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A functional perspective on student thinking about the grammar of multiply quantified statements

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ABSTRACT

Fluency with formal mathematical language is necessary for students in advanced mathematics. Yet, the language has been documented as being particularly challenging for students, motivating the need for more empirical studies that investigate the language and undergraduate students' understanding of it. This study makes progress on this goal for the case of multiply quantified statements. By using Halliday's Systemic Functional Grammar, I identified meanings that students had for quantified variables, connected these meanings to the grammar of the statements, and explained how such meanings impacted the full statement. I argue for the utility of Systemic Functional Grammar when investigating formal mathematical language and students' thinking about it.

1. Introduction

Definitions, theorems, and proofs are integral aspects of undergraduate mathematics courses and are expressed with formal mathematical language. Students' success in these courses depend on their fluency with mathematical language. For instance, students' ability to construct proofs (a typical assessment task according to Weber, 2001) depends on students' understanding of the language (Moore, 1994). Yet, there is growing evidence that written mathematical language is challenging for undergraduate students (e.g., Dawkins & Roh, 2020; Dubinsky & Yiparaki, 2000). For instance, students use mathematical language unconventionally in proof writing (Lew & Mejía-Ramos, 2015) and may not fully comprehend the nuances involved in how mathematicians introduce objects in proofs (Brown, 2017; Lew & Mejía-Ramos, 2019). Given the central role of mathematical language in doing mathematics as well as students' difficulties with it, scholars have pointed to the need for more empirical studies that investigate the language and undergraduate students' understanding of it (Lew & Mejía-Ramos, 2015, 2019, 2020). This study makes progress on this goal for the case of multiply quantified statements and, in doing so, I illustrate a useful analytical tool for studies on such matters.

Multiply quantified statements are characteristic of many theorems and defining properties in advanced mathematics. These statements are used to encode precise mathematical meanings and span different content areas. For instance, the defining property of a set $A\subset\mathbb{R}$ that is bounded above is given by: there exists $b \in \mathbb{R}$ such that for all $a \in A$, $a \leq b$. This type of statement is often referred to as an EA Statement in which the "E" before the "A" signals that the existential quantifier ("there exists") comes before the universal quantifier ("for all"). In algebra, inverse elements of a group G with operation $*$ are described by an AE Statement: for all $a \in G$, there exists $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$. This type of statement is often referred to as an AE Statement to indicate that the universal quantifier comes before the existential quantifier. There are also many concepts that are articulated with more than two

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quantifiers (e.g., the $\varepsilon - N$ definition of sequence convergence).

Multiply quantified statements are also particularly challenging for students despite the central role they play in communicating many advanced mathematical concepts. Prior research has approached investigating students' understandings of AE and EA statements by comparing students' interpretations to the standard interpretations. Such studies have pointed to students' lack of attention to and understanding of the order of quantified variables (Dubinsky & Yiparaki, 2000; Durand-Guerrier & Arsac, 2005; Roh & Lee, 2011; Roh, 2010), though students are likely to become more sensitive to the ordering after they take an Introduction to Proofs course (Dawkins & Roh, 2020). In particular, novice students are likely to interpret EA statements as AE statements (Dubinsky & Yiparaki, 2000) and be more influenced by the mathematical context than the order of the quantifiers (Dawkins & Roh, 2020).

Scholars have also investigated student meanings for quantified variables (Sellers, 2020; Sellers et al., 2017, 2021). Notably, Sellers et al. (2021) took a constructivist lens to investigate the ways that students quantify variables in multiply quantified statements, like the "a" or the "b" in the bounded above defining property previously discussed. They found that students held different meanings for quantified variables. Students described their imagined process of checking whether the predicate of the statement (e.g., $a \leq b$ in the bounded above defining property) holds for at least one value, exactly one value, for all values, or spontaneously chosen values. They also found that some students did not quantify a variable or there was an absence of meaning for the quantified variable. A constructivist perspective afforded the researchers in investigating the mental thought processes that result from students' perspectives of the quantified variables. However, it was limiting in systematically investigating how students see the language as working to communicate their meanings. This understanding concerns not only connecting a student's meaning for a quantified variable (e.g., check if the predicate holds for at least one value of x) to the language itself (e.g., the existentially quantified variable "there exists x in X"), but it also concerns how such a student's meaning of the quantified variable contributes to the student's view of what the full statement conveys. More research is needed to systematically understand whether or not (and why) students connect their meanings to the conventional language choices. For instance, whether students construe 'there exists at least one real number b' with 'there exists $b \in R$'. Further, if students do in fact make unconventional choices, then more research is needed to understand the impacts of their choices in how they construct and interpret statements with quantifiers.

The prior studies discussed are fundamental in documenting that students have difficulties with conventional interpretations of the statements, what might influence the ways that students interpret statements, and students' mental actions when they interpret a quantified variable in a given statement. This study builds on the previous research cited here, aiming to understand how students connect mathematical language to their meanings, and specifically, how they see grammatical choices as functioning to convey meanings. Such an understanding gives important insight into not only how students use language to articulate their thinking, but also provide researchers more insight into *why* students interpret AE and EA statements the way that they do. This study began with investigating students' construction of mathematical statements with multiple quantifiers. As I analyzed the students' thinking, it became evident that a functional approach to the grammar could be a useful lens through which to analyze their ways of thinking about multiply quantified mathematical statements. A functional approach focuses on the meaningful grammatical choices that work to convey specific meanings (I elaborate this in Section 2). In this paper, I seek to answer: How do undergraduate students' think about the grammar of AE and EA statements? By answering this question, I also discuss insights that a functional perspective affords in this analysis.

2. Theoretical grounding

In the following section, I first introduce Halliday's functional perspective on language (Halliday & Matthiessen, 2013), which provides an analytical framework for my investigation of students' thinking about the grammar of AE and EA statements. In Section 2.2, I also used a functional approach to investigate the mathematics community's grammatical patterns and the associated meanings with AE and EA statements. Throughout the manuscript I use the term *mathematics community* to refer to a general set of people who abide by the conventional, or taken as if shared, grammatical choices that I discuss in this paper. By using this term, I do not mean to suggest that this community is stagnant, rather I see this community is expanding and contracting at any given moment.

2.1. Functional perspective on language

Halliday's social semiotic theory of language, known as Systemic Functional Linguistics (SFL), deals with the construction and representation of meaning. SFL is built on the assumption that "language is not realized in the abstract; it is realized as the activity of people in situations" (Halliday, 2009, p. 18). Halliday (1978) notion of *register* highlights that we use language to construct knowledge in different ways across different academic domains. A register is, "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings" (Halliday, 1978, p. 195). In particular, a *mathematical register* is "the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes" (Halliday, 1978, p. 195). Scholars have noted the challenge that students face as they constantly switch registers when moving from subject area to subject area (Schleppegrell, 2007).

Systemic Functional Grammar (SFG) is part of SFL (Halliday & Matthiessen, 2013). When we use language (text or spoken), we use specific patterns for specific purposes. Hallidayan scholars studying the grammar of scientific language (e.g., Lemke, 2003; O'Halloran, 2015), adopt Halliday's view that the difficulty with using and interpreting scientific language "lies more with the grammar than with the vocabulary" (1996, p. 78). Halliday elaborates:

The problems with technical terminology usually arise not from the technical terms themselves but from the complex relationships they have with one another. Technical terms cannot be defined in isolation; each one has to be understood as part of a larger framework, and each one is defined by reference to all the others (1996, p. 78).

To learn mathematical language, one must learn *how* to use the vocabulary, the patterns associated with the terms, in order to construct mathematical concepts.

Halliday's notion of grammar is "functional" which is in contrast to traditional school grammar. Gebhard and Martin (2011) comment that traditional school grammar consists of the study of parts of speech and prescriptive rules for correct usage. By contrast, Halliday's conceptualization of grammar emphasizes the role that something plays in the structure – its function. The 'systemic' in SFG highlights that language is comprehensive in the sense that "what [is] said about any one aspect also contributes to the total picture" (Halliday & Matthiessen, 2013, p. 20). Halliday and Matthiessen explain that "when we [analyze] a text, we show the functional organization of its structure; and we show what meaningful choices have been made" (p. 24).

In this study I aim to take a functional approach to studying AE and EA statements. In Section 2.2 I use SFG to investigate the mathematical community's grammatical choices and their function with AE and EA statements. Then, in Section 4, I use SFG to analyze my participants' grammatical choices and their function in conveying meaning. These analyses consider the *system* (paradigmatic ordering in language) and the *structure* (syntagmatic ordering in language). The system answers "what *could go instead of* what" (Halliday & Matthiessen, 2013, p. 22); whereas the structure is the patterns or regularities in language, answering "what *goes together with* what" (p. 22).

Gebhard and Martin (2011) offered an example that illustrates the system and structure aspects of the grammar of the sentence 'Al wrote an essay'. In terms of the system, one could say 'Al has written an essay' since 'has written' could go instead of 'wrote'. Additionally, 'will write' could go instead of 'wrote' to convey that Al's writing will happen in the future instead of it happening in the past. Alternatively, one could make a structural choice by placing 'has Al' with 'written an essay' to formulate a question of whether Al has written an essay instead of declaring it. Halliday and Matthiessen describe the system and the structure as working together, explaining "each system — each moment of choice — contributes to the formation of the structure" (2004, p. 24). To determine the significance of a particular word or phrase's function, semioticians use what's called a communication test and select a word or phrase and then consider alternatives (Chandler, 2004). The test is used to understand the function and identify what Bateson (1979) refers to as 'differences that make a difference'. For instance, changing 'Al has' to 'has Al' makes a difference to one's construal of 'Al has written an essay' – the declaration changes to a question.

I leveraged communication tests to investigate differences in the system and structure that make a difference to the construals of AE and EA statements, from both the students' perspective and the mathematics community's perspective. It is important to note that while I present differences in the mathematics community's and students' uses of the language, it is *not* meant to be interpreted as identifying students' misconceptions of mathematical language. Rather, I argue that by drawing on SFG, I am able to first recognize when students are considering a meaningful grammatical choice to articulate a conceptual distinction and then explain their choices as a way to communicate (or interpret) meaning with a statement. These differences highlight that students make reasonable choices to articulate their thinking, and in doing so, their thinking gives insight into the complexities of the conventional use of the language.

There are a few undergraduate mathematics education researchers that have drawn on SFL to frame and motivate studies (e.g., Lew & Mejía-Ramos, 2015; Mejía-Ramos & Inglis, 2011). However, Herbel-Eisenman et al. (2017) note that the theory has generally not been taken up by the mathematics education researchers as a major theoretical or methodological framework, explaining one reason is that the theory is "complex, with many different components" in which its "primary purpose has been to describe how language, particularly English, operates, and so it would be unlikely that any mathematics education researcher could make use of all of its components when conducting research" (Herbel-Eisenman et al., 2017, p. 735). A counterargument is that such a comprehensive theory affords mathematics education researchers many different tools to study how language, even mathematical language, operates. Moreover, the theory has proven to be a valuable tool in studying students' thinking about other technical languages such as in the language used to communicate science (e.g., Buxton et al., 2014, 2019; Kim et al., 2017). In this paper, I aim to provide an example of how I used SFG as an analytic tool to investigate statements with multiple quantifiers and students' thinking about them. I argue that the theory can guide researchers to important insights about the teaching and learning of formal mathematical language and the language itself, which I demonstrate through the case of EA and AE statements.

2.2. *Toward a functional approach for statements with multiple quantifiers*

2.2.1. *Approaches for thinking about the meanings of AE and EA statements*

Scholars have described two different, yet compatible, approaches for thinking about the meanings of AE and EA statements. I will draw on the example statements given in Table 1 to illustrate these approaches as well as throughout the remainder of this section (borrowed from Dawkins & Roh, 2019[1]).

First, Dawkins and Roh (2020) introduced *each to some* and *one to every* relationships as ways of thinking about the meaning of an AE and EA statement, respectively. For instance, Example 1 in Table 1 conveys that *each* positive real number s is paired with *some* point C on the ray $\overrightarrow{AB}$ in such a way that the distance between A and the point C is s (each s to some C). By contrast, Example 3 conveys

[1] The original statements in Dawkins and Roh (2019) paper separated the referent from the property. For instance, the paper offered "There exists a real number M such that for all real numbers x, $f(x) < M$" with the referent "$f(x) = 3x + 2$". I confirmed that including the referent in the statement as I have in Table 1 is common practice by searching for this type of statement in several textbooks.

Table 1
Example statements.

Example 1	True AE Statement	For every positive real number s, there exists a point C on a ray $\overrightarrow{AB}$ such that $d(A,C) = s$.
Example 2	False AE Statement	For every positive real number s, there exists a point C on the segment $\overline{AB}$ such that $d(A,C) = s$.
Example 3	True EA Statement	There exists a real number M such that for all real numbers x, $sin(x) < M$.
Example 4	False EA Statement	There exists a real number M such that for all real numbers x, $3x + 2 < M$.

that there is *one* real number M that makes $sin(x) < M$ true for *every* real number x (one M to every x). Dawkins and Roh (2020) also provide accompanying diagrams of these relationships.[2] See Fig. 1.

The way that scholars have written about proving statements with multiple quantifiers suggest a second approach (e.g., Selden & Selden, 1995; Selden et al., 2018). With this approach, one considers an *arbitrary* member in the domain of discourse of a universally quantified variable. For instance, with Example 1, one can imagine an *arbitrary* positive real number s is paired with some point C on the ray $\overrightarrow{AB}$ in such a way that the distance between A and the point C is s. Whereas with Example 3, one can imagine one real number M that makes $sin(x) < M$ true for an *arbitrary* real number x. Such an approach to thinking about AE and EA statements is productive in constructing proofs that prove AE or EA statements, and particularly their proof frameworks (Selden & Selden, 1995).

However, the ways that the students in my study approached thinking about these meanings did not consistently fit with either of these approaches. Rather, throughout the sessions they developed and articulated what I call *AE and EA processes*. I argue that these processes are compatible with Dubinsky and Yiparki's (2000) two-player game that was designed to support students in understanding how one would use the order of quantifiers to make sense of a statement's meaning. In what follows, I first describe the rules of the game and then I will discuss my re-interpretation of the game as a process approach for thinking about EA and AE statements.

To start the game, two players are assigned either a universal quantifier role (Player A) or an existential quantifier role (Player E) and are given an AE or EA statement. The goal of Player E is to establish that the statement is true, while Player A aims to establish that the statement is false. In the case that the players are given an AE statement, such Example 1, Player A takes the first turn by selecting an element from the set that the universally quantified variable belongs to (e.g., an s from the set of positive real numbers) and then Player E selects an element from the set that the existentially quantified variable belongs to (e.g., a point C on a ray $\overrightarrow{AB}$) such that the relation is satisfied (e.g., $d(A,C) = s$). They continue the game by taking alternating turns. Each time Player A takes a turn by selecting an s from the set of positive real numbers, Player E will be able to find a point C on the ray $\overrightarrow{AB}$ in such a way that the distance between A and C is the selected positive real number. Because rays extend to all positive distances away from the endpoint, the game could go on forever. Player A cannot win the game because no matter what positive real number s they play, Player E will be able to find a paired C such that $d(A,C) = s$.

In the case that the players are given an EA statement, such as Example 3, Player E takes the first turn selecting an element from the set to which the existentially quantified variable belongs (e.g., an M from the set of real numbers). During Player A's turn, they select an element from the set to which the universally quantified variable belongs (e.g., an x from the set of real numbers) such that the relation is *not* satisfied (e.g., $sin(x) \geq M$). Since $f(x) = sin(x)$ is bounded above, Player E can end the game in one turn and play an M from the set of real numbers (e.g., $M = 2$) so that no matter what real number Player E selects, $sin(x) < M$. In this case, Player E will win the game.

The rules of the game have some implications for how one can approach thinking about the meanings of AE and EA statements. Consider the following generic AE statement: For all x in X, there exists y in Y such that $R(x,y)$. Each time that it's Player E's turn, Player E attempts to select a y in Y for the previously played x in X such that $R(x,y)$ is satisfied. This is analogous to interpreting an AE statement as follows: the reader iteratively takes an x in X and finds a y in Y in such a way that satisfies $R(x,y)$ until every member of the set X has been paired (or imagines being paired[3]) with a y in Y. If one can create an *AE process* with an AE statement, then the statement is true.

Alternatively, consider the generic EA statement: There exists y in Y such that for all x in X, $R(x,y)$. Player E starts the game by attempting to find a y^* in Y so that no matter what x in X Player A selects in the following turn, $R(x,y^*)$ will still be satisfied. This game with an EA statement is analogous to interpreting an EA statement as follows: the reader attempts to find a y^* in Y such that $R(x,y^*)$ is satisfied for every x in X. If one can create an *EA process* with an EA statement, then the statement is true. See Fig. 2 for detailed descriptions of the *AE* and *EA processes*.

The three different approaches to thinking about AE and EA statements that I describe here complement each other. I view the pairwise relationships as the result of the *AE* and *EA processes*. That is, an AE statement encodes an *AE process* (or an EA statement encodes an *EA process*), and the result of that process is an *each to some* relationship (or a *one to every* relationship) between the quantified variables. Additionally, if there is a pairwise relationship (and corresponding process), then one can imagine an arbitrary member in the domain of discourse of a universally quantified variable rather than every member. That is, each x in X is paired with some y in Y is equivalent to an arbitrary x in X is paired with some y in Y. Similarly, one y in Y is paired with each x in X is equivalent to one y in $Y$is paired with an arbitrary x in X.

[2] The relationships "each y to some x" and "one x to every y" can also be constructed with "for all y in Y, there exists x in X such that $R(x,y)$" and "there exists x in X such that for all y in Y, $R(x,y)$", respectively.

[3] One must *imagine* pairing each member of the set X with a member of the set Y in the case that the cardinality of X is infinite.

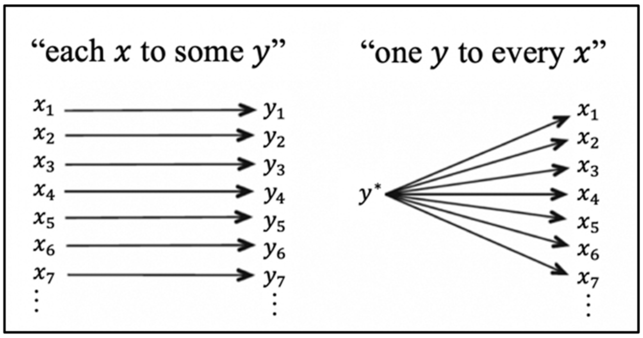

Fig. 1. Pairwise relationships (adapted from Dawkins & Roh, 2020, p. 4).

AE statement: For all x in X, there exists y in Y such that $R(x,y)$.	*EA statement:* There exists y in Y such that for all x in X, $R(x,y)$.
Corresponding AE process: Iteratively take an x in X and find a y in Y such that $R(x,y)$ is true for the selected pair and then continue this process until every member of the set X has been paired with a y in Y. That is, take x_1 in X and then find an y_1 in Y such that $R(x_1,y_1)$, then select x_2 in X and then find an y_2 in Y such that $R(x_2,y_2)$, and continue (or imagine continuing) this process for every member of the set X.	*Corresponding EA process:* Find a y^* in Y so that $R(x,y^*)$ is true for every selected x in X. That is, find a y^* in Y such that if $X = \{x_1, x_2, x_3, \dots x_n, \dots\}$, then $R(x_1,y^*), R(x_2,y^*), R(x_3,y^*), \dots, R(x_n,y^*) \dots$ are satisfied.

Fig. 2. AE and EA processes.

These approaches give important insight into the meaning that the mathematics community intends to convey with AE and EA statements. In the following section, I unpack some of the functional grammar of AE and EA statements to further investigate *why* an AE statement describes an *AE process* and *why* an EA statement describes an *EA process*. While one could reasonably argue that this an arbitrary convention (Hewitt, 1999), SFG suggests that meaningful choices are made with the grammar. I argue that this exploration is important if we are to support students in making sense of the mathematical language rather than ask that they memorize that given sentence structures map to particular processes with pairwise relationships as Hewitt (1999) would recommend with arbitrary conventions.

2.2.2. Investigating the functional grammar of AE and EA statements

In what follows, I will investigate the functional grammar of AE and EA statements as it relates to the current study. One of the most cited grammatical choices of AE and EA statements is about the order of quantifiers with their quantified variables. Both textbooks (Alcock, 2012; Houston, 2009) and research articles (Dawkins & Roh, 2020; Dubinsky & Yiparaki, 2000; Roh & Lee, 2011) have described the order of the quantifiers as the key indicator of the intended meaning. This suggests that a communication test in which the two quantified variables are switched should give important insight into the function of the quantifier order. See Fig. 3 for such a test with the False EA statement (Example 4) and the False AE statement (Example 2) introduced in Table 1.

Notice, the results of literally switching the order of quantified variables (with their quantifiers) does *not* result in the expected AE statement (i.e., For all real numbers x, there exists a real number M such that $3x + 2 < M$) or the expected EA statement (i.e., There exists a point C on the segment $\overline{AB}$ such that for every positive real number s, $d(A,C) = s$). Both results have a different placement of "such that". There seems to be a lack of a transition between the existentially quantified variable and the universally quantified variable in the result of Example 2 and Example 4. This test suggests that the placement of "such that" is a meaningful choice for the mathematics community[4]; "such that" should follow "There exists a real number M" and "there exists a point C on the segment $\overline{AB}$". See

[4] This choice is confirmed in the literature. Houston (2009) explained that when reading a statement using symbols (e.g., $\exists$ x in $\mathbb{Z}$ ($x^2 - 4x + 3 = 0$)) the reader should "insert a 'such that' between the quantifier and the sentence to which it refers" (p. 81). We read this example statement as, 'There exists an integer x such that $x^2 - 4x + 3 = 0$'. Although I found no other explicit mention of the use of 'such that' (or equivalent phrases) with AE and EA statements, all of the example statements in the literature that I reviewed were consistent with this advice. Additionally, while this analysis is beyond the scope of this paper, one could further investigate the function of "such that" is carried in different symbolic representations than the English language.

False EA Statement (Example 4):

There exists a real number M such that for all real numbers x, $3x + 2 < M$.

Example 4 Result:

for all real numbers x, such that There exists a real number M $3x + 2 < M$.

False AE Statement (Example 2):

For every positive real number s, there exists a point C on the segment $\overline{AB}$ such that $d(A, C) = s$.

Example 2 Result:

there exists a point C on the segment $\overline{AB}$ For every positive real number s, such that $d(A, C) = s$.

Fig. 3. Communication Test for order of quantified variables.

Fig. 4. This switch does produce the anticipated EA and AE statements (if one were to adjust capitalization appropriately).

Using these tests, we can also glean some insight regarding the function of "there exists y in Y such that" in AE and EA statements. The function of "there exists a point C on the segment $\overline{AB}$ such that" in Example 2 is to signal the reader to find a particular point on the segment that fits a particular criterion.[5] If "such that" did not follow an existential quantified variable (as exemplified in Fig. 3) then the reader could be left to wonder how to select an appropriate value. Thus, the function of "there exists y in Y such that" is to inform the reader to find a particular y in Y that fits the criterion.

The test in Fig. 4 suggests that the order of "there exists y in Y such that" and "for all x in X" is a meaningful choice for construing an *EA* or *AE process*. In Example 4, $3x + 2 < M$ depends on the universally quantified variable x and the existentially quantified variable M. In Example 2, $d(A, C) = s$ depends on the universally quantified variable s and the existentially quantified variable C (and the given A). How one selects x and M (or s and C) is determined by whether "there exists a real number M such that" comes before or after "for all real numbers x," (or whether "there exists a real number M such that" comes before or after "for all real numbers x,"). Choosing to place "there exists a real number M such that" *before* "for all real numbers x," is meant to emphasize that the reader should look to the criterion that $3x + 2 < M$ in order to find a particular M that is greater than $3x + 2$ for every selected real number x. That is, it functions to describe an *EA process*. Whereas choosing to place "there exists a real number M such that" *after* "for all real numbers x," functions to tell the reader to look to the criterion that $3x + 2 < M$ in order to find a particular M for a given real number x. And the placement of "for all real numbers x," functions to tell the reader to imagine iterating the pairing process for every real number (i.e., an *AE process*).

The system is also important to consider in order to understand the grammar of AE and EA statements. There are some alternatives that would not change the meaning. Some direct substitutions that do not change the meaning of "such that" are "where", "satisfying", or "for which". Additionally, "there is" could go instead of "there exists" and "for every" or "for any" can go instead of "for all". I offer three equivalent statements (that convey the same *AE process*) in Fig. 5. Notice that "For all positive real number s," in place of "For every positive real number s," is not a meaningful choice. That is, when I constructed these statements the choice of using one of these phrases over the other did not change the conveyed meaning. This is interesting in relation to English grammar (using the natural language registrar): the noun "number" is singular when used with "for every" and plural when used with "for all". This is an example in which the English grammar is stricter than the mathematical grammar, highlighting that it is not always the case that mathematical language is more technical than the English language.

One possible explanation for students' difficulty with interpreting statements with multiple quantifiers that Dubinsky and Yiparaki (2000) explored is whether the use of 'all' versus 'every' impacted the way that students interpreted the statements. They posited this as a reasonable conjecture since "the word 'all' seems to be collective" and that "the word 'every' appears to address each case individually" (p. 22). In their study, they interviewed students and asked them to interpret AE and EA Statements in the natural context (e.g., There is a mother for all children) and the mathematical context (e.g., For every positive number a there exists a positive number b such that $b < a$). They hypothesized that if students made a distinction with all instead of every, then they would likely interpret statements with all as an EA statement whereas statements with every would be interpreted as an AE statement. However, they argued that students holding a collective meaning was not supported by their data by comparing students' interpretations of statements with "every" and "all" in the natural context (e.g., "there is a mother for *all* children" vs. "there is a perfect gift for *every* child", emphasis added). Importantly, they did not report on this theory with the statement in the mathematics context (both mathematical statements used "every"). It is plausible that students might interpret different meanings with mathematical statements that use all versus every, since English grammar is different from mathematical grammar.

The second choice that is important to highlight is that using "there exists a real number M" instead of "there exists a unique real

[5] Similarly, the function of "there exists a real number M such that" in Example 4 is to signal finding a particular real value that fits a particular criterion.

False EA Statement (Example 4):

There exists a real number M such that for all real numbers x, $3x + 2 < M$.

Example 4 Result:

for all real numbers x, There exists a real number M such that $3x + 2 < M$.

False AE Statement (Example 2):

For every positive real number s, there exists a point C on the segment $\overline{AB}$ such that $d(A, C) = s$.

Example 2 Result:

there exists a point C on the segment $\overline{AB}$ such that For every positive real number s, $d(A, C) = s$.

Fig. 4. Communication Test for order of quantified variables.

For every positive real number s, there exists a point C on the segment $\overline{AB}$ such that $d(A, C) = s$.

For all positive real numbers s, there is some point C on the segment $\overline{AB}$ satisfying $d(A, C) = s$.

For any positive real number s, there exists at least one point C on the segment $\overline{AB}$ for which $d(A, C) = s$.

Fig. 5. Statements that convey the same AE process.

number *M*" (or "there exists exactly one real number *M*") is a meaningful choice because the latter changes the conveyed meaning of the statement. For instance, changing "For all real numbers x, there exists a real number M such that $3x + 2 < M$" to "For all real numbers x, there exists a unique real number M such that $3x + 2 < M$" changes the meaning from something that is true to something that is false. The mathematics community chooses to use "there exists" to mean there exists at least one (Alcock, 2012) and use phrases like "there exists at most one" or "there exists a unique" when it is important to convey uniqueness (Batty & Woodhouse, 1994).

By focusing on the system and structure choices of AE and EA statements (as used by the mathematics community) like I have here, we can unpack the grammatical choices of the statements. These grammatical choices are not simply arbitrary conventions, rather they work to construe specific meanings.

3. Methods

3.1. Data collection

This exploratory study grew out of a larger design research project that aimed to leverage guided reinvention contexts to support students in learning about formal mathematical language (Vroom, 2020b). The overarching instructional approach is guided by Realistic Mathematics Education in which students are supported to construct mathematics from their informal mathematical ideas (Gravemeijer, 1998). Here, I present a retrospective analysis of data from one round of the design experiment (Cobb & Gravemeijer, 2008) with a pair of students, who I refer to as Ada and Lori.

Ada and Lori were recruited from a community college bridge course (similar to an Introduction to Proofs course) that implemented inquiry-oriented materials in the context of real analysis (for more information about the materials see Strand, 2016). In this particular class, the students conjectured the Intermediate Value Theorem and made progress on proving their conjecture, defining key related concepts towards this goal. Much of their defining activity during this course worked to specify different sequence properties such as increasing and bounded. I chose to invite Ada and Lori to participate in the study since I observed them working well together during class and an exit survey suggested that the students had more to learn about formal mathematical language.

The participants typically excelled in their mathematics course work. Prior to the bridge course, Ada had completed the single-variable calculus courses and Lori had completed all introductory mathematics courses offered by the community college (i.e., the full calculus sequence, Differential Equations, Linear Algebra). The laboratory design experiment lasted 11 teaching sessions, each session lasting approximately 1.5 h. I was the teacher-researcher and there was at least one witness present. All sessions were recorded on multiple devices (audio and video) and I used these recordings between sessions to create detailed content logs (more information about the logs will be described in the next subsection). Between sessions, I debriefed with the witness and another researcher who read the content logs prior to our debriefing meetings. During these meetings, we reviewed our interpretations of what happened and revised my plan for the following session when needed.

The goal of the laboratory design experiment was to leverage Ada and Lori's experience with the inquiry-oriented materials, and specifically with sequences, to support them in learning about formal mathematical language and proof more broadly. Having a rich (shared) meaning for a concept prior to attempting to articulate the concept with a defining property was an important theoretical design decision. Thus, I choose to focus much of their defining work on various sequence properties since the students had experience with sequences that held different properties. Throughout the experiment I regularly engaged the students in writing mathematical statements (writing defining properties or conjecturing) to articulate their thinking. The students were encouraged to write their idea informally and then the students and I would work to refine the statement so that it was more formal and more accurately articulated their idea. Some of the concepts that I asked the students to write defining properties for emerged from their own mathematical exploration. For instance, "eventually constant sequences" emerged from the students generating such a sequence by applying a bisection method that they developed during the bridge course. Other times, I offered more contrived sequence properties that featured a specific relationship that I thought would be beneficial for the students to attempt to articulate with formal mathematical language. For instance, I offered "prime sequences" as those that had every prime number show up at least once in the list of numbers (an *AE process* in which each prime is associated with some index).

Throughout the experiment, I made conjectures about the ways the students were thinking about the language and tested these conjectures in subsequent teaching sessions. In particular, I asked the students to explain the meaning of a given statement with a grammatical choice that I anticipated was meaningful for the students. The given statement was either one that they previously constructed or one that I constructed based on their prior thinking. Then, I proposed a structure or system grammatical changes, and we would discuss whether (and how) it changed the construed meaning. For example, during the experiment I conjectured that the students thought that "there exists" could *not* replace "there exists at least one" without changing the meaning of a statement. To test this conjecture, I asked them to explain the meaning of a statement that they constructed that used "there exists at least one" and then asked whether I could replace it with "there exists". In this case, the students confirmed my conjecture explaining how they imagined the grammatical change affecting the conveyed meaning.

The data presented in this study is quite different from the data analyzed in prior studies on students' understandings of statements with multiple quantifiers. Prior studies have investigated students' extracting meaning from a given statement (Dawkins & Roh, 2020; Dubinsky & Yiparki, 2000; Sellers et al., 2021), whereas I leverage how students attempted to articulate their meanings to make sense of their thinking about the grammar of AE and EA statements. The difference afforded a functional approach to make sense of the students' meaningful grammatical choices that they used to convey *AE* and *EA processes*.

3.2. Data analysis

My main goal for the data analysis that I present in this study was to investigate the ways that my participants thought about the grammar of AE and EA statements (rather than focus on their longitudinal development of their gained fluency with these statements[6]). In order to investigate this, I conducted a retrospective analysis in which I first re-watching the videos and created content logs. The content logs were detailed chronological descriptions of the tasks and the activity that occurred during each session that included transcribed excerpts and pictures of student work. After the teaching sessions ended, I reread the content logs tagging timestamps in which I posed questions related to how the students might answer "what goes together with what" and "what could go instead of what?" (Halliday & Matthiessen, 2013, p. 22) in relation to the quantifiers in multiply quantified statements. Each of the tagged instances were explicit discussions about how, if at all, different structure or system choices changed the meaning of the given statements. Using these discussions, I wrote analytic memos of how and why the students might answer the following guiding questions: what goes with "there exists"?, what goes instead of "there exists"?, and what goes instead of "for all"? Answers to these questions attended to how each grammatical choice was understood as part of the larger framework of the full statement. In this way, the memos were my attempt at describing the functional organization of the multiply quantified statements from the students' perspectives.

I then returned to the full content logs to further understand how (or if) my answers could explain students' construction or interpretations of statements. For example, with my emerging understanding of what the students considered could go instead of "there exists", I searched my full data set for instances when we worked with statements with an existential quantifier. I then analyzed the data to confirm or dispute my conjecture, and then refined my analytic memo of students' thinking about what could go instead of "there exists". This process helped me to both test the validity of my conjectures about the students' thinking as well as gain more understanding about the impacts of their meaningful grammatical choices.

4. Results

The following results are presented in two subsections. In Section 4.1, I elaborate on the students' thinking about the grammar of universally quantified variables in relation to *AE* and *EA processes*. In Section 4.2, I then discuss the students' thinking about the grammar of existentially quantified variables in relation to these processes. The results are not necessarily presented in chronological

[6] Readers interested in the students' defining activity and/or their gained fluency with statements with multiple quantifiers should see Vroom (2020a, 2020b).

order, but lends itself to understanding the students' thinking about the grammar of AE and EA statements. In particular, the results present the meaningful grammatical choices that Ada and Lori made related to the universal and existential quantifiers. These choices are then considered in relation to full statements to understand how those choices impacted the way Ada and Lori interpreted multiply quantified statements and used quantifiers to describe their meanings.

4.1. Students' thinking about the grammar of universally quantified variables

4.1.1. The students saw differences in "for all," "for any," and "for every" and construed different meanings from statements with these phrases

Throughout the teaching sessions, the students continuously engaged in thinking about their language and how it communicated (or did not communicate) their ideas. One common point of discussion was the differences that Ada and Lori saw between "for all," "for any," and "for every" in various statements. Ada explained, "I feel like 'for any' sounds more singular, and I think 'for all' sounds a little bit more plural" (and later explained that, to her, 'for every' also sounded singular). By singular, Ada meant "you can just pick anything," meaning you can consider one arbitrary member of the set. On the other hand, Ada imagined selecting the whole set that the quantified variable belonged to (e.g., X) at once for the phrase "for all x in X". This distinction is similar to Dubinsky and Yiparaki (2000) hypothesized interpretation that the word "all" construes a *collective meaning* whereas the word "every" seems to address each case individually. Unlike Dubinsky and Yiparaki, I found that this grammatical difference changed the conveyed meaning for Ada and Lori.

To illustrate, the *collective meaning* of "for all" made a difference in the conveyed meaning of Ada and Lori's "Prime Sequence" definition. During the ninth session, Ada and Lori explained that prime sequences "must contain all prime numbers," "may have other terms [that are not prime]", and "doesn't have to go in order" (meaning the prime numbers did not have to be listed from least to greatest). They eventually captured this idea in the definition given in Fig. 6A, in which $\boldsymbol{P}$ represented the set of prime numbers. During the last session, I aimed to determine the significance of Ada and Lori's word choice "for every". I asked them if we could replace "every" with "all". The following conversation followed:

> Lori: No, it has to be a "for every" doesn't it?
>
> Teacher-researcher: How so? Or why does it have to be "for every"?
>
> Ada: Because […] it still sounds like they all go to one k-value. They all map to one k-value. So, all the prime numbers are like, all the prime number are sitting on top of each other at just one k-value.
>
> Teacher-researcher: This does? (pointing to "for all p in P" in Fig. 6A).
>
> Ada: Yeah.
>
> Teacher-researcher: This "for every" or "for all"?
>
> Ada: If you were to say "for all". This is what "for all" does for number seven (sketches picture in Fig. 6B).
>
> Teacher-researcher: Can you explain how it is that?
>
> Ada: Because… [.] So it's like for all of them. They all – it's just they all mapped to one. That's what it sounds like. How do I describe that?
>
> Lori: "For every" sounds like a little bit more like you're singling out….
>
> Ada: Each individual.
>
> Lori: Here's one [prime number], here's a k. Here's one [prime number], here's a k. But if you say "for all" (makes a gesture like she is picking up a large object) ….

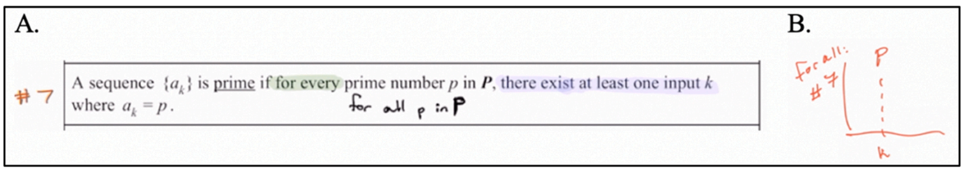

Fig. 6. Prime Sequence definition with "for every" vs. "for all".

Ada: Yeah, exactly! (Laughs).

Lori: And here is a k.

During the above exchange, Lori and Ada explained that one could not replace “for every prime number p in P” with “for all p in P” without changing the meaning of the definition. They intended their definition to convey an *AE process* in which each prime number was mapped to by some natural number. However, replacing “for every prime number p in P” with “for all p in P” did not describe this *AE process*. Instead, the phrase instructed the reader to select the whole set of prime numbers (demonstrated by Lori’s gesture of picking up a large object) and then find one corresponding input (“And here is a k”). This process does not describe a sequence, much less a Prime Sequence, since it required one input k to map to every prime (an *EA process*). “For every” was a meaningful choice in their Prime Sequence definition because it played a key function in describing an *AE process*: it instructed the interpreter to select one prime number at a time allowing the interpreter to iteratively pair prime numbers with input values. From the students’ point of view, the *collective meaning* of “for all” disabled the interpreter to engage in the iterative selection of individual prime numbers and corresponding input values.

In this example, the students’ description of the altered definition’s meaning suggests how they would answer the questions of what could go instead of “for all” differently than the mathematics community. In particular, the mathematics community takes “for all” and “for any” as equivalent phrases (Batty & Woodhouse, 1994). Alternatively, the students considered a *collective meaning* of “for all”. The *collective meaning* essentially changes the meaning of an AE statement to describe an *EA Process*. If students use the *collective meaning* with “for all x in X, there exists y in Y such that $R(x,y)$” then it is likely that they will interpret the statement as claiming there is one y^* in Y such that $R(x,y^*)$ is satisfied for every x in X.

4.1.2. An example of what goes with the collective meaning of “for all”

The next example also comes from the ninth session with Ada and Lori when they started to define Prime Sequence. At this point in the session, the students had created examples and non-examples of prime sequences and offered an informal definition (“A prime sequence is a sequence that must contain the set of prime numbers in its range”) but had not yet written the statement discussed in the previous section (Fig. 6A). During this session, I was also unaware of how the collective use of “for all” could affect the way that the students construed meaning with AE statements. As an attempt to support Ada and Lori in refining their informal definition, I requested that they rewrite their statement using “for all” and/or “there exists.” The idea was that this request might generate discussions of the *AE process* and support them to produce a statement like: for all p in P, there exists k in N such that $a_k = p$. However, such a request is quite challenging if one were to use the *collective meaning* of “for all”. For Ada and Lori, “for all p in P” could not go before “there exists k in N such that $a_k = p$” and still convey their intended meaning. This structure choice was not an option for Ada and Lori to encode their intended meaning since the *collective meaning* of “for all” would not describe an *AE process*.

After my request, Ada quickly responded “I definitely [want to] put the ‘for all’ with the prime numbers like - for all prime numbers”. She expressed that she wanted their statement to convey that, “each prime number has its own natural number.” Then, Ada and Lori produced the statement in Fig. 7. Using the *collective meaning* of “For all prime numbers in P”, I take their definition to mean: there is a step in the sequence that lists a particular prime number and this is the case for the set of prime numbers. While the students eventually saw a need to refine this statement (in which they ultimately chose the wording “for every”), it does illustrate what goes with the collective “for all” in order to convey an *AE process*. In this case, “there exists an input k which maps to a prime number” went before “for all prime numbers in P” as a way to instruct the reader to find an input for a prime number and then continue to do this for each member of the set of prime numbers.

In summary, Ada and Lori’s *collective meaning* of “for all” impacted the way in which they communicated their ideas. In Section 4.1.1, I explained how from Ada and Lori’s point of view “for all” could not go instead of “for every” in their Prime Sequence definition and still describe an *AE process*. In Section 4.1.2, I offered an example of how Ada and Lori structured a statement with “for all” in order to convey an *AE process*. Taking a *collective meaning* of “for all” can be a source of communication issues in at least two ways. First, imagine a reader with a *collective meaning* of “for all” (e.g., Ada and Lori) and a writer who sees “for all” as synonymous with “for every” (e.g., a textbook author). The reader is unlikely to interpret the writer’s intended meaning in the case of an AE statement that uses “for all”. Additionally, imagine a writer like Ada and Lori with a *collective meaning* of “for all” and a reader like a mathematics instructor who does not have a *collective meaning*. The reader is unlikely to interpret the writer’s intended meaning in many cases, such as the case in Fig. 7.

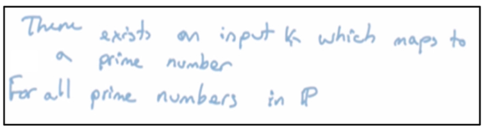

Fig. 7. Proposed property for Prime Sequence definition.

4.2. Students' thinking about the grammar of existentially quantified variables

4.2.1. The students saw differences in "there exists at least one" and "there exists" and made a different grammatical choice than the mathematics community

During the ninth session with Ada and Lori, they refined their Prime Sequence Definition (in Fig. 7) to the definition given in Fig. 6A, which included "there exists at least one input k". In order to determine the significance of "at least one", I asked if they could simply use "there exists an" instead of "there exists at least one". Ada was hesitant, explaining, "Well it doesn't matter too much. But it also kind of insinuates that *each prime number can't be repeated* that was my only concern" (emphasis added). I take this to mean that Ada was concerned that without the phrase "at least one", a reader could interpret their statement to mean that each prime number was paired with *exactly one* input. This was unsettling for Ada since they previously agreed that a prime sequence could list a prime number more than once. For instance, {2, 2, 3, 3, 5, 5, 7, 7, …} was a Prime Sequence even though the prime number 2 was paired with $k = 1$ and $k = 2$.

The two phrases "there exists at least one" and "there exists" are equivalent for the mathematics community. If a student used "there exists at least one" in place of "there exists," then someone who viewed the phrases as synonymous would likely interpret the students' statement as the student intended. However, if a student were to take "there exists" to mean "there exists exactly one" when interpreting a statement such as in a textbook then the student is likely to interpret an unintended meaning. For example, during the tenth session with Ada and Lori, I asked them to interpret an AE statement given the definition in Fig. 8 (which specifies an onto function). In what follows, I describe Ada's interpretation, which can be explained by taking "there exists a x in X" to mean "there exists exactly one x in X".

After the students had time to think privately about the given definition, I followed up with Ada about a picture she drew (see Fig. 9). She explained the parabola did not represent a type-2 function because "y maps to two different values on both sides". I take this to mean that she did not see the function represented in her sketch as type-2 since two unique input values mapped to the same output (e.g., $f(x_2) = f(x_3) = y_2$). She continued to connect this idea to the definition: "it says 'there exists *a* x in X' (emphasizing 'a') it sounds like there is only one x for each y". Again, Ada considered "there exists a x in X" to mean "there exists exactly one x in X". For Ada, "there exists a x in X" in the definition of type-2 functions indicated that each y in Y must be mapped to by a unique x in X. That is, her interpretation suggested that all type-2 (onto) functions were one-to-one. However, this consequence is not the case if one were to think of "there exists a x in X" as "there exists at least one x in X" in the type-2 definition.

The previous two examples show that taking the *exactly one meaning* of "there exists" (rather than the *at least one meaning*) is meaningful for AE Statements. In the Prime Sequence definition, the students needed to allow prime numbers to be paired with more than one index, rationalizing their choice of using "there exists at least one input k". Additionally, using the *exactly one meaning* of "there exists" for the type-2 function definition explained Ada's interpretation: each output value must correspond with exactly one input value. The *exactly one meaning* changes the meaning of an AE statement in such a way that describes a stronger process than an *AE process*. That is, if one were to take the *exactly one meaning* with "for all x in X, there exists y in Y such that $R(x,y)$" then they likely consider the statement as claiming each x in X is paired with *only one* y in Y.

Ada and Lori also gave some insight into how the *exactly one meaning* affects the meaning of EA Statements. During the last session, I explicitly asked the students whether they took "there exists" to mean "there exists at least one" or whether they saw a distinction with those phrases. The students then decided to consider the generic EA statement in the form "there exists a in A… for every b in B…". Lori drew two pictures to capture the two possible meanings (see Fig. 10). With Fig. 10A, Lori explained that "there is *an* a for every b," seemingly highlighting that there was only one a in A associated with every b in B. Alternatively, Lori explained that adding "at least one" to the phrase could "open the potential possibility of there being more [a values]". She then continued to draw the picture in Fig. 10B while stating "a_1 goes to b_1 and to b_2, to b_n, but a_2 also goes to b_1, b_2, and b_n". I take Lori's descriptions and pictures to mean that the *exactly one meaning* changes the meaning of an EA statement. If one were to take the *exactly one meaning* with "there exists y in Y such that for all x in X, $R(x,y)$" then they likely consider the statement as claiming there is *only one* y^* in Y such that $R(x,y^*)$ is satisfied for every x in X. Alternatively, the *at least one meaning* would convey that there is one (though potentially more) y^* in Y such that $R(x,y^*)$ is satisfied for every x in X.

In the previous examples, the students highlighted two meanings for the existentially quantified variable that were important to AE and EA statements: the *exactly one meaning* and the *at least one meaning.* Whether or not they were allowed to have more than one value that fit the criterion has implications for the construed meaning. The students chose to encode this decision by using "there exists a" to suggest uniqueness and "there exists at least one" to indicate when more than one value was allowed to fit the criterion. The mathematics community makes a different (and opposite) choice. When uniqueness is important, the community adds "unique" or "exactly one" to "there exists" (Alcock, 2012; Batty & Woodhouse, 1994). Importantly, both the students and the mathematics community identify the conceptual distinction, but they chose different grammatical functions to convey this distinction. This decision had important implications for the meaning of AE and EA statements. If one took the *exactly one meaning* of "there exists", then an AE statement with the phrase would likely convey an *each to exactly one* relationship (rather than an *each to some* relationship) and an EA

A function, $f: X \to Y$, is type-2 if for all y in Y there exists a x in X such that $y = f(x)$.

Fig. 8. Type-2 (Onto) Function definition.

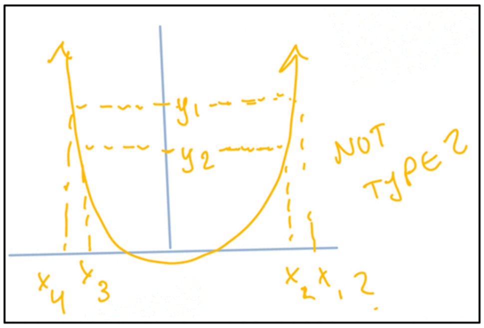

Fig. 9. Ada's non-example of type 2 function.

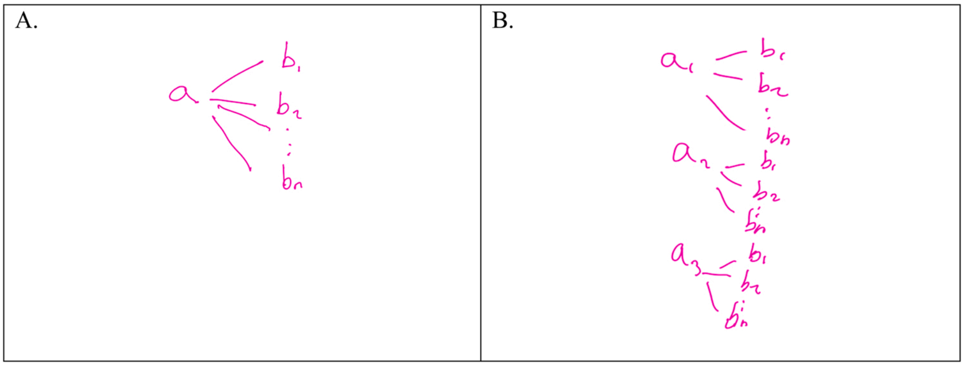

Fig. 10. Relationship in EA statement with (A) exactly one meaning and (B) at least one meaning.

statement would likely convey an *exactly one to every* relationship (rather than a *one to every* relationship).

4.2.2. The students recognized consequences of choosing "there exists some natural number L" to mean "there is at least one ***L*"**. The next example comes from the third session when the students defined Eventually Constant Sequence. See Fig. 11. They intended the definition to describe a sequence that had an index value such that every index value after it mapped to the same real number. Notice that their definition used "there exists some natural number *L*".

After they constructed their definition, I asked them to explain how an example that they previously constructed satisfied their definition (see Fig. 12A).

Ada: I notice that there are a whole bunch of terms over here and they look constant to me so I worked backwards to the beginning. So, if they are all equal to the same value like $y = 7$, and we would be like 'how far back does it equal 7?' Well like this one equals 6 so that one isn't going to work, like $y = 6$ here. But this one equals 7 and it is the first of this groups of 7 s and so I would say this one is *L*. And so that means that for all *k* beyond that *L* it should be constant and it is. (See Fig. 12B).

[…].

Teacher-researcher: What if I picked my *L*-value. (labels another *L* on the horizontal axis, see Fig. 12C).

Ada: Why would you do that?

Teacher-researcher: Well would that be ok?

Ada: It wouldn't because - oh this is awesome - you're like totally picking at this (looks at the definition). Wow! It would work by our definition, so we need to fix that!.

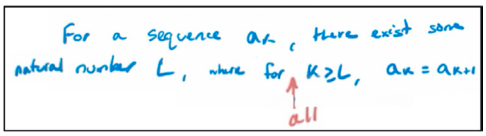

Fig. 11. Eventually Constant Sequence definition.

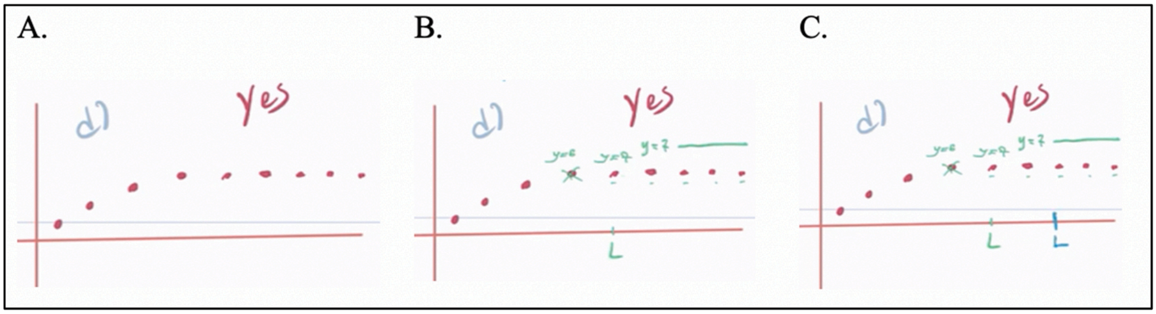

Fig. 12. Potential L-value.

In the above exchange, Ada explained that the example fit their Eventually Constant definition because she could find a *minimal* input value L for which every successive output was equal to a_k. Notice that there is exactly one type of input value; however, Ada considered their definition to allow more than one input value. ("Wow! It would work by our definition…") This suggests that Ada saw "there exists at least one natural number L" as a replacement for "there exists some natural number L". Since she intended the definition to find the minimal input value, she expressed wanting to alter their definition so that the grammar functioned to convey her intended meaning. ("…we need to fix that!") The conversation continued with Lori questioning whether there was a problem specifying Eventually Constant Sequence in this (different) way.

Lori: Do we?

Ada: I think we need to fix it. There exists some natural number. because for all k above this random L that she chose it is all equal to each other so we can totally say that's L. So how do we fix that?

Lori: Does it matter though? Isn't the L just arbitrary to show. because it is infinitely long.

Ada: It is arbitrary, I can understand that, but I think that if we are going to use this L definition, I think that L should be a very specific thing. And I think our original definition was that L was the very first of the constant - the section where it appears constant. Does that make sense?

Lori: Yeah, I just don't know if it is necessary.

Ada: It doesn't sound necessary, I know what you're saying. But I feel like there is something that is telling me that it is necessary.

In the above segment, Lori questioned whether they needed to change their definition to highlight that there was a first value in which the sequence appeared constant, suggesting that the concept could be defined without identifying the minimal L. Choosing "there exists some natural number L" to mean "there is at least one L" consequently conveyed that there is at least one input value L for which every successive output was equal to a_k rather than that there is a *minimal* such input value. To her, the concept could be specified in this different way. At this point, the students returned to the collection of examples and non-examples of Eventually Constant Sequences that they previously created to see if there was an issue with any of their (non-)examples from choosing an L greater than the first index value that satisfies the condition. Ada said, "I am not seeing a problem." Then Lori added, "as long as from that [index] and beyond it satisfies that condition, […] I don't think it would matter where along the constant portion that you pick your L". At the end of the session, Ada and Lori decided to keep their definition in Fig. 11 because there were two equivalent conceptualizations of the concept, which they were able to recognize by considering the consequences of choosing "there exists some natural number L" as a replacement for "there is at least one L".

A functional perspective emphasizes that Ada and Lori saw a conceptual distinction with their initial intended meaning (there exists a minimal input value) and "there exists some natural number L" in their constructed definition. Their constructed definition allowed an interpreter to select an input value larger than the minimal value to show that the property held. Even though using "there exists some natural number L" conveyed a different meaning, the students realized that the different meaning was an alternate way to specify the concept.

The previous two subsections give insight into Lori and Ada's thinking about existentially quantified variables in multiply quantified statements. In Section 4.2.1, I discussed how Ada and Lori did not think that "there exists" could go instead of "there exists at least one" and still convey the same meaning. I argued that this choice could be a source of communication issues because taking "there exists" to mean "there exists a unique" impacts the way one interprets multiply quantified statements (e.g., when Ada interpreted AE statement that specified onto functions). In Section 4.2.2, I offered an example of how Ada and Lori dealt with the consequences of taking "there exists some" to mean "there is at least one" when defining Eventually Constant sequences.

5. Concluding remarks

In what follows, I first discuss the main contributions of this study. In Section 3.1, I will discuss what this study contributes to the existing research literature on students' thinking about multiply quantified statements. In Section 3.2, I will revisit two theoretical contributions of this study – a functional perspective on AE and EA statements and corresponding *AE* and *EA processes*. Then, in Section 3.3 I discuss the practical implications for teaching from my study. I will close the paper in Section 3.4 describing insights that a functional grammar perspective afforded my study, arguing for its utility for research that seeks to understand formal mathematical language and students' thinking about it.

5.1. Students' thinking about the grammar of AE and EA statements

Prior research has found that students have difficulties interpreting multiply quantified statements as intended and have identified factors that influence their interpretations (e.g., Dubinsky & Yiparaki, 2000; Dawkins & Roh, 2020). Additionally, Sellers et al. (2021) identified different student meanings for quantified variables, including checking whether the predicate of the statement holds for at least one value, exactly one value, for all values, or spontaneously chosen values. My findings extend these ways that students think about AE and EA statements in two ways.

First, I identified another student meaning that has not yet been documented in the literature: the *collective meaning*, in which a student checks whether the predicate of the statement simultaneously holds for the whole domain of discourse. The students in this study viewed "for all" differently than "for every" or "for any". To them, with phrases like "for all x in X" they imagined selecting the whole set X that the quantified variable belonged to.

Second, my study systematically connected the students' meanings to the grammar of AE and EA statements, and notably, described how each meaning affected the full meaning of the statements. I emphasized two grammatical choices. First, the students in my study connected the *collective meaning* to "for all x in X." This was attributed to the English grammar of "for all" in which a noun that follows "for all" should be plural (e.g., "for all real numbers"). The *collective meaning* impacted the way that students interpreted AE statements with the phrase. I argued that if one were to have a *collective meaning* of "for all", then they are likely to interpret an AE statement that uses the phrase as an *EA process* rather than an *AE process*.

The second grammatical choice that I highlighted was regarding the existential quantifier. Like Sellers et. al's (2021) participants, the students in this study checked if the predicate held for (a) exactly one value of x in X and (b) at least one value of x in X. In addition, I found that Ada and Lori connected the *exactly one meaning* to phrases like "there exists a x in X" whereas they were more explicit with their wording to convey the *at least one meaning*, using phrases like "there exists at least one x in X". This choice is different from the mathematics community's choice in that the mathematics community chooses to be more explicit with their wording to convey an *exactly one meaning*, using phrases like "there exists a unique x in X" and use phrases like "there exists a x in X" to convey the *at least one meaning*. Both the students and the community use the grammar to convey this conceptual distinction; the students in my study chose a different grammatical choice in order to do so. I also demonstrated how this grammatical choice impacted the way that students might interpret AE and EA statements. I argued that if one were to have an *exactly one meaning* for the existentially quantified variable then they are likely to interpret an AE statement as describing an *each to exactly one* relationship and an EA statement as an *exactly one to every* relationship.

Thus, both the *collective meaning* and the *exactly one meaning* can be a potential source for communication issues. Furthermore, the *collective meaning* may explain why many students interpret AE statements (with "for all") differently than the mathematics community (Dawkins & Roh, 2020; Dubinsky & Yiparaki, 2000). It is important to note that the grammatical choices that I highlight in this study were related to the word choices, or the system, rather than the structure. Earlier work has attended to the influence of the structure, and in particular the order of the quantifiers, in students' interpretations of multiply quantified statements. While the ordering is part of what might influence students' interpretations, this study shows there are other grammatical features that might also play a key role.

5.2. A functional perspective of the grammar of multiply quantified statements

There are two theoretical contributions of this study: the functional grammar of AE and EA statements and the process approach to thinking about the statements. First, this study offers insight into *why* the mathematics community uses AE and EA statements to convey *AE* and *EA processes* (or corresponding pairwise relationships). In Section 2.2.2, I offered a functional perspective of the grammar of the statements. Rather than using prescriptive rules (e.g., the order of the quantifiers indicates a particular relationship between the quantified variables), this study explained some meaningful grammatical choices in conveying particular meanings. I argued that placing "such that" (or equivalent) after an existentially quantified variable "there exists x in X" is a meaningful choice in order to instruct the interpreter to find a particular element from the specified domain of discourse X that fits a particular criterion. Without such a phrase following an existential quantified variable, the interpreter is left to decide how to select an appropriate value. Additionally, I discussed how placing "for all x in X" before "there exists y in Y such that" functions to describe an *AE process*; whereas, "there exists y in Y such that" before "for all x in X" functions to convey an *EA process*.

The process approach to thinking about AE and EA statements is the second theoretical contribution (see Section 2.2.1 and Fig. 2). Articulating these processes grew out of the ways that the students in this study discussed the meanings that they intended to convey with their definitions as well as how they interpreted statements that I gave them. I elaborated how the *AE* and *EA processes* were consistent with Dubinsky and Yiparki's (2000) instructional game, as well as how they complement the ways that other scholars have described approaches to thinking about AE and EA statements (Dawkins & Roh, 2020; Selden & Selden, 1995; Selden et al., 2018).

These processes also allowed me to articulate how each grammatical choice contributed to the total meaning of the AE or EA statements (both from the perspective of the students as well as from the perspective of the mathematics community). My descriptions of the *AE* and *EA processes* can contribute to future work on students' thinking about multiply quantified statements. Like Ada and Lori, I suspect that other students might leverage *AE* and *EA processes* to describe what they intend to convey or what they interpret an AE or EA statement to mean.

5.3. Implications for practice

This study offers some initial steps to improve teaching multiply quantified statements so that it goes beyond requesting students to simply memorizing prescriptive rules about the order of the quantifiers. The contributions of this work should support instructors to be able to answer *why* the conventional grammatical choices in AE and EA statements function to convey their respective processes and relationships (rather than *that* these statements convey these processes). This work should also equip instructors with the ability to notice other grammatical choices their students might make that contradict the conventional choices, allowing for them to address these inconsistencies when needed. I expect that this insight would enable conversations about mathematical language to move away from prescriptive rules, and towards mathematical reasoning involved with the grammar. Additionally, this insight is not limited to benefiting instructors; textbook authors can also benefit from this work. When discussing multiply quantified statements, textbooks can unpack some of the functional grammar and the conventional grammatical choices for students. Future research should investigate students' longitudinal development of their gained fluency with these statements as instructors and/or textbooks take a more functional approach.

Additionally, the data presented in this study shows that engaging students in activities that require them to use language to specify their own meanings, such as defining, paired with communication tests seem to be a productive activity for students. In particular, these contexts are promising for eliciting discussions about different meaningful choices that students could make with the grammar and the impacts of such choices in conveying *AE* and *EA processes*. I view these discussions as productive first steps in supporting students to learn about the grammar of AE and EA statements. More research is needed to understand how to then support students in making the same choices as the mathematics community in courses that are designed to support students in becoming more fluent with the conventional uses of formal mathematical language (e.g., Introduction to Proofs courses) or courses that expect students to abide by these conventions (as in more advanced proof-based courses like Real Analysis). In earlier work, I offer one way an instructor might support students in making the same grammatical choices as the larger mathematics community (Vroom, 2020b). Future research could further explore how an instructor can act as what is referred to as a broker (Lave & Wenger, 1991; Rasmussen et al., 2009; Wenger, 1999) between the students and the mathematics community, strategically sharing conventions when the students make choices that contradict the community's choices.

5.4. Utility of SFG for investigating mathematical language and students' thinking about It

This study exemplifies the utility of SFG as a key theoretical and analytical tool in studying formal mathematical language and students' thinking about it. Here, I attended to the system and structure of AE and EA statements, which, admittedly, is one of the more simplistic tools compared to others that Halliday and Matthiessen (2013) describe. Even still, the theory proved useful in two distinct ways.

First, SFG enabled me to make sense of the students' thinking about mathematical language. By focusing on how the students might answer the system and structure questions, I was able to make sense of how they could reasonably construct and interpret statements. Such an approach provided a non-deficit lens for explaining students' thinking about the language and was especially productive when the students used grammar differently than the mathematics community. The approach allowed me to identify when students considered important conceptual distinctions and connect these distinctions with how they choose to convey their meanings. For instance, choosing the *exactly one meaning* for "there exists a x in X" in the type-2 (onto) function definition explained why, for Ada, the property required one-to-one. A perspective that enables a researcher to attach student meaning to the wording and then investigate how that choice contributes to the total picture can explain student interpretations. In this case, the different interpretation was not based on some "misconception" of the type-2 (onto) concept, but rather was based on a different (and reasonable) grammatical choice. A functional approach would be productive for future research investigating students' thinking of formal mathematical language such as the language used in specific types of proofs.

Second, SFG also helped me to better understand why the mathematics community chooses to use the grammar in particular ways to convey *AE* and *EA processes*. If we are to support students in using the nuanced ways that the mathematics community communicates mathematical ideas, such as in formal mathematical statements or in formal proof, then we must question the purpose of these nuances and how they function to convey meaning. Future work that aims to investigate this matter would also benefit from utilizing SFG to investigate the conventional use of formal mathematical language.

As I have exemplified with the case of AE and EA Statements, the ways in which mathematical meanings are conveyed depend on more than arbitrary norms. The intentional grammatical choices that make up mathematical language purposefully function to convey precise mathematical meanings. I emphasize that instruction that aims to support students in learning about formal mathematical language must do more than request that students memorize how to use and interpret the language. Instead, instruction should support students to recognize mathematical distinctions and understand the choices that the mathematics community have made in order to discern such meanings.

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