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CATEGORICAL CONES AND QUADRATIC HOMOLOGICAL PROJECTIVE DUALITY

BY ALEXANDER KUZNETSOV AND ALEXANDER PERRY

ABSTRACT. – We introduce the notion of a categorical cone, which provides a categorification of the classical cone over a projective variety, and use our work on categorical joins to prove that the homologically projectively dual category of a categorical cone is equivalent to a categorical cone of the homologically projectively dual category. We check that the categorical cone construction provides well-behaved categorical resolutions of singular quadrics, which we use to obtain an explicit quadratic version of the main theorem of homological projective duality. As applications, we prove the duality conjecture for Gushel-Mukai varieties, and produce interesting examples of conifold transitions between noncommutative and honest Calabi-Yau threefolds.

RÉSUMÉ. – Nous introduisons la notion de cône catégorique, qui fournit une catégorification du cône classique au-dessus d'une variété projective, et nous utilisons notre travail sur les joints catégoriques pour prouver que le dual projectif homologique d'un cône catégorique est équivalent au cône catégorique de la catégorie duale projective homologique. Nous vérifions que la construction du cône catégorique fournit des résolutions catégoriques qui se comportent bien de quadriques singulières, que nous utilisons pour obtenir une version quadratique explicite du théorème principal de la dualité projective homologique. Comme applications, nous prouvons la conjecture de dualité pour les variétés de Gushel-Mukai, et produisons des exemples intéressants de transitions conifoldes entre des variétés de Calabi-Yau noncommutatives et de vraies variétés de Calabi-Yau de dimension trois.

1. Introduction

This paper is a sequel to [25], where we introduced categorical joins in the context of homological projective duality (HPD). Building on that work, our goals here are to study a categorical version of the classical cone over a projective variety, to use categorical quadratic cones to give a powerful method for studying derived categories of quadratic sections of varieties, and to give several applications.

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1.1. Background

The basic object of HPD is a *Lefschetz variety*, which consists of a variety mapping to a projective space $X \to \mathbf{P}(V)$ equipped with a *Lefschetz decomposition* of its derived category (a special type of semiorthogonal decomposition). The theory in this form was introduced and developed in [12]. At that point it was already clear that the theory is more categorical in nature, and that for applications it is useful to replace the (perfect) derived category $\mathrm{Perf}(X)$ of X by a more general (suitably enhanced) triangulated category $\mathcal{A}$ equipped with a Lefschetz decomposition; the structure of a map $X \to \mathbf{P}(V)$ is then replaced by a $\mathbf{P}(V)$-linear structure (an action of the monoidal category $\mathrm{Perf}(\mathbf{P}(V))$ on $\mathcal{A}$. We call such data a *Lefschetz category* over $\mathbf{P}(V)$ and think of it as of a *noncommutative* Lefschetz variety. The reader is encouraged to focus on the case where $X \to \mathbf{P}(V)$ is an ordinary morphism of varieties for this introduction, and to consult [32, 25] for more details on the noncommutative situation.

The HPD of a (noncommutative) Lefschetz variety $X \to \mathbf{P}(V)$ is another (noncommutative) Lefschetz variety

$$X^\natural \to \mathbf{P}(V^\vee)$$

over the dual projective space, which governs the derived categories of linear sections of X and can be thought of as a categorical version of the classical projective dual. For details and applications of this theory, see [12, 32, 17, 35].

In [25] given a pair of (noncommutative) Lefschetz varieties $X_1 \to \mathbf{P}(V_1)$ and $X_2 \to \mathbf{P}(V_2)$, we constructed a (noncommutative) Lefschetz variety

$$\mathcal{J}(X_1, X_2) \to \mathbf{P}(V_1 \oplus V_2),$$

called their *categorical join*, which can be thought of as a noncommutative resolution of singularities of the classical join of X_1 and X_2. Moreover, we proved that various classical properties of joins can be lifted to this level; in particular, (under suitable assumptions) the main result of [25] states that there is an equivalence of Lefschetz varieties

$$\mathcal{J}(X_1, X_2)^\natural \simeq \mathcal{J}(X_1^\natural, X_2^\natural) \tag{1.1}$$

over $\mathbf{P}(V_1^\vee \oplus V_2^\vee)$, i.e., the HPD of a categorical join is the categorical join of the HPDs. This leads to numerous applications, including a nonlinear HPD theorem (see also [10]) giving an equivalence between the "essential parts" of the derived categories of the fiber products

$$X_1 \times_{\mathbf{P}(V)} X_2 \qquad \text{and} \qquad X_1^\natural \times_{\mathbf{P}(V^\vee)} X_2^\natural.$$

The simplest case of this result—when X_2 is a linear subspace of $\mathbf{P}(V)$ and hence $X_2^\natural$ is its orthogonal linear subspace of $\mathbf{P}(V^\vee)$—reduces to the main theorem of HPD, and other examples of HPD pairs $(X_2, X_2^\natural)$ provide extensions of this theorem. Such extensions are most useful in cases when both X_2 and $X_2^\natural$ have a nice geometric description. One of the goals of this paper is to produce such pairs where both X_2 and $X_2^\natural$ are *categorical resolutions* of singular quadrics and to relate in this way quadratic sections of X_1 and $X_1^\natural$. Allowing the quadrics to be singular is crucial for applications, as we will explain below in §1.4.

1.2. Categorical cones

Assume given an exact sequence of vector spaces

$$0 \to V_0 \to V \to \bar{V} \to 0 \tag{1.2}$$

and a closed subvariety X of $\mathbf{P}(\bar{V})$. Recall that the *classical cone* over X with vertex $\mathbf{P}(V_0)$ is the strict transform

$$\mathbf{C}_{V_0}(X) \subset \mathbf{P}(V)$$

of X under the linear projection $\mathbf{P}(V) \dashrightarrow \mathbf{P}(\bar{V})$ from $\mathbf{P}(V_0)$. Note that $\mathbf{C}_{V_0}(X)$ is usually highly singular along its vertex $\mathbf{P}(V_0) \subset \mathbf{C}_{V_0}(X)$.

In this paper, given a (noncommutative) Lefschetz variety $X \to \mathbf{P}(\bar{V})$, we construct a (noncommutative) Lefschetz variety

$$\mathcal{C}_{V_0}(X) \to \mathbf{P}(V)$$

called the *categorical cone* which provides (if X is smooth) a categorical resolution of $\mathbf{C}_{V_0}(X)$. The basic idea of the construction is to first replace the classical cone with the *resolved cone* $\tilde{\mathbf{C}}_{V_0}(X) \to \mathbf{P}(V)$ given by the blowup along $\mathbf{P}(V_0) \subset \mathbf{C}_{V_0}(X)$; the resolved cone is the projectivization of the pullback to X of a natural vector bundle on $\mathbf{P}(\bar{V})$, and hence makes sense even when $X \to \mathbf{P}(\bar{V})$ is not an embedding. The categorical cone is then defined as a certain triangulated subcategory of $\mathrm{Perf}(\tilde{\mathbf{C}}_{V_0}(X))$ following a construction in [14], and can be thought of as a noncommutative birational modification of $\tilde{\mathbf{C}}_{V_0}(X)$ along its exceptional divisor.

As we will show, the categorical cone has several advantages over its classical counterpart:

— $\mathcal{C}_{V_0}(X)$ naturally has the structure of a Lefschetz variety over $\mathbf{P}(V)$ induced by that of X (Theorem 3.21).

— $\mathcal{C}_{V_0}(X)$ is smooth and proper if X is (Lemma 3.11).

— $\mathcal{C}_{V_0}(X)$ is defined when $X \to \mathbf{P}(\bar{V})$ is not an embedding, and even when X is noncommutative (Definition 3.6).

For us, however, the main advantage of the categorical cone is its compatibility with HPD: our first main result is the identification of the HPD of a categorical cone with another categorical cone.

In fact, we work in a more general setup than above, that simultaneously allows for extensions of the ambient projective space, because this extra generality is useful in applications (see §1.4). Namely, let V be a vector space and assume given a pair of subspaces

$$V_0 \subset V \qquad \text{and} \qquad V_\infty \subset V^\vee$$

such that $V_0 \subset V_\infty^\perp$, or equivalently $V_\infty \subset V_0^\perp$, where the orthogonals are taken with respect to the natural pairing between V and $V^\vee$. Let

$$\bar{V} = V_\infty^\perp / V_0, \qquad \text{so that} \qquad \bar{V}^\vee \cong V_0^\perp / V_\infty.$$

For $V_\infty = 0$ this reduces to the situation (1.2) above. Let $X \to \mathbf{P}(\bar{V})$ be a Lefschetz variety, with HPD variety $X^\natural \to \mathbf{P}(\bar{V}^\vee)$. The categorical cone $\mathcal{C}_{V_0}(X)$ is then a Lefschetz variety over $\mathbf{P}(V_\infty^\perp)$. Via the inclusion $\mathbf{P}(V_\infty^\perp) \to \mathbf{P}(V)$ we can regard $\mathcal{C}_{V_0}(X)$ as a Lefschetz variety over $\mathbf{P}(V)$, which we write as $\mathcal{C}_{V_0}(X)/\mathbf{P}(V)$ for emphasis. Similarly, we have a Lefschetz variety $\mathcal{C}_{V_\infty}(X^\natural)/\mathbf{P}(V^\vee)$ over $\mathbf{P}(V^\vee)$.

THEOREM 1.1 (Theorem 4.1). – *In the above situation if X is a right strong, moderate Lefschetz variety over $\mathbf{P}(\bar{V})$ and $X^\natural$ is its HPD over $\mathbf{P}(\bar{V}^\vee)$, then there is an equivalence*

$$(\mathcal{C}_{V_0}(X)/\mathbf{P}(V))^\natural \simeq \mathcal{C}_{V_\infty}(X^\natural)/\mathbf{P}(V^\vee)$$

of Lefschetz varieties over $\mathbf{P}(V^\vee)$, i.e., $\mathcal{C}_{V_\infty}(X^\natural)$ is the HPD of $\mathcal{C}_{V_0}(X)$ over $\mathbf{P}(V)$.

In the statement of the theorem "right strong" and "moderate" refer to technical assumptions on a Lefschetz variety (see Definitions 2.5 and 2.7) which are essentially always satisfied in practice. The theorem categorifies an analogous classical relation between cones and projective duality: for a variety $X \subset \mathbf{P}(\bar{V})$ we have

$$(\mathbf{C}_{V_0}(X) \subset \mathbf{P}(V))^\vee = \mathbf{C}_{V_\infty}(X^\vee) \subset \mathbf{P}(V^\vee),$$

where $(-)^\vee$ denotes the operation of classical projective duality.

Our proof of Theorem 1.1 is based on the HPD Result (1.1) for categorical joins mentioned above, and the following relation between categorical cones and joins. Given an exact sequence (1.2) with $V_0 \neq 0$, we show that if $X \to \mathbf{P}(\bar{V})$ is a Lefschetz variety, then the choice of a splitting of (1.2) induces a natural equivalence

$$\mathcal{C}_{V_0}(X) \simeq \mathcal{J}(\mathbf{P}(V_0), X) \tag{1.3}$$

of Lefschetz categories over $\mathbf{P}(V)$ (Proposition 3.15).

REMARK 1.2. – For $V_0 = 0$ the identification (1.3) fails, since then $\mathcal{C}_{V_0}(X) \simeq X$ whereas $\mathcal{J}(\mathbf{P}(V_0), X) = 0$. Moreover, even if $V_0 \neq 0$, we need to choose a splitting of (1.2) to be able to form $\mathcal{J}(\mathbf{P}(V_0), X)$. When working over a field (as we tacitly do in the introduction) this is not a problem, but it is typically not possible when working over a general base scheme, as we do in the body of the paper with a view toward applications. Finally, when (1.3) holds, there is an advantage of working with the categorical cone description: $\mathcal{C}_{V_0}(X)$ becomes isomorphic to the classical cone $\mathbf{C}_{V_0}(X)$ over an a priori bigger Zariski open locus than the categorical join $\mathcal{J}(\mathbf{P}(V_0), X)$, which is also important for geometric applications.

1.3. Quadratic HPD

We use categorical cones and results from [26] to develop HPD for singular quadrics. By a quadric, we mean an integral scheme isomorphic to a degree 2 hypersurface in a projective space. Any quadric Q can be expressed as a classical cone $Q = \mathbf{C}_K(\bar{Q})$ over a smooth quadric $\bar{Q}$, where $\mathbf{P}(K) = \mathrm{Sing}(Q)$. We consider the categorical cone

$$\mathfrak{Q} = \mathcal{C}_K(\bar{Q}),$$

where $\bar{Q}$ is equipped with a natural Lefschetz decomposition involving spinor bundles, see Lemma 5.9. This $\mathfrak{Q}$ is in fact a crepant categorical resolution of singularities of Q, see Lemma 5.11. We call it the *standard categorical resolution* of Q.

We deduce from Theorem 1.1 and [26, Theorem 1.1] that the class of standard categorical resolutions of quadrics is closed under HPD. Namely, we consider pairs (Q, f) where Q is a quadric and $f\colon Q \to \mathbf{P}(V)$ is a *standard morphism*, i.e., such that $f^*\mathcal{O}_{\mathbf{P}(V)}(1)$ is the ample line bundle that realizes Q as a quadric hypersurface in a projective space. In other words, f is either an embedding as a quadric hypersurface into a linear subspace of $\mathbf{P}(V)$, or a double

covering of a linear subspace of $\mathbf{P}(V)$ branched along a quadric hypersurface. We define in Definition 5.17 a *generalized duality* operation

$$(Q, f) \mapsto (Q^\natural, f^\natural)$$

on such pairs, where the target of $f^\natural\colon Q^\natural \to \mathbf{P}(V^\vee)$ is the dual projective space. This generalized duality reduces to classical projective duality when Q has even rank and $f\colon Q \to \mathbf{P}(V)$ is an embedding, and involves passing to a double covering or branch divisor in other cases.

THEOREM 1.3 (Theorem 5.20). – *Let (Q, f) and $(Q^\natural, f^\natural)$ be a generalized dual pair as above. Then the HPD of the standard categorical resolution of Q over $\mathbf{P}(V)$ is equivalent to the standard categorical resolution of $Q^\natural$ over $\mathbf{P}(V^\vee)$.*

By combining Theorem 1.3 with our nonlinear HPD Theorem from [25], we prove the following quadratic HPD theorem.

THEOREM 1.4 (Theorem 5.21 and Lemma 5.13). – *Let $X \to \mathbf{P}(V)$ be a right strong, moderate Lefschetz variety. Let $f\colon Q \to \mathbf{P}(V)$ and $f^\natural\colon Q^\natural \to \mathbf{P}(V^\vee)$ be a generalized dual pair of quadrics, with standard categorical resolutions $\mathfrak{Q}$ and $\mathfrak{Q}^\natural$, respectively. Then there are induced semiorthogonal decompositions of*

$$\mathrm{Perf}(X) \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathfrak{Q} \qquad \textit{and} \qquad \mathrm{Perf}(X^\natural) \otimes_{\mathbf{P}(V^\vee)} \mathfrak{Q}^\natural$$

which have a distinguished component in common.

Moreover, when X and $X^\natural$ are supported away from the singular loci of Q and $Q^\natural$, the above tensor product categories are identified with the derived categories of the (derived) fiber products

$$X \times_{\mathbf{P}(V)} Q \qquad \textit{and} \qquad X^\natural \times_{\mathbf{P}(V^\vee)} Q^\natural$$

of X and $X^\natural$ with the underlying quadrics Q and $Q^\natural$.

The semiorthogonal decompositions mentioned above are described in Theorem 5.21.

1.4. The importance of being singular

An interesting feature of generalized duality of quadrics is that the dimension of $Q^\natural$ may be very different from the dimension of Q; in fact, the dimension of $Q^\natural$ decreases as the dimension of the singular locus of Q increases, see (5.4). This observation has interesting consequences.

Indeed, imagine we are interested in a fiber product

$$Z = X \times_{\mathbf{P}(V)} Q, \tag{1.4}$$

where X is a Lefschetz variety over $\mathbf{P}(V)$, whose HPD variety $X^\natural$ is known, and $Q \subset \mathbf{P}(V)$ is a quadric hypersurface (the case where $Q \to \mathbf{P}(V)$ is a standard morphism of a quadric of other type works similarly). Imagine also that X itself is an intersection of quadrics (or at least there is a big family of quadrics in $\mathbf{P}(V)$ containing the image of X); note that this assumption is usually satisfied in applications, since most varieties for which the HPD is known are homogeneous, and every homogeneous variety (in an equivariant embedding) is an intersection of quadrics.

Under this assumption the quadric Q such that Z is defined by a fiber product (1.4) is not unique; indeed, it can be replaced by any quadric in the affine space of quadrics which

contain Z but not X. Typically, the rank of Q varies in this family. From this we obtain a family of "dual fiber products" $X^\natural \times_{\mathbf{P}(V^\vee)} Q^\natural$ parameterized by the same affine space, which have varying dimension, but all contain the distinguished component $\mathcal{K}(Z) \subset \mathrm{Perf}(Z)$. If we want to use these varieties to understand the structure of $\mathcal{K}(Z)$, it is natural to choose a fiber product $X^\natural \times_{\mathbf{P}(V^\vee)} Q^\natural$ of smallest possible dimension (hence the most singular quadric Q defining Z) and use its geometry.

To show how this works consider for example the Fermat quartic surface

$$Z = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbf{P}^3 =: \mathbf{P}(W).$$

Note that it can be realized as a fiber product (1.4), where $X = \mathbf{P}(W)$, $V = \mathrm{Sym}^2 W$, the map $X \to \mathbf{P}(V)$ is the double Veronese embedding, and $Q \subset \mathbf{P}(V)$ is any quadric hypersurface, corresponding to a point of an affine space over the vector space

$$\ker(\mathrm{Sym}^2 \mathrm{Sym}^2 W^\vee \to \mathrm{Sym}^4 W^\vee)$$

of quadrics containing X. The most singular quadric among these is the quadric

$$Q_0 = \{x_{00}^2 + x_{11}^2 + x_{22}^2 + x_{33}^2 = 0\} \subset \mathbf{P}^9 = \mathbf{P}(V),$$

where x_{ij} is the coordinate on $\mathbf{P}(V)$ corresponding to the quadratic function $x_i \cdot x_j$ on $\mathbf{P}(W)$. Note that the kernel space K of the corresponding quadratic form on V is 6-dimensional. In this case, the generalized dual $Q_0^\natural$ of Q_0 coincides with the classical projective dual $Q_0^\vee$ of Q_0, which is a smooth quadric surface in the linear space

$$\mathbf{P}(\langle x_{00}, x_{11}, x_{22}, x_{33}\rangle) = \mathbf{P}(K^\perp) \subset \mathbf{P}(V^\vee) = \mathbf{P}^9$$

of codimension 6. In this case, $X^\natural = (\mathbf{P}(V^\vee), \mathsf{Cliff}_0)$ is the noncommutative variety whose derived category is the category of coherent sheaves of Cliff_0-modules on $\mathbf{P}(V^\vee)$, where Cliff_0 is the universal sheaf of even parts of Clifford algebras on $\mathbf{P}(V^\vee)$, see [13]. Therefore, the dual fiber product can be rewritten as

$$X^\natural \times_{\mathbf{P}(V^\vee)} Q_0^\natural = \left(Q_0^\vee, \mathsf{Cliff}_0|_{Q_0^\vee}\right)$$

and Theorem 1.4 gives an equivalence of categories

$$\mathrm{Perf}(Z) \simeq \mathrm{Perf}\left(Q_0^\vee, \mathsf{Cliff}_0|_{Q_0^\vee}\right). \tag{1.5}$$

(See Remark 1.5 below for a more precise description of the right hand side).

Note that if we replace Q_0 with a general quadric cutting out Z in X, then instead of the above equivalence we would obtain a fully faithful embedding of $\mathrm{Perf}(Z)$ into the derived category of sheaves of Cliff_0-modules over an 8-dimensional quadric in $\mathbf{P}(V^\vee)$, which is definitely less effective.

REMARK 1.5. – In fact, the equivalence (1.5) can be made more precise as follows. Consider the union of coordinate hyperplanes in the above space $\mathbf{P}(K^\perp) = \mathbf{P}^3$ (this is a reducible quartic hypersurface) and the double covering $Z' \to Q_0^\vee$ branched along the intersection of $Q_0^\vee$ with these hyperplanes. Then Z' is a K3 surface with 12 ordinary double points, the sheaf of algebras Cliff_0 defines a Brauer class of order 2 on the resolution of singularities of Z', and the right hand side of (1.5) is equivalent to the corresponding twisted derived category.

1.5. Duality of Gushel-Mukai varieties

As an application of our results, we prove the duality conjecture for *Gushel-Mukai* (GM) varieties from [24]. Abstractly, the class of smooth GM varieties consists of smooth Fano varieties of Picard number 1, coindex 3, and degree 10, together with Brill-Noether general polarized K3 surfaces of degree 10; concretely, any such variety can be expressed as an intersection of the cone over the Plücker embedded Grassmannian $\mathrm{Gr}(2,5) \subset \mathbf{P}^9$ with a linear space and a quadric Q, or equivalently, as a fiber product of $\mathrm{Gr}(2,5)$ with a standard morphism $Q \to \mathbf{P}^9$.

In [4, Definitions 3.22 and 3.26] the notions of *period partnership* and *duality* for a pair of GM varieties of the same dimension were introduced, and in [4, Proposition 3.28] the notion of duality was related to projective duality of quadrics. Moreover, in [4, Corollary 4.16 and Theorem 4.20] it was shown that smooth period partners and dual GM varieties are always birational. Finally, [5, Theorem 5.1 and Remark 5.28] combined with [31, Theorem 1.3] proved that period partners of any smooth GM variety of even dimension form the fiber of the period map from the moduli space of smooth GM varieties [7] to the appropriate period domain, and a similar result for GM varieties of odd dimension was partially proved in [6, Theorem 1.3].

[24, Proposition 2.3] gives a semiorthogonal decomposition of the derived category $\mathrm{Perf}(Y)$ of a smooth GM variety Y consisting of exceptional vector bundles and the *GM category* $\mathcal{K}(Y) \subset \mathrm{Perf}(Y)$. The GM category was shown to be a noncommutative K3 or Enriques surface according to whether $\dim(Y)$ is even or odd. In [24, Definition 3.5] the notions of period partnership and duality were generalized to allow GM varieties of different dimension (but of the same parity!). The following result settles the duality conjecture [24, Conjecture 3.7], which previously was only known in a very special case by [24, Theorem 4.1 and Corollary 4.2].

THEOREM 1.6 (Corollary 6.5). – *Let Y_1 and Y_2 be smooth GM varieties whose associated Lagrangian subspaces do not contain decomposable vectors. If Y_1 and Y_2 are generalized partners or duals, then there is an equivalence $\mathcal{K}(Y_1) \simeq \mathcal{K}(Y_2)$.*

For the notion of the Lagrangian subspace associated to a GM variety see [4, §3] and the discussion in §6.1 below. For now we just note that, with the exception of some GM surfaces, the assumption of the theorem holds for all smooth GM varieties.

Let us explain some consequences of this result. In combination with the period results from [5] and [6] mentioned above, Theorem 1.6 shows that the assignment $Y \rightsquigarrow \mathcal{K}(Y)$ is constant on the fibers of the period morphism; since these fibers are positive-dimensional, this is an interesting phenomenon connecting Hodge theory to derived categories. Moreover, in combination with the birationality results from [4] also mentioned above, Theorem 1.6 gives strong evidence for the following conjecture.

CONJECTURE 1.7. – *If Y_1 and Y_2 are GM varieties of the same dimension at least 3 such that there is an equivalence $\mathcal{K}(Y_1) \simeq \mathcal{K}(Y_2)$, then Y_1 and Y_2 are birational.*

Because of the tight parallels between GM fourfolds, cubic fourfolds, and their K3 categories (see [24, Theorem 1.3]), Theorem 1.6 can also be considered as evidence for the analogous conjecture for cubic fourfolds suggested by Huybrechts (see [29, Question 3.25]). We note that every GM fivefold or sixfold is rational [4, Proposition 4.2], so Conjecture 1.7 is of interest specifically for GM threefolds and fourfolds. As explained in [24, §3.3], Theorem 1.6 also verifies cases of the derived category heuristics for rationality discussed in [15, 18].

Finally, we note that Theorem 1.6 implies that for certain special GM fourfolds and sixfolds Y, there exists a K3 surface T such that $\mathcal{K}(Y) \simeq \mathrm{Perf}(T)$ (see [24, §3.2]). In fact, for some GM fourfolds this is the main result of [24], and our proof of Theorem 1.6 gives an extension and a conceptual new proof of this result. We expect this fact that GM categories of even-dimensional GM varieties are "deformation equivalent" to an ordinary K3 surface to be very important for future applications. In fact, following the case of cubic fourfolds handled in [1], this was recently exploited in [33] to prove a structure theorem for moduli spaces of Bridgeland stable objects in such categories, giving (among other results) infinitely many new locally-complete unirational families of polarized hyperkähler varieties.

1.6. Other applications

For another application of the quadratic HPD theorem, we introduce a class of *spin GM varieties*. Roughly speaking, these varieties are obtained by replacing the role of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbf{P}^9$ in the definition of GM varieties with the connected component $\mathrm{OGr}_+(5,10) \subset \mathbf{P}^{15}$ of the orthogonal Grassmannian $\mathrm{OGr}(5,10)$ in its spinor embedding. The *spin GM category* $\mathcal{K}(Y) \subset \mathrm{Perf}(Y)$ corresponding to such a variety can be thought of as a 3-dimensional analogue of a GM category, as it is (fractional) Calabi-Yau of dimension 3. In this setting, we deduce from Theorem 1.4 a spin analogue of Theorem 1.6 (see Theorem 6.9).

Going further, we consider the case where Y is a fivefold, which is particularly interesting from the perspective of rationality. The heuristics of [15, 18] lead to the following conjecture: if such a Y is rational, then $\mathcal{K}(Y) \simeq \mathrm{Perf}(M)$ for a smooth Calabi-Yau threefold M. We show that such an equivalence cannot exist if Y is smooth (Lemma 6.10), and hence we expect Y to be irrational. We use Theorem 1.4 to prove the following result (stated somewhat imprecisely here), which verifies the conjecture in a mildly degenerate case.

THEOREM 1.8 (Theorem 6.11 and Corollary 6.13). – *For certain nodal spin GM fivefolds Y, the variety Y is rational and there exists a smooth Calabi-Yau threefold M which gives a crepant categorical resolution of $\mathcal{K}(Y)$.*

Finally, we note that Theorem 1.8 can be regarded as giving a noncommutative conifold transition from a smooth spin GM category to the Calabi-Yau threefold M. This suggests a noncommutative version of Reid's fantasy [34]: by degenerations and crepant resolutions, can we connect any noncommutative Calabi-Yau threefold to the derived category of a smooth projective Calabi-Yau threefold? When the answer to this question is positive, it opens the way to proving results by deforming to a geometric situation. For instance, using the methods of [1], this gives a potential way to reduce the construction of stability conditions on noncommutative Calabi-Yau threefolds to the geometric case. Further, once stability conditions are known to exist, one can try to analyze the corresponding moduli spaces of

semistable objects by relating them to the case of geometric Calabi-Yau threefolds; this would be a higher-dimensional version of the approach to studying moduli spaces of objects in the K3 category of a cubic fourfold carried out in [1].

1.7. Conventions

In this paper, we follow the conventions laid out in [25, §1.7], which we briefly summarize here. All schemes are quasi-compact and separated, and we work relative to a fixed base scheme S. For the applications in §5 and §6, we assume the base scheme S is the spectrum of an algebraically closed field $\mathbf{k}$ of characteristic 0. A vector bundle V on a scheme T means a finite locally free $\mathcal{O}_T$-module of constant rank; we use the convention that

$$\mathbf{P}(V) = \mathbf{P}_T(V) = \operatorname{Proj}(\operatorname{Sym}^\bullet(V^\vee)) \to T$$

with $\mathcal{O}_{\mathbf{P}(V)}(1)$ normalized so that its pushfoward to T is $V^\vee$. A subbundle $W \subset V$ is an inclusion of vector bundles whose cokernel is a vector bundle. Given such a $W \subset V$, its orthogonal is the subbundle of $V^\vee$ given by

$$W^\perp = \ker(V^\vee \to W^\vee).$$

By abuse of notation, given a line bundle $\mathcal{L}$ or a divisor class D on a scheme T, we denote still by $\mathcal{L}$ or D its pullback to any variety mapping to T. Similarly, if $X \to T$ is a morphism and V is a vector bundle on T, we sometimes write $V \otimes \mathcal{O}_X$ for the pullback of V to X.

Given morphisms of schemes $X \to T$ and $Y \to T$, the symbol $X \times_T Y$ denotes their *derived* fiber product (see [28, 9]), which agrees with the usual fiber product of schemes whenever X and Y are Tor-independent over T. We write fiber products over our fixed base S as absolute fiber products, i.e., $X \times Y := X \times_S Y$.

We work with linear categories as reviewed in [25, §1.6 and Appendix A]. In particular, given a scheme X over T, we denote by $\operatorname{Perf}(X)$ its category of perfect complexes and by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ its bounded derived category of coherent sheaves, regarded as T-linear categories.

If $\mathcal{C}$ is a T-linear category and $T' \to T$ is a morphism of schemes, we denote by

$$\mathcal{C}_{T'} = \mathcal{C} \otimes_{\operatorname{Perf}(T)} \operatorname{Perf}(T')$$

the base change of $\mathcal{C}$ along $T' \to T$. If $Z \subset T$ is a closed subset, we say $\mathcal{C}$ is *supported over* Z if $\mathcal{C}_U \simeq 0$, where $U = T \setminus Z$. If $U \subset T$ is an open subset, we say $\mathcal{C}$ is *supported over* U if the restriction functor $\mathcal{C} \to \mathcal{C}_U$ is an equivalence.

All functors considered in this paper (pullback, pushforward, tensor product) will be taken in the derived sense. Recall that for a morphism of schemes $f\colon X \to Y$ the pushforward f_* is right adjoint to the pullback f^*. Sometimes, we need other adjoint functors as well. Provided they exist, we denote by $f^!$ the right adjoint of $f_*\colon \operatorname{Perf}(X) \to \operatorname{Perf}(Y)$ and by $f_!$ the left adjoint of $f^*\colon \operatorname{Perf}(Y) \to \operatorname{Perf}(X)$, so that $(f_!, f^*, f_*, f^!)$ is an adjoint sequence.

REMARK 1.9. – The above adjoint functors all exist if $f\colon X \to Y$ is a morphism between schemes which are smooth and projective over S (see [25, Remark 1.9]); this will be satisfied in all of the cases where we need $f^!$ and $f_!$ in the paper.

1.8. Organization of the paper

In §2 we review preliminaries on HPD. In §3 we define categorical cones, study their basic properties, and relate them to categorical joins. In §4 we prove Theorem 1.1 on HPD for categorical cones. In §5 we introduce standard categorical resolutions of quadrics, and prove the HPD Result Theorem 1.3 for them and the quadratic HPD theorem stated as Theorem 1.4 above. Finally, in §6 we establish the applications discussed in §1.5 and §1.6. In the appendix we prove some results in the context of HPD that are used in the paper.

1.9. Acknowledgements

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2. Preliminaries on HPD

In this section, we discuss preliminary material on HPD that will be needed in the rest of the paper. We fix a vector bundle V over our base scheme S. We denote by N the rank of V and by H the relative hyperplane class on the projective bundle $\mathbf{P}(V)$ such that $\mathcal{O}(H) = \mathcal{O}_{\mathbf{P}(V)}(1)$.

2.1. Lefschetz categories

The fundamental objects of HPD are Lefschetz categories. We summarize the basic definitions following [32, §6].

DEFINITION 2.1. – Let T be a scheme over S with a line bundle $\mathcal{L}$. Let $\mathcal{A}$ be a T-linear category. An admissible S-linear subcategory $\mathcal{A}_0 \subset \mathcal{A}$ is called a *Lefschetz center* of $\mathcal{A}$ with respect to $\mathcal{L}$ if the subcategories $\mathcal{A}_i \subset \mathcal{A}$, $i \in \mathbf{Z}$, determined by

$$\mathcal{A}_i = \mathcal{A}_{i-1} \cap {}^{\perp}(\mathcal{A}_0 \otimes \mathcal{L}^{-i}) \quad , \quad i \geq 1 \tag{2.1}$$

$$\mathcal{A}_i = \mathcal{A}_{i+1} \cap \quad (\mathcal{A}_0 \otimes \mathcal{L}^{-i})^{\perp}, \quad i \leq -1 \tag{2.2}$$

are right admissible in $\mathcal{A}$ for $i \geq 1$, left admissible in $\mathcal{A}$ for $i \leq -1$, vanish for all i of sufficiently large absolute value, say for $|i| \geq m$, and provide S-linear semiorthogonal decompositions

$$\mathcal{A} = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes \mathcal{L}, \dots, \mathcal{A}_{m-1} \otimes \mathcal{L}^{m-1} \rangle, \tag{2.3}$$

$$\mathcal{A} = \langle \mathcal{A}_{1-m} \otimes \mathcal{L}^{1-m}, \dots, \mathcal{A}_{-1} \otimes \mathcal{L}^{-1}, \mathcal{A}_0 \rangle. \tag{2.4}$$

The categories $\mathcal{A}_i$, $i \in \mathbf{Z}$, are called the *Lefschetz components* of the Lefschetz center $\mathcal{A}_0 \subset \mathcal{A}$. The semiorthogonal decompositions (2.3) and (2.4) are called *the right Lefschetz decomposition* and *the left Lefschetz decomposition* of $\mathcal{A}$. The minimal m above is called the *length* of the Lefschetz decompositions.

The Lefschetz components form two (different in general) chains of (left or right) admissible subcategories

$$0 \subset \mathcal{A}_{1-m} \subset \cdots \subset \mathcal{A}_{-1} \subset \mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_{m-1} \supset 0. \tag{2.5}$$

Note that the assumption of right or left admissibility of $\mathcal{A}_i$ in $\mathcal{A}$ is equivalent to the assumption of right or left admissibility in $\mathcal{A}_0$.

DEFINITION 2.2. – A *Lefschetz category* $\mathcal{A}$ over $\mathbf{P}(V)$ is a $\mathbf{P}(V)$-linear category equipped with a Lefschetz center $\mathcal{A}_0 \subset \mathcal{A}$ with respect to $\mathcal{O}(H)$. The *length* of $\mathcal{A}$ is the length of its Lefschetz decompositions, and is denoted by length($\mathcal{A}$).

Given Lefschetz categories $\mathcal{A}$ and $\mathcal{B}$ over $\mathbf{P}(V)$, an *equivalence of Lefschetz categories* or a *Lefschetz equivalence* is a $\mathbf{P}(V)$-linear equivalence $\mathcal{A} \simeq \mathcal{B}$ which induces an S-linear equivalence $\mathcal{A}_0 \simeq \mathcal{B}_0$ of centers.

REMARK 2.3. – By [32, Lemma 6.3], if the subcategories $\mathcal{A}_i \subset \mathcal{A}$ are admissible for all $i \geq 0$ or all $i \leq 0$, then the length m defined above satisfies

$$m = \min\{\, i \geq 0 \mid \mathcal{A}_i = 0 \,\} = \min\{\, i \geq 0 \mid \mathcal{A}_{-i} = 0 \,\}.$$

REMARK 2.4. – If $\mathcal{A}$ is smooth and proper over S, then in order for a subcategory $\mathcal{A}_0 \subset \mathcal{A}$ to be a Lefschetz center, it is enough to give only one of the semiorthogonal decompositions (2.3) or (2.4). This follows from [32, Lemmas 4.15 and 6.3].

For $i \geq 1$ the i-th *right primitive component* $\mathfrak{a}_i$ of a Lefschetz center is defined as the right orthogonal to $\mathcal{A}_{i+1}$ in $\mathcal{A}_i$, i.e.,

$$\mathfrak{a}_i = \mathcal{A}_{i+1}^{\perp} \cap \mathcal{A}_i,$$

so that

$$\mathcal{A}_i = \langle \mathfrak{a}_i, \mathcal{A}_{i+1} \rangle = \langle \mathfrak{a}_i, \mathfrak{a}_{i+1}, \ldots, \mathfrak{a}_{m-1} \rangle. \tag{2.6}$$

Similarly, for $i \leq -1$ the i-th *left primitive component* $\mathfrak{a}_i$ of a Lefschetz center is the left orthogonal to $\mathcal{A}_{i-1}$ in $\mathcal{A}_i$, i.e.,

$$\mathfrak{a}_i = {}^{\perp}\mathcal{A}_{i-1} \cap \mathcal{A}_i,$$

so that

$$\mathcal{A}_i = \langle \mathcal{A}_{i-1}, \mathfrak{a}_i \rangle = \langle \mathfrak{a}_{1-m}, \ldots, \mathfrak{a}_{i-1}, \mathfrak{a}_i \rangle. \tag{2.7}$$

For $i = 0$, we have both right and left primitive components, defined by

$$\mathfrak{a}_{+0} = \mathcal{A}_1^{\perp} \cap \mathcal{A}_0 \quad \text{and} \quad \mathfrak{a}_{-0} = {}^{\perp}\mathcal{A}_{-1} \cap \mathcal{A}_0,$$

and then (2.6) and (2.7) hold true for $i = 0$ with $\mathfrak{a}_{+0}$ taking the place of $\mathfrak{a}_0$ for the first and $\mathfrak{a}_{-0}$ for the second.

For HPD we will need to consider Lefschetz categories that satisfy certain "strongness" and "moderateness" conditions, defined below.

DEFINITION 2.5. – A Lefschetz category $\mathcal{A}$ is called *right strong* if all of its right primitive components $\mathfrak{a}_{+0}, \mathfrak{a}_i$, $i \geq 1$, are admissible in $\mathcal{A}$, *left strong* if all of its left primitive components $\mathfrak{a}_{-0}, \mathfrak{a}_i$, $i \leq -1$, are admissible in $\mathcal{A}$, and *strong* if all of its primitive components are admissible.

REMARK 2.6. – If $\mathcal{A}$ is smooth and proper over S, then any Lefschetz structure on $\mathcal{A}$ is automatically strong, see [32, Remark 6.7].

By [32, Corollary 6.19(1)], the length of a Lefschetz category $\mathcal{A}$ over $\mathbf{P}(V)$ satisfies

$$\operatorname{length}(\mathcal{A}) \leq \operatorname{rank}(V). \tag{2.8}$$

DEFINITION 2.7. – A Lefschetz category $\mathcal{A}$ over $\mathbf{P}(V)$ is called *moderate* if its length satisfies the strict inequality

$$\operatorname{length}(\mathcal{A}) < \operatorname{rank}(V).$$

Moderateness of a Lefschetz category $\mathcal{A}$ is a very mild condition, see [25, Remark 2.12].

There are many examples of interesting Lefschetz categories, see [17] for a survey; the most basic is the following.

EXAMPLE 2.8. – Let $0 \neq W \subset V$ be a subbundle of rank $m > 0$. The morphism $\mathbf{P}(W) \to \mathbf{P}(V)$ induces a $\mathbf{P}(V)$-linear structure on $\operatorname{Perf}(\mathbf{P}(W))$. Pullback along the projection $\mathbf{P}(W) \to S$ gives an embedding $\operatorname{Perf}(S) \subset \operatorname{Perf}(\mathbf{P}(W))$; its image is a Lefschetz center in $\operatorname{Perf}(\mathbf{P}(W))$ and provides it with the structure of a strong Lefschetz category over $\mathbf{P}(V)$. The corresponding right and left Lefschetz decompositions are given by Orlov's projective bundle formulas:

$$\begin{aligned}
\operatorname{Perf}(\mathbf{P}(W)) &= \langle \operatorname{Perf}(S), \operatorname{Perf}(S)(H), \dots, \operatorname{Perf}(S)((m-1)H) \rangle, \\
\operatorname{Perf}(\mathbf{P}(W)) &= \langle \operatorname{Perf}(S)((1-m)H), \dots, \operatorname{Perf}(S)(-H), \operatorname{Perf}(S) \rangle.
\end{aligned}$$

We call this the *standard Lefschetz structure* on $\mathbf{P}(W)$. Note that the length of $\operatorname{Perf}(\mathbf{P}(W))$ is m, so it is a moderate Lefschetz category as long as $W \neq V$.

2.2. The HPD category

Let H' denote the relative hyperplane class on $\mathbf{P}(V^\vee)$ such that $\mathcal{O}(H') = \mathcal{O}_{\mathbf{P}(V^\vee)}(1)$. Let

$$\delta \colon \mathbf{H}(\mathbf{P}(V)) \to \mathbf{P}(V) \times \mathbf{P}(V^\vee).$$

be the natural incidence divisor. We think of $\mathbf{H}(\mathbf{P}(V))$ as the universal hyperplane in $\mathbf{P}(V)$. If X is a scheme with a morphism $X \to \mathbf{P}(V)$, then the universal hyperplane section of X is defined by

$$\mathbf{H}(X) = X \times_{\mathbf{P}(V)} \mathbf{H}(\mathbf{P}(V)).$$

This definition extends directly to linear categories as follows.

DEFINITION 2.9. – Let $\mathcal{A}$ be a $\mathbf{P}(V)$-linear category. The *universal hyperplane section* of $\mathcal{A}$ is defined by

$$\mathbf{H}(\mathcal{A}) = \mathcal{A} \otimes_{\operatorname{Perf}(\mathbf{P}(V))} \operatorname{Perf}(\mathbf{H}(\mathbf{P}(V))).$$

We sometimes use the more elaborate notation

$$\mathbf{H}(X/\mathbf{P}(V)) = \mathbf{H}(X) \qquad \text{and} \qquad \mathbf{H}(\mathcal{A}/\mathbf{P}(V)) = \mathbf{H}(\mathcal{A})$$

to emphasize the universal hyperplane section is being taken with respect to $\mathbf{P}(V)$.

There is a commutative diagram

(2.9)
$$\begin{array}{ccccc} & & \mathbf{H}(\mathbf{P}(V)) & & \\ & \overset{\pi}{\swarrow} & \downarrow{\scriptstyle \delta} & \overset{h}{\searrow} & \\ \mathbf{P}(V) & \xleftarrow[\mathrm{pr}_1]{} & \mathbf{P}(V)\times\mathbf{P}(V^\vee) & \xrightarrow[\mathrm{pr}_2]{} & \mathbf{P}(V^\vee). \end{array}$$

Here we follow the notation of [25, §2.2] and deviate slightly from the notation of [32], where the morphisms π, δ, and h are instead denoted p, ι, and f. For a $\mathbf{P}(V)$-linear category $\mathcal{A}$ there are canonical identifications

$$\mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathrm{Perf}(\mathbf{P}(V)\times\mathbf{P}(V^\vee)) \simeq \mathcal{A} \otimes \mathrm{Perf}(\mathbf{P}(V^\vee)), \quad \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathrm{Perf}(\mathbf{P}(V)) \simeq \mathcal{A},$$

by which we will regard the functors induced by morphisms in (2.9) as functors

$$\delta_*\colon \mathbf{H}(\mathcal{A}) \to \mathcal{A} \otimes \mathrm{Perf}(\mathbf{P}(V^\vee)), \quad \pi_*\colon \mathbf{H}(\mathcal{A}) \to \mathcal{A},$$

and so on. The following definition differs from the original in [12], but is equivalent to it by [25, Lemma 2.22].

Definition 2.10. – Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(V)$. Then the *HPD category* $\mathcal{A}^\natural$ of $\mathcal{A}$ is the full $\mathbf{P}(V^\vee)$-linear subcategory of $\mathbf{H}(\mathcal{A})$ defined by

$$\mathcal{A}^\natural = \{\, C \in \mathbf{H}(\mathcal{A}) \mid \delta_*(C) \in \mathcal{A}_0 \otimes \mathrm{Perf}(\mathbf{P}(V^\vee)) \,\}. \tag{2.10}$$

We sometimes use the notation

$$(\mathcal{A}/\mathbf{P}(V))^\natural = \mathcal{A}^\natural$$

to emphasize the dependence on the $\mathbf{P}(V)$-linear structure.

Remark 2.11. – The HPD category $\mathcal{A}^\natural$ depends on the choice of the Lefschetz center $\mathcal{A}_0 \subset \mathcal{A}$, although this is suppressed in the notation. For instance, for the "stupid" Lefschetz center $\mathcal{A}_0 = \mathcal{A}$ we have $\mathcal{A}^\natural = \mathbf{H}(\mathcal{A})$.

A less trivial example of HPD is the following.

Example 2.12. – Consider the Lefschetz category $\mathrm{Perf}(\mathbf{P}(W))$ of Example 2.8 and assume $0 \subsetneq W \subsetneq V$. Then by [12, Corollary 8.3] there is a Lefschetz equivalence

$$\mathrm{Perf}(\mathbf{P}(W))^\natural \simeq \mathrm{Perf}(\mathbf{P}(W^\perp)).$$

This is usually referred to as *linear* HPD.

If $\mathcal{A}$ is a Lefschetz category over $\mathbf{P}(V)$ of length m, there is a $\mathbf{P}(V^\vee)$-linear semiorthogonal decomposition

$$\mathbf{H}(\mathcal{A}) = \Big\langle \mathcal{A}^\natural, \delta^*(\mathcal{A}_1(H) \otimes \mathrm{Perf}(\mathbf{P}(V^\vee))), \dots, \delta^*(\mathcal{A}_{m-1}((m-1)H) \otimes \mathrm{Perf}(\mathbf{P}(V^\vee)))\Big\rangle. \tag{2.11}$$

Moreover, $\mathcal{A}^\natural$ is an admissible subcategory in $\mathbf{H}(\mathcal{A})$, i.e., its inclusion functor

$$\gamma\colon \mathcal{A}^\natural \to \mathbf{H}(\mathcal{A}) \tag{2.12}$$

has both left and right adjoints $\gamma^*, \gamma^! \colon \mathbf{H}(\mathcal{A}) \to \mathcal{A}^\natural$. Further, if $\mathcal{A}$ is a right strong moderate Lefschetz category, then $\mathcal{A}^\natural$ is equipped with a natural left strong moderate Lefschetz structure over $\mathbf{P}(V^\vee)$ with center $\mathcal{A}_0^\natural = \gamma^*\pi^*(\mathcal{A}_0)$, see [32, Theorem 8.7].

REMARK 2.13. – When $\mathcal{A}$ is smooth and proper, the HPD operation is an involution; in other words, the double dual category $\mathcal{A}^{\natural\natural}$ is naturally Lefschetz equivalent to $\mathcal{A}$. In a more general situation, the inverse operation to HPD duality is called "left HPD", see [32, Definition 7.1]. The left HPD category ${}^\natural\mathcal{A}$ is defined analogously to Definition 2.10, one just needs to replace the right adjoint functor δ_* of δ^* by its left adjoint $\delta_!$ in (2.10), see [32, (7.4)]. Alternatively, one can replace in (2.11) the right orthogonal to the components coming from the ambient variety by the left orthogonal, see [32, (7.2)]. Then there are natural Lefschetz equivalences

$$ {}^\natural(\mathcal{A}^\natural) \simeq \mathcal{A} \simeq ({}^\natural\mathcal{A})^\natural. $$

See [32, Theorem 8.9] for the first equivalence; the second is analogous. In particular, these equivalences imply that showing a Lefschetz equivalence $\mathcal{A}^\natural \simeq \mathcal{B}$ is equivalent to showing $\mathcal{A} \simeq {}^\natural\mathcal{B}$. We will use this observation in the paper.

2.3. Categorical joins

In this section, we summarize some of our results on categorical joins from [25]. Let V_1 and V_2 be vector bundles on S. Denote by H_i the relative hyperplane class of $\mathbf{P}(V_i)$ such that $\mathcal{O}(H_i) \cong \mathcal{O}_{\mathbf{P}(V_i)}(1)$.

The *universal resolved join* is defined as the $\mathbf{P}^1$-bundle

$$ \tilde{\mathbf{J}}(\mathbf{P}(V_1), \mathbf{P}(V_2)) = \mathbf{P}_{\mathbf{P}(V_1)\times\mathbf{P}(V_2)}(\mathcal{O}(-H_1) \oplus \mathcal{O}(-H_2)). \tag{2.13} $$

The canonical embedding of vector bundles on $\tilde{\mathbf{J}}(\mathbf{P}(V_1), \mathbf{P}(V_2))$

$$ \mathcal{O}(-H_1) \oplus \mathcal{O}(-H_2) \hookrightarrow (V_1 \otimes \mathcal{O}) \oplus (V_2 \otimes \mathcal{O}) = (V_1 \oplus V_2) \otimes \mathcal{O} $$

induces a morphism

$$ f \colon \tilde{\mathbf{J}}(\mathbf{P}(V_1), \mathbf{P}(V_2)) \to \mathbf{P}(V_1 \oplus V_2) $$

which can be identified with a blowup along $\mathbf{P}(V_1) \sqcup \mathbf{P}(V_2) \subset \mathbf{P}(V_1 \oplus V_2)$ with exceptional divisors

$$ \mathbf{E}_1 = \mathbf{P}_{\mathbf{P}(V_1)\times\mathbf{P}(V_2)}(\mathcal{O}(-H_1)) \cong \mathbf{P}(V_1) \times \mathbf{P}(V_2) \overset{\varepsilon_1}{\hookrightarrow} \tilde{\mathbf{J}}(\mathbf{P}(V_1), \mathbf{P}(V_2)), $$

$$ \mathbf{E}_2 = \mathbf{P}_{\mathbf{P}(V_1)\times\mathbf{P}(V_2)}(\mathcal{O}(-H_2)) \cong \mathbf{P}(V_1) \times \mathbf{P}(V_2) \overset{\varepsilon_2}{\hookrightarrow} \tilde{\mathbf{J}}(\mathbf{P}(V_1), \mathbf{P}(V_2)). $$

This situation is summarized in the following commutative diagram

$$ \begin{array}{ccccc} & & \mathbf{P}(V_1) \times \mathbf{P}(V_2) & & \\ & \overset{\sim}{\nearrow} & \uparrow{\scriptstyle p} & \overset{\sim}{\nwarrow} & \\ \mathbf{E}_1 & \xrightarrow[\varepsilon_1]{} & \tilde{\mathbf{J}}(\mathbf{P}(V_1), \mathbf{P}(V_2)) & \xleftarrow[\varepsilon_2]{} & \mathbf{E}_2 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \mathbf{P}(V_1) & \longrightarrow & \mathbf{P}(V_1 \oplus V_2) & \longleftarrow & \mathbf{P}(V_2), \end{array} \tag{2.14} $$

where p is the canonical projection morphism.

DEFINITION 2.14. – Let $\mathcal{A}^1$ be a $\mathbf{P}(V_1)$-linear category and $\mathcal{A}^2$ a $\mathbf{P}(V_2)$-linear category. The *resolved join* of $\mathcal{A}^1$ and $\mathcal{A}^2$ is the category

$$\tilde{\mathbf{J}}(\mathcal{A}^1, \mathcal{A}^2) = \big(\mathcal{A}^1 \otimes \mathcal{A}^2\big) \otimes_{\mathrm{Perf}(\mathbf{P}(V_1)\times\mathbf{P}(V_2))} \mathrm{Perf}(\tilde{\mathbf{J}}(\mathbf{P}(V_1), \mathbf{P}(V_2))).$$

Further, for $k = 1, 2$, we define

$$\mathbf{E}_k(\mathcal{A}^1, \mathcal{A}^2) = \big(\mathcal{A}^1 \otimes \mathcal{A}^2\big) \otimes_{\mathrm{Perf}(\mathbf{P}(V_1)\times\mathbf{P}(V_2))} \mathrm{Perf}(\mathbf{E}_k) \simeq \mathcal{A}^1 \otimes \mathcal{A}^2.$$

We define the categorical join of Lefschetz categories over $\mathbf{P}(V_1)$ and $\mathbf{P}(V_2)$ as a certain subcategory of the resolved join.

DEFINITION 2.15. – Let $\mathcal{A}^1$ and $\mathcal{A}^2$ be Lefschetz categories over $\mathbf{P}(V_1)$ and $\mathbf{P}(V_2)$ with Lefschetz centers $\mathcal{A}^1_0$ and $\mathcal{A}^2_0$. The *categorical join* $\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2)$ of $\mathcal{A}^1$ and $\mathcal{A}^2$ is defined by

$$\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2) = \left\{ C \in \tilde{\mathbf{J}}(\mathcal{A}^1, \mathcal{A}^2) \;\middle|\; \begin{array}{l} \varepsilon_1^*(C) \in \mathcal{A}^1 \otimes \mathcal{A}^2_0 \subset \mathbf{E}_1(\mathcal{A}^1, \mathcal{A}^2), \\ \varepsilon_2^*(C) \in \mathcal{A}^1_0 \otimes \mathcal{A}^2 \subset \mathbf{E}_2(\mathcal{A}^1, \mathcal{A}^2) \end{array} \right\}.$$

The categorical join is an admissible subcategory in the resolved join; its orthogonal complements are supported on the exceptional divisors $\mathbf{E}_k$ and can be explicitly described in terms of Lefschetz components of $\mathcal{A}^1$ and $\mathcal{A}^2$, see [25, Lemma 3.12]. Furthermore, $\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2)$ is smooth and proper as soon as both $\mathcal{A}^1$ and $\mathcal{A}^2$ are [25, Lemma 3.14]. Note also that the categorical join depends on the choice of Lefschetz centers for $\mathcal{A}^1$ and $\mathcal{A}^2$, although this is suppressed in the notation. Finally, by [25, Theorem 3.21], $\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2)$ has a natural Lefschetz structure with center

$$\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2)_0 = p^*\big(\mathcal{A}^1_0 \otimes \mathcal{A}^2_0\big) \subset \mathcal{J}(\mathcal{A}_1, \mathcal{A}_2). \tag{2.15}$$

It is right or left strong if both $\mathcal{A}^1$ and $\mathcal{A}^2$ are, its length is equal to $\mathrm{length}(\mathcal{A}^1) + \mathrm{length}(\mathcal{A}^2)$, and its Lefschetz and primitive components can be explicitly described, see [25, (3.14), (3.15), (3.16), and Lemma 3.24]. The main property of categorical joins is that they commute with HPD in the following sense.

THEOREM 2.16 ([25, Theorem 4.1]). – *Let $\mathcal{A}^1$ and $\mathcal{A}^2$ be right strong, moderate Lefschetz categories over $\mathbf{P}(V_1)$ and $\mathbf{P}(V_2)$. Then there is an equivalence*

$$\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2)^\natural \simeq \mathcal{J}((\mathcal{A}^1)^\natural, (\mathcal{A}^2)^\natural)$$

of Lefschetz categories over $\mathbf{P}(V_1^\vee \oplus V_2^\vee)$.

By [25, Proposition 3.17] the fiber product of $\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2)$ with any $\mathbf{P}(V_1 \oplus V_2)$-linear category supported over the complement of $\mathbf{P}(V_1) \sqcup \mathbf{P}(V_2)$ is equivalent to the fiber product of the resolved join with the same category. If $\xi\colon V_1 \xrightarrow{\sim} V_2$ is an isomorphism, the graph of ξ in $\mathbf{P}(V_1 \oplus V_2)$ is contained in the complement of $\mathbf{P}(V_1) \sqcup \mathbf{P}(V_2)$ and its fiber product with $\tilde{\mathbf{J}}(\mathcal{A}^1, \mathcal{A}^2)$ is equivalent to $\mathcal{A}^1 \otimes_{\mathrm{Perf}(\mathbf{P}(V_1))} \mathcal{A}^2$. A combination of this observation with Theorem 2.16 and the main theorem of HPD gives the following result, which we call the Nonlinear HPD Theorem.

THEOREM 2.17 ([25, Theorem 5.5]). – *Let $\mathcal{A}^1$ and $\mathcal{A}^2$ be right strong, moderate Lefschetz categories over projective bundles $\mathbf{P}(V_1)$ and $\mathbf{P}(V_2)$, where V_1 and V_2 have the same rank*

$$r = \operatorname{rank}(V_1) = \operatorname{rank}(V_2).$$

Let W be a vector bundle on S equipped with isomorphisms $\xi_k \colon W \xrightarrow{\sim} V_k$ for $k = 1, 2$, and let $(\xi_k^\vee)^{-1} \colon W^\vee \xrightarrow{\sim} V_k^\vee$ be the inverse dual isomorphisms. Set

$$m = \operatorname{length}(\mathcal{A}^1) + \operatorname{length}(\mathcal{A}^2) \qquad \text{and} \qquad m^\natural = \operatorname{length}((\mathcal{A}^1)^\natural) + \operatorname{length}((\mathcal{A}^2)^\natural).$$

For $i, j \in \mathbf{Z}$ let $\mathcal{J}_i$ and $\mathcal{J}_j^\natural$ be the Lefschetz components of $\mathcal{J}(\mathcal{A}^1, \mathcal{A}^2)$ and $\mathcal{J}((\mathcal{A}^1)^\natural, (\mathcal{A}^2)^\natural)$ respectively. Denote by H and H' the relative hyperplane classes on $\mathbf{P}(W)$ and $\mathbf{P}(W^\vee)$. Then there are semiorthogonal decompositions

$$\begin{aligned} &\mathcal{A}^1_{\mathbf{P}(W)} \otimes_{\operatorname{Perf}(\mathbf{P}(W))} \mathcal{A}^2_{\mathbf{P}(W)} \\ &\qquad = \big\langle \mathcal{K}_W(\mathcal{A}^1, \mathcal{A}^2), \mathcal{J}_r(H), \dots, \mathcal{J}_{m-1}((m-r)H) \big\rangle, \end{aligned} \tag{2.16}$$

$$\begin{aligned} &(\mathcal{A}^1)^\natural_{\mathbf{P}(W^\vee)} \otimes_{\operatorname{Perf}(\mathbf{P}(W^\vee))} (\mathcal{A}^2)^\natural_{\mathbf{P}(W^\vee)} \\ &\qquad = \Big\langle \mathcal{J}^\natural_{1-m^\natural}((r-m^\natural)H'), \dots, \mathcal{J}^\natural_{-r}(-H'), \mathcal{K}'_{W^\vee}((\mathcal{A}^1)^\natural, (\mathcal{A}^2)^\natural) \Big\rangle, \end{aligned} \tag{2.17}$$

and an S-linear equivalence

$$\mathcal{K}_W(\mathcal{A}^1, \mathcal{A}^2) \simeq \mathcal{K}'_{W^\vee}((\mathcal{A}^1)^\natural, (\mathcal{A}^2)^\natural).$$

In the case where $\mathcal{A}^2$ and $(\mathcal{A}^2)^\natural$ is an HPD pair from Example 2.12, this recovers the main theorem of HPD.

2.4. Categorical resolutions

Finally, we recall the notion of a categorical resolution of singularities developed in [14], which we will need later.

DEFINITION 2.18. – Given a projective variety Y over a field $\mathbf{k}$, a $\mathbf{k}$-linear category $\mathcal{C}$ is called a *categorical resolution* of Y if $\mathcal{C}$ is smooth and proper and there exists a pair of functors

$$\pi_* \colon \mathcal{C} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) \qquad \text{and} \qquad \pi^* \colon \operatorname{Perf}(Y) \to \mathcal{C},$$

such that π^* is left adjoint to π_*, and π^* is fully faithful. If further π^* is both left and right adjoint to π_*, then the categorical resolution $\mathcal{C}$ is called *weakly crepant*.

If $\pi \colon X \to Y$ is a morphism from a smooth proper scheme such that $\pi_* \mathcal{O}_X \cong \mathcal{O}_Y$ (for instance, if Y has rational singularities and π is a resolution of singularities) then the pullback and the pushforward functors provide $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \operatorname{Perf}(X)$ with a structure of a categorical resolution of Y (and if the singularities of Y are worse than rational, a categorical resolution of Y was constructed in [22]). Such a resolution is weakly crepant if and only if $K_{X/Y} = 0$, i.e., if and only if the morphism π is crepant.

REMARK 2.19. – The notion of a categorical resolution and weak crepancy extends to the noncommutative case where Y is replaced with an admissible subcategory $\mathcal{A} \subset \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$; then $\mathcal{A}^{\mathrm{perf}} = \mathcal{A} \cap \operatorname{Perf}(Y)$ plays the role of $\operatorname{Perf}(Y)$. In this case, we say that $\mathcal{C}$ is a categorical resolution (or weakly crepant categorical resolution) of $\mathcal{A}^{\mathrm{perf}}$.

3. Categorical cones

In this section, we introduce the operation of taking the categorical cone of a Lefschetz category. This operation is closely related to that of a categorical join reviewed in §2.3; in fact, in §3.3 we show that under a splitting assumption, categorical cones can be described in terms of categorical joins.

We fix an exact sequence

$$0 \to V_0 \to V \to \bar{V} \to 0 \tag{3.1}$$

of vector bundles on S. We write H_0, H, and $\bar{H}$ for the relative hyperplane classes on the projective bundles $\mathbf{P}(V_0)$, $\mathbf{P}(V)$, and $\mathbf{P}(\bar{V})$, and denote by N_0 the rank of V_0.

3.1. Resolved cones

Let $\mathcal{V}$ be the vector bundle on $\mathbf{P}(\bar{V})$ defined as the preimage of the line subbundle $\mathcal{O}(-\bar{H}) \subset \bar{V} \otimes \mathcal{O}_{\mathbf{P}(\bar{V})}$ under the surjection $V \otimes \mathcal{O}_{\mathbf{P}(\bar{V})} \to \bar{V} \otimes \mathcal{O}_{\mathbf{P}(\bar{V})}$, so that on $\mathbf{P}(\bar{V})$ we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_0 \otimes \mathcal{O}_{\mathbf{P}(\bar{V})} & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{O}(-\bar{H}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V_0 \otimes \mathcal{O}_{\mathbf{P}(\bar{V})} & \longrightarrow & V \otimes \mathcal{O}_{\mathbf{P}(\bar{V})} & \longrightarrow & \bar{V} \otimes \mathcal{O}_{\mathbf{P}(\bar{V})} & \longrightarrow & 0 \end{array} \tag{3.2}$$

with exact rows. If (3.1) is split then $\mathcal{V} \cong V_0 \otimes \mathcal{O} \oplus \mathcal{O}(-\bar{H})$.

Now let $X \to \mathbf{P}(\bar{V})$ be a morphism of schemes. Then the *resolved cone* over X with vertex $\mathbf{P}(V_0)$ is defined as the projective bundle

$$\tilde{\mathbf{C}}_{V_0}(X) = \mathbf{P}_X(\mathcal{V}_X), \tag{3.3}$$

where $\mathcal{V}_X$ denotes the pullback of $\mathcal{V}$ to X. The embedding $\mathcal{V}_X \hookrightarrow V \otimes \mathcal{O}_X$ induced by the middle vertical arrow in (3.2) gives a morphism

$$\tilde{\mathbf{C}}_{V_0}(X) \to \mathbf{P}(V).$$

If $X \to \mathbf{P}(\bar{V})$ is an embedding, then this morphism factors birationally through the classical cone $\mathbf{C}_{V_0}(X) \subset \mathbf{P}(V)$, and provides a resolution of singularities if X is smooth.

Note that there is an isomorphism

$$\tilde{\mathbf{C}}_{V_0}(X) \cong X \times_{\mathbf{P}(\bar{V})} \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})). \tag{3.4}$$

Motivated by this, we call $\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) = \mathbf{P}_{\mathbf{P}(\bar{V})}(\mathcal{V})$ the *universal resolved cone* with vertex $\mathbf{P}(V_0)$. Denote by

$$\bar{p}\colon \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) \to \mathbf{P}(\bar{V})$$

the canonical projection morphism. Note that the rank of $\mathcal{V}$ is $N_0 + 1$, so $\bar{p}$ is a $\mathbf{P}^{N_0}$-bundle. Further, denote by

$$f\colon \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) \to \mathbf{P}(V)$$

the morphism induced by the canonical embedding $\mathcal{V} \hookrightarrow V \otimes \mathcal{O}_{\mathbf{P}(\bar{V})}$ from (3.2). Define

$$\mathbf{E} = \mathbf{P}_{\mathbf{P}(\bar{V})}(V_0 \otimes \mathcal{O}_{\mathbf{P}(\bar{V})}) \cong \mathbf{P}(V_0) \times \mathbf{P}(\bar{V})$$

and let

$$\varepsilon\colon \mathbf{E} \to \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))$$

be the canonical divisorial embedding induced by the first map in the top row of (3.2). We have a commutative diagram

$$\begin{array}{ccccc} \mathbf{E} & \xrightarrow{\varepsilon} & \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) & \xrightarrow{\bar{p}} & \mathbf{P}(\bar{V}) \\ \downarrow & & \downarrow f & & \\ \mathbf{P}(V_0) & \longrightarrow & \mathbf{P}(V), & & \end{array} \tag{3.5}$$

where the bottom arrow is the natural embedding. The isomorphism $\mathbf{E} \cong \mathbf{P}(V_0) \times \mathbf{P}(\bar{V})$ is induced by the product of the vertical arrow and $\bar{p} \circ \varepsilon$.

The next result follows easily from the definitions.

LEMMA 3.1. – *The following hold:*

(1) *The morphism* $f\colon \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) \to \mathbf{P}(V)$ *is the blowup of* $\mathbf{P}(V)$ *in* $\mathbf{P}(V_0)$*, with exceptional divisor* $\mathbf{E}$*.*

(2) *The* $\mathcal{O}(1)$ *line bundle for the projective bundle* $\bar{p}\colon \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) \to \mathbf{P}(\bar{V})$ *is* $\mathcal{O}(H)$*.*

(3) *We have the following equality of divisors modulo linear equivalence:*

$$\mathbf{E} = H - \bar{H}, \qquad H|_{\mathbf{E}} = H_0.$$

(4) *The relative dualizing complex of the morphism* $\bar{p}$ *is given by*

$$\omega_{\bar{p}} \cong \det(\bar{p}^*\mathcal{V}^\vee)(-(N_0+1)H)[N_0].$$

Following (3.4) we define the resolved cone of a category linear over $\mathbf{P}(\bar{V})$ by base change from the universal resolved cone.

DEFINITION 3.2. – Let $\mathcal{A}$ be a $\mathbf{P}(\bar{V})$-linear category. The *resolved cone* over $\mathcal{A}$ with vertex $\mathbf{P}(V_0)$ is the category

$$\tilde{\mathbf{C}}_{V_0}(\mathcal{A}) = \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(\bar{V}))} \mathrm{Perf}(\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))).$$

Further, we define

$$\mathbf{E}(\mathcal{A}) = \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(\bar{V}))} \mathrm{Perf}(\mathbf{E}).$$

REMARK 3.3. – The isomorphism $\mathbf{E} \cong \mathbf{P}(V_0) \times \mathbf{P}(\bar{V})$ induces a canonical equivalence

$$\mathbf{E}(\mathcal{A}) \simeq \mathrm{Perf}(\mathbf{P}(V_0)) \otimes \mathcal{A}.$$

We identify these categories via this equivalence; in particular, below we will regard subcategories of the right side as subcategories of the left. Furthermore, using this identification the morphism ε from (3.5) induces functors between $\mathrm{Perf}(\mathbf{P}(V_0)) \otimes \mathcal{A}$ and $\tilde{\mathbf{C}}_{V_0}(\mathcal{A})$.

REMARK 3.4. – If X is a scheme over $\mathbf{P}(\bar{V})$, then by the isomorphism (3.4) and [2, Theorem 1.2] the resolved cone satisfies

$$\tilde{\mathbf{C}}_{V_0}(\mathrm{Perf}(X)) \simeq \mathrm{Perf}(\tilde{\mathbf{C}}_{V_0}(X)).$$

REMARK 3.5. – Resolved cones are functorial in the same way as resolved joins, see [25, Lemma 3.5]. Namely, given a $\mathbf{P}(\bar{V})$-linear functor $\gamma\colon \mathcal{A} \to \mathcal{B}$, its base change along the morphism $\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) \to \mathbf{P}(\bar{V})$ gives a $\mathbf{P}(V)$-linear functor

$$\tilde{\mathbf{C}}_{V_0}(\gamma)\colon \tilde{\mathbf{C}}_{V_0}(\mathcal{A}) \to \tilde{\mathbf{C}}_{V_0}(\mathcal{B}).$$

Moreover, if $\gamma^*\colon \mathcal{B} \to \mathcal{A}$ is a left adjoint functor to γ, then $\tilde{\mathbf{C}}_{V_0}(\gamma^*)$ is left adjoint to $\tilde{\mathbf{C}}_{V_0}(\gamma)$, and similarly for right adjoints, see [32, Lemma 2.12].

3.2. Categorical cones

We define the categorical cone of a Lefschetz category over $\mathbf{P}(\bar{V})$ as a certain subcategory of the resolved cone, similarly to Definition 2.15 of a categorical join.

DEFINITION 3.6. – Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$ with Lefschetz center $\mathcal{A}_0$. The *categorical cone* $\mathcal{C}_{V_0}(\mathcal{A})$ over $\mathcal{A}$ with vertex $\mathbf{P}(V_0)$ is the subcategory of $\tilde{\mathbf{C}}_{V_0}(\mathcal{A})$ defined by

$$\mathcal{C}_{V_0}(\mathcal{A}) = \left\{ C \in \tilde{\mathbf{C}}_{V_0}(\mathcal{A}) \mid \varepsilon^*(C) \in \mathrm{Perf}(\mathbf{P}(V_0)) \otimes \mathcal{A}_0 \subset \mathbf{E}(\mathcal{A}) \right\}.$$

Here, we have used the identification of Remark 3.3. If $\mathcal{A} = \mathrm{Perf}(X)$ for a scheme X over $\mathbf{P}(\bar{V})$, we abbreviate notation by writing

$$\mathcal{C}_{V_0}(X) = \mathcal{C}_{V_0}(\mathrm{Perf}(X)).$$

REMARK 3.7. – The categorical cone depends on the choice of a Lefschetz center for $\mathcal{A}$, although this is suppressed in the notation. For instance, for the "stupid" Lefschetz center $\mathcal{A}_0 = \mathcal{A}$, the condition in the definition is void, so $\mathcal{C}_{V_0}(\mathcal{A}) = \tilde{\mathbf{C}}_{V_0}(\mathcal{A})$.

We note that if $V_0 = 0$, then taking the categorical cone does nothing:

LEMMA 3.8. – *Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. If $V_0 = 0$ then $\mathcal{C}_{V_0}(\mathcal{A}) \simeq \mathcal{A}$.*

Proof. – If $V_0 = 0$ then $\mathcal{V} \cong \mathcal{O}_{\mathbf{P}(\bar{V})}(-\bar{H})$ by (3.2), hence $\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) = \mathbf{P}_{\mathbf{P}(\bar{V})}(\mathcal{V}) \cong \mathbf{P}(\bar{V})$ and $\tilde{\mathbf{C}}_{V_0}(\mathcal{A}) \simeq \mathcal{A}$. Furthermore, the divisor $\mathbf{E}$ is empty in this case, hence the defining condition of $\mathcal{C}_{V_0}(\mathcal{A}) \subset \tilde{\mathbf{C}}_{V_0}(\mathcal{A})$ is void and $\mathcal{C}_{V_0}(\mathcal{A}) = \tilde{\mathbf{C}}_{V_0}(\mathcal{A})$. □

LEMMA 3.9. – *Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$ of length m. Then the categorical cone $\mathcal{C}_{V_0}(\mathcal{A})$ is an admissible $\mathbf{P}(V)$-linear subcategory of $\tilde{\mathbf{C}}_{V_0}(\mathcal{A})$, and there are $\mathbf{P}(V)$-linear semiorthogonal decompositions*

(3.6)
$$\tilde{\mathbf{C}}_{V_0}(\mathcal{A}) = \left\langle \mathcal{C}_{V_0}(\mathcal{A}), \varepsilon_!\big(\mathrm{Perf}(\mathbf{P}(V_0)) \otimes \mathcal{A}_1(\bar{H})\big), \dots, \varepsilon_!\big(\mathrm{Perf}(\mathbf{P}(V_0)) \otimes \mathcal{A}_{m-1}((m-1)\bar{H})\big) \right\rangle,$$

(3.7)
$$\tilde{\mathbf{C}}_{V_0}(\mathcal{A}) = \left\langle \varepsilon_*\left(\mathrm{Perf}(\mathbf{P}(V_0)) \otimes \mathcal{A}_{1-m}((1-m)\bar{H})\right), \dots, \varepsilon_*\left(\mathrm{Perf}(\mathbf{P}(V_0)) \otimes \mathcal{A}_{-1}(-\bar{H})\right), \mathcal{C}_{V_0}(\mathcal{A}) \right\rangle,$$

where $\varepsilon_!$ denotes the left adjoint of ε^*.

Proof. – Apply [25, Proposition 3.11] with $T = \mathbf{P}(\bar{V})$, $Y = \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))$, and $E = \mathbf{E}$. Then in the notation of that proposition, $\mathcal{A}_Y = \tilde{\mathbf{C}}_{V_0}(\mathcal{A})$ and $\mathcal{A}_E = \mathbf{E}(\mathcal{A}) = \mathrm{Perf}(\mathbf{P}(V_0)) \otimes \mathcal{A}$, and the result follows. □

EXAMPLE 3.10. – Let $\bar{W} \subset \bar{V}$ be a subbundle, so that $\mathbf{P}(\bar{W}) \subset \mathbf{P}(\bar{V})$. The classical cone over $\mathbf{P}(\bar{W})$ with vertex $\mathbf{P}(V_0)$ is given by $\mathbf{C}_{V_0}(\mathbf{P}(\bar{W})) = \mathbf{P}(W)$, where $W \subset V$ is the preimage of $\bar{W}$ under the epimorphism $V \to \bar{V}$. Consider the Lefschetz structure on $\mathbf{P}(\bar{W})$ defined in Example 2.8. Then it follows from Lemma 3.1 and Orlov's blowup formula that the pullback functor $f^*\colon \mathrm{Perf}(\mathbf{P}(W)) \to \mathrm{Perf}(\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{W})))$ induces an equivalence

$$\mathrm{Perf}(\mathbf{P}(W)) \simeq \mathcal{C}_{V_0}(\mathbf{P}(\bar{W})).$$

Further, Theorem 3.21 below equips $\mathcal{C}_{V_0}(\mathbf{P}(\bar{W}))$ with a canonical Lefschetz structure, with respect to which this equivalence is easily seen to be a Lefschetz equivalence.

LEMMA 3.11. – *Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$ which is smooth and proper over S. Then the categorical cone $\mathcal{C}_{V_0}(\mathcal{A})$ is smooth and proper over S.*

Proof. – Being the base change of $\mathcal{A}$ along the projective bundle $\bar{p}\colon \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) \to \mathbf{P}(\bar{V})$, the resolved cone $\tilde{\mathbf{C}}_{V_0}(\mathcal{A})$ is smooth and proper over S by [32, Lemma 4.11]. Hence the result follows from Lemma 3.9 and [32, Lemma 4.15]. □

PROPOSITION 3.12. – *Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. Let $T \to \mathbf{P}(V)$ be a morphism of schemes which factors through the complement of $\mathbf{P}(V_0)$ in $\mathbf{P}(V)$. Then there are T-linear equivalences*

$$\mathcal{C}_{V_0}(\mathcal{A})_T \simeq \tilde{\mathbf{C}}_{V_0}(\mathcal{A})_T \simeq \mathcal{A}_T,$$

where the base change of $\mathcal{A}$ is taken along the morphism $T \to \mathbf{P}(\bar{V})$ obtained by composing $T \to \mathbf{P}(V)$ with the linear projection from $\mathbf{P}(V_0) \subset \mathbf{P}(V)$.

Proof. – By Lemma 3.1, the morphism $f\colon \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) \to \mathbf{P}(V)$ is an isomorphism over the complement of $\mathbf{P}(V_0)$. Hence there is an isomorphism $\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))_T \cong T$. The equivalence $\tilde{\mathbf{C}}_{V_0}(\mathcal{A})_T \simeq \mathcal{A}_T$ then follows from the definition of the resolved cone. Further, the components to the right of $\mathcal{C}_{V_0}(\mathcal{A})$ in (3.6) are supported over $\mathbf{P}(V_0)$, hence their base changes along $T \to \mathbf{P}(\bar{V})$ vanish. This shows $\mathcal{C}_{V_0}(\mathcal{A})_T \simeq \tilde{\mathbf{C}}_{V_0}(\mathcal{A})_T$. □

For future use we fix the following immediate corollary of the proposition.

COROLLARY 3.13. – *Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. Let $T \to \mathbf{P}(V)$ be a morphism of schemes which factors through the complement of $\mathbf{P}(V_0)$ in $\mathbf{P}(V)$, and such that the composition $T \to \mathbf{P}(\bar{V})$ is an isomorphism. Then there is an equivalence*

$$\mathcal{C}_{V_0}(\mathcal{A})_T \simeq \mathcal{A}.$$

3.3. Relation to categorical joins

In this subsection, we assume $V_0 \neq 0$ and we are given a splitting of (3.1):

$$V = V_0 \oplus \bar{V}.$$

Under these assumptions, we relate the cone operations discussed above (classical, resolved, and categorical) to taking a join (in the corresponding senses) with $\mathbf{P}(V_0)$.

The relation between the classical operations is easy: if $X \subset \mathbf{P}(\bar{V})$ is a closed subscheme, then the classical join of X with $\mathbf{P}(V_0)$ coincides with the cone over X with vertex $\mathbf{P}(V_0)$, i.e.,

$$\mathbf{J}(\mathbf{P}(V_0), X) = \mathbf{C}_{V_0}(X) \subset \mathbf{P}(V).$$

Note that the assumption $V_0 \neq 0$ is necessary for this equality; if $V_0 = 0$ then $\mathbf{P}(V_0) = \emptyset$ and hence $\mathbf{J}(\mathbf{P}(V_0), X) = \emptyset$, while $\mathbf{C}_{V_0}(X) = X$.

Next we compare the universal resolved join (2.13) to the universal resolved cone

$$\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) = \mathbf{P}_{\mathbf{P}(\bar{V})}((V_0 \otimes \mathcal{O}_{\mathbf{P}(\bar{V})}) \oplus \mathcal{O}(-\bar{H})).$$

The natural embedding $\mathcal{O}(-H_0) \hookrightarrow V_0 \otimes \mathcal{O}_{\mathbf{P}(\bar{V})}$ induces a morphism

$$\beta\colon \tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathbf{P}(\bar{V})) \to \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})).$$

Denoting $Z = \mathbf{P}_{\mathbf{P}(\bar{V})}(\mathcal{O}(-\bar{H})) \cong \mathbf{P}(\bar{V}) \subset \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))$, the diagram (2.14) (with $V_1 = V_0$ and $V_2 = \bar{V}$) and the diagram (3.5) merge to a commutative diagram

$$\begin{array}{ccccc}
\mathbf{E}_1 & \longrightarrow & \tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathbf{P}(\bar{V})) & \longleftarrow & \mathbf{E}_2 \\
\wr\downarrow & & \beta\downarrow & & \downarrow \\
\mathbf{E} & \longrightarrow & \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) & \longleftarrow & Z \\
\downarrow & & \downarrow & & \wr\downarrow \\
\mathbf{P}(V_0) & \longrightarrow & \mathbf{P}(V_0 \oplus \bar{V}) & \longleftarrow & \mathbf{P}(\bar{V}),
\end{array}$$

where under the isomorphisms $\mathbf{E}_2 \cong \mathbf{P}(V_0) \times \mathbf{P}(\bar{V})$ and $Z \cong \mathbf{P}(\bar{V})$, the map $\mathbf{E}_2 \to Z$ is identified with the projection.

LEMMA 3.14. – *The morphism* $\beta\colon \tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathbf{P}(\bar{V})) \to \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))$ *defined above is the blowup of* $\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))$ *in* Z*, with exceptional divisor* $\mathbf{E}_2$.

Proof. – Follows from Lemma 3.1(1) and [25, Lemma 3.1(1)]. □

Using this, we can finally compare categorical joins and cones. We consider the categorical join $\mathcal{J}(\mathbf{P}(V_0), \mathcal{A})$ of $\mathrm{Perf}(\mathbf{P}(V_0))$ (with the standard Lefschetz structure from Example 2.8) and a Lefschetz category $\mathcal{A}$ over $\mathbf{P}(\bar{V})$.

PROPOSITION 3.15. – *Let* $\mathcal{A}$ *be a Lefschetz category over* $\mathbf{P}(\bar{V})$*, and let* V_0 *be a nonzero vector bundle on* S*. Then there is an equivalence*

$$\mathcal{C}_{V_0}(\mathcal{A}) \simeq \mathcal{J}(\mathbf{P}(V_0), \mathcal{A})$$

of $\mathbf{P}(V_0 \oplus \bar{V})$*-linear categories. More precisely, pullback and pushforward along the blowup morphism* $\beta\colon \tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathbf{P}(\bar{V})) \to \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))$ *give functors*

$$\beta^*\colon \tilde{\mathbf{C}}_{V_0}(\mathcal{A}) \to \tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathcal{A}),$$
$$\beta_*\colon \tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathcal{A}) \to \tilde{\mathbf{C}}_{V_0}(\mathcal{A}),$$

which induce mutually inverse equivalences between the subcategories

$$\mathcal{C}_{V_0}(\mathcal{A}) \subset \tilde{\mathbf{C}}_{V_0}(\mathcal{A}) \quad \textit{and} \quad \mathcal{J}(\mathbf{P}(V_0), \mathcal{A}) \subset \tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathcal{A}).$$

Moreover, for any $\mathbf{P}(\bar{V})$-linear functor $\gamma\colon \mathcal{A} \to \mathcal{B}$ there are commutative diagrams

$$\begin{array}{ccc} \tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathcal{A}) & \xrightarrow{\tilde{\mathbf{J}}(\mathrm{id},\gamma)} & \tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathcal{B}) \\ {\scriptstyle\beta_*}\downarrow & & \downarrow{\scriptstyle\beta_*} \\ \tilde{\mathbf{C}}_{V_0}(\mathcal{A}) & \xrightarrow{\tilde{\mathbf{C}}_{V_0}(\gamma)} & \tilde{\mathbf{C}}_{V_0}(\mathcal{B}) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathcal{A}) & \xrightarrow{\tilde{\mathbf{J}}(\mathrm{id},\gamma)} & \tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathcal{B}) \\ \uparrow{\scriptstyle\beta^*} & & \uparrow{\scriptstyle\beta^*} \\ \tilde{\mathbf{C}}_{V_0}(\mathcal{A}) & \xrightarrow{\tilde{\mathbf{C}}_{V_0}(\gamma)} & \tilde{\mathbf{C}}_{V_0}(\mathcal{B}), \end{array} \tag{3.8}$$

where the functor $\tilde{\mathbf{J}}(\mathrm{id},\gamma)$ is defined for resolved joins in the same way as $\tilde{\mathbf{C}}_{V_0}(\gamma)$ for cones, see [25, (3.4)].

Proof. – Diagrams (3.8) are obtained from the functor γ by base change along the morphism β.

Lemma 3.14 together with Orlov's blowup formula implies $\beta^*\colon \tilde{\mathbf{C}}_{V_0}(\mathcal{A}) \to \tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathcal{A})$ is fully faithful and gives an equivalence onto the subcategory

$$\beta^*(\tilde{\mathbf{C}}_{V_0}(\mathcal{A})) = \left\{ C \in \tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathcal{A}) \mid \varepsilon_2^*(C) \in \mathrm{Perf}(S)\otimes\mathcal{A} \subset \mathbf{E}_2(\mathrm{Perf}(\mathbf{P}(V_0)),\mathcal{A}) \right\}$$

with the inverse functor given by β_*. Since β maps $\mathbf{E}_1$ isomorphically onto $\mathbf{E}$, it thus follows from Definition 3.6 that β^* induces an equivalence from $\mathcal{C}_{V_0}(\mathcal{A})$ onto the subcategory

$$\beta^*(\mathcal{C}_{V_0}(\mathcal{A})) = \left\{ C \in \tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathcal{A}) \;\middle|\; \begin{array}{ll} \varepsilon_1^*(C) \in \mathrm{Perf}(\mathbf{P}(V_0))\otimes\mathcal{A}_0 & \subset \mathbf{E}_1(\mathrm{Perf}(\mathbf{P}(V_0)),\mathcal{A}), \\ \varepsilon_2^*(C) \in \mathrm{Perf}(S)\otimes\mathcal{A} & \subset \mathbf{E}_2(\mathrm{Perf}(\mathbf{P}(V_0)),\mathcal{A}) \end{array} \right\},$$

with the inverse equivalence induced by β_*. But by Definition 2.15 this subcategory coincides with $\mathcal{J}(\mathbf{P}(V_0),\mathcal{A})$ since $\mathrm{Perf}(S)$ is the Lefschetz center of $\mathrm{Perf}(\mathbf{P}(V_0))$. □

REMARK 3.16. – Proposition 3.15 does not apply if $V_0 = 0$. Indeed, if $V_0 = 0$ then $\mathbf{P}(V_0) = \emptyset$ and hence $\mathcal{J}(\mathbf{P}(V_0),\mathcal{A}) = 0$, while $\mathcal{C}_{V_0}(\mathcal{A}) \simeq \mathcal{A}$ by Lemma 3.8.

REMARK 3.17. – In Theorem 3.21 we will equip any categorical cone with a canonical Lefschetz structure in such a way that the equivalence $\mathcal{C}_{V_0}(\mathcal{A}) \simeq \mathcal{J}(\mathbf{P}(V_0),\mathcal{A})$ of Proposition 3.15 is an equivalence of Lefschetz categories.

REMARK 3.18. – Let $\mathcal{A}^1$ and $\mathcal{A}^2$ be Lefschetz categories over $\mathbf{P}(V_1)$ and $\mathbf{P}(V_2)$, where V_1 and V_2 are nonzero. Then there is a $\mathbf{P}(V_1\oplus V_2)$-linear equivalence

$$\mathcal{J}(\mathcal{A}^1,\mathcal{A}^2) \simeq \mathcal{C}_{V_1}(\mathcal{A}^2) \otimes_{\mathrm{Perf}(\mathbf{P}(V_1\oplus V_2))} \mathcal{C}_{V_2}(\mathcal{A}^1).$$

This can be proved either directly, or (in its dual form) by combining Corollary 4.3 below and [25, Corollary B.4]. The right side can be endowed with a semiorthogonal decomposition by an application of [25, Corollary 5.3], which can be shown to be a Lefschetz decomposition compatible with the Lefschetz structure of the left side. Note also that the equivalence of Proposition 3.15 is a special case of this. Indeed, take $\mathcal{A}^1 = \mathbf{P}(V_1)$ and use the equivalence $\mathcal{C}_{V_2}(\mathbf{P}(V_1)) \simeq \mathrm{Perf}(\mathbf{P}(V_1\oplus V_2))$ of Example 3.10. We omit further details as we shall not need this.

3.4. The Lefschetz structure of a categorical cone

Our goal in this subsection is to equip any categorical cone with a canonical Lefschetz structure.

LEMMA 3.19. – *Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. Then the image of $\mathcal{A}_0$ under the functor $\bar{p}^*\colon \mathcal{A} \to \tilde{\mathbf{C}}_{V_0}(\mathcal{A})$ is contained in the categorical cone $\mathcal{C}_{V_0}(\mathcal{A})$. If $\mathcal{A}_i$ are the Lefschetz components of $\mathcal{A}$, then $\bar{p}^*(\mathcal{A}_i) \subset \mathcal{C}_{V_0}(\mathcal{A})$ is left admissible for $i < 0$, admissible for $i = 0$, and right admissible for $i > 0$.*

Proof. – Because the morphism $\bar{p}\colon \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) \to \mathbf{P}(\bar{V})$ is a projective bundle, the pull-back functor $\bar{p}^*\colon \mathrm{Perf}(\mathbf{P}(\bar{V})) \to \mathrm{Perf}(\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})))$ is fully faithful and admits left and right adjoints. Thus the same holds for its base change $\bar{p}^*\colon \mathcal{A} \to \tilde{\mathbf{C}}_{V_0}(\mathcal{A})$ (see [32, Lemma 2.12]). Further, by Definition 3.6 we see that the image of $\mathcal{A}_0$ under $\bar{p}^*$ is contained in $\mathcal{C}_{V_0}(\mathcal{A})$. The result follows. □

DEFINITION 3.20. – Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. For $i \in \mathbf{Z}$, we define a subcategory $\mathcal{C}_{V_0}(\mathcal{A})_i \subset \mathcal{C}_{V_0}(\mathcal{A})$ by

$$\mathcal{C}_{V_0}(\mathcal{A})_i = \begin{cases} \bar{p}^*(\mathcal{A}_{i+N_0}) & \text{if } i \le -N_0, \\ \bar{p}^*(\mathcal{A}_0) & \text{if } -N_0 \le i \le N_0, \\ \bar{p}^*(\mathcal{A}_{i-N_0}) & \text{if } i \ge N_0, \end{cases} \tag{3.9}$$

where N_0 is the rank of V_0.

Note that the containment $\mathcal{C}_{V_0}(\mathcal{A})_i \subset \mathcal{C}_{V_0}(\mathcal{A})$ holds by Lemma 3.19.

THEOREM 3.21. – *Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\bar{V})$. Then the categorical cone $\mathcal{C}_{V_0}(\mathcal{A})$ has the structure of a Lefschetz category over $\mathbf{P}(V)$ with Lefschetz components $\mathcal{C}_{V_0}(\mathcal{A})_i$ given by* (3.9)*. If $\mathcal{A}$ is either right or left strong, then so is $\mathcal{C}_{V_0}(\mathcal{A})$. Moreover, we have*

$$\mathrm{length}(\mathcal{C}_{V_0}(\mathcal{A})) = \mathrm{length}(\mathcal{A}) + N_0,$$

and $\mathcal{C}_{V_0}(\mathcal{A})$ is moderate if and only if $\mathcal{A}$ is moderate.

One could prove this directly by an analogue of the argument of [25, §3.4]. However, we prefer to reduce to the case of categorical joins using Proposition 3.15 and the local-to-global result of Lemma A.6.

Proof. – We may also assume $V_0 \neq 0$, otherwise the result is trivial. The key claim is that we have semiorthogonal decompositions

$$\mathcal{C}_{V_0}(\mathcal{A}) = \langle \mathcal{C}_0, \mathcal{C}_1(H), \dots, \mathcal{C}_{m+N_0-1}((m+N_0-1)H) \rangle, \tag{3.10}$$

$$\mathcal{C}_{V_0}(\mathcal{A}) = \langle \mathcal{C}_{1-m-N_0}((1-m-N_0)H), \dots, \mathcal{C}_{-1}(-H), \mathcal{C}_0 \rangle, \tag{3.11}$$

where $\mathcal{C}_i = \mathcal{C}_{V_0}(\mathcal{A})_i$ and $m = \mathrm{length}(\mathcal{A})$. By Lemma A.6—whose hypotheses are satisfied by Lemma 3.19— it is enough to prove (3.10) and (3.11) after base change to any fpqc cover of S. Therefore, we may assume that we have a splitting $V = V_0 \oplus \bar{V}$ of (3.1). Then Proposition 3.15 gives an equivalence

$$\mathcal{C}_{V_0}(\mathcal{A}) \simeq \mathcal{J}(\mathbf{P}(V_0), \mathcal{A}).$$

By [25, Theorem 3.21] the categorical join $\mathcal{J}(\mathbf{P}(V_0), \mathcal{A})$ has the structure of a Lefschetz category of length $\mathrm{length}(\mathcal{A}) + N_0$. By Example 2.8 the nonzero primitive components $\mathfrak{p}_i$ of $\mathrm{Perf}(\mathbf{P}(V_0))$ are $\mathfrak{p}_{\pm(N_0-1)} = \mathrm{Perf}(S)$, hence the second formula of [25, Lemma 3.24] shows that the Lefschetz components $\mathcal{J}_i \subset \mathcal{J}(\mathbf{P}(V_0), \mathcal{A})$ are equal to

$$\mathcal{J}_i = \begin{cases} p^*\mathrm{pr}_2^*(\mathcal{A}_{i+N_0}) & \text{if } i \le -N_0, \\ p^*\mathrm{pr}_2^*(\mathcal{A}_0) & \text{if } -N_0 \le i \le N_0, \\ p^*\mathrm{pr}_2^*(\mathcal{A}_{i-N_0}) & \text{if } i \ge N_0, \end{cases}$$

Using the commutative diagram

$$\begin{array}{ccc} \tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathbf{P}(\bar{V})) & \xrightarrow{p} & \mathbf{P}(V_0) \times \mathbf{P}(\bar{V}) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) & \xrightarrow{\bar{p}} & \mathbf{P}(\bar{V}) \end{array}$$

it is easy to see the equivalence $\mathcal{C}_{V_0}(\mathcal{A}) \simeq \mathcal{J}(\mathbf{P}(V_0), \mathcal{A})$ identifies $\mathcal{C}_i$ with $\mathcal{J}_i$; thus the decompositions (3.10) and (3.11) hold.

By [25, Lemma 2.4] and Lemma 3.19, we thus deduce that $\mathcal{C}_0 \subset \mathcal{C}_{V_0}(\mathcal{A})$ is a Lefschetz center with $\mathcal{C}_i, i \in \mathbf{Z}$, the corresponding Lefschetz components. The strongness claims follow from the definitions and Lemma 3.19, and the claims about the length and moderateness of $\mathcal{C}_{V_0}(\mathcal{A})$ follow from the definitions. □

4. HPD for categorical cones

In this section we show that (under suitable hypotheses) the formation of categorical cones commutes with HPD. We formulate the theorem in a way that allows for extensions of the base projective bundle (in the sense of Definition A.7), because this extra generality is useful in applications.

THEOREM 4.1. – *Let V be a vector bundle on S, let*

$$V_0 \subset V \quad \text{and} \quad V_\infty \subset V^\vee$$

be subbundles such that the natural pairing $V \otimes V^\vee \to \mathcal{O}_S$ is zero on $V_0 \otimes V_\infty$, so that we have a pair of filtrations

$$0 \subset V_0 \subset V_\infty^\perp \subset V \quad \text{and} \quad 0 \subset V_\infty \subset V_0^\perp \subset V^\vee. \tag{4.1}$$

Set

$$\bar{V} := V_\infty^\perp / V_0, \qquad \text{so that} \quad V_0^\perp / V_\infty \cong \bar{V}^\vee. \tag{4.2}$$

Let $\mathcal{A}$ be a right strong, moderate Lefschetz category over $\mathbf{P}(\bar{V})$. Then there is an equivalence

$$(\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural \simeq \mathcal{C}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee)$$

of Lefschetz categories over $\mathbf{P}(V^\vee)$.

REMARK 4.2. – Let us explain the structure of the categories appearing in Theorem 4.1. By Theorem 3.21 the categorical cone $\mathcal{C}_{V_0}(\mathcal{A})$ is a right strong, moderate Lefschetz category over $\mathbf{P}(V_\infty^\perp)$. By extending the base along the inclusion $\mathbf{P}(V_\infty^\perp) \to \mathbf{P}(V)$, we obtain by Remark A.8 a right strong, moderate Lefschetz category $\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V)$ over $\mathbf{P}(V)$. Hence by [32, Theorem 8.7(1)], the HPD category $(\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural$ has the structure of a Lefschetz category over $\mathbf{P}(V^\vee)$. The structure of $\mathcal{C}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee)$ as a Lefschetz category over $\mathbf{P}(V^\vee)$ is similarly obtained by a combination of [32, Theorem 8.7(1)], Theorem 3.21, and base extension.

In the case $V_0 = 0$ (we also choose a subbundle $W \subset V$ and take $V_\infty = W^\perp \subset V^\vee$) we obtain the following corollary, mentioned in Remark 3.18 above.

COROLLARY 4.3. – *Let $W \subset V$ be an inclusion of vector bundles on S. Let $\mathcal{A}$ be a right strong, moderate Lefschetz category over $\mathbf{P}(W)$. Then there is an equivalence*

$$(\mathcal{A}/\mathbf{P}(V))^\natural \simeq \mathcal{C}_{W^\perp}(\mathcal{A}^\natural)$$

of Lefschetz categories over $\mathbf{P}(V^\vee)$.

Our strategy for proving Theorem 4.1 is the following. First, we use the relation between categorical cones and categorical joins described in Proposition 3.15 and linear HPD of Example 2.12 to deduce the theorem when both V_0 and V_∞ are nonzero and the filtrations (4.1) are split (which always holds locally over the base scheme S) from Theorem 2.16. Then we use a local-to-global argument analogous to the one used in the proof of Theorem 3.21 to deduce the theorem without the splitting assumption. Finally, we use a relation between HPD and hyperplane sections (Proposition A.10) and duality to deduce the theorem in full generality.

For the local-to-global argument it is important to define a functor between the categories $\mathcal{C}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee)$ and $(\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural$ in general. This is what we start with in §4.1, where we define a functor

$$\gamma_{\tilde{\mathbf{C}}}\colon \tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee) \to \mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathcal{A})/\mathbf{P}(V))$$

via a double cone construction, an analogue of the double join construction from [25, §4.1]. Next, in §4.2 we check its compatibility with the analogous functor between resolved joins, deduce the theorem in the split nonzero case, and then by the local-to-global argument remove the splitting assumption. Finally, in §4.3 we prove the general case.

4.1. Double resolved cones and the HPD functor for categorical cones

Throughout this section we fix filtrations (4.1), and use (4.2) to identify their quotients with $(V_0, \bar{V}, V_\infty^\vee)$ and $(V_\infty, \bar{V}^\vee, V_0^\vee)$ respectively.

Let Y be a scheme equipped with a morphism $Y \to \mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee)$. In this situation, we can form two resolved cones, $\tilde{\mathbf{C}}_{V_0}(Y)$ and $\tilde{\mathbf{C}}_{V_\infty}(Y)$, using the projection to $\mathbf{P}(\bar{V})$ for the first and the projection to $\mathbf{P}(\bar{V}^\vee)$ for the second. We define the *double resolved cone* over Y as the fiber product

$$\widetilde{\mathbf{CC}}_{V_0,V_\infty}(Y) = \tilde{\mathbf{C}}_{V_0}(Y) \times_Y \tilde{\mathbf{C}}_{V_\infty}(Y), \tag{4.3}$$

which is a $\mathbf{P}^{N_0} \times \mathbf{P}^{N_\infty}$-bundle over Y, where $N_0 = \dim V_0$ and $N_\infty = \dim V_\infty$. In particular, we can consider the universal double resolved cone with its natural projection

$$\widetilde{\mathbf{CC}}_{V_0,V_\infty}(\mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee)) \to \mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee). \tag{4.4}$$

Now, given a category $\mathcal{B}$ which has a $\mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee)$-linear structure, we define the *double resolved cone* $\widetilde{\mathbf{CC}}_{V_0,V_\infty}(\mathcal{B})$ over $\mathcal{B}$ as

$$\widetilde{\mathbf{CC}}_{V_0,V_\infty}(\mathcal{B}) = \mathcal{B} \otimes_{\mathrm{Perf}(\mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee))} \mathrm{Perf}(\widetilde{\mathbf{CC}}_{V_0,V_\infty}(\mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee))),$$

that is the base change of $\mathcal{B}$ along (4.4).

The key case for us is when Y is the universal hyperplane in $\mathbf{P}(\bar{V})$, which we denote by

$$\bar{\mathbf{H}} = \mathbf{H}(\mathbf{P}(\bar{V})). \tag{4.5}$$

Note that $\bar{\mathbf{H}}$ indeed naturally maps to $\mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee)$, hence we can form the double resolved cone $\widetilde{\mathbf{CC}}_{V_0,V_\infty}(\bar{\mathbf{H}})$ over $\bar{\mathbf{H}}$. We also write $\mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))/\mathbf{P}(V))$ for the universal hyperplane section of $\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))$ with respect to the morphism $\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V})) \to \mathbf{P}(V_\infty^\perp) \to \mathbf{P}(V)$.

LEMMA 4.4. – *We have a diagram*

$$\begin{array}{ccc} & \widetilde{\mathbf{CC}}_{V_0,V_\infty}(\bar{\mathbf{H}}) & \\ {\scriptstyle \tilde{p}}\swarrow & & \searrow{\scriptstyle \alpha} \\ \tilde{\mathbf{C}}_{V_\infty}(\bar{\mathbf{H}}) & & \mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))/\mathbf{P}(V)) \end{array} \tag{4.6}$$

of schemes over $\mathbf{P}(V^\vee)$, where all schemes appearing are smooth and projective over S.

Proof. – By definition the double resolved cone is the subvariety

$$\widetilde{\mathbf{CC}}_{V_0,V_\infty}(\bar{\mathbf{H}}) \subset \mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee) \times \mathbf{P}(V_\infty^\perp) \times \mathbf{P}(V_0^\perp)$$

defined by the incidence conditions in $\mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee)$, $\mathbf{P}(\bar{V}) \times \mathbf{P}(V_\infty^\perp)$, and $\mathbf{P}(\bar{V}^\vee) \times \mathbf{P}(V_0^\perp)$. Its image along the projection

$$\mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee) \times \mathbf{P}(V_\infty^\perp) \times \mathbf{P}(V_0^\perp) \to \mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee) \times \mathbf{P}(V_0^\perp)$$

satisfies the incidence conditions in $\mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee)$ and $\mathbf{P}(\bar{V}^\vee) \times \mathbf{P}(V_0^\perp)$, hence is contained in $\tilde{\mathbf{C}}_{V_\infty}(\bar{\mathbf{H}})$; this defines the morphism $\tilde{p}$.

Similarly, the image of $\widetilde{\mathbf{CC}}_{V_0,V_\infty}(\bar{\mathbf{H}})$ under the map

$$\mathbf{P}(\bar{V}) \times \mathbf{P}(\bar{V}^\vee) \times \mathbf{P}(V_\infty^\perp) \times \mathbf{P}(V_0^\perp) \to \mathbf{P}(\bar{V}) \times \mathbf{P}(V_\infty^\perp) \times \mathbf{P}(V^\vee)$$

satisfies the incidence conditions in $\mathbf{P}(\bar{V}) \times \mathbf{P}(V_\infty^\perp)$ and $\mathbf{P}(V_\infty^\perp) \times \mathbf{P}(V^\vee)$. Indeed, the first is clear and the second follows because the restriction of the pairing $V_\infty^\perp \otimes V^\vee \subset V \otimes V^\vee \to \mathcal{O}_S$ to the subbundle $V_\infty^\perp \otimes V_0^\perp$ factors as the composition $V_\infty^\perp \otimes V_0^\perp \to \bar{V} \otimes \bar{V}^\vee \to \mathcal{O}_S$ of the projections from (4.2) and the natural pairing. This defines the morphism α.

Smoothness of all these schemes is evident. □

Let $\mathcal{A}$ be a $\mathbf{P}(\bar{V})$-linear category. Then

$$\mathbf{H}(\mathcal{A}) = \mathbf{H}(\mathcal{A}/\mathbf{P}(\bar{V})) = \mathrm{Perf}(\bar{\mathbf{H}}) \otimes_{\mathrm{Perf}(\mathbf{P}(\bar{V}))} \mathcal{A}$$

according to the notation (4.5). Recall the canonical $\mathbf{P}(\bar{V}^\vee)$-linear inclusion functor (2.12)

$$\gamma\colon \mathcal{A}^\natural = (\mathcal{A}/\mathbf{P}(\bar{V}))^\natural \to \mathbf{H}(\mathcal{A}/\mathbf{P}(\bar{V})) = \mathrm{Perf}(\bar{\mathbf{H}}) \otimes_{\mathrm{Perf}(\mathbf{P}(\bar{V}))} \mathcal{A}.$$

This induces a $\mathbf{P}(V_0^\perp)$-linear functor (see Remark 3.5)

$$\tilde{\mathbf{C}}_{V_\infty}(\gamma)\colon \tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural) \to \tilde{\mathbf{C}}_{V_\infty}(\mathbf{H}(\mathcal{A})),$$

which can be regarded as a $\mathbf{P}(V^\vee)$-linear functor

$$\tilde{\mathbf{C}}_{V_\infty}(\gamma)\colon \tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee) \to \tilde{\mathbf{C}}_{V_\infty}(\mathbf{H}(\mathcal{A}))/\mathbf{P}(V^\vee).$$

Here, we have written $\tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee)$ and $\tilde{\mathbf{C}}_{V_\infty}(\mathbf{H}(\mathcal{A}))/\mathbf{P}(V^\vee)$ to emphasize that we regard the resolved cones $\tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural)$ and $\tilde{\mathbf{C}}_{V_\infty}(\mathbf{H}(\mathcal{A}))$ as $\mathbf{P}(V^\vee)$-linear categories, via the inclusion $\mathbf{P}(V_0^\perp) \subset \mathbf{P}(V^\vee)$.

By base change from diagram (4.6) we obtain a diagram of $\mathbf{P}(V^\vee)$-linear functors

$$\begin{array}{ccccc} & & \widetilde{\mathbf{CC}}_{V_0,V_\infty}(\mathbf{H}(\mathcal{A})) & & \\ & \overset{\tilde{p}^*}{\nearrow} & & \overset{\alpha_*}{\searrow} & \\ \tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee) & \xrightarrow{\tilde{\mathbf{C}}_{V_\infty}(\gamma)} & \tilde{\mathbf{C}}_{V_\infty}(\mathbf{H}(\mathcal{A}))/\mathbf{P}(V^\vee) & & \mathbf{H}(\tilde{\tilde{\mathbf{C}}}_{V_0}(\mathcal{A})/\mathbf{P}(V)). \end{array}$$

We define a $\mathbf{P}(V^\vee)$-linear functor as the composition

$$\gamma_{\tilde{\mathbf{C}}} = \alpha_* \circ \tilde{p}^* \circ \tilde{\mathbf{C}}_{V_\infty}(\gamma)\colon \tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee) \to \mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathcal{A})/\mathbf{P}(V)). \tag{4.7}$$

The following fact will be needed later.

LEMMA 4.5. – *The functor $\gamma_{\tilde{\mathbf{C}}}$ has both left and right adjoints.*

Proof. – The functor γ has both left and right adjoints by [32, Lemma 7.2], hence so does $\tilde{\mathbf{C}}_{V_\infty}(\gamma)$ (see Remark 3.5). Further, α_* and $\tilde{p}^*$ have both left and right adjoints, by Lemma 4.4 and Remark 1.9. □

REMARK 4.6. – The functor $\gamma_{\tilde{\mathbf{C}}}$ can also be described in terms of Fourier-Mukai kernels, similarly to [25, Remark 4.5]. We leave this as an exercise.

Note that the HPD category $(\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural$ is naturally a $\mathbf{P}(V^\vee)$-linear subcategory of the target $\mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathcal{A})/\mathbf{P}(V))$ of the functor $\gamma_{\tilde{\mathbf{C}}}$. Indeed, by definition $(\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural$ is a $\mathbf{P}(V^\vee)$-linear subcategory of $\mathbf{H}(\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))$, and if $\eta\colon \mathcal{C}_{V_0}(\mathcal{A}) \to \tilde{\mathbf{C}}_{V_0}(\mathcal{A})$ denotes the admissible embedding (see Lemma 3.9) then the natural $\mathbf{P}(V^\vee)$-linear functor

$$\begin{aligned} \eta \otimes \mathrm{id}\colon \mathbf{H}(\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V)) &= \mathcal{C}_{V_0}(\mathcal{A}) \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathrm{Perf}(\mathbf{H}(\mathbf{P}(V))) \\ &\to \tilde{\mathbf{C}}_{V_0}(\mathcal{A}) \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathrm{Perf}(\mathbf{H}(\mathbf{P}(V))) = \mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathcal{A})/\mathbf{P}(V)) \end{aligned}$$

is fully faithful by [32, Lemma 2.12(2)].

4.2. The nonzero case

The goal of this subsection is to prove the following more precise version of Theorem 4.1 when V_0 and V_∞ are nonzero.

PROPOSITION 4.7. – *Let $\mathcal{A}$ be a right strong, moderate Lefschetz category over $\mathbf{P}(\bar{V})$. Assume V_0 and V_∞ are nonzero. Then the functor*

$$\gamma_{\tilde{\mathbf{C}}}\colon \tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee) \to \mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathcal{A})/\mathbf{P}(V))$$

defined in (4.7) *induces a Lefschetz equivalence between the subcategories*

$$\mathcal{C}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee) \subset \tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee) \quad \textit{and} \quad (\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural \subset \mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathcal{A})/\mathbf{P}(V)).$$

The proof takes the rest of the subsection. First we will prove the claim of Proposition 4.7 when the filtrations (4.1) split. In this case, we fix a splitting

$$V = V_0 \oplus \bar{V} \oplus V_\infty^\vee \tag{4.8}$$

and set $V_1 = V_0 \oplus V_\infty^\vee$. Then the orthogonal to $V_0 \subset V_1$ is $V_\infty \subset V_1^\vee$, so by Example 2.12 there is an equivalence

$$\mathrm{Perf}(\mathbf{P}(V_0))^\natural \simeq \mathrm{Perf}(\mathbf{P}(V_\infty)) \tag{4.9}$$

of Lefschetz categories over $\mathbf{P}(V_1^\vee)$. Hence we have a commutative diagram of equivalences of Lefschetz categories over $\mathbf{P}(V^\vee)$:

$$\begin{array}{ccc} \mathcal{J}(\mathbf{P}(V_\infty), \mathcal{A}^\natural)/\mathbf{P}(V^\vee) & \xrightarrow{\sim} & (\mathcal{J}(\mathbf{P}(V_0), \mathcal{A})/\mathbf{P}(V))^\natural \\ \wr\uparrow & & \downarrow\wr \\ \mathcal{C}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee) & \xrightarrow{\sim} & (\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural, \end{array} \tag{4.10}$$

where the vertical equivalences are consequences of Proposition 3.15, the top equivalence is given by Theorem 2.16 (note that $V = V_1 \oplus \bar{V}$) combined with (4.9), and the bottom equivalence is the composition of the other three. In the following proposition, we prove that the bottom equivalence is induced by $\gamma_{\tilde{\mathbf{C}}}$. We freely use notation from [25, §4.1], in particular the notation $\tilde{\mathbf{J}}$, $\tilde{\mathbf{J}}^\vee$, and $\widetilde{\mathbf{JJ}}$ for resolved and double resolved joins.

REMARK 4.8. – The above argument (even without checking the bottom arrow is induced by $\gamma_{\tilde{\mathbf{C}}}$) already proves Theorem 4.1 under the assumptions that V_0 and V_∞ are nonzero and the filtrations (4.1) are split. However, for our local-to-global proof of Proposition 4.7 below, it is essential to know the equivalence is given by a functor which is defined independently of the choice of a splitting of (4.1).

PROPOSITION 4.9. – *Let $\mathcal{A}$ be a right strong, moderate Lefschetz category over $\mathbf{P}(\bar{V})$. Assume V_0 and V_∞ are nonzero and that the filtrations* (4.1) *split, and choose a splitting* (4.8)*. Then there is a commutative diagram*

$$\begin{array}{ccc} \tilde{\mathbf{J}}^\vee(\mathbf{P}(V_\infty), \mathcal{A}^\natural)/\mathbf{P}(V^\vee) & \xrightarrow{\gamma_{\tilde{\mathbf{J}}_\infty}} & \mathbf{H}(\tilde{\mathbf{J}}(\mathbf{P}(V_0), \mathcal{A})/\mathbf{P}(V)) \\ {\scriptstyle\beta_\infty^*}\uparrow & & \downarrow{\scriptstyle\beta_{0*}} \\ \tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee) & \xrightarrow{\gamma_{\tilde{\mathbf{C}}}} & \mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathcal{A})/\mathbf{P}(V)), \end{array} \tag{4.11}$$

where β_0 and β_∞ are the blowup morphisms from Lemma 3.14 *of the cones with vertices $\mathbf{P}(V_0)$ and $\mathbf{P}(V_\infty)$, respectively, and $\gamma_{\tilde{\mathbf{J}}_\infty}$ is the composition*

$$\gamma_{\tilde{\mathbf{J}}_\infty}\colon \tilde{\mathbf{J}}^\vee(\mathbf{P}(V_\infty),\mathcal{A}^\natural)/\mathbf{P}(V^\vee) \xrightarrow{\sim} \tilde{\mathbf{J}}^\vee(\mathrm{Perf}(\mathbf{P}(V_0))^\natural,\mathcal{A}^\natural)/\mathbf{P}(V^\vee) \xrightarrow{\gamma_{\tilde{\mathbf{J}}}} \mathbf{H}(\tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathcal{A})/\mathbf{P}(V))$$

where the equivalence is induced by (4.9) *and $\gamma_{\tilde{\mathbf{J}}}$ is the functor from* [25, Theorem 4.9] (*with $V_2 = \bar{V}$*). *Moreover, diagram* (4.11) *restricts to the diagram* (4.10) *from above; in particular, the conclusion of Proposition* 4.7 *holds in this case.*

Proof. – First, we unwind the definition of the functor $\gamma_{\tilde{\mathbf{J}}_\infty}$. Consider the diagram

$$\begin{array}{ccc} \mathbf{P}(V_0)\times\mathbf{P}(V_\infty) & \xhookrightarrow{\ \iota\ } & \mathbf{H}(\mathbf{P}(V_0)) \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow \\ \mathbf{P}(V_\infty) & \longrightarrow & \mathbf{P}(V_1^\vee), \end{array}$$

where $\mathbf{H}(\mathbf{P}(V_0)) = \mathbf{H}(\mathbf{P}(V_0)/\mathbf{P}(V_1))$ is the universal hyperplane section of the morphism $\mathbf{P}(V_0)\to\mathbf{P}(V_1)$ and $\iota\colon \mathbf{P}(V_0)\times\mathbf{P}(V_\infty) \to \mathbf{H}(\mathbf{P}(V_0))$ is the natural embedding. By [12, Corollary 8.3] the functor

$$\iota_*\circ \mathrm{pr}_2^*\colon \mathrm{Perf}(\mathbf{P}(V_\infty)) \to \mathrm{Perf}(\mathbf{H}(\mathbf{P}(V_0)))$$

induces the HPD between $\mathbf{P}(V_0)$ and $\mathbf{P}(V_\infty)$. It follows from the definitions that we have

$$\gamma_{\tilde{\mathbf{J}}_\infty} \simeq \alpha_{0*}\circ\tilde{p}_0^*\circ\tilde{\mathbf{J}}(\iota_*\circ\mathrm{pr}_2^*,\gamma),$$

where the morphisms

$$\tilde{\mathbf{J}}^\vee(\mathbf{H}(\mathbf{P}(V_0)),\bar{\mathbf{H}}) \xleftarrow{\tilde{p}_0} \widetilde{\mathbf{JJ}}(\mathbf{H}(\mathbf{P}(V_0)),\bar{\mathbf{H}}) \xrightarrow{\alpha_0} \mathbf{H}(\tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathbf{P}(\bar{V}))/\mathbf{P}(V))$$

are the base change along $\mathbf{P}(V_0)\to\mathbf{P}(V_1)$ of [25, diagram (4.4)] (with $V_2 = \bar{V}$), and $\tilde{\mathbf{J}}(\iota_*\circ\mathrm{pr}_2^*,\gamma)$ is the join of the functors $\iota_*\circ\mathrm{pr}_2^*$ and γ, where γ is the inclusion (2.12).

Further, note that we can write $\tilde{\mathbf{J}}(\iota_*\circ\mathrm{pr}_2^*,\gamma)$ as a composition

$$\begin{aligned}\tilde{\mathbf{J}}^\vee(\mathbf{P}(V_\infty),\mathcal{A}^\natural) \xrightarrow{\tilde{\mathbf{J}}(\mathrm{id},\gamma)} \tilde{\mathbf{J}}^\vee(\mathbf{P}(V_\infty),\mathbf{H}(\mathcal{A})) &\xrightarrow{\tilde{\mathbf{J}}(\mathrm{pr}_2^*,\mathrm{id})} \tilde{\mathbf{J}}^\vee(\mathbf{P}(V_0)\times\mathbf{P}(V_\infty),\mathbf{H}(\mathcal{A})) \\ &\xrightarrow{\tilde{\mathbf{J}}(\iota_*,\mathrm{id})} \tilde{\mathbf{J}}^\vee(\mathbf{H}(\mathbf{P}(V_0)),\mathbf{H}(\mathcal{A})),\end{aligned}$$

and hence

$$\gamma_{\tilde{\mathbf{J}}_\infty} \simeq \alpha_{0*}\circ\tilde{p}_0^*\circ\tilde{\mathbf{J}}(\iota_*,\mathrm{id})\circ\tilde{\mathbf{J}}(\mathrm{pr}_2^*,\mathrm{id})\circ\tilde{\mathbf{J}}(\mathrm{id},\gamma). \tag{4.12}$$

By definition $\gamma_{\tilde{\mathbf{C}}}$ is the composition (4.7) of three functors analogous to $\tilde{\mathbf{J}}(\mathrm{id},\gamma)$, $\tilde{p}_0^*$, and α_{0*} in (4.12). To prove the proposition, we will relate the analogous functors appearing in these compositions, using the blowup morphisms β_∞ and β_0 and the morphisms ι and pr_2.

The relation between $\tilde{\mathbf{J}}(\mathrm{id},\gamma)$ and $\tilde{\mathbf{C}}_{V_\infty}(\gamma)$ is provided by the commutative diagram (3.8), that in our case takes the form

$$\begin{array}{ccc} \tilde{\mathbf{J}}^\vee(\mathbf{P}(V_\infty),\mathcal{A}^\natural) & \xrightarrow{\tilde{\mathbf{J}}(\mathrm{id},\gamma)} & \tilde{\mathbf{J}}^\vee(\mathbf{P}(V_\infty),\mathbf{H}(\mathcal{A})) \\ \uparrow{\scriptstyle \beta_\infty^*} & & \uparrow{\scriptstyle \beta_\infty^*} \\ \tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural) & \xrightarrow{\tilde{\mathbf{C}}_{V_\infty}(\gamma)} & \tilde{\mathbf{C}}_{V_\infty}(\mathbf{H}(\mathcal{A})). \end{array} \tag{4.13}$$

To relate the other functors, we write down diagrams of schemes that induce diagrams of functors by base change. First note that we have a fiber square

$$\begin{array}{ccc} \tilde{\mathbf{J}}^{\vee}(\mathbf{P}(V_0)\times\mathbf{P}(V_\infty),\bar{\mathbf{H}}) & \xleftarrow{\tilde{p}_{0\infty}} & \widetilde{\mathbf{JJ}}(\mathbf{P}(V_0)\times\mathbf{P}(V_\infty),\bar{\mathbf{H}}) \\ \downarrow{\scriptstyle \tilde{\mathbf{J}}(\iota,\mathrm{id})} & & \downarrow{\scriptstyle \widetilde{\mathbf{JJ}}(\iota,\mathrm{id})} \\ \tilde{\mathbf{J}}^{\vee}(\mathbf{H}(\mathbf{P}(V_0)),\bar{\mathbf{H}}) & \xleftarrow{\tilde{p}_0} & \widetilde{\mathbf{JJ}}(\mathbf{H}(\mathbf{P}(V_0)),\bar{\mathbf{H}}), \end{array} \tag{4.14}$$

where $\tilde{\mathbf{J}}(\iota,\mathrm{id})$ and $\widetilde{\mathbf{JJ}}(\iota,\mathrm{id})$ denote the morphisms between the (double) resolved joins induced by the morphisms $\iota\colon \mathbf{P}(V_0)\times\mathbf{P}(V_\infty)\to\mathbf{H}(\mathbf{P}(V_0))$ and $\mathrm{id}\colon\bar{\mathbf{H}}\to\bar{\mathbf{H}}$. Next observe that there is an isomorphism

$$\widetilde{\mathbf{JJ}}(\mathbf{P}(V_0)\times\mathbf{P}(V_\infty),\bar{\mathbf{H}})\cong\tilde{\mathbf{J}}(\mathbf{P}(V_0),\bar{\mathbf{H}})\times_{\bar{\mathbf{H}}}\tilde{\mathbf{J}}^{\vee}(\mathbf{P}(V_\infty),\bar{\mathbf{H}}),$$

which can be seen as in the proof of Lemma 4.4 (by identifying both sides as $\mathbf{P}^1\times\mathbf{P}^1$-bundles over subvarieties of $\mathbf{P}(V_0)\times\mathbf{P}(V_\infty)\times\mathbf{P}(\bar{V})\times\mathbf{P}(\bar{V}^{\vee})$). We also have by definition

$$\widetilde{\mathbf{CC}}_{V_0,V_\infty}(\bar{\mathbf{H}})=\tilde{\mathbf{C}}_{V_0}(\bar{\mathbf{H}})\times_{\bar{\mathbf{H}}}\tilde{\mathbf{C}}_{V_\infty}(\bar{\mathbf{H}}).$$

The blowup morphisms $\beta_0\colon\tilde{\mathbf{J}}(\mathbf{P}(V_0),\bar{\mathbf{H}})\to\tilde{\mathbf{C}}_{V_0}(\bar{\mathbf{H}})$ and $\beta_\infty\colon\tilde{\mathbf{J}}^{\vee}(\mathbf{P}(V_\infty),\bar{\mathbf{H}})\to\tilde{\mathbf{C}}_{V_\infty}(\bar{\mathbf{H}})$ from Lemma 3.14 thus combine to give a morphism

$$\beta_{0\infty}\colon\widetilde{\mathbf{JJ}}(\mathbf{P}(V_0)\times\mathbf{P}(V_\infty),\bar{\mathbf{H}})\to\widetilde{\mathbf{CC}}_{V_0,V_\infty}(\bar{\mathbf{H}}).$$

It is easy to see that the morphism $\beta_{0\infty}$ makes the diagrams

$$\begin{array}{ccccc} \tilde{\mathbf{J}}^{\vee}(\mathbf{P}(V_\infty),\bar{\mathbf{H}}) & \xleftarrow{\tilde{\mathbf{J}}(\mathrm{pr}_2,\mathrm{id})} & \tilde{\mathbf{J}}^{\vee}(\mathbf{P}(V_0)\times\mathbf{P}(V_\infty),\bar{\mathbf{H}}) & \xleftarrow{\tilde{p}_{0\infty}} & \widetilde{\mathbf{JJ}}(\mathbf{P}(V_0)\times\mathbf{P}(V_\infty),\bar{\mathbf{H}}) \\ \downarrow{\scriptstyle\beta_\infty} & & & & \downarrow{\scriptstyle\beta_{0\infty}} \\ \tilde{\mathbf{C}}_{V_\infty}(\bar{\mathbf{H}}) & & \xleftarrow{\qquad\qquad\tilde{p}\qquad\qquad} & & \widetilde{\mathbf{CC}}_{V_0,V_\infty}(\bar{\mathbf{H}}) \end{array} \tag{4.15}$$

and

$$\begin{array}{ccccc} \widetilde{\mathbf{JJ}}(\mathbf{P}(V_0)\times\mathbf{P}(V_\infty),\bar{\mathbf{H}}) & \xrightarrow{\widetilde{\mathbf{JJ}}(\iota,\mathrm{id})} & \widetilde{\mathbf{JJ}}(\mathbf{H}(\mathbf{P}(V_0)),\bar{\mathbf{H}}) & \xrightarrow{\alpha_0} & \mathbf{H}(\tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathbf{P}(\bar{V}))/\mathbf{P}(V)) \\ \downarrow{\scriptstyle\beta_{0\infty}} & & & & \downarrow{\scriptstyle\beta_0} \\ \widetilde{\mathbf{CC}}_{V_0,V_\infty}(\bar{\mathbf{H}}) & & \xrightarrow{\qquad\qquad\alpha\qquad\qquad} & & \mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))/\mathbf{P}(V)) \end{array} \tag{4.16}$$

commutative, where in (4.16) we abusively write β_0 for the morphism induced by the blowup $\beta_0\colon\tilde{\mathbf{J}}(\mathbf{P}(V_0),\mathbf{P}(\bar{V}))\to\tilde{\mathbf{C}}_{V_0}(\mathbf{P}(\bar{V}))$. Note also that since $\beta_{0\infty}$ is a product of two blowup morphisms, the functor $\beta_{0\infty}^*$ is fully faithful, so we have an isomorphism of functors

$$\beta_{0\infty *}\circ\beta_{0\infty}^*\simeq\mathrm{id}. \tag{4.17}$$

Combining the above ingredients and taking into account that $\tilde{\mathbf{J}}(\iota_*,\mathrm{id})\simeq\tilde{\mathbf{J}}(\iota,\mathrm{id})_*$ and $\tilde{\mathbf{J}}(\mathrm{pr}_2^*,\mathrm{id})\simeq\tilde{\mathbf{J}}(\mathrm{pr}_2,\mathrm{id})^*$, we can rewrite the composition of the three upper arrows in (4.11)

as

$$\begin{aligned}
\beta_{0*}\circ\gamma_{\tilde{\mathbf{J}}_\infty}\circ\beta_\infty^* &\simeq \beta_{0*}\circ\alpha_{0*}\circ\tilde{p}_0^*\circ\tilde{\mathbf{J}}(\iota_*,\mathrm{id})\circ\tilde{\mathbf{J}}(\mathrm{pr}_2^*,\mathrm{id})\circ\tilde{\mathbf{J}}(\mathrm{id},\gamma)\circ\beta_\infty^* && (4.12)\\
&\simeq \beta_{0*}\circ\alpha_{0*}\circ\tilde{p}_0^*\circ\tilde{\mathbf{J}}(\iota_*,\mathrm{id})\circ\tilde{\mathbf{J}}(\mathrm{pr}_2^*,\mathrm{id})\circ\beta_\infty^*\circ\tilde{\mathbf{C}}_{V_\infty}(\gamma) && (4.13)\\
&\simeq \beta_{0*}\circ\alpha_{0*}\circ\widetilde{\mathbf{JJ}}(\iota_*,\mathrm{id})\circ\tilde{p}_{0\infty}^*\circ\tilde{\mathbf{J}}(\mathrm{pr}_2^*,\mathrm{id})\circ\beta_\infty^*\circ\tilde{\mathbf{C}}_{V_\infty}(\gamma) && (4.14)\\
&\simeq \beta_{0*}\circ\alpha_{0*}\circ\widetilde{\mathbf{JJ}}(\iota_*,\mathrm{id})\circ\beta_{0\infty}^*\circ\tilde{p}^*\circ\tilde{\mathbf{C}}_{V_\infty}(\gamma) && (4.15)\\
&\simeq \alpha_*\circ\beta_{0\infty*}\circ\beta_{0\infty}^*\circ\tilde{p}^*\circ\tilde{\mathbf{C}}_{V_\infty}(\gamma) && (4.16)\\
&\simeq \alpha_*\circ\tilde{p}^*\circ\tilde{\mathbf{C}}_{V_\infty}(\gamma) && (4.17)\\
&= \gamma_{\tilde{\mathbf{C}}}, && (4.7)
\end{aligned}$$

which completes the proof of commutativity of diagram (4.11).

The claim that diagram (4.11) restricts to diagram (4.10) follows directly from the definitions. Finally, Proposition 4.7 under the assumptions we took follows from commutativity of diagram (4.10). □

Proof of Proposition 4.7. – If the filtrations (4.1) split, then the result holds by Proposition 4.9. In the nonsplit case we use Proposition A.1 with $\mathcal{C} = \tilde{\mathbf{C}}_{V_\infty}(\mathcal{A}^\natural)$, $\mathcal{D} = \mathbf{H}(\tilde{\mathbf{C}}_{V_0}(\mathcal{A})/\mathbf{P}(V))$, $\phi = \gamma_{\tilde{\mathbf{C}}}$, $\mathcal{A} = \mathcal{C}_{V_\infty}(\mathcal{A}^\natural)$, and $\mathcal{B} = (\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural$. We note that the assumptions of the proposition are satisfied by Lemmas 3.9 and 4.5. We take an fpqc cover of our base scheme S over which the filtrations (4.1) split. By the argument above the functor $\gamma_{\tilde{\mathbf{C}}}$ induces the desired Lefschetz equivalence after base change to this cover. Hence by Proposition A.1 the functor $\gamma_{\tilde{\mathbf{C}}}$ induces an equivalence between $\mathcal{C}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee)$ and $(\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural$, which is in fact a Lefschetz equivalence by Corollary A.5. □

4.3. The general case

In this subsection we bootstrap from Proposition 4.7 to the general case of Theorem 4.1.

LEMMA 4.10. – *The claim of Theorem* 4.1 *holds if* V_0 *is nonzero.*

Proof. – By Proposition 4.7 we only need to consider the case where $V_\infty = 0$. Take an auxiliary nonzero vector bundle $\tilde{V}_\infty$ on S, and set $\tilde{V} = V \oplus \tilde{V}_\infty^\vee$. Then $V_0 \subset \tilde{V}$ and $\tilde{V}_\infty \subset \tilde{V}^\vee$ are such that the pairing $\tilde{V} \otimes \tilde{V}^\vee \to \mathcal{O}_S$ is zero on $V_0 \otimes \tilde{V}_\infty$. Hence Proposition 4.7 gives an equivalence

$$\mathcal{C}_{\tilde{V}_\infty}(\mathcal{A}^\natural)/\mathbf{P}(\tilde{V}^\vee) \simeq (\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(\tilde{V}))^\natural \tag{4.18}$$

of Lefschetz categories over $\mathbf{P}(\tilde{V}^\vee)$. By base change along the embedding $\mathbf{P}(V^\vee) \to \mathbf{P}(\tilde{V}^\vee)$ we obtain a $\mathbf{P}(V^\vee)$-linear equivalence

$$\left(\mathcal{C}_{\tilde{V}_\infty}(\mathcal{A}^\natural)/\mathbf{P}(\tilde{V}^\vee)\right) \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee)) \simeq (\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(\tilde{V}))^\natural \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee)),$$

We have $\mathbf{P}(V^\vee)$-linear equivalences

$$\left(\mathcal{C}_{\tilde{V}_\infty}(\mathcal{A}^\natural)/\mathbf{P}(\tilde{V}^\vee)\right) \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee)) \simeq \mathcal{A}^\natural/\mathbf{P}(V^\vee) \simeq \mathcal{C}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee), \tag{4.19}$$

where the first holds by Corollary 3.13 and the second by Lemma 3.8 since $V_\infty = 0$. Furthermore, note that $\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(\tilde{V})$ is supported over the open subset $\mathbf{P}(\tilde{V}) \setminus \mathbf{P}(\tilde{V}_\infty^\vee)$ since this category's $\mathbf{P}(\tilde{V})$-linear structure is induced from a $\mathbf{P}(V)$-linear structure via the

morphism $\mathbf{P}(V) \to \mathbf{P}(\tilde{V})$ and $\mathbf{P}(V) \cap \mathbf{P}(V_\infty^\vee) = \emptyset$. Hence by Proposition A.10 we have a $\mathbf{P}(V^\vee)$-linear equivalence

$$(\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(\tilde{V}))^\natural \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee)) \simeq (\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural. \tag{4.20}$$

Combining the above equivalences, we thus obtain a $\mathbf{P}(V^\vee)$-linear equivalence

$$\mathcal{C}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee) \simeq (\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V))^\natural. \tag{4.21}$$

Finally, tracing through the equivalences (4.19) and (4.20) and using Remark A.12 and the fact that (4.18) is a Lefschetz equivalence, one verifies that (4.21) identifies the Lefschetz centers on each side. □

Now we can handle the general case.

Proof of Theorem 4.1. – By Lemma 4.10 it remains to consider the case where $V_0 = 0$. We may assume $V_\infty \neq 0$, otherwise there is nothing to prove. Using Remark 2.13 we reduce the claim of Theorem 4.1 to the existence of a Lefschetz equivalence

$$\mathcal{C}_{V_0}(\mathcal{A})/\mathbf{P}(V) \simeq {}^\natural\left(\mathcal{C}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee)\right).$$

Since $V_\infty \neq 0$ we can apply (the left version of) Lemma 4.10 to obtain a Lefschetz equivalence

$${}^\natural\left(\mathcal{C}_{V_\infty}(\mathcal{A}^\natural)/\mathbf{P}(V^\vee)\right) \simeq \mathcal{C}_{V_0}({}^\natural(\mathcal{A}^\natural))/\mathbf{P}(V).$$

We conclude by noting that ${}^\natural(\mathcal{A}^\natural) \simeq \mathcal{A}$, again by Remark 2.13. □

5. HPD for quadrics

In this section, we use categorical cones to describe HPD for quadrics. We assume the base scheme S is the spectrum of an algebraically closed field $\mathbf{k}$ of characteristic not equal to 2. The main reason for this assumption is that our results depend on our work [26], reviewed in §5.1 below, where we described HPD for *smooth* quadrics over such a field $\mathbf{k}$.

We study the following class of morphisms from a quadric to a projective space.

Definition 5.1. – Let Q be a quadric, i.e., an integral scheme over $\mathbf{k}$ which admits a closed immersion into a projective space as a quadric hypersurface. We denote by $\mathcal{O}_Q(1)$ the restriction of the line bundle $\mathcal{O}(1)$ from this ambient space. A morphism $f\colon Q \to \mathbf{P}(V)$ is *standard* if there is an isomorphism

$$f^*\mathcal{O}_{\mathbf{P}(V)}(1) \cong \mathcal{O}_Q(1).$$

In other words, f is either an embedding as a quadric hypersurface into a linear subspace of $\mathbf{P}(V)$, or a double covering of a linear subspace of $\mathbf{P}(V)$ branched along a quadric hypersurface. We call f *non-degenerate* if its image is not contained in a hyperplane of $\mathbf{P}(V)$.

Note that Q is not required to be smooth, but is required to be integral. In the preliminary §5.1, we recall the results of [26] saying that if Q *is* smooth then it has a natural Lefschetz structure, and if $f\colon Q \to \mathbf{P}(V)$ is a non-degenerate standard morphism the HPD category can be described in terms of classical projective duality. In §5.2 we use categorical cones to construct for a general standard morphism $f\colon Q \to \mathbf{P}(V)$ a Lefschetz category $\mathfrak{Q}$ over $\mathbf{P}(V)$—called the *standard categorical resolution* of Q— which is smooth and proper over $\mathbf{k}$ and agrees with $\mathrm{Perf}(Q)$ over the complement of $f(\mathrm{Sing}(Q)) \subset \mathbf{P}(V)$. In §5.3 we introduce a "generalized duality" operation that associates to a standard morphism $f\colon Q \to \mathbf{P}(V)$ of a quadric another such morphism $f^\natural\colon Q^\natural \to \mathbf{P}(V^\vee)$. We use HPD for categorical cones to prove that this notation is compatible with the notation for the HPD category, i.e., that the HPD of the standard categorical resolution of Q is Lefschetz equivalent to the standard categorical resolution of the generalized dual $Q^\natural$ (Theorem 5.20). By combining this with the nonlinear HPD theorem, we prove in §5.4 a quadratic HPD theorem (Theorem 5.21).

5.1. HPD for smooth quadrics

In this subsection, we briefly review HPD for smooth quadrics following [26]. Given a smooth quadric Q, we will denote by $\mathcal{S}$ a chosen spinor bundle on it. Note that there is either one or two choices for $\mathcal{S}$ depending on whether $\dim(Q)$ is odd or even.

LEMMA 5.2 ([26, Lemma 2.4]). – *Let $f\colon Q \to \mathbf{P}(V)$ be a standard morphism of a smooth quadric Q. Let $\mathcal{S}$ denote a spinor bundle on Q. Then* $\mathrm{Perf}(Q)$ *is smooth and proper over* $\mathbf{k}$*, and has the structure of a strong, moderate Lefschetz category over* $\mathbf{P}(V)$ *with Lefschetz center*

$$\mathfrak{Q}_0 = \langle \mathcal{S}, \mathcal{O} \rangle \subset \mathrm{Perf}(Q)$$

and length $\dim(Q)$*. Further, if $p \in \{0,1\}$ is the parity of* $\dim(Q)$*, i.e., $p = \dim(Q) \pmod 2$, then the nonzero Lefschetz components of* $\mathrm{Perf}(Q)$ *are given by*

$$\mathfrak{Q}_i = \begin{cases} \langle \mathcal{S}, \mathcal{O} \rangle & \text{for } |i| \leq 1-p, \\ \langle \mathcal{O} \rangle & \text{for } 1-p < |i| \leq \dim(Q)-1. \end{cases}$$

REMARK 5.3. – If $\dim Q$ is even there are two choices of $\mathcal{S}$, but up to equivalence, the Lefschetz structure on $\mathrm{Perf}(Q)$ does not depend on this choice, see [26, Remark 2.5]. Further, the Lefschetz center $\mathfrak{Q}_0$ of $\mathrm{Perf}(Q)$ can be also written as

$$\mathfrak{Q}_0 = \langle \mathcal{O}, \mathcal{S}'^\vee \rangle$$

where $\mathcal{S}' = \mathcal{S}$ if $\dim(Q)$ is not divisible by 4, and the other spinor bundle otherwise. The nonzero primitive Lefschetz components (as defined in §2.1) of $\mathrm{Perf}(Q)$ are given by

$$\mathfrak{q}_i = \begin{cases} \langle \mathcal{O} \rangle & \text{if } i = -(\dim(Q)-1), \\ \langle \mathcal{S}'^\vee \rangle & \text{if } i = -(1-p), \\ \langle \mathcal{S} \rangle & \text{if } i = 1-p, \\ \langle \mathcal{O} \rangle & \text{if } i = \dim(Q)-1. \end{cases}$$

The next result describes HPD for non-degenerate standard morphisms of smooth quadrics. This will be generalized to arbitrary standard morphisms of quadrics in Theorem 5.20. Recall that the classical projective dual of a smooth quadric hypersurface $Q \subset \mathbf{P}(V)$ is itself a smooth quadric hypersurface $Q^\vee \subset \mathbf{P}(V^\vee)$.

THEOREM 5.4 ([26, Theorem 1.1]). – *Let $f\colon Q \to \mathbf{P}(V)$ be a non-degenerate standard morphism of a smooth quadric Q. Then there is an equivalence*

$$\mathrm{Perf}(Q)^\natural \simeq \mathrm{Perf}(Q^\natural)$$

of Lefschetz categories over $\mathbf{P}(V^\vee)$, where the left-hand side is the HPD category, and

(1) *if f is a divisorial embedding and $\dim(Q)$ is even, then $Q^\natural = Q^\vee$ is the classical projective dual of Q and $Q^\natural \to \mathbf{P}(V^\vee)$ is its natural embedding;*

(2) *if f is a divisorial embedding and $\dim(Q)$ is odd, then $Q^\natural \to \mathbf{P}(V^\vee)$ is the double cover branched along the classical projective dual $Q^\vee \subset \mathbf{P}(V^\vee)$ of Q;*

(3) *if f is a double covering and $\dim(Q)$ is even, then $Q^\natural \to \mathbf{P}(V^\vee)$ is the classical projective dual of the branch locus of f;*

(4) *if f is a double covering and $\dim(Q)$ is odd, then $Q^\natural \to \mathbf{P}(V^\vee)$ is the double cover branched along the classical projective dual of the branch locus of f.*

5.2. Standard categorical resolutions of quadrics

In this subsection, we will obtain a categorical resolution of a singular quadric by expressing it as a cone over a smooth quadric, and then taking a categorical cone. To start with, we analyze the general structure of a standard morphism of quadrics.

LEMMA 5.5. – *Let $Q \subset \mathbf{P}(\widetilde{W})$ be a quadric hypersurface. Then there are a unique subspace $K \subset \widetilde{W}$ and a smooth quadric $\bar{Q} \subset \mathbf{P}(\widetilde{W}/K)$ such that*

$$Q \cong \mathbf{C}_K(\bar{Q}).$$

Moreover, if $f\colon Q \to \mathbf{P}(V)$ is a standard morphism of Q, there is a unique commutative diagram of vector spaces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \hookrightarrow & \widetilde{W} & \twoheadrightarrow & \widetilde{W}/K & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mathbf{C}_K(\bar{f})} & & \downarrow{\scriptstyle \bar{f}} & & \\
0 & \longrightarrow & K & \hookrightarrow & W & \twoheadrightarrow & W/K & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & V & & & &
\end{array}$$

with surjective morphism $\bar{f}$ such that f is the composition

$$Q \hookrightarrow \mathbf{P}(\widetilde{W}) \xrightarrow{\mathbf{C}_K(\bar{f})} \mathbf{P}(W) \hookrightarrow \mathbf{P}(V).$$

Moreover, one of the following two possibilities holds:

(1) *The map $\bar{f}$ is an isomorphism. In this case, f is an embedding.*

(2) *The spaces* $\ker(\bar{f})$ *and* $\ker(\mathbf{C}_K(\bar{f}))$ *are* 1*-dimensional, and the corresponding points of the projective spaces* $\mathbf{P}(\widetilde{W})$ *and* $\mathbf{P}(\widetilde{W}/K)$ *do not lie on the quadrics* Q *and* $\bar{Q}$ *respectively. In this case,* f *is a double covering onto* $\mathbf{P}(W) \subset \mathbf{P}(V)$.

Proof. – We define K to be the kernel of the quadratic form on $\widetilde{W}$ corresponding to Q and $\bar{Q}$ to be the quadric corresponding to the induced quadratic form on $\widetilde{W}/K$. We set

$$W = \operatorname{im}\left(V^\vee = H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1)) \xrightarrow{f^*} H^0(Q, \mathcal{O}_Q(1)) = \widetilde{W}^\vee\right)^\vee.$$

This gives a factorization of f^* as a composition $V^\vee \to W^\vee \to \widetilde{W}^\vee$ and we define the maps in the middle column of the diagram as the dual maps. The rest is clear. □

We call the quadric $\bar{Q}$ above the *base quadric* of Q. Moreover, if (1) holds we say f is of *embedding type*, and if (2) holds we say f is of *covering type*.

Next we define some useful numerical invariants of a standard morphism of a quadric. In the definition below we use the notation introduced in Lemma 5.5.

DEFINITION 5.6. – Let $f: Q \to \mathbf{P}(V)$ be a standard morphism of a quadric. Then:

— $r(Q) = \dim \widetilde{W} - \dim K$ denotes the rank of Q, i.e., the rank of the quadratic form on $\widetilde{W}$ corresponding to Q.

— $p(Q) \in \{0, 1\}$ denotes the *parity* of $r(Q)$, i.e., $p(Q) = r(Q) \pmod 2$.

— $k(Q) = \dim K$.

— $c(f) = \dim V - \dim W$ denotes the codimension of the linear span $\langle f(Q)\rangle \subset \mathbf{P}(V)$.

— $t(f) = \dim \widetilde{W} - \dim W \in \{0, 1\}$ denotes the *type* of Q, defined by

$$t(f) = \begin{cases} 0 & \text{if } f \text{ is of embedding type,} \\ 1 & \text{if } f \text{ is of covering type.} \end{cases}$$

Note that our convention that Q is integral is equivalent to $r(Q) \geq 3$.

REMARK 5.7. – As indicated by the notation, $r(Q)$, $p(Q)$, and $k(Q)$ depend only on Q, while $c(f)$ and $t(f)$ are invariants of the morphism f. We note the relations

$$\dim(Q) = r(Q) + k(Q) - 2, \tag{5.1}$$

$$\dim(V) = r(Q) + k(Q) + c(f) - t(f). \tag{5.2}$$

Moreover, if $\bar{Q}$ is the base quadric of Q, we have $r(Q) = r(\bar{Q})$ and $p(Q) = p(\bar{Q})$.

Using the identification of Lemma 5.5 of a quadric Q with the cone over a smooth quadric, we see that the corresponding resolved cone gives a resolution of Q. We call the induced map

$$\pi: \tilde{\mathbf{C}}_K(\bar{Q}) \to \mathbf{C}_K(\bar{Q}) = Q \tag{5.3}$$

the *standard geometric resolution* of Q. Note that this map is nothing but the blowup of Q in its singular locus $\operatorname{Sing}(Q) = \mathbf{P}(K)$.

DEFINITION 5.8. – Let $f\colon Q \to \mathbf{P}(V)$ be a standard morphism of a quadric. Using the above notation, the *standard categorical resolution* of Q over $\mathbf{P}(V)$ is the Lefschetz category $\mathfrak{Q}$ over $\mathbf{P}(V)$ defined as the categorical cone over the base quadric $\bar{Q}$:

$$\mathfrak{Q} = \mathcal{C}_K(\bar{Q})/\mathbf{P}(V)$$

taken with respect to the Lefschetz structure of $\bar{Q}$ described in Lemma 5.2.

Note that the categorical resolution $\mathfrak{Q}$ of Q depends on the choice of a spinor bundle $\mathcal{S}$ on $\bar{Q}$, although this is suppressed in the notation; there is little harm in this, as by Remark 5.3 the categorical resolutions for different choices of spinor bundle are equivalent.

In Lemma 5.9 we explicitly describe the Lefschetz components of $\mathfrak{Q}$, and in Lemma 5.11 we justify calling $\mathfrak{Q}$ a categorical resolution.

LEMMA 5.9. – *Let $f\colon Q \to \mathbf{P}(V)$ be a standard morphism of a quadric. Let $\mathcal{S}$ be a spinor bundle on the base quadric of Q. Then the standard categorical resolution $\mathfrak{Q}$ of Q over $\mathbf{P}(V)$ associated with $\mathcal{S}$ is smooth and proper over $\mathbf{k}$, and it is endowed with a strong, moderate Lefschetz structure of length* $\dim(Q)$. *If $k = k(Q)$ and $p = p(Q)$, then its nonzero Lefschetz components are given by*

$$\mathfrak{Q}_i = \begin{cases} \langle \mathcal{S}, \mathcal{O} \rangle = \langle \mathcal{O}, \mathcal{S}'^{\vee} \rangle & \text{for } |i| \le k+1-p, \\ \langle \mathcal{O} \rangle & \text{for } k+1-p < |i| \le \dim(Q)-1, \end{cases}$$

and its nonzero primitive Lefschetz components are given by

$$\mathfrak{q}_i = \begin{cases} \langle \mathcal{O} \rangle & \text{if } i = -(\dim(Q)-1), \\ \langle \mathcal{S}'^{\vee} \rangle & \text{if } i = -(k+1-p), \\ \langle \mathcal{S} \rangle & \text{if } i = k+1-p, \\ \langle \mathcal{O} \rangle & \text{if } i = \dim(Q)-1, \end{cases}$$

where $\mathcal{S}'$ is described in Remark 5.3.

Proof. – Combine Theorem 3.21, Lemma 3.11, Lemma 5.2, and (5.1). □

REMARK 5.10. – In Lemma 5.9 and below, we tacitly identify the objects $\mathcal{O}$, $\mathcal{S}$, and $\mathcal{S}'$ on the base quadric $\bar{Q}$ of Q with their pullbacks to $\mathfrak{Q} \subset \tilde{\mathbf{C}}_K(\bar{Q})$.

The next lemma justifies our terminology by showing that $\mathfrak{Q}$ is a weakly crepant categorical resolution of Q in the sense of Definition 2.18. Recall that by definition the standard categorical resolution of a quadric is a subcategory of the derived category of the standard geometric resolution (5.3).

LEMMA 5.11. – *Let $f\colon Q \to \mathbf{P}(V)$ be a standard morphism of a quadric, with standard geometric resolution $\pi\colon \widetilde{Q} = \tilde{\mathbf{C}}_K(\bar{Q}) \to Q$ and standard categorical resolution $\mathfrak{Q}$ over $\mathbf{P}(V)$. Then π_* and π^* restrict to functors*

$$\pi_*\colon \mathfrak{Q} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Q) \qquad \text{and} \qquad \pi^*\colon \mathrm{Perf}(Q) \to \mathfrak{Q},$$

which give $\mathfrak{Q}$ the structure of a weakly crepant categorical resolution of Q.

Furthermore, the object $\mathcal{R} = \pi_* \mathcal{E}nd(\mathcal{O} \oplus \mathcal{S}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Q)$ *is a coherent sheaf of* $\mathcal{O}_Q$*-algebras on* Q *of finite homological dimension, and there is an equivalence*

$$\mathfrak{Q} \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Q, \mathcal{R}) \simeq \mathrm{Perf}(Q, \mathcal{R}),$$

where $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Q, \mathcal{R})$ *and* $\mathrm{Perf}(Q, \mathcal{R})$ *denote the bounded derived category of coherent* $\mathcal{R}$*-modules on* Q *and the corresponding perfect category.*

Proof. – Recall that the morphism π is the blowup with center at $\mathbf{P}(K)$. Consider the blowup diagram

$$\begin{array}{ccc} E & \longrightarrow & \tilde{Q} \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{P}(K) & \longrightarrow & Q, \end{array}$$

where E is the exceptional divisor; note moreover that $E \cong \mathbf{P}(K) \times \bar{Q}$, where $\bar{Q}$ is the base quadric of Q. It is easy to see that π is a resolution of rational singularities (recall that the rank of $\bar{Q}$ is assumed to be greater than 2). Moreover, $\pi_E^*(\mathrm{Perf}(\mathbf{P}(K)))$ is contained in the pullback to E of the Lefschetz center

$$\mathrm{Perf}(\mathbf{P}(K)) \otimes \langle \mathcal{S}, \mathcal{O} \rangle \subset \mathrm{Perf}(\mathbf{P}(K) \times \bar{Q}) \simeq \mathrm{Perf}(E).$$

Hence by [14, Theorem 4.4] the functors π^* and π_* indeed give $\mathfrak{Q}$ the structure of a categorical resolution of Q.

Moreover, Q is Gorenstein and a direct computation shows that

$$K_{\tilde{Q}} = \pi^*(K_Q) + (\dim(\bar{Q}) - 1)E.$$

Note that $\dim(\bar{Q})$ is the length of the Lefschetz decomposition of $\mathrm{Perf}(E)$ above. Hence [14, Proposition 4.5] shows that $\mathfrak{Q}$ is a weakly crepant categorical resolution of Q. By an argument similar to [14, Proposition 7.1], the bundle $\mathcal{O} \oplus \mathcal{S}$ on $\tilde{Q}$ is tilting over Q (i.e., the derived pushforward $\pi_* \mathcal{E}nd(\mathcal{O} \oplus \mathcal{S})$ is a pure sheaf) if and only if for all $t \geq 0$ we have

$$\mathrm{H}^{>0}(\bar{Q}, \mathcal{E}nd(\mathcal{O} \oplus \mathcal{S})(t\bar{H})) = 0.$$

A computation shows that this vanishing holds, and then the rest of the lemma follows from [14, Theorem 5.2]. □

REMARK 5.12. – The last statement of Lemma 5.11 shows that $\mathfrak{Q}$ can also be considered as a noncommutative resolution in the sense of Van den Bergh [37, 36].

The following lemma relates standard categorical resolutions of quadrics to geometry and shows that $\mathfrak{Q}$ is "birational" to Q over $\mathbf{P}(V)$.

LEMMA 5.13. – *Let* $f : Q \to \mathbf{P}(V)$ *be a standard morphism of a quadric. Let* $\mathfrak{Q}$ *be the standard categorical resolution of* Q *over* $\mathbf{P}(V)$. *Let* $U = \mathbf{P}(V) \setminus f(\mathrm{Sing}(Q))$.

(1) *The base change to* U *of the* $\mathbf{P}(V)$*-linear functor* $\pi^* : \mathrm{Perf}(Q) \to \mathfrak{Q}$ *gives an equivalence*

$$\mathrm{Perf}(Q_U) \simeq \mathfrak{Q}_U.$$

(2) *Let $\mathcal{A}$ be a $\mathbf{P}(V)$-linear category supported over U. Then there is an equivalence*

$$\mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathfrak{Q} \simeq \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathrm{Perf}(Q).$$

In particular, if $\mathcal{A} = \mathrm{Perf}(X)$ for a scheme X over $\mathbf{P}(V)$ supported over U, then

$$\mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathfrak{Q} \simeq \mathrm{Perf}\big(X \times_{\mathbf{P}(V)} Q\big).$$

Note that the fiber product in the right side of the last formula should be understood in the derived sense.

Proof. – Part (1) follows from Proposition 3.12 because the morphism $\tilde{\mathbf{C}}_K(\bar{Q}) \to \mathbf{P}(V)$ factorizes as

$$\tilde{\mathbf{C}}_K(\bar{Q}) \to \mathbf{C}_K(\bar{Q}) = Q \xrightarrow{f} \mathbf{P}(V),$$

where the first map is the blowup in $\mathbf{P}(K) = \mathrm{Sing}(Q)$. Part (2) follows from Lemma A.9 and part (1). □

5.3. Generalized quadratic duality and HPD

Our goal in this subsection is to define a geometric duality operation on standard morphisms of quadrics, which after passing to standard categorical resolutions corresponds to the operation of taking the HPD category.

The desired duality operation will be defined using a combination of the following three operations.

DEFINITION 5.14. – Let $f\colon Q \to \mathbf{P}(V)$ be a standard morphism of a quadric.

— If $f\colon Q \to \mathbf{P}(V)$ is of embedding type, we denote by

$$f^\vee : Q^\vee \to \mathbf{P}(V^\vee)$$

the embedding of the classical projective dual of $Q \subset \mathbf{P}(V)$.

— If $f\colon Q \to \mathbf{P}(V)$ is of embedding type, we define

$$f_{\mathrm{cov}}\colon Q_{\mathrm{cov}} \to \mathbf{P}(V)$$

as the composition of the double cover $Q_{\mathrm{cov}} \to \langle Q \rangle$ of the linear span of Q in $\mathbf{P}(V)$ branched along $Q \subset \langle Q \rangle$ with the embedding $\langle Q \rangle \hookrightarrow \mathbf{P}(V)$.

— If $f\colon Q \to \mathbf{P}(V)$ is of covering type, we define

$$f_{\mathrm{br}}\colon Q_{\mathrm{br}} \to \mathbf{P}(V)$$

as the composition of the inclusion $Q_{\mathrm{br}} \hookrightarrow \langle f(Q) \rangle$ of the branch divisor of the double cover $Q \to \langle f(Q) \rangle$ with the embedding $\langle f(Q) \rangle \hookrightarrow \mathbf{P}(V)$.

REMARK 5.15. – Let $f\colon Q \to \mathbf{P}(V)$ be a standard morphism of a quadric, and recall the canonical diagram of vector spaces associated to f in Lemma 5.5. The operations of Definition 5.14 affect this diagram as follows.

— If $f\colon Q \to \mathbf{P}(V)$ is of embedding type, then its classical projective dual can be described as follows. The filtration $0 \subset K \subset W \subset V$ gives a filtration $0 \subset W^\perp \subset K^\perp \subset V^\vee$ by taking orthogonals. The pairing between V and $V^\vee$ induces a nondegenerate pairing between W/K and $K^\perp/W^\perp$, and hence an isomorphism $K^\perp/W^\perp \cong (W/K)^\vee$. Via this isomorphism, the base quadric $\bar{Q} \subset \mathbf{P}(W/K)$ of Q corresponds to a quadric in $\bar{Q}^\vee \subset \mathbf{P}(K^\perp/W^\perp)$ (its projective dual), and then $Q^\vee = \mathbf{C}_{W^\perp}(\bar{Q}^\vee) \subset \mathbf{P}(V^\vee)$. Thus, the operation $f \mapsto f^\vee$ replaces W by $K^\perp$ and K by $W^\perp$.

— If $f\colon Q \to \mathbf{P}(V)$ is of embedding type, then $\widetilde{W} = W$. The operation $f \mapsto f_{\mathrm{cov}}$ replaces $\widetilde{W}$ by $\widetilde{W} \oplus \mathbf{k}$, and keeps W and K fixed.

— Similarly, if $f\colon Q \to \mathbf{P}(V)$ is of covering type, then the operation $f \mapsto f_{\mathrm{br}}$ replaces $\widetilde{W}$ by W, and keeps W and K fixed.

REMARK 5.16. – The operations of Definition 5.14 preserve the integrality of Q, except for the branch divisor operation in case $r(Q) = 3$ and f is a morphism of covering type. Indeed, this follows from the formulas:

$$r(Q^\vee) = r(Q), \quad r(Q_{\mathrm{cov}}) = r(Q) + 1, \quad r(Q_{\mathrm{br}}) = r(Q) - 1.$$

Note, however, that the operations of Definition 5.14 are defined even for non-integral quadrics.

The next definition is modeled on the cases considered in Theorem 5.4.

DEFINITION 5.17. – Let $f\colon Q \to \mathbf{P}(V)$ be a standard morphism of a quadric. The *generalized dual* of f is the standard morphism

$$f^\natural\colon Q^\natural \to \mathbf{P}(V^\vee)$$

of the quadric $Q^\natural$ defined as follows:

— If $f\colon Q \to \mathbf{P}(V)$ is of embedding type, then:

◇ If $r(Q)$ is even, we set $Q^\natural = Q^\vee$ and $f^\natural = f^\vee\colon Q^\natural \to \mathbf{P}(V^\vee)$.

◇ If $r(Q)$ is odd, we set $Q^\natural = (Q^\vee)_{\mathrm{cov}}$ and $f^\natural = (f^\vee)_{\mathrm{cov}}\colon Q^\natural \to \mathbf{P}(V^\vee)$.

— If $f\colon Q \to \mathbf{P}(V)$ is of covering type, then:

◇ If $r(Q)$ is even, we set $Q^\natural = (Q_{\mathrm{br}})^\vee$ and $f^\natural = (f_{\mathrm{br}})^\vee\colon Q^\natural \to \mathbf{P}(V^\vee)$.

◇ If $r(Q)$ is odd, we set $Q^\natural = ((Q_{\mathrm{br}})^\vee)_{\mathrm{cov}}$ and $f^\natural = ((f_{\mathrm{br}})^\vee)_{\mathrm{cov}}\colon Q^\natural \to \mathbf{P}(V^\vee)$.

In other words, we first pass to a morphism of the embedding type (by taking the branch divisor if necessary), then apply classical projective duality, and then if necessary go to the double covering.

REMARK 5.18. – Using the description of Remark 5.15 it is easy to check that generalized duality affects the numerical invariants of f described in Definition 5.6 as follows:

$$r(Q^\natural) = r(Q) + p(Q) - t(f), \quad p(Q^\natural) = t(f), \quad k(Q^\natural) = c(f),$$
$$c(f^\sharp) = k(Q), \quad t(f^\natural) = p(Q).$$

In particular, note that generalized duality preserves the integrality of Q. Note also that by (5.1) we have

$$\dim(Q^\natural) = r(Q) + p(Q) + c(f) - t(f) - 2. \tag{5.4}$$

REMARK 5.19. – By (5.1), (5.4), and (5.2) we have

$$\begin{aligned}\dim(Q^\natural) + \dim(Q) &= (r(Q) + k(Q) + c(f) - t(f)) + (r(Q) + p(Q) - 4)\\ &= \dim(V) + (r(Q) + p(Q) - 4),\end{aligned}$$

which is congruent to $\dim(V)$ mod 2 since by definition $p(Q) \equiv r(Q)$ mod 2. This means that if $\dim(V)$ is even, then the parities of the dimensions of $Q^\natural$ and Q are the same, and if $\dim(V)$ is odd, then the parities are opposite.

Now we can bootstrap from Theorem 5.4 to a result for arbitrary standard morphisms.

THEOREM 5.20. – *Let $f: Q \to \mathbf{P}(V)$ be a standard morphism of a quadric. Let $\mathfrak{Q}$ be the standard categorical resolution of Q over $\mathbf{P}(V)$ as defined in Definition* 5.8. *Then the HPD category $\mathfrak{Q}^\natural$ is Lefschetz equivalent to the standard categorical resolution of the quadric $Q^\natural$ defined in Definition* 5.17 *over $\mathbf{P}(V^\vee)$.*

Proof. – Follows from Theorem 5.4, Theorem 4.1, and the definitions. □

5.4. The quadratic HPD theorem

Now we can prove our quadratic HPD theorem, by combining the above results with the nonlinear HPD Theorem 2.17.

THEOREM 5.21. – *Let $\mathcal{A}$ be a right strong, moderate Lefschetz category over $\mathbf{P}(V)$. Let*

$$f: Q \to \mathbf{P}(V) \qquad \text{and} \qquad f^\natural: Q^\natural \to \mathbf{P}(V^\vee)$$

be a standard map of a quadric and its generalized dual. Let $\mathfrak{Q}$ be the standard categorical resolution of Q over $\mathbf{P}(V)$, and let $\mathfrak{Q}^\natural$ be the standard categorical resolution of $Q^\natural$ over $\mathbf{P}(V^\vee)$. Let $\mathcal{S} \in \mathfrak{Q}$ and $\mathcal{S}^\natural \in \mathfrak{Q}^\natural$ be the pullbacks of the corresponding spinor bundles on the base quadrics of Q and $Q^\natural$. Let H and H' denote the hyperplane classes on $\mathbf{P}(V)$ and $\mathbf{P}(V^\vee)$. Let

$$N = \dim(V), \quad m = \operatorname{length}(\mathcal{A}), \quad m^\natural = \operatorname{length}(\mathcal{A}^\natural), \quad d = \dim(Q), \quad d^\natural = \dim(Q^\natural).$$

Then there are semiorthogonal decompositions

$$\begin{aligned}\mathcal{A} \otimes_{\operatorname{Perf}(\mathbf{P}(V))} \mathfrak{Q} = \Big\langle & \mathcal{K}_Q(\mathcal{A}),\\ & \mathcal{A}_{d^\natural}(H)\otimes\langle\mathcal{S}\rangle, \dots, \mathcal{A}_{m-1}((m-d^\natural)H)\otimes\langle\mathcal{S}\rangle,\\ & \mathcal{A}_{N-d}(H)\otimes\langle\mathcal{O}\rangle, \dots, \mathcal{A}_{m-1}((m+d-N)H)\otimes\langle\mathcal{O}\rangle\Big\rangle.\end{aligned}$$

$$\begin{aligned}\mathcal{A}^\natural \otimes_{\operatorname{Perf}(\mathbf{P}(V^\vee))} \mathfrak{Q}^\natural = \Big\langle & \mathcal{A}^\natural_{1-m^\natural}((N-d^\natural-m^\natural)H')\otimes\langle\mathcal{O}\rangle, \dots, \mathcal{A}^\natural_{d^\natural-N}(-H')\otimes\langle\mathcal{O}\rangle,\\ & \mathcal{A}^\natural_{1-m^\natural}((d-m^\natural)H')\otimes\langle(\mathcal{S}^\natural)^\vee\rangle, \dots, \mathcal{A}^\natural_{-d}(-H')\otimes\langle(\mathcal{S}^\natural)^\vee\rangle,\\ & \mathcal{K}'_{Q^\natural}(\mathcal{A}^\natural)\Big\rangle,\end{aligned}$$

and an equivalence of categories $\mathcal{K}_Q(\mathcal{A}) \simeq \mathcal{K}'_{Q^\natural}(\mathcal{A}^\natural)$.

Proof. – We apply the nonlinear HPD Theorem 2.17 in case $\mathcal{A}^1 = \mathcal{A}$, $\mathcal{A}^2 = \mathfrak{Q}$; it gives semiorthogonal decompositions (2.16) and (2.17) and the equivalence; so we only have to check that the components $\mathcal{J}_i = \mathcal{J}(\mathcal{A}, \mathfrak{Q})_i$ of (2.16) and $\mathcal{J}_j^\natural = \mathcal{J}(\mathcal{A}^\natural, \mathfrak{Q}^\natural)_j$ of (2.17) have the prescribed form.

By [25, Lemma 3.24] and Lemma 5.9, for $i \geq N$ we have

$$\mathcal{J}_i = \left\langle \mathcal{A}_{i-k+p-2} \otimes \langle \mathcal{S} \rangle, \mathcal{A}_{i-d} \otimes \langle \mathcal{O} \rangle \right\rangle \subset \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathfrak{Q}.$$

Combined with the observation that $k - p + 2 = N - d^\natural$ by (5.2) and (5.4), it follows that the semiorthogonal decomposition (2.16) takes the claimed form. Using the expression for the numerical invariants of $Q^\natural$ in terms of those of Q (Remark 5.18), it follows similarly that the semiorthogonal decomposition (2.17) takes the claimed form. □

It is natural to combine Theorem 5.21 with the result of Lemma 5.13(2) that provides the left hand sides of the semiorthogonal decompositions with a clear geometric meaning. In the next section we use this to derive the applications promised in §1.5.

6. Applications

In this section we collect some applications of the quadratic HPD Theorem 5.21 obtained above. In §6.1 we prove the duality conjecture for Gushel-Mukai varieties, in §6.2 we discuss and prove its spin analogue, and in §6.3 we discuss a noncommutative conifold transition for a certain nodal spin GM fivefold. We continue to assume the base scheme S is the spectrum of an algebraically closed field $\mathbf{k}$ of characteristic not equal to 2.

6.1. Duality of Gushel-Mukai varieties

We will prove [24, Conjecture 3.7]. For context and consequences of this conjecture, see the discussion in §1.5.

The definition of Gushel-Mukai varieties from [4] can be rephrased as follows; note that unlike [4], by convention we require GM varieties to have dimension at least 2.

Definition 6.1. – A *Gushel-Mukai* (*GM*) *variety* is a dimensionally transverse fiber product

$$X = \mathrm{Gr}(2, V_5) \times_{\mathbf{P}(\wedge^2 V_5)} Q,$$

where V_5 is a 5-dimensional vector space, $\mathrm{Gr}(2, V_5) \to \mathbf{P}(\wedge^2 V_5)$ is the Plücker embedding of the Grassmannian of 2-dimensional subspaces of V_5, and $Q \to \mathbf{P}(\wedge^2 V_5)$ is a standard morphism of a quadric with $\dim Q \geq 5$.

The dimensional transversality assumption means that the above fiber product is underived. Note also that $\dim(X) \leq 6$ and that a representation of X as the above fiber product is far from unique, see [4] for details.

In [24], for any smooth GM variety a semiorthogonal decomposition of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \mathrm{Perf}(X)$ was constructed (appearing as (6.2) below), and in particular, an interesting subcategory

$$\mathcal{K}(X) \subset \mathrm{Perf}(X),$$

(called the *GM category* of X) was defined. GM categories were thoroughly studied, and in particular, it was shown that a GM category $\mathcal{K}(X)$ is either a K3 category or an Enriques type category, depending on whether $\dim(X)$ is even or odd.

In [4] GM varieties were classified. In particular, in [4, Theorem 3.6] to every GM variety was associated its *Lagrangian data set*, a triple of vector spaces $(V_6(X), V_5(X), A(X))$, where:

— $V_6(X)$ is a 6-dimensional vector space;

— $V_5(X) \subset V_6(X)$ is a hyperplane; and

— $A(X) \subset \wedge^3 V_6(X)$ is a Lagrangian subspace.

Here we endow the 20-dimensional space $\wedge^3 V_6(X)$ with a symplectic form given by wedge product $\wedge^3 V_6(X) \otimes \wedge^3 V_6(X) \to \wedge^6 V_6(X) \cong \mathbf{k}$. The form depends on a choice of isomorphism above, but the property of being Lagrangian does not.

Conversely, to every triple (V_6, V_5, A) as above two *GM intersections* X^{ord}_{A,V_5} and X^{spe}_{A,V_5} were associated. Both X^{ord}_{A,V_5} and X^{spe}_{A,V_5} have the form

$$X_{A,V_5} = \mathrm{Gr}(2, V_5) \times_{\mathbf{P}(\wedge^2 V_5)} Q,$$

and their type (*ordinary* or *special*) corresponds to the type of the morphism $Q \to \mathbf{P}(\wedge^2 V_5)$ (embedding or covering). Note that a GM intersection X_{A,V_5} is not necessarily dimensionally transverse (so it is not necessarily a GM variety).

Furthermore, in [4, Theorem 3.16] it was shown that if X is a smooth GM variety of dimension $d_X \geq 3$ then the Lagrangian $A(X) \subset \wedge^3 V_6(X)$ *contains no decomposable vectors*, i.e., $\mathbf{P}(A(X))$ does not intersect $\mathrm{Gr}(3, V_6(X)) \subset \mathbf{P}(\wedge^3 V_6(X))$. Conversely, if $A \subset \wedge^3 V_6$ contains no decomposable vectors, then for any $V_5 \subset V_6$ both GM intersections X_{A,V_5} are smooth GM varieties. Note that the dimension of the two types of X_{A,V_5} differs by 1 and depends on V_5.

This already shows that the Lagrangian $A(X)$ controls many important properties of a GM variety X. Motivated by this and a birationality result [4, Corollary 4.16 and Theorem 4.20], we introduced in [24, Definition 3.5] the notions of generalized duality and partnership of GM varieties (generalizing [4, Definition 3.22 and 3.26]).

Definition 6.2. – Let X_1 and X_2 be GM varieties such that

$$\dim(X_1) \equiv \dim(X_2) \pmod 2.$$

— X_1 and X_2 are *generalized partners* if there exists an isomorphism $V_6(X_1) \cong V_6(X_2)$ identifying $A(X_1) \subset \wedge^3 V_6(X_1)$ with $A(X_2) \subset \wedge^3 V_6(X_2)$.

— X_1 and X_2 are *generalized dual* if there exists an isomorphism $V_6(X_1) \cong V_6(X_2)^\vee$ identifying $A(X_1) \subset \wedge^3 V_6(X_1)$ with $A(X_2)^\perp \subset \wedge^3 V_6(X_2)^\vee$.

The duality conjecture [24, Conjecture 3.7] predicted that for (smooth) GM varieties whose associated Lagrangians contain no decomposable vectors, being generalized partners or duals implies an equivalence of GM categories. A special case was proved in [24, Theorem 4.1]; below we prove the conjecture in full generality.

The idea of our proof is as follows. First, we note that the $\mathrm{Gr}(2, V_5)$ factor in the fiber product defining a GM variety in Definition 6.1 is homologically projectively self-dual, so one can use the quadratic HPD theorem to relate the derived categories of two GM

varieties associated to generalized dual standard morphisms of quadrics. Second, we note that generalized duality of the quadric factors corresponds to generalized duality of the corresponding GM intersections. Finally, we iterate equivalences of GM categories obtained in this way to prove the conjecture.

Now we start implementing this approach. First, recall the homological projective self-duality of $\mathrm{Gr}(2, V_5)$:

THEOREM 6.3 ([11, Section 6.1 and Theorem 1.2]). – *Let $\mathcal{U}$ and $\mathcal{U}'$ be the tautological rank 2 subbundles on $\mathrm{Gr}(2, V_5)$ and $\mathrm{Gr}(2, V_5^\vee)$. Then $\mathrm{Perf}(\mathrm{Gr}(2, V_5))$ and $\mathrm{Perf}(\mathrm{Gr}(2, V_5^\vee))$ have the structure of strong, moderate Lefschetz categories over $\mathbf{P}(\wedge^2 V_5)$ and $\mathbf{P}(\wedge^2 V_5^\vee)$, respectively, of length 5, with Lefschetz components given by*

$$\mathcal{A}_i = \langle \mathcal{O}, \mathcal{U}^\vee \rangle \subset \mathrm{Perf}(\mathrm{Gr}(2, V_5)) \quad \text{and} \quad \mathcal{A}'_i = \langle \mathcal{U}', \mathcal{O} \rangle \subset \mathrm{Perf}(\mathrm{Gr}(2, V_5^\vee))$$

for $|i| \leq 4$. Moreover, there is an equivalence

$$\mathrm{Perf}(\mathrm{Gr}(2, V_5))^\natural \simeq \mathrm{Perf}(\mathrm{Gr}(2, V_5^\vee))$$

of Lefschetz categories over $\mathbf{P}(\wedge^2 V_5^\vee)$.

Now we apply Theorem 5.21 to GM varieties.

THEOREM 6.4. – *Let*

$$X = \mathrm{Gr}(2, V_5) \times_{\mathbf{P}(\wedge^2 V_5)} Q \quad \text{and} \quad Y = \mathrm{Gr}(2, V_5^\vee) \times_{\mathbf{P}(\wedge^2 V_5^\vee)} Q^\natural \tag{6.1}$$

be smooth GM varieties of dimensions $d_X \geq 2$ and $d_Y \geq 2$, where $Q \to \mathbf{P}(\wedge^2 V_5)$ is a standard morphism of a quadric and $Q^\natural \to \mathbf{P}(\wedge^2 V_5^\vee)$ is its generalized dual defined in Definition 5.17. Let $\mathcal{U}_X$ and $\mathcal{U}'_Y$ denote the pullbacks of $\mathcal{U}$ and $\mathcal{U}'$ to X and Y, and let $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ denote the pullbacks of the $\mathcal{O}(1)$ line bundles on $\mathbf{P}(\wedge^2 V_5)$ and $\mathbf{P}(\wedge^2 V_5^\vee)$. Then there are semiorthogonal decompositions

$$\mathrm{Perf}(X) = \langle \mathcal{K}(X), \mathcal{O}_X(1), \mathcal{U}_X^\vee(1), \dots, \mathcal{O}_X(d_X - 2), \mathcal{U}_X^\vee(d_X - 2) \rangle, \tag{6.2}$$

$$\mathrm{Perf}(Y) = \langle \mathcal{U}'_Y(2 - d_Y), \mathcal{O}_Y(2 - d_Y), \dots, \mathcal{U}'_Y(-1), \mathcal{O}_Y(-1), \mathcal{K}'(Y) \rangle, \tag{6.3}$$

and an equivalence $\mathcal{K}(X) \simeq \mathcal{K}'(Y)$.

Before giving a proof note that if $d_X \leq 1$ then Y is necessarily singular. Indeed, in this case we have $\dim(Q) \leq 4$, hence $c(Q) \geq 4$, hence $k(Q^\natural) \geq 4$, hence $\mathrm{Gr}(2, V_5^\vee)$ intersects the image of the singular locus of $Q^\natural$, hence Y is singular. This is one of the reasons why we restrict to the case $d_X, d_Y \geq 2$.

Proof. – This is a combination of Theorem 5.21, Theorem 6.3, and Lemma 5.13(2). Indeed, the smoothness of X and Y implies that the Grassmannians in (6.1) do not intersect the singular loci of the quadrics, so by Lemma 5.13(2) we have

$$\mathrm{Perf}(\mathrm{Gr}(2, V_5)) \otimes_{\mathrm{Perf}(\mathbf{P}(\wedge^2 V_5))} \mathfrak{Q} \simeq \mathrm{Perf}(X),$$

$$\mathrm{Perf}(\mathrm{Gr}(2, V_5^\vee)) \otimes_{\mathrm{Perf}(\mathbf{P}(\wedge^2 V_5^\vee))} \mathfrak{Q}^\natural \simeq \mathrm{Perf}(Y),$$

where $\mathfrak{Q}$ and $\mathfrak{Q}^\natural$ are the standard categorical resolutions of Q and $Q^\natural$. We just need to show the semiorthogonal decompositions of Theorem 5.21 take the prescribed form.

The length of the Lefschetz decompositions of $\mathrm{Perf}(\mathrm{Gr}(2, V_5))$ is $m = 5$, and the codimension of $\mathrm{Gr}(2, V_5)$ in $\mathbf{P}(\wedge^2 V_5)$ is 3, so by dimensional transversality

$$d^\natural = \dim(Q^\natural) = d_Y + 3 \geq 5.$$

Thus $m - d^\natural \leq 0$, hence $\mathcal{S}$ does not show up in the semiorthogonal decomposition of $\mathrm{Perf}(X)$. The same argument shows that $(\mathcal{S}^\natural)^\vee$ does not show up in the decomposition of $\mathrm{Perf}(Y)$. Similarly, $N = \dim \wedge^2 V_5 = 10$ and $d = \dim(Q) = d_X + 3$, hence

$$m + d - N = 5 + (d_X + 3) - 10 = d_X - 2,$$

and so the component $\langle \mathcal{O}, \mathcal{U}_X^\vee \rangle$ coming from $\mathrm{Perf}(\mathrm{Gr}(2, V_5))$ appears $d_X - 2$ times in the decomposition of $\mathrm{Perf}(X)$. The same argument shows that the component $\langle \mathcal{U}'_Y, \mathcal{O}_Y \rangle$ appears $d_Y - 2$ times in the decomposition of $\mathrm{Perf}(Y)$. Hence the semiorthogonal decompositions of Theorem 5.21 take the prescribed form. □

Now we are ready to prove the duality conjecture ([24, Conjecture 3.7]).

COROLLARY 6.5. – *Let X and Y be smooth GM varieties whose associated Lagrangian subspaces $A(X)$ and $A(Y)$ do not contain decomposable vectors. If X and Y are generalized partners or duals, then there is an equivalence $\mathcal{K}(X) \simeq \mathcal{K}(Y)$.*

By [4, Theorem 3.16] the assumption that the Lagrangian subspace $A(X)$ does not contain decomposable vectors holds automatically unless X is a special GM surface or an ordinary GM surface with singular Grassmannian hull.

Proof. – First assume X and Y are generalized duals. Then, under the isomorphism $V_6(X) \cong V_6(Y)^\vee$, the hyperplane $V_5(Y) \subset V_6(Y)$ corresponds to a point $\mathbf{q}_Y \in \mathbf{P}(V_6(X))$. Further assume that

$$\mathbf{q}_Y \notin \mathbf{P}(V_5(X)). \tag{6.4}$$

By [4, Proposition 3.28] we can write

$$X = \mathrm{Gr}(2, V_5) \times_{\mathbf{P}(\wedge^2 V_5)} Q, \qquad Y = \mathrm{Gr}(2, V_5^\vee) \times_{\mathbf{P}(\wedge^2 V_5^\vee)} Q',$$

where if $Q_0 \subset \mathbf{P}(\wedge^2 V_5)$ is defined by

$$Q_0 = \begin{cases} Q, & \text{if } Q \to \mathbf{P}(\wedge^2 V_5) \text{ is of embedding type,} \\ Q_{\mathrm{br}}, & \text{if } Q \to \mathbf{P}(\wedge^2 V_5) \text{ is of covering type,} \end{cases}$$

and $Q'_0 \subset \mathbf{P}(\wedge^2 V_5^\vee)$ is defined analogously, then Q'_0 is classically projectively dual to Q_0. Since $\dim(X)$ and $\dim(Y)$ have the same parity, the same is true for $\dim(Q)$ and $\dim(Q')$, and recalling Definition 5.17 and using Remark 5.19 (note that $\dim(\wedge^2 V_5)$ is even) we conclude that $Q' \to \mathbf{P}(\wedge^2 V_5^\vee)$ is generalized dual to $Q \to \mathbf{P}(\wedge^2 V_5)$. Thus, X and Y are obtained as in Theorem 6.4 from an appropriate pair of generalized dual quadrics. Twisting the decomposition (6.2) by $\mathcal{O}_X(-1)$ shows that $\mathcal{K}(X)$ is equivalent to the GM category of X, as defined in [24, Definition 2.5]. Twisting the decomposition (6.3) by $\mathcal{O}_Y(1)$ and using [24, (2.20) and (2.21)] shows that $\mathcal{K}'(Y)$ is equivalent to the GM category of Y. Thus Theorem 6.4 gives the result under our above assumptions.

Next assume X and Y are generalized partners. Choose $\mathbf{p} \in \mathbf{P}(V_6(X)) = \mathbf{P}(V_6(Y))$ a point away from the hyperplanes $\mathbf{P}(V_5(X))$ and $\mathbf{P}(V_5(Y))$. Let $V_5 \subset V_6 := V_6(X)^\vee$ be the

corresponding hyperplane, and let $A = A(X)^{\perp} \subset \wedge^3 V_6$. Then by [4, Theorem 3.10] there is a GM variety Z such that

$$\dim(Z) \equiv \dim(X) \equiv \dim(Y) \bmod 2 \qquad \text{and} \qquad (V_6(Z), V_5(Z), A(Z)) = (V_6, V_5, A).$$

Since A does not contain decomposable vectors, Z is smooth by [4, Theorem 3.16]. By construction, X and Y are both generalized duals of Z satisfying the extra assumption considered in the previous paragraph. Thus $\mathcal{K}(X) \simeq \mathcal{K}(Z)$ and $\mathcal{K}(Y) \simeq \mathcal{K}(Z)$, hence $\mathcal{K}(X) \simeq \mathcal{K}(Y)$.

The final case to consider is when X and Y are generalized duals, and under the isomorphism $V_6(X) \cong V_6(Y)^{\vee}$, the hyperplane $V_5(Y) \subset V_6(Y)$ corresponds to a point $\mathbf{q} \in \mathbf{P}(V_6(X))$ which lies in $\mathbf{P}(V_5(X))$. In this case, arguing as in the previous paragraph we can construct a GM variety Y' generalized dual to X, such that under the isomorphism $V_6(X) \cong V_6(Y')^{\vee}$, the hyperplane $V_5(Y') \subset V_6(Y')$ corresponds to a point $\mathbf{q}' \in \mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X))$. Then the GM category of X is equivalent to that of Y' by the first paragraph. By construction Y' and Y are generalized partners, so their GM categories are equivalent by the previous paragraph. □

REMARK 6.6. – As we already mentioned, the duality conjecture does not cover the case of smooth GM varieties X whose Lagrangians contain decomposable vectors, i.e., all special GM surfaces and some ordinary GM surfaces. Note that such X have neither smooth generalized partners nor duals of dimension greater than 2 and moreover $\mathcal{K}(X) = \mathrm{Perf}(X)$. Thus, extending [24, Conjecture 3.7] it is natural to ask: If X and Y are smooth GM surfaces which are generalized partners or duals and whose Lagrangians contain decomposable vectors, then is there an equivalence $\mathrm{Perf}(X) \simeq \mathrm{Perf}(Y)$?

We expect that the answer is positive, although the argument of Corollary 6.5 does not work as the crucial assumption (6.4) never holds for smooth generalized partners when the corresponding Lagrangian has decomposable vectors. There are two possible strategies to work around this.

First, one can also consider (mildly) singular GM varieties and prove that if X and Y are generalized dual with X singular and Y smooth, then $\mathcal{K}(Y)$ is a categorical resolution of $\mathcal{K}(X)$ (cf. the proof of Theorem 6.11 below). Then, however, one will have to check that the resolutions $\mathcal{K}(Y_1)$ and $\mathcal{K}(Y_2)$ obtained from two smooth generalized duals Y_1 and Y_2 of the same X are equivalent. This seems possible, but does not fit into the scope of this paper.

Another possibility is to extend the arguments of [4, Theorem 4.7 and Propositions 4.13 and 4.19] to show that if X and Y are two smooth generalized partners or duals, then X is birational to Y, and hence $X \cong Y$.

6.2. Duality of spin GM varieties

It is well known that the Grassmannian $\mathrm{Gr}(2, V_5)$ shares many properties with its elder brother, the spinor tenfold $\mathrm{OGr}_+(5, V_{10})$. The content of this subsection provides yet another confirmation of this principle.

Let V_{10} be a 10-dimensional vector space. Recall the orthogonal Grassmannian $\mathrm{OGr}(5, V_{10})$ of 5-dimensional isotropic subspaces for a nondegenerate quadratic form on V_{10} has two connected components, $\mathrm{OGr}_+(5, V_{10})$ and $\mathrm{OGr}_-(5, V_{10})$, which are abstractly isomorphic.

The Plücker embedding $\mathrm{OGr}_+(5, V_{10}) \to \mathrm{Gr}(5, V_{10}) \to \mathbf{P}(\wedge^5 V)$ is given by the square of the generator of $\mathrm{Pic}(\mathrm{OGr}_+(5, V_{10}))$; the generator itself gives an embedding

$$\mathrm{OGr}_+(5, V_{10}) \subset \mathbf{P}(\mathsf{S}_{16}),$$

where S_{16} is the 16-dimensional half-spinor representation of $\mathrm{Spin}(V_{10})$. Note that

$$\dim \mathrm{OGr}_+(5, V_{10}) = 10 \qquad \text{and} \qquad \mathrm{codim}_{\mathbf{P}(\mathsf{S}_{16})} \mathrm{OGr}_+(5, V_{10}) = 5.$$

The spinor tenfold $\mathrm{OGr}_+(5, V_{10}) \subset \mathbf{P}(\mathsf{S}_{16})$ shares a very special property with the Grassmannian $\mathrm{Gr}(2, V_5) \subset \mathbf{P}(\wedge^2 V_5)$: both are projectively self-dual, and even homologically projectively self-dual. More precisely, the classical projective dual variety of $\mathrm{OGr}_+(5, V_{10}) \subset \mathbf{P}(\mathsf{S}_{16})$ is given by the spinor embedding $\mathrm{OGr}_-(5, V_{10}) \subset \mathbf{P}(\mathsf{S}_{16}^\vee)$. This lifts to the homological level as follows.

THEOREM 6.7 ([11, Section 6.2 and Theorem 1.2]). – *Let $\mathcal{U}$ and $\mathcal{U}'$ be the tautological rank 5 subbundles on $\mathrm{OGr}_+(5, V_{10})$ and $\mathrm{OGr}_-(5, V_{10})$. Then $\mathrm{Perf}(\mathrm{OGr}_+(5, V_{10}))$ and $\mathrm{Perf}(\mathrm{OGr}_-(5, V_{10}))$ have the structure of strong, moderate Lefschetz categories over the spinor spaces $\mathbf{P}(\mathsf{S}_{16})$ and $\mathbf{P}(\mathsf{S}_{16}^\vee)$, respectively, of length 8, with Lefschetz components given by*

$$\mathcal{A}_i = \langle \mathcal{O}, \mathcal{U}^\vee \rangle \quad \text{and} \quad \mathcal{A}'_i = \langle \mathcal{U}', \mathcal{O} \rangle$$

for $|i| \leq 7$. Moreover, there is an equivalence

$$\mathrm{Perf}(\mathrm{OGr}_+(5, V_{10}))^\natural \simeq \mathrm{Perf}(\mathrm{OGr}_-(5, V_{10}))$$

of Lefschetz categories over $\mathbf{P}(\mathsf{S}_{16}^\vee)$.

This parallel between $\mathrm{Gr}(2, V_5)$ and $\mathrm{OGr}_+(5, V_{10})$ motivates the following definition.

DEFINITION 6.8. – A *spin GM variety* is a dimensionally transverse fiber product

$$X = \mathrm{OGr}_+(5, V_{10}) \times_{\mathbf{P}(\mathsf{S}_{16})} Q,$$

where $Q \to \mathbf{P}(\mathsf{S}_{16})$ is a standard morphism of a quadric with $\dim Q \geq 8$, so that $\dim(X) \geq 3$.

We note that if X is a smooth spin GM variety of dimension d, then X is a Fano variety of Picard number 1, coindex 4, and degree 24 for $d \geq 4$, and X is a polarized Calabi-Yau threefold of Picard number 1 and degree 24 for $d = 3$. We also remark that $\dim(X) \leq 10$ for any spin GM variety and that its representation as the above fiber product is far from unique.

Using Theorem 6.7 in place of Theorem 6.3, the argument of Theorem 6.9 proves the following spin analogue.

THEOREM 6.9. – *Let*

$$X = \mathrm{OGr}_+(5, V_{10}) \times_{\mathbf{P}(\mathsf{S}_{16})} Q \quad \text{and} \quad Y = \mathrm{OGr}_-(5, V_{10}) \times_{\mathbf{P}(\mathsf{S}_{16}^\vee)} Q^\natural$$

be smooth spin GM varieties of dimensions $d_X \geq 4$ and $d_Y \geq 4$, where $Q \to \mathbf{P}(\mathsf{S}_{16})$ is a standard morphism of a quadric and $Q^\natural \to \mathbf{P}(\mathsf{S}_{16}^\vee)$ is its generalized dual as defined in Definition 5.14. Let $\mathcal{U}_X$ and $\mathcal{U}'_Y$ denote the pullbacks of $\mathcal{U}$ and $\mathcal{U}'$ to X and Y, and let $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ denote

the pullbacks of the $\mathcal{O}(1)$ *line bundles on* $\mathbf{P}(\mathsf{S}_{16})$ *and* $\mathbf{P}(\mathsf{S}_{16}^\vee)$. *Then there are semiorthogonal decompositions*

$$\mathrm{Perf}(X) = \langle \mathcal{K}(X), \mathcal{O}_X(1), \mathcal{U}_X^\vee(1), \dots, \mathcal{O}_X(d_X - 3), \mathcal{U}_X^\vee(d_X - 3)\rangle, \tag{6.5}$$

$$\mathrm{Perf}(Y) = \langle \mathcal{U}'_Y(3 - d_Y), \mathcal{O}_Y(3 - d_Y), \dots, \mathcal{U}'_Y(-1), \mathcal{O}_Y(-1), \mathcal{K}'(Y)\rangle, \tag{6.6}$$

and an equivalence $\mathcal{K}(X) \simeq \mathcal{K}'(Y)$.

The assumptions $d_X \geq 4$ and $d_Y \geq 4$ may seem strange, but they are necessary for smoothness of Y and X; indeed, if $d_X \leq 3$ then $\dim(Q) \leq 8$, hence the codimension of the linear span of the image of Q in $\mathbf{P}(\mathsf{S}_{16})$ is at least 6, hence $\dim(\mathrm{Sing}(Q^\natural)) \geq 5$, hence Y intersects $\mathrm{Sing}(Q^\natural)$, hence is singular.

We call the category $\mathcal{K}(X)$ occurring in (6.5) a *spin GM category*. Spin GM categories should be thought of as 3-dimensional counterparts of GM categories. Indeed, whereas a GM category is always (fractional) Calabi-Yau of dimension 2, a spin GM category is (fractional) Calabi-Yau of dimension 3 by [20, Remark 4.9]. More precisely, if X is odd-dimensional then $\mathcal{K}(X)$ is 3-Calabi-Yau, while if X is even-dimensional then the Serre functor of $\mathcal{K}(X)$ is given by $\mathrm{S}_{\mathcal{K}(X)} = \sigma \circ [3]$ where σ is an involutive autoequivalence of $\mathcal{K}(X)$.

Furthermore, one can develop the notion of a Lagrangian data set for spin GM varieties: this should consist of triples (V_{12}, V_{10}, A), where V_{12} is a 12-dimensional space endowed with a non-degenerate quadratic form, $V_{10} \subset V_{12}$ a 10-dimensional subspace to which the quadratic form restricts non-degenerately, and $A \subset \mathsf{S}_{32}(V_{12})$ is a Lagrangian subspace in the 32-dimensional half-spinor representation of $\mathrm{Spin}(V_{12})$ (note that $\mathsf{S}_{32}(V_{12})$ has a natural $\mathrm{Spin}(V_{12})$-invariant symplectic form). Then the notion of generalized spin partnership and duality for spin GM varieties can be defined analogously to Definition 6.2, and the argument of Corollary 6.5 would prove that spin GM categories of generalized spin partners or duals are equivalent.

It would be interesting to investigate the rationality question for spin GM varieties in relation to Theorem 6.9, following the GM case discussed in [24, §3]. The critical case is when X has dimension 5; then $\mathcal{K}(X)$ is a 3-Calabi-Yau category, which is conjecturally equivalent to the derived category of a Calabi-Yau threefold if and only if X is rational. Theorem 6.9, however, does *not* give examples of this sort. Indeed, if Y is a smooth spin GM variety of dimension 3, then X must be singular as we explained above. More generally, we have the following result.

LEMMA 6.10. – *Let* X *be a smooth spin GM variety whose dimension is odd and at least* 5.

(1) *The* 0*-th Hochschild homology of* $\mathcal{K}(X)$ *is given by* $\mathrm{HH}_0(\mathcal{K}(X)) \cong \mathbf{k}^2$.

(2) *There does not exist an equivalence* $\mathcal{K}(X) \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(M)$ *for any projective variety* M.

Proof. – We first note that $\mathrm{OGr}_+(5, V_{10})$ has cohomology of Tate type, and Poincaré polynomial given by

$$1 + t^2 + t^4 + 2t^6 + 2t^8 + 2t^{10} + 2t^{12} + 2t^{14} + t^{16} + t^{18} + t^{20},$$

see [30, §2.2] or [19, Corollary 3.8]. The Lefschetz hyperplane theorem combined with the Hochschild-Kostant-Rosenberg (HKR) theorem then determines $\mathrm{HH}_0(X)$, and the

claimed formula for $\mathrm{HH}_0(\mathcal{K}(X))$ follows from the additivity of Hochschild cohomology [21, Theorem 7.3].

If $\mathcal{K}(X) \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(M)$ for a projective variety M, then M is smooth by [11, Lemma 10.19]. Moreover, M must have dimension 3 since $\mathcal{K}(X)$ is 3-Calabi-Yau. The HKR theorem then implies $\dim \mathrm{HH}_0(M) \geq 4$, contradicting part (1). □

6.3. A noncommutative conifold transition

Despite the negative result of Lemma 6.10, by considering a mild degeneration of the situation of Theorem 6.9, we can find spin GM fivefolds whose category $\mathcal{K}(X)$ admits a geometric resolution of singularities. Recall from Definition 2.18 and Remark 2.19 the notion of a weakly crepant categorical resolution.

THEOREM 6.11. – *Let* $K \subset W \subset \mathsf{S}_{16}$ *be generic subspaces with* $\dim(K) = 6$ *and* $\dim(W) = 12$, *and let* $\bar{Q} \subset \mathbf{P}(W/K)$ *be a general smooth quadric. Set* $Q = \mathbf{C}_K(\bar{Q})$ *and let*

$$f\colon Q \to \mathbf{P}(W) \to \mathbf{P}(\mathsf{S}_{16})$$

be the induced morphism. Let

$$X = \mathrm{OGr}_+(5, V_{10}) \times_{\mathbf{P}(\mathsf{S}_{16})} Q \quad \text{and} \quad Y = \mathrm{OGr}_-(5, V_{10}) \times_{\mathbf{P}(\mathsf{S}_{16}^\vee)} Q^\natural.$$

Then X *is a spin GM fivefold with* 12 *nodal singularities and* Y *is a smooth spin GM threefold. Moreover, there is a semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}}(X), \mathcal{O}_X(1), \mathcal{U}_X^\vee(1), \mathcal{O}_X(2), \mathcal{U}_X^\vee(2)\rangle, \tag{6.7}$$

and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ *is a weakly crepant categorical resolution of* $\mathcal{K}(X) = \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}}(X) \cap \mathrm{Perf}(X)$.

Proof. – The spinor embedding $\mathrm{OGr}_+(5, V_{10}) \subset \mathbf{P}(\mathsf{S}_{16})$ has degree 12 and codimension 5. Thus for general K the intersection $Z = \mathrm{OGr}_+(5, V_{10}) \cap \mathbf{P}(K)$ consists of 12 reduced points, say $z_1, \dots, z_{12}$, and the dual intersection $\mathrm{OGr}_-(5, V_{10}) \cap \mathbf{P}(K^\perp)$ is a smooth fourfold. Furthermore, for general W containing K the intersection $\mathrm{OGr}_-(5, V_{10}) \cap \mathbf{P}(W^\perp)$ is empty, and the intersection $\mathrm{OGr}_+(5, V_{10}) \cap \mathbf{P}(W)$ is a smooth sixfold containing Z.

The embedded tangent space to $\mathrm{OGr}_+(5, V_{10})$ at the point z_i corresponds to an 11-dimensional subspace $T_i \subset \mathsf{S}_{16}$ such that $\dim(T_i \cap K) = 1$. The intersection $T_i \cap W$ corresponds to the embedded tangent space to $\mathrm{OGr}_+(5, V_{10}) \cap \mathbf{P}(W)$ at z_i, hence $\dim(T_i \cap W) = 7$ and the natural map $T_i \cap W \to W/K$ is surjective with kernel $T_i \cap K$. For any smooth quadric $\bar{Q} \subset \mathbf{P}(W/K)$ its strict preimage in $\mathbf{P}(T_i \cap W)$ is the cone over $\bar{Q}$ with vertex $z_i = \mathbf{P}(T_i \cap K)$ and it is identified with the normal cone to X at z_i, hence z_i is a node. This proves that for K and W chosen as above and any smooth $\bar{Q}$ the intersection X has nodes at points of Z. Also, for general $\bar{Q}$ by Bertini's theorem X is smooth away from Z and Y is smooth. Thus Y is a smooth spin GM threefold.

The semiorthogonal decomposition (6.7) is induced by the Lefschetz decomposition of the spinor tenfold $\mathrm{OGr}_+(5, V_{10})$, cf. [23, Lemma 5.5].

Let $\mathfrak{Q}$ denote the standard categorical resolution of Q over $\mathbf{P}(\mathsf{S}_{16})$. Then arguing as in Theorem 6.9, we see that Theorem 5.21 gives a semiorthogonal decomposition

$$\mathrm{Perf}(\mathrm{OGr}_+(5, V_{10})) \otimes_{\mathrm{Perf}(\mathbf{P}(\mathsf{S}_{16}))} \mathfrak{Q} = \left\langle \widetilde{\mathcal{K}}(X), \langle \mathcal{O}(1), \mathcal{U}^\vee(1)\rangle \otimes \langle \mathcal{O}\rangle, \langle \mathcal{O}(2), \mathcal{U}^\vee(2)\rangle \otimes \langle \mathcal{O}\rangle \right\rangle \tag{6.8}$$

and an equivalence $\mathrm{Perf}(Y) \simeq \widetilde{\mathcal{K}}(X)$. Note that $\mathrm{Perf}(Y) = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ since Y is smooth. Thus to finish it suffices to show that $\widetilde{\mathcal{K}}(X)$ is a weakly crepant categorical resolution of $\mathcal{K}(X)$.

The functors $\pi_*\colon \mathfrak{Q} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Q)$ and $\pi^*\colon \mathrm{Perf}(Q) \to \mathfrak{Q}$ of Lemma 5.11 induce by base change along $\mathrm{OGr}_+(5, V_{10}) \to \mathbf{P}(\mathsf{S}_{16})$ functors

$$\pi_*\colon \mathrm{Perf}(\mathrm{OGr}_+(5, V_{10})) \otimes_{\mathrm{Perf}(\mathbf{P}(\mathsf{S}_{16}))} \mathfrak{Q} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X),$$
$$\pi^*\colon \mathrm{Perf}(X) \to \mathrm{Perf}(\mathrm{OGr}_+(5, V_{10})) \otimes_{\mathrm{Perf}(\mathbf{P}(\mathsf{S}_{16}))} \mathfrak{Q},$$

such that π^* is left and right adjoint to π_* and $\pi_* \circ \pi^* \simeq \mathrm{id}$. Thus, these functors provide the category $\mathrm{Perf}(\mathrm{OGr}_+(5, V_{10})) \otimes_{\mathrm{Perf}(\mathbf{P}(\mathsf{S}_{16}))} \mathfrak{Q}$ with the structure of a weakly crepant categorical resolution of X.

Furthermore, (6.7) also induces a semiorthogonal decomposition

$$\mathrm{Perf}(X) = \langle \mathcal{K}(X), \mathcal{O}_X(1), \mathcal{U}_X^\vee(1), \mathcal{O}_X(2), \mathcal{U}_X^\vee(2) \rangle, \tag{6.9}$$

where $\mathcal{K}(X) = \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}}(X) \cap \mathrm{Perf}(X)$. Indeed, by [16, Proposition 4.1] it is enough to show that the components of (6.7) are admissible; this is clear for the exceptional objects that appear, and then follows for $\mathcal{K}^{\mathrm{b}}_{\mathrm{coh}}(X)$ by Serre duality and the fact that X is Gorenstein. Clearly, the functor π^* takes the four exceptional objects from (6.9) to the four exceptional objects in (6.8). Therefore, from full faithfulness it follows that π^* takes the right orthogonal $\mathcal{K}(X)$ of the former to the right orthogonal $\widetilde{\mathcal{K}}(X)$ of the latter, and thus defines a functor

$$\pi^*\colon \mathcal{K}(X) \to \widetilde{\mathcal{K}}(X).$$

Similarly, by adjunction it follows that the right adjoint functor π_* takes $\widetilde{\mathcal{K}}(X)$ to $\mathcal{K}^{\mathrm{b}}_{\mathrm{coh}}(X)$, and hence defines a functor

$$\pi_*\colon \widetilde{\mathcal{K}}(X) \to \mathcal{K}^{\mathrm{b}}_{\mathrm{coh}}(X).$$

Since we have already shown that π_* and π^* provide $\mathrm{Perf}(\mathrm{OGr}_+(5, V_{10})) \otimes_{\mathrm{Perf}(\mathbf{P}(\mathsf{S}_{16}))} \mathfrak{Q}$ with the structure of a weakly crepant categorical resolution of X, it follows that $\widetilde{\mathcal{K}}(X)$ is a weakly crepant categorical resolution of $\mathcal{K}(X)$ via these functors. □

The proof of the theorem shows that the resolution $\widetilde{X} \to X$ given by blowing up the singular points of X has a semiorthogonal decomposition consisting of exceptional objects and the derived category of the Calabi-Yau threefold Y. Thus, the philosophy of [15, 18] suggests that $\widetilde{X}$ (and therefore X) should be rational. We will prove this as a consequence of the following.

LEMMA 6.12. – *If X is as in Theorem* 6.11*, then there is a resolution of singularities $X' \to X$ and a morphism $X' \to \mathbf{P}^2$ whose general fiber is a smooth Fano threefold of Picard number* 1*, degree* 12*, and index* 1*. Moreover, the morphism $X' \to \mathbf{P}^2$ has* 12 *sections.*

Proof. – The following argument is inspired by [4, Lemma 4.1].

Recall that the kernel space K of the quadric Q defining X is 6-dimensional and its span W is 12-dimensional. Therefore, the maximal isotropic spaces for Q are 9-dimensional. Let $I \supset K$ be a generic such space. Then linear projection from $\mathbf{P}^8 = \mathbf{P}(I) \subset \mathbf{P}(W)$ induces a morphism

$$q\colon X' \to \mathbf{P}(W/I) = \mathbf{P}^2,$$

where X' is the blowup of X along

$$X \cap \mathbf{P}(I) = \mathrm{OGr}_+(5, V_{10}) \cap \mathbf{P}(I).$$

The genericity of I guarantees that X' is smooth.

The fibers of q can be described as follows: a point $b \in \mathbf{P}^2$ corresponds to a $\mathbf{P}^9_b \subset \mathbf{P}(W)$ containing $\mathbf{P}(I)$; we have $Q \cap \mathbf{P}^9_b = \mathbf{P}(I) \cup \mathbf{P}(I_b)$ where I_b is the residual isotropic space for Q; and the fiber over b is $q^{-1}(b) = \mathrm{OGr}_+(5, V_{10}) \cap \mathbf{P}(I_b)$. Thus the general fiber of q is a smooth threefold given as a codimension 7 linear section of $\mathrm{OGr}_+(5, V_{10}) \subset \mathbf{P}(\mathsf{S}_{16})$, i.e., a threefold of the claimed type.

Furthermore, since any maximal isotropic subspace in Q contains K, we have $K \subset I_b$, hence

$$\mathrm{OGr}_+(5, V_{10}) \cap \mathbf{P}(K) \subset \mathrm{OGr}_+(5, V_{10}) \cap \mathbf{P}(I_b).$$

It remains to note that the left side is a set of 12 reduced points; each of these points gives a section of the morphism $X' \to \mathbf{P}^2$. □

COROLLARY 6.13. – *If X is as in Theorem* 6.11 *and the base field* $\mathbf{k}$ *is algebraically closed of characteristic* 0*, then X is rational.*

Proof. – By Lemma 6.12, it suffices to show that a smooth Fano threefold of Picard number 1, degree 12, and index 1 is rational if it has a rational point. This holds by [27, Theorem 1.1(ii)]. □

We note that Theorem 6.11 can be thought of as giving a conifold transition from the noncommutative Calabi-Yau threefold $\mathcal{K}(X)$ to the Calabi-Yau threefold Y. In the spirit of Reid's fantasy [34], we pose the following (loosely formulated) question:

QUESTION 6.14. – Can any noncommutative Calabi-Yau threefold be connected to a geometric Calabi-Yau threefold via a sequence of degenerations and crepant resolutions?

Theorem 6.11 gives a positive answer to this question for spin GM categories of spin GM fivefolds, and similar arguments also give a positive answer for spin GM varieties of dimension 7 or 9. The results of [8] give a positive answer for noncommutative Calabi-Yau threefolds associated to cubic sevenfolds (using, however, degenerations with worse-than-nodal singularities). It would be interesting to investigate more examples, in particular the list of noncommutative Calabi-Yau threefolds given in [20, §4.5].

REMARK 6.15. – If Y is a smooth *strict* Calabi-Yau threefold in the sense that $\omega_Y \cong \mathcal{O}_Y$ and $\mathrm{H}^j(Y, \mathcal{O}_Y) = 0$ for $j = 1, 2$, then the HKR theorem shows that $\mathrm{HH}^2(Y) \cong \mathrm{H}^1(\mathrm{T}_Y)$, so $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ has no noncommutative infinitesimal deformations. Thus to have a hope of connecting a noncommutative Calabi-Yau threefold to a geometric Calabi-Yau threefold, we should indeed allow more operations than deformations.

Appendix

HPD results

In this appendix we provide some material on semiorthogonal decompositions and HPD that is used in the body of the paper. In §A.1 we establish a local criterion for an equivalence of T-linear categories. In §A.2 we describe the behavior of HPD under linear projections.

A.1. A local criterion for an equivalence

The main result of this subsection is the following proposition.

PROPOSITION A.1. – *Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a T-linear functor and let $\mathcal{A} \subset \mathcal{C}$ be a T-linear subcategory. Assume that either ϕ has a left adjoint and $\mathcal{A}$ is left admissible, or ϕ has a right adjoint and $\mathcal{A}$ is right admissible. Let also $\mathcal{B} \subset \mathcal{D}$ be a T-linear subcategory which is either right or left admissible. Let $U \to T$ be an fpqc cover, and let $\phi_U\colon \mathcal{C}_U \to \mathcal{D}_U$ denote the induced functor obtained by base change along $U \to T$. Then ϕ induces an equivalence $\mathcal{A} \simeq \mathcal{B}$ if and only if ϕ_U induces an equivalence $\mathcal{A}_U \simeq \mathcal{B}_U$.*

As we observe in Corollary A.5, the proposition also implies a local criterion for a functor between Lefschetz categories to be a Lefschetz equivalence.

We build up some preliminary results before giving the proof. If $\mathcal{C}$ is a T-linear category and $T' \to T$ is a morphism, we write $C|_{T'}$ for the image of $C \in \mathcal{C}$ under the canonical functor $\mathcal{C} \to \mathcal{C}_{T'}$ induced by pullback.

LEMMA A.2. – *Let $\mathcal{C}$ be a T-linear category, and let $C \in \mathcal{C}$. Let $U \to T$ be an fpqc cover. Then $C \simeq 0$ if and only if $C|_U \simeq 0$.*

Proof. – The forward implication is obvious. Conversely, by the Künneth formula in the form of [32, Lemma 2.10], we have

$$\mathcal{H}om_T(C, C)|_U \simeq \mathcal{H}om_U(C|_U, C|_U),$$

where $\mathcal{H}om_T(C, C) \in \mathrm{D}_{\mathrm{qc}}(T)$ is the mapping object defined in [32, §2.3.1]. Hence if $C|_U \simeq 0$, we have $\mathcal{H}om_T(C, C)|_U \simeq 0$. Then $\mathcal{H}om_T(C, C) \simeq 0$ since the vanishing of an object in $\mathrm{D}_{\mathrm{qc}}(T)$ can be checked fpqc locally, and therefore $C \simeq 0$. □

COROLLARY A.3. – *Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a T-linear functor. Let $U \to T$ be an fpqc cover. Then $\phi \simeq 0$ if and only if $\phi_U \simeq 0$.*

Proof. – The forward implication is obvious. Conversely, we must show that $\phi(C) \simeq 0$ for all $C \in \mathcal{C}$ if $\phi_U \simeq 0$. For this, just note that $\phi(C)|_U \simeq \phi_U(C|_U)$ and apply Lemma A.2. □

LEMMA A.4. – *Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a T-linear functor. Let $\mathcal{B} \subset \mathcal{D}$ be a T-linear subcategory which is left or right admissible. Let $U \to T$ be an fpqc cover. Then ϕ factors through the inclusion $\mathcal{B} \subset \mathcal{D}$ if and only if ϕ_U factors through the inclusion $\mathcal{B}_U \subset \mathcal{D}_U$.*

Proof. – We consider the case where $\mathcal{B}$ is left admissible; the right admissible case is similar. Since $\mathcal{B}$ is left admissible, its left orthogonal ${}^{\perp}\mathcal{B}$ is right admissible, hence its inclusion functor $j\colon {}^{\perp}\mathcal{B} \to \mathcal{D}$ has a right adjoint $j^!$ whose kernel is $\mathcal{B}$. Therefore ϕ factors through $\mathcal{B} \subset \mathcal{D}$ if and only if the composition $j^! \circ \phi$ vanishes. By Corollary A.3, this composition vanishes if and only if its base change to U vanishes. But this base change identifies with $j_U^! \circ \phi_U$ where $j_U^!$ is the right adjoint to the inclusion ${}^{\perp}\mathcal{B}_U \subset \mathcal{D}_U$ (see [32, Lemma 3.15]), and hence vanishes if and only if ϕ_U factors through $\mathcal{B}_U \subset \mathcal{D}_U$. □

Proof of Proposition A.1. – We consider the left adjoints case of the proposition; the right adjoints case is similar. First assume $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \mathcal{D}$. Note that a functor with a left adjoint is an equivalence if and only if the cones of the unit and counit of the adjunction vanish. If ψ denotes the cone of the unit or counit for the adjoint pair (ϕ, ϕ^*), then ψ_U is the cone of the unit or counit for the adjoint pair (ϕ_U, ϕ_U^*) (cf. [32, Lemma 2.12] or [11, §2.6]). Hence applying Corollary A.3 proves the lemma in this case.

Now consider the case of general $\mathcal{A}$ and $\mathcal{B}$. Denote by $\alpha\colon \mathcal{A} \to \mathcal{C}$ and $\beta\colon \mathcal{B} \to \mathcal{D}$ the inclusions. If ϕ_U induces an equivalence $\mathcal{A}_U \simeq \mathcal{B}_U$, then by Lemma A.4 the composition of functors $\phi \circ \alpha\colon \mathcal{A} \to \mathcal{D}$ factors through $\mathcal{B} \subset \mathcal{D}$, i.e., there is a functor $\phi_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{B}$ such that $\phi \circ \alpha = \beta \circ \phi_{\mathcal{A}}$. We want to show $\phi_{\mathcal{A}}$ is an equivalence. But $\phi_{\mathcal{A}}$ admits a left adjoint, namely $\alpha^* \circ \phi^* \circ \beta$, and $(\phi_{\mathcal{A}})_U\colon \mathcal{A}_U \to \mathcal{B}_U$ is an equivalence, so we conclude by the case handled above. □

Let $S' \to S$ be a morphism of schemes, and let $V_{S'}$ denote the pullback of a vector bundle V on S to S'. Then if $\mathcal{A}$ is a Lefschetz category over $\mathbf{P}(V)$, the base change $\mathcal{A}_{S'}$ is naturally a Lefschetz category over $\mathbf{P}(V_{S'})$ with Lefschetz center given by the base change $(\mathcal{A}_0)_{S'} \subset \mathcal{A}_{S'}$. This follows from a combination of [25, Lemma 2.4] and [32, Lemmas 3.15 and 3.17]. Proposition A.1 then implies the following.

COROLLARY A.5. – *Let $\mathcal{A}$ and $\mathcal{B}$ be Lefschetz categories over $\mathbf{P}(V)$. Let $\phi\colon \mathcal{A} \to \mathcal{B}$ be a $\mathbf{P}(V)$-linear functor which admits a left or right adjoint. Let $U \to S$ be an fpqc cover of S, and let $\phi_U\colon \mathcal{A}_U \to \mathcal{B}_U$ denote the induced functor. Then ϕ is an equivalence of Lefschetz categories over $\mathbf{P}(V)$ if and only if ϕ_U is an equivalence of Lefschetz categories over $\mathbf{P}(V_U)$.*

The following related result is useful for establishing the existence of a semiorthogonal decomposition, by reduction to a local situation.

LEMMA A.6. – *Let $\mathcal{C}$ be a T-linear category, and let $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{C}$ be a sequence of right or left admissible T-linear subcategories. Let $U \to T$ be an fpqc cover. Then $\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ if and only if $\mathcal{C}_U = \langle \mathcal{A}_{1U}, \dots, \mathcal{A}_{nU} \rangle$.*

Proof. – The forward implication holds by [32, Lemma 3.15]. Conversely, assume we have a semiorthogonal decomposition $\mathcal{C}_U = \langle \mathcal{A}_{1U}, \dots, \mathcal{A}_{nU} \rangle$. Then the argument of Lemma A.2 shows that the categories $\mathcal{A}_i \subset \mathcal{C}$ are semiorthogonal. Assume the categories $\mathcal{A}_i \subset \mathcal{C}$ are right admissible (a similar argument works in the left admissible case). Then setting $\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle^{\perp}$ we have a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{D}, \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$. But $\mathcal{D} = 0$ by Lemma A.2. □

A.2. HPD over quotients and subbundles

Given a surjective morphism $\tilde{V} \to V$ of vector bundles with kernel K, we consider the corresponding rational map $\mathbf{P}(\tilde{V}) \dashrightarrow \mathbf{P}(V)$ and denote by $U = \mathbf{P}(\tilde{V}) \setminus \mathbf{P}(K) \subset \mathbf{P}(\tilde{V})$ the open subset on which it is regular. If $\mathcal{A}$ is a $\mathbf{P}(\tilde{V})$-linear category supported over U (i.e., if the restriction functor $\mathcal{A} \to \mathcal{A}_U$ is an equivalence), then it inherits a natural $\mathbf{P}(V)$-linear structure via the linear projection map. In this situation, we can ask for a relation between HPD with respect to the two linear structures on $\mathcal{A}$. Before answering this, we make some preliminary observations.

DEFINITION A.7. – Let $\mathcal{C}$ be a T-linear category, and let $T \to T'$ be a morphism of schemes. We write $\mathcal{C}/T'$ for $\mathcal{C}$ regarded as a T'-linear category via the pullback functor $\mathrm{Perf}(T') \to \mathrm{Perf}(T)$, and say $\mathcal{C}/T'$ is obtained from $\mathcal{C}$ by *extending the base scheme* along $T \to T'$.

REMARK A.8. – If $\mathcal{A}$ is a Lefschetz category over $\mathbf{P}(V)$ and $V \to V'$ is an embedding of vector bundles, then the category $\mathcal{A}/\mathbf{P}(V')$ is naturally a Lefschetz category over $\mathbf{P}(V')$, with the same center. Moreover, this operation preserves (right or left) strongness and moderateness of Lefschetz categories.

LEMMA A.9. – *Let T be a scheme and let $U \subset T$ be an open subscheme. Let $\mathcal{C}$ be a T-linear category which is supported over U. Then for any T-linear category $\mathcal{D}$, there is a canonical T-linear equivalence*

$$\mathcal{C} \otimes_{\mathrm{Perf}(T)} \mathcal{D} \simeq \mathcal{C} \otimes_{\mathrm{Perf}(T)} \mathcal{D}_U.$$

Proof. – We have equivalences

$$\mathcal{C} \otimes_{\mathrm{Perf}(T)} \mathcal{D} \simeq \mathcal{C}_U \otimes_{\mathrm{Perf}(T)} \mathcal{D} \simeq \mathcal{C} \otimes_{\mathrm{Perf}(T)} \mathrm{Perf}(U) \otimes_{\mathrm{Perf}(T)} \mathcal{D} \simeq \mathcal{C} \otimes_{\mathrm{Perf}(T)} \mathcal{D}_U. \qquad \square$$

Now we can answer the question posed above about HPD under linear projection. Note that the surjection $\tilde{V} \to V$ induces an embedding of bundles $V^\vee \to \tilde{V}^\vee$, so that $\mathbf{P}(V^\vee) \subset \mathbf{P}(\tilde{V}^\vee)$.

PROPOSITION A.10. – *Let $\mathcal{A}$ be a Lefschetz category over $\mathbf{P}(\tilde{V})$ with center $\mathcal{A}_0$. Assume $\tilde{V} \to V$ is a surjection of vector bundles with kernel K such that $\mathcal{A}$ is supported over $\mathbf{P}(\tilde{V}) \setminus \mathbf{P}(K)$. Then $\mathcal{A}$ has the structure of a Lefschetz category over $\mathbf{P}(V)$ (with the same center $\mathcal{A}_0$), and there is a $\mathbf{P}(V^\vee)$-linear equivalence*

$$(\mathcal{A}/\mathbf{P}(V))^\natural \simeq (\mathcal{A}/\mathbf{P}(\tilde{V}))^\natural \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee)).$$

REMARK A.11. – The proposition can be generalized to the case where $\mathcal{A}$ is not assumed to be supported over $\mathbf{P}(\tilde{V}) \setminus \mathbf{P}(K)$, by working with a suitable "blowup" of $\mathcal{A}$. In the situation where $\mathcal{A}$ is geometric, this is the main result of [3]; for general Lefschetz categories, see [25, Proposition B.1]. For convenience, we supply the proof in the simpler case needed in the paper.

Proof. – Let $U = \mathbf{P}(\tilde{V}) \setminus \mathbf{P}(K)$. Then by the support assumption, $\mathcal{A}$ has a U-linear structure such that the $\mathbf{P}(\tilde{V})$-linear structure is induced by pullback along $U \to \mathbf{P}(\tilde{V})$. Via the morphism $U \to \mathbf{P}(V)$ given by linear projection, $\mathcal{A}$ also carries a $\mathbf{P}(V)$-linear structure. Let H and $\tilde{H}$ denote the relative hyperplane classes on $\mathbf{P}(V)$ and $\mathbf{P}(\tilde{V})$. Note that $\mathcal{O}(H)$ and $\mathcal{O}(\tilde{H})$ both pull back to the same object of $\mathrm{Perf}(U)$, and hence their actions on $\mathcal{A}$ coincide. From this, it follows that the given Lefschetz center $\mathcal{A}_0 \subset \mathcal{A}$ is also a Lefschetz center with respect to the $\mathbf{P}(V)$-linear structure with the same Lefschetz components.

Consider the induced embedding $V^\vee \hookrightarrow \tilde{V}^\vee$. There is a canonical isomorphism

$$U \times_{\mathbf{P}(\tilde{V})} \mathbf{H}(\mathbf{P}(\tilde{V})) \times_{\mathbf{P}(\tilde{V}^\vee)} \mathbf{P}(V^\vee) \cong U \times_{\mathbf{P}(V)} \mathbf{H}(\mathbf{P}(V)). \tag{A.1}$$

Using this, we deduce

$$\begin{aligned}
&\mathbf{H}(\mathcal{A}/\mathbf{P}(\tilde{V})) \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee)) \\
&\qquad = \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}))} \mathrm{Perf}(\mathbf{H}(\mathbf{P}(\tilde{V}))) \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee)) \\
&\qquad \simeq \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}))} \mathrm{Perf}(U) \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}))} \mathrm{Perf}(\mathbf{H}(\mathbf{P}(\tilde{V}))) \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee)) \\
&\qquad \simeq \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}))} \mathrm{Perf}\left(U \times_{\mathbf{P}(\tilde{V})} \mathbf{H}(\mathbf{P}(\tilde{V})) \times_{\mathbf{P}(\tilde{V}^\vee)} \mathbf{P}(V^\vee)\right) \\
&\qquad \simeq \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}))} \mathrm{Perf}\left(U \times_{\mathbf{P}(V)} \mathbf{H}(\mathbf{P}(V))\right) \\
&\qquad \simeq \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}))} \mathrm{Perf}(U) \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathrm{Perf}(\mathbf{H}(\mathbf{P}(V))) \\
&\qquad \simeq \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathrm{Perf}(\mathbf{H}(\mathbf{P}(V))) \\
&\qquad = \mathbf{H}(\mathcal{A}/\mathbf{P}(V)).
\end{aligned}$$

Indeed, the second line holds by definition of $\mathbf{H}(\mathcal{A}/\mathbf{P}(\tilde{V}))$, the third and the seventh follow from the fact that $\mathcal{A}$ is supported over U (see Lemma A.9), the fourth and the sixth hold by [2, Theorem 1.2], the fifth holds by (A.1), and the last holds by definition. Using the semiorthogonal decomposition (2.11) defining the HPD category, it is easy to check that this equivalence induces an equivalence between the subcategories

$$(\mathcal{A}/\mathbf{P}(\tilde{V}))^\natural \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee)) \subset \mathbf{H}(\mathcal{A}/\mathbf{P}(\tilde{V})) \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee))$$

and

$$(\mathcal{A}/\mathbf{P}(V))^\natural \subset \mathbf{H}(\mathcal{A}/\mathbf{P}(V)).$$

This completes the proof. □

REMARK A.12. – In the situation of Proposition A.10, note that we have $K = (V^\vee)^\perp$ and $\mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}))} \mathrm{Perf}(\mathbf{P}(K)) = 0$ by the support assumption for $\mathcal{A}$. Assume that $\mathcal{A}$ is right strong and moderate as a Lefschetz category over $\mathbf{P}(V)$ (and hence also over $\mathbf{P}(\tilde{V})$). Then [32, Theorem 8.7] implies there is a semiorthogonal decomposition

$$(\mathcal{A}/\mathbf{P}(\tilde{V}))^\natural \otimes_{\mathrm{Perf}(\mathbf{P}(\tilde{V}^\vee))} \mathrm{Perf}(\mathbf{P}(V^\vee)) = \left\langle \mathcal{A}^\natural_{1-n}((1+r-n)H'), \dots, \mathcal{A}^\natural_{-r} \right\rangle,$$

where $n = \mathrm{length}(\mathcal{A}^\natural)$ and $r = \mathrm{rank}(K)$. This provides the left side with a Lefschetz structure of length $n-r$ and center $\mathcal{A}^\natural_{-r}$, with respect to which the equivalence of Proposition A.10 is a

Lefschetz equivalence. We also note that $\mathcal{A}^{\natural}_{-r} = \mathcal{A}^{\natural}_0$; indeed, by the left HPD version of [32, Theorem 8.7(1)] we have

$$\operatorname{length}(\mathcal{A}) = \operatorname{rank}(\tilde{V}) - \#\{\, i \leq 0 \mid \mathcal{A}^{\natural}_i = \mathcal{A}^{\natural}_0 \,\}.$$

By moderateness of $\mathcal{A}$ over $\mathbf{P}(V)$ we also have

$$\operatorname{length}(\mathcal{A}) < \operatorname{rank}(V) = \operatorname{rank}(\tilde{V}) - r.$$

Hence $\#\{\, i \leq 0 \mid \mathcal{A}^{\natural}_i = \mathcal{A}^{\natural}_0 \,\} > r$.

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