Coherence and Transfer of Complex Learning with Fourier Analysis Learning Trajectories for Engineering Mathematics Education

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***Abstract*—Fourier analysis learning trajectories are investigated in this full paper as a joint interdisciplinary construct for a scholarly collaboration among engineering and mathematics faculty. This is a dynamic and recursive construct for aligning, developing, and sharing research based innovative practices for engineering mathematics education. Towards building more coherence and transfer of learning between engineering and mathematics courses, these trajectories offer experimental practice templates for the interdisciplinary community of practice for engineering mathematics education. Conjectured learning trajectories for Fourier analysis thinking are here articulated and experimented in three courses - Trigonometry, Linear Algebra, and Signal Processing. Informed by the interdisciplinary perspectives from the team, these trajectories help to design instruction to support the complex learning of the mathematical, and engineering foundations for the advanced mathematical concepts and practices such as Fourier Analysis for engineers. The results highlight the impact of collaborative, interdisciplinary, and innovative practices within and across courses to purposefully build and refine instruction to foster coherence and transfer with learning trajectories across mathematics and engineering courses for engineering majors. This offers a transformative process towards an interdisciplinary engineering mathematics education. The valid assessment and measurement of complex learning outcomes along learning trajectories are discussed for engineering mathematics education, paving the pathway for our future research direction.**

***Index Terms*—learning trajectories, Fourier analysis, transfer, engineering mathematics education, coherence, complex learning**

I. Introduction

One of the major challenges of engineering mathematics education is teaching and learning of advanced mathematical concepts such as Fourier analysis heavily used in upper level engineering courses. Due to the gaps among engineering and mathematics courses, the development and transfer of such advanced applied mathematics perspectives are often inadequately supported and integrated along the required mathematics and engineering courses for undergraduates. Fourier methods have found a significant place in engineering mathematics education programs around the world [1]. There is a need for articulating and supporting the emergence and progression of advanced mathematical practices for engineers across the courses and curricula.

Linking the theory and practice in engineering mathematics education requires educational research constructs that can recognize the complexity of learning and applying advanced mathematical content such as Fourier analysis. Learning trajectories are research constructs adopted here to inform, guide, and build research-based collaborations among faculty to innovate instructional practices for engineering mathematics education [2]. Learning trajectories are dynamic constructs, initially conjectured from practice, then validated and refined by experts from relevant disciplines including experiments with students and faculty.

Fourier analysis learning trajectory starts early and progress across the mathematics courses such as Trigonometry, Calculus sequence, Differential Equations, Linear Algebra, and engineering courses such as analysis of waves in Signal Processing, Control systems, Electromagnetic Theory. Research identified that the multiplicity of representations is a major source of the coherence problem in upper level math courses for engineers. Students struggle to make connections between different framings (geometrical, algebraic, abstract), between different registers of representations (graphical, algebraic, symbolic, tabular), and analytic-arithmetic and arithmetic-structural modes of reasoning when they are learning undergraduate mathematics education courses including calculus [3], [4].

Mathematical modeling is a practice that allows the connection of knowledge across disciplines such as engineering, mathematics, and technology by responding to real world challenges. Mathematical models and modeling activities as a sequence can be used to develop student understanding of deeper content and connections for engineering mathematics education along a targeted learning trajectory [6]–[8]. Modeling in engineering education can be implemented as a prototype for interdisciplinary mathematics education demanding coordination and assessment of interdisciplinary complex learning outcomes [9].

Fourier analysis as a branch of functional analysis is a core mathematical idea extensively utilized in engineering mathematics education. Its applications include modeling and processing of signals for electrical and mechanical engineers, as well as for solving differential equations. Functional analysis has been the key conceptualization in modernizing linear algebra extending its study of arbitrary vector spaces beyond $\mathbb{R}^2$, $\mathbb{R}^3$ to include infinite dimensional linear spaces and vector spaces of functions [10]. Fourier analysis methods require engineering students to develop an understanding of functional analysis perspective that they can apply in signal and data processing. On the other hand, Linear Algebra and Engineering Analysis courses generally fall short of providing a clear understanding of function spaces spanned by orthonormal functions as Hilbert Spaces, Banach spaces as complete, normed vector spaces.

Fourier's idea goes back to ancient astronomy where empirical models of planetary motions were build upon deferents and epicycles. Fourier's work offered a novel formulation of heat diffusion through Fourier series and integrals establishing a process that a continuous function can be represented by a trigonometric series [11].

II. Problem and Background

Learning trajectories provide an integrated approach to develop mathematical knowledge for teaching engineering mathematics through design and analysis of mathematical modeling tasks for engineering majors and consequently, students' learning experience, connecting, anticipating, attending to learning discourse and assessment [12].

Students are observed to have difficulty in recognizing conceptual shifts while making transitions from algebra $\rightarrow$ functions$\rightarrow$ analysis [13] where they need to conceptualize functions as mathematical objects to operate and build function spaces such as the Hilbert space as needed in Fourier Analysis.

While there are examples of Learning Trajectories in the research literature [5], [14]–[16], there is a gap in research in the development of Fourier Analysis thinking in undergraduate or engineering mathematics education. There is prior work on linear transformations [16]. We examine learning trajectories for research-oriented teaching and learning of Fourier analysis, as it builds across aligned mathematics and engineering courses. As a part of Fourier Analysis Learning Trajectory, its early foundation in Trigonometry is here investigated for coherent learning of complex and continuous trigonometric functions. This is one of the novel contributions made here to engineering mathematics education.

This represents an innovative interdisciplinary approach to reform engineering mathematics education curricula and practices to align and build an interdisciplinary mathematical knowledge content for engineering students. It builds a scholarly community of practice approach by designing, implementing and refining the instruction across mathematics and engineering courses with an interdisciplinary team of mathematics, engineering, and STEM education faculty [17]. Upper level engineering courses demand mathematical background that may not be addressed adequately in lower level courses such as Differential Equations and Linear Algebra. Instructors also feel the pressure to get students develop a strong mathematical foundation for learning engineering courses such as Signal Processing, that are heavily relying on advanced mathematical methods such as Fourier analysis. To support faculty and students, learning trajectories provide conjectures regarding the organization and the development of advanced mathematical concepts. Nested conceptual hierarchies support instructors purposefully design and experiment learning modules to build their local practices. It provides a dynamic framework for collaboration to clarify the concepts, connections, and instructional support to help students move along conceptual scaffolds in developing Fourier analysis perspectives in mathematics and engineering courses.

Learning Trajectories are studied here as research constructs for a scholarly collaborative practice of engineering mathematics education. Building on mathematics education research [18], [14], these trajectories allow experimentation on a graded and a fine-grained conceptual analysis of the development and transfer of Fourier and broader functional analysis thinking in engineering mathematics courses. Learning trajectories offer viable paths for instructors and learners for the conceptual development and assessment of the higher order mathematical concepts and practices such as Fourier analysis for engineers. Through design based research and mathematical modeling orientation, the conjectured learning trajectories on Fourier analysis are introduced and refined to build coherence and transfer of complex learning for engineering mathematics education.

We discuss how Fourier analysis can be intuitively introduced with College Trigonometry, evolve across Calculus sequence courses, and Differential Equations, then formalized in the Linear Algebra course for engineers with the introduction of inner product spaces. We examine the conceptual barriers in the multiple frameworks of its fundamental mathematical elements such as vectors and function spaces in developing Fourier analysis perspective for students.

A. Assessing Interdisciplinary Complex Learning Trajectories

The process of assessment should be aligned with the practice. Learning trajectories offer the pathways for building disciplinary connections and interdisciplinary connections helping to build advanced mathematical content. The assessment of interdisciplinary knowledge and competences for complex learning trajectories for engineering mathematics is critical to support the development and integration of multiple competencies for students with partner disciplines.

B. Questions

This innovative practice help us develop scholarly knowledge on practice for engineering mathematics education around the following questions:

- What mathematical concepts and connections should be targeted as hypothetical learning trajectories for Fourier analysis for undergraduates?
- How does the collaborative process of developing learning trajectories on Fourier analysis contribute to coherence and transfer of learning between and across courses on mathematics and engineering?

C. Theoretical Framework

This study is descriptive first to critically analyze the undergraduate content and practices as an interdisciplinary community of practice for the emergence and development of Fourier analysis thinking engaging Mathematics, Engineering, and Mathematics Education Faculty and an educational psychometrician. Design-based research with a sequence of learning activities in the aligned course modules are conducted. The faculty met to discuss the refinements of learning trajectory-aligned modules work based on the experience of students along the course of semester-long classroom teaching of trigonometry and linear algebra courses. In a span of three semesters, the relevant content and practice on Fourier Analysis Learning Trajectories were collected from Trigonometry, Engineering Analysis/Linear Algebra courses, and Signal processing. Document analysis and retrospective analysis of teaching/learning episodes are conducted to articulate alignments using course syllabus, lessons, and artifact review across mathematics and engineering courses with aligned content on the Fourier Analysis. Three mathematics and three engineering faculty provided input on the teaching and learning Fourier analysis for engineering majors. While the teacher self-reports, and reflections on their practice and student learning were predominant source. The interdisciplinary community of practice had periodic weekly meetings in Engineering analysis/Linear algebra and trigonometry course had external observers during the experimentation with Fourier related modules.

D. Setting and Background

The authors have been leading a community of practice on scholarship of teaching and learning engineering mathematics education towards learning Fourier analysis thinking as required by the advanced engineering courses.

The design research as described in Prediger *et al.* [19] is conducted to develop local instructional theories for teaching and learning of Fourier analysis as a core engineering mathematics education practice in engineering programs.

This study is undertaken at a midsize state university with large cohorts of engineering majors. Interdisciplinary team of faculty with Mathematics/STEM education, applied mathematics, and engineering backgrounds has been collaborating on building a scholarship of teaching and learning of engineering mathematics with locally effective practices. In this collaborative, pedagogical innovations involved Inquiry-based Learning (IBL) of engineering mathematics with its multiple framings in mathematics and engineering disciplines, adopting multiple representations, and experimenting with modeling tools utilized by engineering mathematics community.

Engineering Analysis as a mathematical course for engineers is dependent on the needs of a number of engineering disciplines for a linear algebra course. The first course in linear algebra is a service course for electrical, mechanical, aerospace and systems engineering among other disciplines. After taking the linear algebra course, students often come away knowing how to perform certain algorithms and computations but they have not acquired the intuition relating knowledge of the mathematics to selection of the method for analysis, design, and control of physical systems. Students in these various disciplines often only take one course in linear algebra.

The interdisciplinary tension observed here is the balance of mathematical abstraction and concretization of mathematical ideas in engineering context. We ask instructors how much an engineering student should learn about the process of mathematical reasoning and abstraction through proofs, and implications in learning upper level engineering courses.

While learning linear algebra there is a tendency to build instruction on two or three dimensional objects and applications. Relying on geometric intuitions is highlighted as a danger of practice repeatedly observed throughout history on teaching/learning linear algebra such as [10]. This has implications for learners to perceive vectors as 2 dimensional or 3 dimensional number strings. It is also observed in our practice that students struggle in making a transition from 2D and 3D vectors to n-dimensional and infinite dimensional vectors, and higher abstractions of vectors as functions with finitely or infinitely many components. Understanding Fourier Methods will require learner a higher conception of vectors beyond 2D/3D vectors and vector arithmetic. It demanding a conceptual transition into higher abstractions of vectors and vector spaces harder for advanced engineering practices. This transition is a critical conceptual transition to occur along Fourier Analysis Learning trajectory in a linear algebra course for engineers (see Figure 2 on Page 7).

Engineering students need coordinated support towards developing an understanding of these ideas during their mathematical content preparation. Learning trajectory for Fourier analysis starts in College Trigonometry by modeling musical sounds or periodic orbits using trigonometric functions helping students understand the Fourier series more accessible for freshmen students. Fourier Analysis Learning Trajectory then builds across Trigonometry, Calculus, Differential Equations, Linear Algebra, and Signal Processing courses.

It is a clear challenge to expect students to develop and integrate their understanding of complex numbers and functions into trigonometry and linear algebra to build their understanding of Fourier analysis.

III. Analysis and Results from Fourier Analysis Learning Trajectory across Courses

In Fourier analysis, continuous functions are represented in terms of an orthogonal set of functions in its vector space with orthonormal properties. Understanding those functions represented as vectors become more accessible for engineering students only during or after they take the Linear Algebra course. Further analysis of functions as vectors helps them to extend their understanding about the vector spaces that students learned in Linear algebra. This orientation helps to construct the founding ideas of functional analysis as heavily implemented by engineers when using Fourier transform methods, and also other transformations as operators or mappings into spaces spanned by orthonormal functions, for instance $\cos(x)$ and $\sin(x)$. Fourier Transform, $\mathscr{F}$, of a signal $f(t)$ in C as $t \in (-\infty, \infty)$ is defined as $\mathscr{F}(w) = \int_{\infty}^{\infty} f(t)e^{-i\omega t}dt$. Therefore $\mathscr{F}$ is a function of ω, the frequency, transforming function *f(t)* from time domain to a function in frequency domain represented as $\mathscr{F}(\omega)$. This is an operation between functions with certain characteristics. Students require a knowledge of orthonormal bases and inner product spaces to have a solid fundamental understanding of the mathematical reasoning behind Fourier Transform. While Fourier Series are formed by summations of basic sinusoidal functions, the Fourier Transform uses integrals. Fourier series represent a periodic function $f(x)$ with period L as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2\pi}{L}x + b_k \sin \frac{2\pi k}{L}x \right),$$

where the coefficients a_k, b_k are defined as

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with k = 0,1,$\cdots$ and $b_0 = 0$.

Not all students start with a strong background of complex representation of periodic behavior using Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$. As described in [21] students can be supported to connect complex representations and the trigonometric functions in orbital modeling. This connection is a core part of Fourier Analysis Learning Trajectory which can be accessed by freshmen engineering students.

A. Fourier Analysis Learning Trajectory in Lower Level Mathematics Courses: Trigonometry

In the Trigonometry course, aligned instructional design experiments were conducted for three semesters by the first author to develop an intuitive understanding of Fourier Analysis. Integrated teaching of Fourier analysis with its modeling, and assessment activities were shared with other mathematics, engineering faculty, STEM Education faculty and an educational psychometrician [4], [21].

Learning trajectories in Trigonometry course culminated in re-framing modeling with trigonometric functions using a series of complex functions in the form $ke^{2\pi\omega ti}$, where $k \in R$ in order to simplify the parametric modeling of the periodic trajectories $(x(t), y(t))$ as t increase in the time domain [21].

Technology integration facilitated transitions among multiple representations. Students modeled sound, musical tunes, and periodic orbits using a series of sinusoidal functions with different amplitude and periods as their term projects. Technological tools allowed students to experiment with a series of trigonometric functions as objects in modeling periodic orbits and seeking patterns in transforming multiplicative to additive structures with sinusoidal forms.

As part of the learning trajectory, the conceptual connections between the addition of sine waves and their multiplication is established. Operating with trigonometric functions, students observe patterns in additive and multiplicative structures with trigonometric functions. This is to be revisited in Calculus when it will help students to see the building ideas of integral transformations. This connection is a crucial part of learning trajectory in understanding orthogonal properties of sinusoidal functions in inner product spaces as introduced in Linear Algebra.

Here, trigonometric functions are practiced as objects (bases) to build more advanced periodic functions to develop the foundations of Fourier analysis approach. In Trigonometry course, using an orbital modeling as model to extend circle trigonometry and blend with complex trigonometry, students build more advanced orbital paths that can be modeled by manipulating and superposing circular functions with different frequencies and amplitudes. The culminating idea at the end is a conceptual orientation towards looking into a periodic function as a composition of a circular functions. This is to provide readiness for more formal approach towards the Fourier Transform as a process to find the set of cycle periods, amplitudes and phases to match any periodic function, *f(t)*. At the end students have done a couple of simple examples for this reverse conceptual orientation that is "How do we decompose a complex periodic function or a signal into its founding circular functions with their amplitudes, frequencies, and phases?"

Extending the modeling of a Ferris Wheel, students designed more complex double and triple Ferris wheel rides by developing mathematical/ trigonometric models of periodic orbits formed by the trajectory of a rider located on several circles simultaneously rotating with different periods and radii (See Figure 1). Experimentally, star and loop patterns are investigated looking into the relationship of periods of the circles and the relationships among amplitudes. As depicted in Figure 1, the star-like periodic orbit is decomposed into a trigonometric sum expressed by the parametric curve, representing the locus of point tracing the orbital pattern with its x and y positions are expressed with the following sums.

$$\Big(r_1 \cos(a_1 t) + r_2 \cos(a_2 t) + r_3 \cos(a_3 t) + r_4 \cos(a_4 t),$$

$$r_1 \sin(a_1 t) + r_2 \sin(a_2 t) + r_3 \sin(a_3 t) + r_4 \sin(a_4 t) \Big)$$

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learn more about series and their convergence in Calculus III. Students learn to represent and approximate with polynomial functions, and series of trigonometric functions.

Students find opportunities to build on their understanding of convergence of the sum of trigonometric series. This is a sharp contrast to the practice of representation of trigonometric functions as polynomial series. Conceiving trigonometric sine functions as building blocks or bases to represent any given function is the core idea to build to understand Fourier analysis as a branch of functional analysis, which further requires the notion of vector spaces, inner product spaces, trigonometric functions as bases to be conceptualized in Linear Algebra.

B. Fourier Analysis Learning Trajectory in Linear Algebra

Based on our conceptual analysis and the review of teaching/learning artifacts, we offer a conjectured learning trajectory across linear algebra required for a formal development of the foundations of functional analysis thinking. Figure 2 on page 7 demonstrates the proposed Learning Trajectory towards Fourier Analysis along the linear algebra course for engineers. As seen from Figure 2 on page 7, it is clear that Fourier analysis requires the idea of vector spaces, basis, subspaces, orthogonal functions as well as inner products with periodic and non-periodic functions. The concept of inner product is the founding idea leading to Fourier series and transform. While the inner products can be defined in different ways, integral based definition is critical to construct Fourier transformation, series and its coefficients. Below we provide a brief description of the local practice on introducing the idea of inner product. Students start with the notion of dot product. The concept of inner product extends the dot product further as a projection of a given function onto bases consisting of polynomial or periodic functions. The dot product and scalar products in the Euclidean space automatically satisfy the conditions of inner product as special cases. An inner product on the set of all continuous functions $C[a, b]$ can be defined as an integral in the form

$$< f, g >= \int_a^b f(x)g(x)dx,$$

where $f(x), g(x) \in C[a, b]$. In particular, if in $C[-1, 1]$ the inner product becomes

$$< f, g >= \int_{-1}^1 f(x)g(x)dx.$$

For the Fourier Analysis Learning Trajectory, the crucial inner product for trigonometric functions/polynomials relevant to Fourier analysis is

$$< f, g >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx,$$

in the function space $C[-\pi, \pi]$, and $w(x) = \frac{1}{\pi}$ is taken to be the weight function. With this definition we have $< \cos x, \sin x >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \sin x dx = 0,$

$$< \cos x, \cos x >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \cos x dx = 1, \text{ and}$$

$$< \sin x, \sin x >= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin x \sin x dx = 1.$$

The first of these implies that $\cos x$ and $\sin x$ are orthogonal, that is, $\cos x \perp \sin x$, in the space of functions, $C[-\pi, \pi]$. The last two scalars are used to illustrate the orthonormal vectors in functional spaces. An orthonormal set is a set of orthogonal unit vectors. The cosine and sine functions are orthonormal bases in representing a given function by Fourier series.

C. Practices with Fourier Analysis in Upper Level Engineering Courses: Signal Processing

1) Crosscutting Engineering Connections to Signal Processing: Fourier analysis of sounds and images with data modeling and compression tools are revisited in Signal Processing course. The students start the course with the visualization of the frequency content of simple signals using MATLAB and Fast Fourier Transform algorithm. The students observe frequency content of signals combined by adding signals with a single frequency each, including low and high frequencies (see Figure 3 on page 7). The students are asked to design a simple digital filter to remove the low frequencies, or the high frequencies which ride on the signal as noise, depending on the purpose of the application. This is depicted in Figure 4 on page 7. At the end of the semester the students take on a final project that involves filtering a signal, such as removing a random truck horn sound from an audio file with conversation, or removing a buzzing noise in the background, among other requirements.

Again in electromagnetic theory course, the propagation of electromagnetic waves are further studied and analyzed by super-positioning of plane waves which builds on the previous mathematical concepts as well as engineering applications.

In signal processing context, the tools used for Fourier Analysis include Fourier Series, Fourier Transformation (FT), Fast FT (FFT), Discrete FT (DFT), and Discrete Time FT (DTFT). While Fourier Series and DFT are finite extent in time with discrete frequency variable, FT and DTFT are infinite extent in time domain with a continuous frequency domain. Discrete Cosine Transform (DCT) is a Fourier-related transform similar to DFT, but using only real numbers.

2) Complex Learning in Modeling and Processing Signals with Fourier Methods: Modeling and processing signals require students' attention to the problem constraints and the data models based on the knowledge and assumptions of the desirable characteristics of the engineering context. In order to model the sound data perceivable for human ear, an engineering design constraint is naturally the audible range for human ear. This range determines the filtering of the sound data that can be attenuated as noise which will be irrelevant to human ear. This in turn leads to the selection criteria based on the dominant audible frequencies to model relevant sounds for human hearing. Humans are most sensitive over a range of frequencies approximately between 500 to 4000

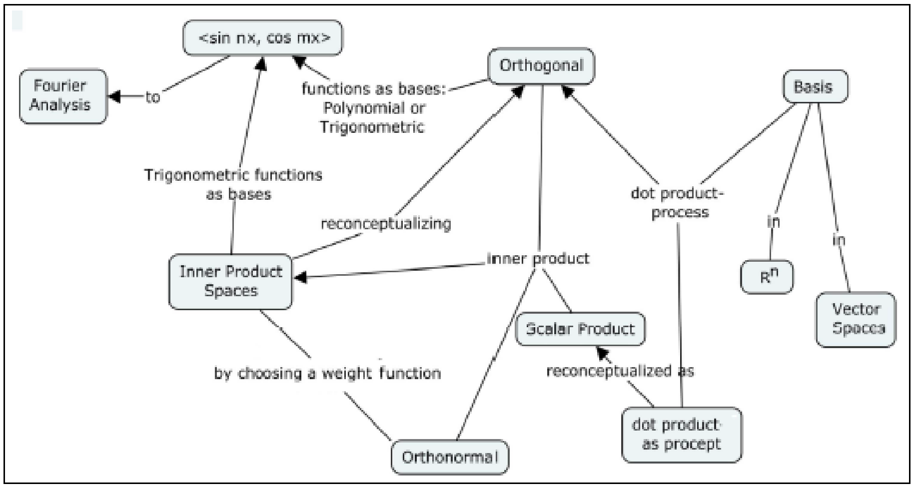

Fig. 2. Learning Trajectory for Fourier Analysis in Linear Algebra

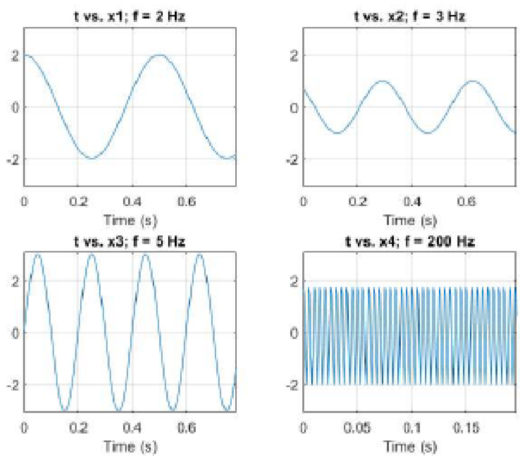

Fig. 3. Signals with different single frequency content

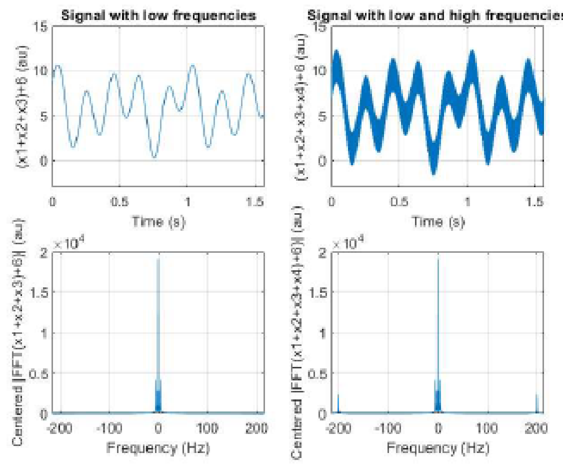

Fig. 4. Combined signals & their Fast Fourier Transform representation

Hz, and sound intensity or amplitude levels approximately between 35 dB and 80 dB or sound pressure level (SPL). In signal processing, sensitivity becomes another contextual constraint in interpreting the models. There is no true model when modeling sound data for human ear; all would be based on the biological characteristics of a human ear which can only discriminate a change of about one decibel in sound level and about a half of a percent change in tonal frequency [20]. Modeling and interpretation of signal data is highly contextual demanding engineers to draw from interdisciplinary knowledge across bio-physics, physiology, bio-mathematics as well as other disciplines.

In musical sound processing, a problem context can call for the extraction of the audible bass sounds as desirable outputs that should range between 32 to 512 Hz frequencies. Engineering students experiment with filters to extract desirable data and detecting noise based on the contextual demands of the underlying engineering problem. Modeling data using signal processing is a complex process requiring research and formulation of assumptions and choice of parameters determined by knowledge from multiple disciplinary backgrounds. In modeling musical tones, the frequency of a sound signal corresponds to the pitch of an auditory musical tone as one of four auditory attributes along with duration, loudness, and timbre.

In signal processing, another common problem that students investigate is the compression methods for image data which requires Fourier transformation method. JPEG compression employs a Fourier transform known as Discrete Cosine Transform on small squared partitions of a digital image and then re-synthesis of the image from the superposition of waves. This transform of small partitions of image data extracts the frequency components so that the re-synthesis can compress the data using not all frequencies. Indeed, filtering high frequencies and only using low frequency components of a signal allow us to express the signal with less information depending on the desired accuracy.

Discrete Cosine Transform (DCT) is a widely used signal processing technique building on representing an image signal as a sum of sinusoids from cosines functions with varying magnitudes and frequencies. The constraint for this DCT transform real data with even symmetry due to the evenness of cosine functions. Not having to deal with the complex components in a Fourier Transform is a strength but the symmetry assumption is critical for the learner. DCT as a data analysis and compression method becomes accessible for engineering students once they learn series in Calculus III and inner product spaces in Linear Algebra. Discrete Cosine Transform is a real valued transformation only building on the even part of Fourier series with cosines. On the other hand, Discrete Fourier Transform seeks to represent a time series signal as a finite sum of complex trigonometric functions by finding amplitudes, λ_ks, and frequencies, ω_ks, to approximate the time data $\sum_{k=1}^{n} \lambda_k e^{i2\pi\omega_k t}$.

Engineering students are expected to develop competency using spectrum analyzers for analysis of the frequency spectrum of a signal. Spectral analysis allows an engineer to discover underlying frequency components for a given time series x_t with its Fourier Series representation by figuring out how to construct it using sines and cosines with the coefficients for each frequency and amplitude. Fourier analysis helps students when designing filters by identifying high-frequency noise in signals that becomes obvious when signals are separated into their frequency content mathematically and visually. The students can then identify the necessary design criteria for their filters to attenuate undesirable frequencies in their signal. Examples in musical signals have been given. In addition, examples of analysis of and filtering noise from other signals are also possible, such as signals from electrocardiogram (ECG) significant for biomedical engineers; radar signal are important for signal propagation analysis in antennas. Fourier analysis provides a tool in anomaly detection by identifying discrepancies and unexpected content in otherwise characterized signals. Besides these one-dimensional signals, in 2D image data, Fourier analysis is used to determine spatial frequencies in the image. For example, edges appear as high frequencies, whereas solid colors appear as low or no frequency regions. Fourier analysis, including functional analysis offers content interpretation of images, and provides a tool to identify and remove noise through high-pass, low-pass or band-pass filtering, once the undesirable frequencies are identified. It is possible to extend Fourier analysis to three dimensions when analyzing point cloud data from lidar.

Besides providing mathematical formulation and background skills needed in engineering courses, engineering mathematics, scaffolding through mathematics courses throughout the engineering curriculum, allows students to characterize and interpret signals associated with engineering systems when devising engineering solutions to related problems.

IV. Conclusions and Future Directions

Functional analysis historically modernized the linear algebra to generalized vector spaces including infinite dimensional spaces, generalized inner product spaces extending the dot and scalar product of vectors with functions. This study provides a pathway to modernize engineering mathematics education by incorporating Fourier analysis as a functional analysis thinking. This study provides a conceptual analysis of a learning trajectory for Fourier analysis across mathematical courses in engineering disciplines. Fourier analysis skills are fundamental in engineering education. Analysis not only requires the students to understand the parameters of equations and input-output relationships, but also in modeling an engineering system or a system's behavior. Fourier transform functions provide tools to establish signal content, and to approximate any signal of any shape. Mathematical scaffolding that begins with lower division math courses are fundamental in capturing these skills necessary in engineering courses. Furthermore, as students begin to use complex equipment for analysis purposes, such as spectrum analyzers, oscilloscopes, as well as other high-technology tools, it is absolutely necessary for them to understand what is behind these results and how they are obtained through the equipment. Engineering mathematics, with addressed learning trajectories, ensures that the students are in a position to interpret data and measurement results from complex equipment.

Interdisciplinary collaboration on improving engineering mathematics education needs a strong and overarching focus driving the innovative teaching experiments across courses. Learning trajectory for Fourier analysis provides this focus yielding more coherence within and across mathematics and engineering courses. This collaboration builds an ongoing dialogue between instructors from mathematics and engineering towards teaching and learning mathematics for engineers. The Fourier Analysis Learning Trajectory will further guide the progressive development and iterative refinement of conceptual scaffolds as students re-frame and transfer their concepts in new mathematical or engineering contexts and build and synthesize new knowledge constructs.

The assessment of interdisciplinary knowledge and competences in complex learning trajectories for engineering mathematics education requires timely support for their emergence and blending, and integration of emergent multiple competencies across disciplines and time. The next phase of this research is the validation of Fourier Analysis Learning Trajectory by the aligned complex learning assessment across courses building on previous research [22], [23]. This line of interdisciplinary research will help to develop research based practices to model and support complex learning of advanced mathematical concepts for engineering and computational sciences. Collaborative innovative practices aligned by conjectured and validated learning trajectories have potential to build scholarly practices with their joint and temporal results allowing to gauge progress, and diagnose and adequately address the difficulties in engineering mathematics education.

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