the object explained why starting with the desired equation was not a flaw in the logic. PPG-11 explained:

"No, they don't start with the thing that they're trying to prove. They're trying to prove that x needs to be an element in G. Right. That's what we're trying to prove, that there exists an x in G such that the statement is true."

While this student seemed to view the property $a * x = b$ as a way to describe a desired characteristic of x, the student focused on showing that such a group element existed. PPF-16 explained that it was appropriate to start by "assuming that there exists a real number t that makes the function equal to zero" because "we know that whatever number we're going to solve t to be is always going to make that [$f(t)$] zero". To this student, the prover could show that there existed a real number t such that $f(t) = 0$ by solving the desired equation for t because the value that they solve for must be the value that makes $f(t) = 0$ true[1].

There were other students who deemed the Invalid Argument as a proof who pointed to the logical necessity of showing how to find the mathematical object.

[1] If the reader interpreted the implicit warrants in the proof to be bi-directional, then they would feel confident that the solved for object would have to be a solution.

PUG-4 explained that she preferred the Invalid Argument because it made sense to her and she could see the statement connecting to the argument. She said:

"I think it makes more sense mathematically. But also in context of the problem that we're supposed to show 'for every a there exists an x…', you know? So, I like that it gets, you know, it starts here, this is the equation that we're given. This is what we have and then here's the x that exists and that's like proof. That's why I like it, because it's kind of in the order of the proof..."

We interpret her comment to mean that she not only could follow the algebraic steps, but she saw it as fitting the logical structure of the given statement. PNG-6 further articulated why from her perspective the Invalid Argument was a proof. When asked how she saw the argument connected to the statement she explained that the prover showed that "no matter what the a is, we're able to figure out an x, that will equal b when you do that [operation]." To these students, an appropriate approach to show that one could find a desired mathematical object was to show how to find it.

Four of the 5 students who did not view showing how to find the object as a proof explained that it assumes there exists the desired object when it solves the desired equation for the value. Some of these students commented on the utility of the algebraic work even though it should not be part of the proof. For instance, NPG-5 said: "That ['Then $a * x = b'$] assumes the conclusion and then works from that. Which is like one of those good strategies you use to write a proof, but you do it separately and then try and write it the other way.". Interestingly, the fifth student who did not see the Invalid Argument as a proof viewed the argument as a necessary part of the proof, but alone was incomplete. To NNF-15, the Invalid Argument showed why $x = -b/m$ and that it was a real number but "it has not proven that $f(x) = 0$. It is just stating 'hey, this is what we're trying to solve for'…".

In summary, we saw most of our students as identifying the prover of the Invalid Argument as showing the process of finding the desired mathematical object. We viewed the students in our study as thinking about the argument in at least one of the following ways:

- The argument was a proof since it showed the existence of the desired object by explaining why such an object fit the desired property,
- The argument was a proof since it fits the logical structure of the statement,
- The argument was not a proof since it assumed the existence of the desired object, or
- The argument was not a proof since it had only solved for the desired object but also needed to show the desired property was true.

The Valid Argument Shows an Instance of the Mathematical Object

All the students in our study (N = 16) were able to identify that the Valid Argument introduced an instance of the mathematical object. For instance, when asked to walk through the prover's logic in the group theory context, NPG-8 explained "they're just choosing an x, which is essentially what the problem is asking for. It's just saying, just find, just find one of them".

While all students saw the Valid Argument as identifying a particular instance of the mathematical object, some students did not view it as a proof (N = 5)[2]. We identified two reasons that these students had this view. First, four students explained that by starting with an instance of the mathematical statement, the prover incorrectly assumed the existence of the object instead of deducing its existence. For instance, when discussing why the Valid Argument may not be a proof, PUG-4 stated: "because it starts with the x - that's a little confusing to be like 'oh, for every a, an x exists' but that's what they're saying right here, this is the x that exists". For this student, like the other three, the prover could not start with $x = a^{-1} * b$ because by doing so they assume that the object exists. The fifth student who did not view the argument as a proof had a different reason. Similar to his reasoning for why the Invalid Argument was not a proof, NNF-15 saw the Valid Argument as only providing half of the necessary argument, saying "you would also have to prove why x is equal to $-b/m$, you can't just state that". Unlike the other four students, NNF-15 viewed the Valid Argument as a necessary part of the proof since the prover showed the mathematical object met the desired equation (i.e., $f(x) = 0$). To NNF-15, you would need the two arguments together to make a proof.

Among the 11 students who thought the Valid Argument was a proof, one salient reasoning was that it was logical but had a jump in explanation. For example,

"The second [argument], just like tosses that up there. So, they maybe did some side work or maybe were just able to see it. But, uh, but they kind of leave a lot of their reasoning off the page" (PPG-11).

While these students also acknowledged that the prover omitted relevant reasoning from the reader by introducing an object up front, we see these students as thinking that the proof only needed to convince rather than explain to the reader how to find the object.

There were several students who explained that the Valid Argument was a proof since it only needed to show the existence of one element that met the desired property. NPF-12 explained:

"So, since we only have to prove there exist one number we don't have to prove for all real numbers, we just have to find that one real number for which the equation is equal to zero, so I guess, we could say that, like to let it be equal to $-b/m$ and then use that to show that the equation is zero."

Several of these students (among others) did not see the line that the mathematical object was an element of the claimed set (i.e., "By closure, x is in G since a^{-1} is in G and b is in G" or "since $m \neq 0$, t is a real number) as being necessary for the argument. For instance, NPG-9 explained:

"I don't think it's necessary, but I think it's helpful. [...] Explaining why x is in there like, to me, it was just kind of intuitive because you define it in terms of things that are already in G…"

[2] This count includes PUG-4 who offered a reason why the Valid Argument was not a proof, but did not ultimately commit to her decision.

NPG-9 as well as others saw this line as simply adding further explanation and clarity for the reader rather than a necessary part of the conclusion.

In summary, we saw all the students as thinking the Valid Argument introduced a specific mathematical object. We viewed the students in our study as thinking about the argument in at least one of the following ways:

- The argument was not a proof because it assumed the existence of the object instead of deducing its existence,
- The argument was not a proof because it only showed that the object met the desired property but not how they found the object,
- The argument was a proof since it showed the existence of the mathematical object but had a jump in explanation when the mathematical object was introduced,
- The argument was a proof since it only needed to show the existence of one element that met the desired property, or
- The argument was a proof since it showed the existence of the mathematical object even without arguing the mathematical object was in the desired set.

The Arguments are Structured in Different Ways

Most of the students in our study identified the different logical structures of the arguments (N = 15). Many of these students described them as "opposite" of each other or explained how the first deduced information from the desired equation whereas the second concluded the desired equation. For instance, when comparing the two arguments PNG-3 stated that "this one [the Invalid Argument] shows how they got to that x and this one [the Valid Argument] just like here's the x and you get b when you multiply by a". The one student's thinking that did not fit in this category saw both arguments as proofs that were structured in the same way.

Some students indicated that the structural differences mattered to them, that is, only one of the frameworks structured the argument to prove the claim. This depended on the students' view of the goal of the argument. These students discussed two different goals. First, some students thought that a proof should conclude with the mathematical object instead of assuming it at the start - these students tended to be the ones who thought a) the Invalid Argument was a proof because it explained how to find the mathematical object and b) the Valid Argument was not a proof because it assumed the existence of the object instead of deducing its existence. PNG-3 explained:

"Well, we're trying to conclude that there exists an x. Okay yeah, so I don't think proof two [the Valid Argument] is [a proof], because that is the conclusion, that there exists an x such that $a * x = b$. Whereas in this one [the Invalid Argument] we're definitely looking for that x, in the first one. Because I think that the conclusion should be that there exists an x in G such that $a * x = b$, and our givens are that a is in G and that b is in G, essentially and that's it."

She later confirmed that to her the Valid Argument assumed the conclusion by introducing the element $x = a^{-1} * b$ at the beginning of the argument. Second, there were some students who thought that the argument should conclude that the desired property held for an instance of the mathematical object. These students were the ones that a) did not think the Invalid Argument was a proof since it assumed the existence of the desired object and b) thought the Valid Argument was a proof since it only needed to show the existence of one element that met the desired property.

Other students indicated that the structural differences did *not* matter to them. These students either explained that neither argument effectively proved the claim (N = 2) or else viewed both arguments as proofs (N = 3). The students who thought neither were a proof thought so because neither had acceptable or complete frameworks. NNG-2 identified that both arguments had issues with how they started. She discussed that a proof of the existence statement needed to deduce the mathematical object rather than assume it, leading to a clear rejection of the Valid Argument due to it starting "with our conclusion and not with our beginning". She preferred the Invalid Argument saying it was "the proof in the correct direction" but explained that it needed one fix. She explained:

"I wouldn't just say 'then' I would say, like I think that's more of like a 'consider' or 'if'. [...] Yeah *if* $a * x = b$ (emphasis added)."

This suggests she also viewed the Invalid Argument as making the error by starting with an ineffective assumption. To her, the Invalid Argument no longer assumed unknown knowledge when she changed the beginning of the argument to a conditional statement by adding "if". NNF-15 was the other student who viewed neither as a proof. As we have previously discussed, he thought the two arguments were structurally different in that they accomplished different, but necessary parts. To him, the frameworks were incomplete on their own.

The other three students who did not think the structural differences mattered viewed both arguments as proofs. These students acknowledged that one could construct different proofs to show the existence of the mathematical object. They saw a) the Invalid Argument as a proof since it showed the existence of the desired object by explaining why such an object fit the desired property and b) the Valid Argument as a proof with a jump in explanation when the mathematical object is introduced. We interpret this as them seeing one proof as being more explanatory to certain readers, while another may only convince. For these students, the first proof provided additional information that the second proof left out. While both arguments were valid, the intended reader differed. For instance, PPF-13 explained:

"So, this one [the Invalid Argument] feels like it's more like explain it like your five and the other one [the Valid Argument] is more like assume they already know what it means to go from there and that they're both valid, but they're both different ways of doing it."

To PPF-13, and the other two students whose thinking fit in this category, the two arguments accomplished the task of finding the object, they just did so in different ways. As PPG-11 stated: "It's just the construction of it [the two arguments] that's different […] They [the provers] both understand that x needs to be equal to $a^{-1} * b$ whatever a is." Unlike these students, a fourth student, PPF-14, who viewed both arguments as proofs, thought that they followed a parallel structure. Like the three students in this category, PPF-14 also saw the explanatory power of the first argument, but saw it as an expanded version of the second argument. He explained:

"They start the same and then this one [the Valid Argument] jumps straight to choose $t = -b/m$. So, it's assuming [...] that hey you were able to complete these steps on your own, essentially. [The Invalid Argument], it's showing you, step by step. And this one's [the Valid Argument] just assuming - okay, we start with A [pointing to the first line] we go to B [pointing to the "Choose" line] we hit C [pointing to the last line with the string of equalities] and there's our answer. This one [the Invalid Argument] is - we started A [pointing to the first line] here's how we get to B [highlighting the algebraic work that leads to $t = -b/m$] here's an example of it and then there's C [pointing to the last line], which is also our answer so it's just giving those extra steps to somebody."

In summary, we saw that all but one of the students in our study identified that the arguments were structured in different ways. We viewed these students as thinking about the arguments in at least one of the following ways:

- Only the Invalid Argument had an effective framework since it was the only one that concluded the mathematical object instead of assuming it at the start,
- Only the Valid Argument had an effective framework since it was the only one that concluded the desired property held for an instance of the mathematical object,
- Neither of the arguments functioned as proofs since neither had acceptable or complete frameworks,
- Both of the arguments functioned as proofs since the Invalid Argument was more explanatory to certain readers while the Valid Argument only convinced.

Conclusion and Discussion

Students' Thinking About Constructive Existence Proofs

Returning to our research question, our study found that students thought about arguments for the existence of a mathematical object in several different ways. Some students identified that an argument that started with the desired property and then solved for the object was not a proof, while other students offered reasons why to them such an argument proved the claim. Additionally, there were also students who identified that an argument that introduced the object as a candidate early on and then showed that it fit the desired criteria was a proof. Yet, there were other students who offered reasons why to them such an argument was not a proof. Despite the different ways that the students validated the arguments, their reasons tended to attend to the logical structure of the arguments. This is in contrast to prior studies that have documented that students ignore the structure of proofs or think that they do not matter (Selden & Selden, 2003; Weber, 2009, 2010). Of particular interest, the students provided reasons for why they thought the structure of the Invalid Argument functioned to prove the claim and why they thought the structure of the Valid

Argument did not function to prove the claim. This offers insights into why constructive existence proofs might be challenging for students.

In reference to the Invalid Argument, we found two reasons for why some students in our study saw starting with the desired property (e.g., $a * x = b$) as acceptable. First, some of the students viewed the Invalid Argument's framework as fitting the logical structure of the existence statement. For them, the statement required that one show how to find the desired object and thus needed to start with the property and end with the object itself. Second, other students viewed it as a way to add transparency when showing the existence of the desired object by explaining how one would find it. We saw these students as valuing the explanatory nature of Invalid Argument while acknowledging it was not the only way to prove the existence of the object. These ways of reasoning can explain why students may produce or endorse an argument that starts with a desired property and follows with solving for the object: students may think that (a) the structure of the proof should logically show how to find the mathematical object or (b) a reader should gain insight about how the mathematical object was derived.

Additionally, we found there were students who did not view the Valid Argument as a proof of the existence claim. In particular, there were students who saw the line that introduced the object (e.g., Choose $x = a^{-1} * b$) as assuming the existence of the mathematical object. In other words, the argument did not prove the existence of the object since it assumed it from the start. Such a view explains why students may not think that proofs of existence claims can first introduce the object as a candidate: students may think that such an introduction is logically invalid as it assumes the conclusion.

Implications and Future Directions

Our findings motivate additional areas for research. First, we echo Brown's (2017) recommendation: there is a need for research on student's conceptions of proof and proof-activity from the students' point of view. By prioritizing the students' point of view, we were able to describe important mathematical thinking from the students that might have gone unnoticed if we conducted our analysis using preconceived notions about what is communicated with the given proofs. Potential important insights can be gained from future research that takes such an asset lens on students' thinking about proof and proof-activity focusing on what students *can* do rather than a deficit lens focusing on what students cannot do.

From our work, we see two potential avenues to further study students' thinking about existence proofs. Our findings suggest that students might find value in arguments that give insight into the informal reasoning used to construct it. More research is needed to understand how students think about explanatory proofs. In particular, a limitation of our study is that we did not ask students to validate a valid and explanatory proof. It could be insightful to better understand how students think about valid explanatory proofs in addition to invalid explanatory proofs like the ones in our study. An extension of our study could be to ask students to validate four categories of arguments: 1) a valid explanatory proof, 2) a valid non-explanatory proof,

3) an invalid explanatory proof, and 4) an invalid non-explanatory proof. To see how students' views of explanatory proofs extends beyond the context of constructive existence proofs, future research might study students' views of proofs of claims that typically employ additional proof techniques. It might also be insightful to further investigate students thinking about the labels used in existence proofs, or proofs in general. For instance, one might investigate the influence on the choice of the letter x in the statement and the Valid Argument on the students' view that the argument assumed the conclusion by presenting students with the Valid Argument without the using x (e.g., "Consider$a^{-1} * b$" instead of "Choose $x = a^{-1} * b$").

Importantly, there is also a need to find ways to base instruction on student thinking to support them in learning about proof, including supporting students in learning about proof frameworks. As our findings show, there is important mathematical work in the formal-rhetorical part of a proof, and we argue instructors should support students new to proof to engage in it. We see that discussions that elicit the possible functions of the frameworks as deeply mathematical in the sense that such a discussion would unpack nuanced ways to convey specific mathematical ideas. We see such instruction far more productive than a procedural approach to proof frameworks that simply asks novice students to practice matching the given statement's assumptions and conclusions to the structure of the argument without any discussion related to the meaningful choices of the framework.

Engaging students in comparing the two arguments in this study seemed to be a productive activity for eliciting possible student interpretations of the prover's argument and how that argument did or did not prove the given statement. We suspect such an activity would lead to productive classroom discussions that could support students in better understanding constructive existence proofs. We emphasize several points of discussion that our findings suggest would be particularly advantageous. First, while finding the mathematical object is a critical step in constructing the proof, it is not the goal of the argument. Instructors should take care to support their students in recognizing *that* starting from a desired property can be a productive problem-solving strategy to find a claimed object but also *why* this work may not show-up in a proof, or if it does, how the prover must carefully do so. This is related to our second point: we see that there is a need to support students in understanding how one might construct a valid explanatory constructive existence proof. That is, if students have this desire, then instructors should support them in adding insight into how they found the object while preserving the connection between the logical structures of the associated claim and their argument. Third, instructors could act as a broker between the classroom community and the larger mathematics community (Rasmussen et al., 2009; Vroom, 2020) to support students in understanding how introducing an object as a candidate early in the proof does not assume the conclusion and how the argument establishes the existence of the desired object.

We see our study as important steps in supporting students in engaging in proof-activity related to existence claims by providing different ways in which students might think about constructive existence proofs and their frameworks. There is important mathematical work in understanding the formal-rhetorical part of a proof that students can, and should, engage in, including how a proof framework functions to prove the claim. We hope that future work will investigate the affordances of instruction that goes beyond procedural frameworks in supporting students in constructing, comprehending, and validating proofs of existence claims.

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Declarations

Ethics Approval The authors report the study was approved by an Internal Review Board.

Consent to Participate The authors report that all participants gave informed consent.

Consent for Publication All authors give consent for the publication of this manuscript.

Conflicts of Interest We report that there is no conflict of interest.

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