

Attributed Graph Alignment

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Abstract—Motivated by various data science applications including de-anonymizing user identities in social networks, we consider the graph alignment problem, where the goal is to identify the vertex/user correspondence between two correlated graphs. Existing work mostly recovers the correspondence by exploiting the user-user connections. However, in many real-world applications, additional information about the users, such as user profiles, might be publicly available. In this paper, we introduce the attributed graph alignment problem, where additional user information, referred to as attributes, is incorporated to assist graph alignment. We establish sufficient and necessary conditions for recovering vertex correspondence exactly, where the conditions match for a wide range of practical regimes. Our results recover existing tight information-theoretic limits for models where only the user-user connections are available, and further span the full spectrum between these models and models where only attribute information is available.

I. INTRODUCTION

The graph alignment problem, also known as the graph matching problem or the noisy graph isomorphism problem, has received growing attention in recent years, brought into prominence by applications in a wide range of areas [1, 2, 3]. For instance, in social network de-anonymization [4, 5], one is given two graphs, each of which represents the user relationship in a social network (e.g., Twitter, Facebook, Flickr, etc). One graph is anonymized and the other graph has user identities as public information. Then the graph alignment problem, whose goal is to find the best correspondence of two graphs with respect to a certain criterion, can be used to de-anonymize users in the anonymous graph by finding the correspondence between them and the users with public identities in the other graph.

The graph alignment problem has been studied under various random graph models, among which the most popular one is the *Erdős-Rényi graph pair* model (see, e.g., [6, 7]). In particular, two Erdős-Rényi graphs on the same vertex set, G_1 and G_2 , are generated in a way such that their edges are correlated. Then G_1 and an anonymous version of G_2 , denoted as G'_2 , are made public, where G'_2 is modeled as a vertex-permuted G_2 with an unknown permutation. Under this model, typically the goal is to achieve the so-called *exact alignment*, i.e., recovering the unknown permutation and thus revealing the correspondence for all vertices exactly.

A fundamental question in the graph alignment problem is: *when is exact alignment possible?* More specifically, *what are the conditions on graph statistics for achieving exact alignment when given unbounded computational resources?* Such conditions, usually referred to as *information-theoretic*

limits, have been established for the Erdős-Rényi graph pair in a line of work [6, 7, 8]. In the most recent study [8], Cullina and Kiyavash established matching sufficient and necessary conditions of exact alignment for a large range of parameters.

In many real-world applications, additional information about the anonymized vertices might be available. For example, Facebook has user profiles on their website about each user's age, birthplace, hobbies, etc. Such associated information is referred to as (vertex) attributes [9], which, unlike user identities, are often publicly available. Then a natural question to ask is: *Can the attribute information help recover the vertex correspondence?* If so, *can we quantify the amount of benefit brought by the attribute information?* The value of attribute information has been demonstrated in the work of aligning Netflix and IMDb users by Narayanan and Shmatikov [10]. They successfully recovered some of the user identities in the anonymized Netflix dataset based only on users' ratings of movies, without any information on the relationship among users. In this paper, we incorporate attribute information to generalize the graph alignment problem. We call this problem the *attributed graph alignment* problem.

To study attributed graph alignment, we extend the current Erdős-Rényi graph pair by adding a set of vertices publicly labeled by attributes. We refer to this set of vertices as *attribute vertices* and assume they are aligned between the graph pair. For distinction, we refer to the original set of vertices in the Erdős-Rényi graph pair as *user vertices*. Edges between user vertices and attribute vertices represent their relationship, and there are no edges between attribute vertices. Similar to the Erdős-Rényi graph pair, user-attributed edges in G_1 and G_2 are also correlated. Then a random permutation is applied on the *user vertices* of G_2 to create the anonymized graph G'_2 . This new model is referred to as *attributed Erdős-Rényi graph pair*. The goal of attributed graph alignment is to recover the unknown permutation from G_1 and G'_2 .

In this paper, we focus on characterizing the information-theoretic limits for graph alignment under the attributed Erdős-Rényi graph pair. We establish sufficient and necessary conditions for achieving exact alignment, where the conditions match for regimes that are typical and interesting in practice. These achievability and converse results allow us to better understand how the attribute information can be integrated with the structural information of the user relationship network, and then to quantify the benefit brought by the attribute information. Our results span the full spectrum between the traditional Erdős-Rényi pair model where only the user rela-

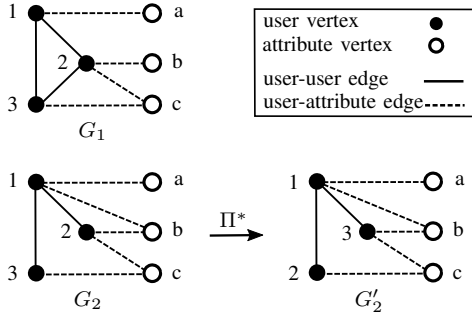


Figure 1: Example of attributed Erdős-Rényi graph pair: Graph G_1 and G_2 are generated on the same set of vertices. Anonymized graph G'_2 is obtained through applying $\Pi^* = (1)(2,3)$ only on $\mathcal{V}_a$ of G_2 (permutation Π^* is written in cycle notation).

relationship network is available and models where only attribute information is available [11, 12], unifying existing results in each of these settings.

We comment that the proposed attributed alignment problem can also be viewed as a graph alignment problem with part of vertices correctly pre-aligned, known as the *seeded graph alignment* problem. Efficient algorithms and achievable regions for seeded graph alignment have been studied in [5, 13, 14, 15]. However, the model assumptions in these existing work are typically not directly comparable with our model assumptions.

II. MODEL

In this section, we describe the attributed Erdős-Rényi graph pair model. We propose and formally define the attributed graph alignment problem under this model. An illustration of the model is given in Figure 1.

User vertices and attribute vertices. We first generate two graphs, G_1 and G_2 , on the same vertex set $\mathcal{V}$. The vertex set $\mathcal{V}$ consists of two disjoint sets of vertices, the *user vertex set* $\mathcal{V}_u$ and the *attribute vertex set* $\mathcal{V}_a$, i.e., $\mathcal{V} = \mathcal{V}_u \cup \mathcal{V}_a$. Assume that the user vertex set $\mathcal{V}_u$ consists of n vertices, labeled as $[n] \triangleq \{1, 2, 3, \dots, n\}$. Assume that the attribute vertex set $\mathcal{V}_a$ consists of m vertices, and m scales as a function of n .

Correlated edges. To describe the probabilistic model for edges in G_1 and G_2 , we first consider the set of user-user vertex pairs $\mathcal{E}_u \triangleq \mathcal{V}_u \times \mathcal{V}_u$ and the set of user-attribute vertex pairs $\mathcal{E}_a \triangleq \mathcal{V}_u \times \mathcal{V}_a$. Then for each vertex pair $e \in \mathcal{E}_u \cup \mathcal{E}_a$, we write $G_1(e) = 1$ (resp. $G_2(e) = 1$) if there is an edge connecting the two vertices in the pair in G_1 (resp. G_2), and write $G_1(e) = 0$ (resp. $G_2(e) = 0$) otherwise. Since we often consider the same vertex pair in both G_1 and G_2 , we write $(G_1, G_2)(e)$ as a shortened form of $(G_1(e), G_2(e))$.

The edges of G_1 and G_2 are then correlatedly generated in the following way. For each user-user vertex pair $e \in \mathcal{E}_u$, $(G_1, G_2)(e)$ follows the joint distribution specified by

$$(G_1, G_2)(e) = \begin{cases} (1, 1) & \text{w.p. } p_{11}, \\ (1, 0) & \text{w.p. } p_{10}, \\ (0, 1) & \text{w.p. } p_{01}, \\ (0, 0) & \text{w.p. } p_{00}, \end{cases} \quad (1)$$

where $p_{11}, p_{10}, p_{01}, p_{00}$ are probabilities that sum up to 1. For

each user-attribute vertex pair $e \in \mathcal{E}_a$, $(G_1, G_2)(e)$ follows the joint probability distribution specified by

$$(G_1, G_2)(e) = \begin{cases} (1, 1) & \text{w.p. } q_{11}, \\ (1, 0) & \text{w.p. } q_{10}, \\ (0, 1) & \text{w.p. } q_{01}, \\ (0, 0) & \text{w.p. } q_{00}, \end{cases} \quad (2)$$

where $q_{11}, q_{10}, q_{01}, q_{00}$ are probabilities that sum up to 1. The correlation between $G_1(e)$ and $G_2(e)$ is measured by the correlation coefficient defined as

$$\rho(e) \triangleq \frac{\text{Cov}(G_1(e), G_2(e))}{\sqrt{\text{Var}[G_1(e)]} \sqrt{\text{Var}[G_2(e)]}},$$

where $\text{Cov}(G_1(e), G_2(e))$ is the covariance between $G_1(e)$ and $G_2(e)$ and $\text{Var}[G_1(e)]$ and $\text{Var}[G_2(e)]$ are the variances. We assume that $G_1(e)$ and $G_2(e)$ are positively correlated, i.e., $\rho(e) > 0$ for every vertex pair e . Across different vertex pair e 's, the $(G_1, G_2)(e)$'s are independent. Finally, recall that there are no edges between attribute vertices in our model.

For compactness of notation, we represent the joint distributions in (1) and (2) in the following matrix form:

$$\mathbf{p} = \begin{pmatrix} p_{11} & p_{10} \\ p_{01} & p_{00} \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} q_{11} & q_{10} \\ q_{01} & q_{00} \end{pmatrix}.$$

We refer to the graph pair (G_1, G_2) as an attributed Erdős-Rényi pair $\mathcal{G}(n, \mathbf{p}, m, \mathbf{q})$. Note that this model is equivalent to the subsampling model in the literature [6].

Anonymization and exact alignment. In the attributed graph alignment problem, we are given G_1 and an anonymized version of G_2 , denoted as G'_2 . The anonymized graph G'_2 is generated by applying a random permutation Π^* on the user vertex set of G_2 , where the permutation Π^* is unknown. More explicitly, each user vertex i in G_2 is re-labeled as $\Pi^*(i)$ in G'_2 . The permutation Π^* is chosen uniformly at random from $\mathcal{S}_n$, where $\mathcal{S}_n$ is the set of all permutations on $[n]$. Since G_1 and G'_2 are observable, we refer to (G_1, G'_2) as the *observable pair* generated from the attributed Erdős-Rényi pair $\mathcal{G}(n, \mathbf{p}, m, \mathbf{q})$.

Then the graph alignment problem, i.e., the problem of recovering the identities/original labels of user vertices in the anonymized graph G'_2 , can be formulated as a problem of estimating the underlying permutation Π^* . The goal of graph alignment is to design an estimator $\hat{\Pi}(G_1, G'_2)$ as a function of G_1 and G'_2 to best estimate Π^* . We say $\hat{\Pi}(G_1, G'_2)$ achieves *exact alignment* if $\hat{\Pi}(G_1, G'_2) = \Pi^*$. The probability of error for exact alignment is defined as $P(\hat{\Pi}(G_1, G'_2) \neq \Pi^*)$. We say exact alignment is achievable *with high probability (w.h.p.)* if there exists $\hat{\Pi}$ such that $\lim_{n \rightarrow \infty} P(\hat{\Pi}(G_1, G'_2) \neq \Pi^*) = 0$.

Reminder of the Landau notation.

Notation	Definition
$f(n) = \omega(g(n))$	$\lim_{n \rightarrow \infty} \frac{ f(n) }{g(n)} = \infty$
$f(n) = o(g(n))$	$\lim_{n \rightarrow \infty} \frac{ f(n) }{g(n)} = 0$
$f(n) = O(g(n))$	$\limsup_{n \rightarrow \infty} \frac{ f(n) }{g(n)} < \infty$
$f(n) = \Omega(g(n))$	$\liminf_{n \rightarrow \infty} \frac{ f(n) }{g(n)} > 0$
$f(n) = \Theta(g(n))$	$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

III. MAIN RESULT

Now we are ready to state our results. To better illustrate the benefit of attribute information in graph alignment, we present in Theorem 1 a simplified version of our achievability result by adding mild assumptions on user-user edges motivated by practical applications. This simplified result also makes it easier to compare the achievability result to the converse result in Theorem 2, which will be illustrated in Figure 2. Note that these additional assumptions are not needed for technical proofs. The more general achievability results without these assumptions are presented in Lemmas 2 and 3 in Section IV.

In a typical social network, the degree of a vertex is much smaller than the total number of users. Based on this observation, we assume that the marginal probabilities of an edge in both G_1 and G_2 are not going to 1, i.e.,

$$1 - (p_{11} + p_{10}) = \Theta(1), \quad 1 - (p_{11} + p_{01}) = \Theta(1). \quad (3)$$

Moreover, two social networks on the same set of users are typically highly correlated. Based on this, we assume that the correlation coefficient of user-user edges, denoted as ρ_u , is not vanishing, i.e.,

$$\rho_u = \Theta(1). \quad (4)$$

Theorem 1 (Achievability). *Consider the attributed Erdős-Rényi pair $\mathcal{G}(n, \mathbf{p}; m, \mathbf{q})$ under conditions (3) and (4). If*

$$np_{11} + m\psi_a - \log n \rightarrow \infty, \quad (5)$$

where $\psi_a = (\sqrt{q_{11}q_{00}} - \sqrt{q_{10}q_{01}})^2$, then there exists an algorithm that achieves exact alignment w.h.p.¹

Theorem 2 (Converse). *Consider the attributed Erdős-Rényi pair $\mathcal{G}(n, \mathbf{p}, m, \mathbf{q})$. If*

$$np_{11} + mq_{11} - \log n \rightarrow -\infty, \quad (6)$$

then no algorithm guarantees exact alignment w.h.p.

Now to better compare the achievability and the converse, we further assume

$$1 - (q_{11} + q_{10}) = \Theta(1), \quad 1 - (q_{11} + q_{01}) = \Theta(1), \quad (7)$$

$$\rho_a = \Theta(1),$$

where ρ_a is the correlation coefficient of user-attribute edges.

Corollary 1. *Consider the attributed Erdős-Rényi pair $\mathcal{G}(n, \mathbf{p}; m, \mathbf{q})$ under conditions (3), (4), and (7). If $m = \Omega((\log n)^3)$ and*

$$np_{11} + mq_{11} - \log n \rightarrow \infty, \quad (8)$$

then there exists an algorithm that achieves exact alignment w.h.p. If $m = o((\log n)^3)$ and

$$np_{11} + mq_{11} - O(mq_{11}^{3/2}) - \log n \rightarrow \infty, \quad (9)$$

then there exists an algorithm that achieves exact alignment w.h.p.

¹Here and throughout the paper, $\log(\cdot) = \log_e(\cdot)$.

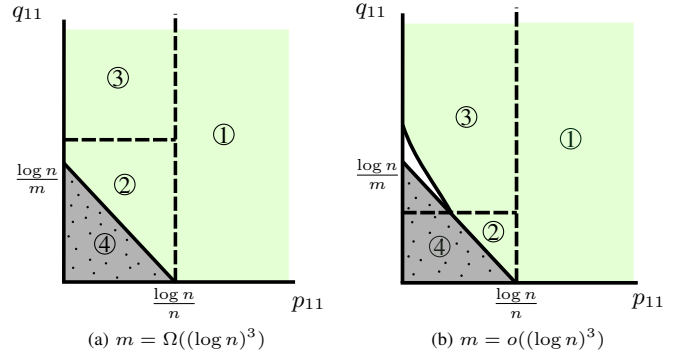


Figure 2: Feasible region (①, ② and ③) and infeasible region (④). Those regions are specified by ①: $p_{11} \geq \frac{\log n + \omega(1)}{n}$; ②: $p_{11} + \frac{m}{n}q_{11} \geq \frac{\log n + \omega(1)}{n}$ and $q_{11} = O(m^{-\frac{2}{3}})$; ③: $p_{11} + \frac{m}{n}\psi_a \geq \frac{\log n + \omega(1)}{n}$ and $q_{11} = \omega(m^{-\frac{2}{3}})$; ④: $p_{11} + \frac{m}{n}q_{11} \leq \frac{\log n - \omega(1)}{n}$. In (b), the gap between ③ and ④ is $\Theta(q_{11}^{3/2})$ that comes from the difference between ψ_a and q_{11} .

Figure 2 illustrates the achievability conditions (8) and (9). Figure 2a shows the tightness of condition (8) when $m = \Omega((\log n)^3)$. Figure 2b demonstrates the difference between the achievability condition (9) and the converse condition (6) when $m = o((\log n)^3)$.

Note that in the traditional Erdős-Rényi pair without attributes, the tight achievability condition for exact alignment is $np_{11} - \log n \rightarrow \infty$ [8, 13]. Now in our attributed Erdős-Rényi pair, the additional attribute information allows us to relax the achievability condition to (8) or (9). We illustrate how this expands the achievability region in Figure 2.

When setting $q_{00} = 1$, i.e., specializing the attributed Erdős-Rényi pair to the traditional Erdős-Rényi pair, our achievability result in Theorem 1 (and its more general version in Lemmas 2 and 3) and converse result in Theorem 2 recover the state-of-the-art information-theoretic limits in [6, 7, 8]. When setting $p_{00} = 1$, i.e., removing the user relationship network, our results improve the information-theoretic limit in [16] for the graph alignment problem on bipartite random graphs.

IV. GENERAL ACHIEVABILITY

In this section, we present the general achievability results. We obtain exact alignment by applying the *maximum a posteriori probability* (MAP) estimator, which minimizes the probability of error. Lemma 1 states that the MAP estimator simplifies to a minimum weighted distance estimator. Due to space limitations, we present the achievability in Lemmas 2 and 3 without proofs (see [17] for proofs). Key proof techniques include tools from enumerative combinatorics, such as generating functions, which are inspired by [7, 8].

Lemma 1 (MAP estimator). *Let (G_1, G'_2) be an observable pair generated from the attributed Erdős-Rényi pair $\mathcal{G}(n, \mathbf{p}; m, \mathbf{q})$. The MAP estimator of the permutation Π^* based on (G_1, G'_2) simplifies to*

$$\hat{\pi}_{\text{MAP}}(G_1, G'_2) = \underset{\pi \in \mathcal{S}_n}{\operatorname{argmin}} \{w_1 \Delta^u(G_1, \pi^{-1}(G'_2)) + w_2 \Delta^a(G_1, \pi^{-1}(G'_2))\},$$

where $w_1 = \log \left(\frac{p_{11}p_{00}}{p_{10}p_{01}} \right)$, $w_2 = \log \left(\frac{q_{11}q_{00}}{q_{10}q_{01}} \right)$, and

$$\Delta^u(G_1, \pi^{-1}(G'_2)) = \sum_{(i,j) \in \mathcal{E}_u} \mathbb{1}\{G_1((i,j)) \neq G'_2((\pi(i), \pi(j)))\},$$

$$\Delta^a(G_1, \pi^{-1}(G'_2)) = \sum_{(i,v) \in \mathcal{E}_a} \mathbb{1}\{G_1((i,v)) \neq G'_2((\pi(i), v))\}.$$

Lemma 2 (General achievability). *Consider the attributed Erdős–Rényi pair $\mathcal{G}(n, \mathbf{p}; m, \mathbf{q})$. If*

$$\frac{n\psi_u}{2} + m\psi_a - \log n = \omega(1), \quad (10)$$

where $\psi_u = (\sqrt{p_{11}p_{00}} - \sqrt{p_{10}p_{01}})^2$ and $\psi_a = (\sqrt{q_{11}q_{00}} - \sqrt{q_{10}q_{01}})^2$, then the MAP estimator achieves exact alignment w.h.p.

Lemma 3 (Achievability in sparse region). *Consider the attributed Erdős–Rényi pair $\mathcal{G}(n, \mathbf{p}; m, \mathbf{q})$. If*

$$p_{11} = O\left(\frac{\log n}{n}\right), \quad (11)$$

$$p_{10} + p_{01} = O\left(\frac{1}{\log n}\right), \quad (12)$$

$$\frac{p_{10}p_{01}}{p_{11}p_{00}} = O\left(\frac{1}{(\log n)^3}\right), \quad (13)$$

$$np_{11} + m\psi_a - \log n = \omega(1), \quad (14)$$

then the MAP estimator achieves exact alignment w.h.p.

Proof of Theorem 1. We discuss two regimes $p_{11} = \omega\left(\frac{\log n}{n}\right)$ and $p_{11} = O\left(\frac{\log n}{n}\right)$. First suppose $p_{11} = \omega\left(\frac{\log n}{n}\right)$. When (3) is satisfied, we have $\psi_u = \Theta(p_{11})$ [17, Fact 4]. Then $\frac{n\psi_u}{2} + m\psi_a - \log n = \omega(\log n) + m\psi_a - \log n = \omega(1)$. By Lemma 2, exact alignment is achievable. Now suppose $p_{11} = O\left(\frac{\log n}{n}\right)$. When (4) is satisfied, we have $p_{10} = O(p_{11})$ and $p_{01} = O(p_{11})$ [17, Fact 4]. It follows that (11) (12) (13) in Lemma 3 are all satisfied. Now if $np_{11} + m\psi_a - \log n = \omega(1)$, then exact alignment is achievable by Lemma 3. $\square$

V. PROOF OF CONVERSE

In this section, we give a detailed proof for Theorem 2. Let (G_1, G_2) be an attributed Erdős–Rényi pair $\mathcal{G}(n, \mathbf{p}; m, \mathbf{q})$. In this proof, we will focus on the intersection graph of G_1 and G_2 , denoted as $G_1 \wedge G_2$, which is the graph on the vertex set $\mathcal{V} = \mathcal{V}_u \cup \mathcal{V}_a$ whose edge set is the intersection of the edge sets of G_1 and G_2 . We say a permutation π on the vertex set $\mathcal{V}$ is an automorphism of $G_1 \wedge G_2$ if a vertex pair (i, j) is in the edge set of $G_1 \wedge G_2$ if and only if $(\pi(i), \pi(j))$ is in the edge set of $G_1 \wedge G_2$, i.e., if π is *edge-preserving*. Note that an identity permutation is always an automorphism. Let $\text{Aut}(G_1 \wedge G_2)$ denote the set of automorphisms of $G_1 \wedge G_2$. By Lemma 4 below, exact alignment cannot be achieved w.h.p. if $\text{Aut}(G_1 \wedge G_2)$ contains permutations other than the identity permutation. This allows us to establish conditions for not achieving exact alignment w.h.p. by analyzing automorphisms of $G_1 \wedge G_2$.

Lemma 4. *Let (G_1, G_2) be an attributed Erdős–Rényi pair $\mathcal{G}(n, \mathbf{p}; m, \mathbf{q})$. Given $|\text{Aut}(G_1 \wedge G_2)|$, the probability that MAP estimator succeeds is at most $\frac{1}{|\text{Aut}(G_1 \wedge G_2)|}$.*

In the proof of Theorem 2, we will further focus on the automorphisms given by swapping two user vertices. To this end, we first define the following equivalence relation between a pair of user vertices. We say two user vertices i and j ($i \neq j$) are *indistinguishable* in $G_1 \wedge G_2$, denoted as $i \equiv j$, if $(G_1 \wedge G_2)((i, v)) = (G_1 \wedge G_2)((j, v))$ for all $v \in \mathcal{V}$. It is not hard to see that swapping two indistinguishable vertices is an automorphism of $G_1 \wedge G_2$, and thus $|\text{Aut}(G_1 \wedge G_2) \setminus \{\text{identity permutation}\}| \geq |\{\text{indistinguishable vertex pairs}\}|$. Therefore, in the proof below, we show that the number of such indistinguishable vertex pairs is positive with a large probability, which suffices for proving Theorem 2.

Proof of Theorem 2. Let G_1 and G_2 be an attributed Erdős–Rényi pair $\mathcal{G}(n, \mathbf{p}; m, \mathbf{q})$ and let $G = G_1 \wedge G_2$. Let X denote the number of indistinguishable user vertex pairs in G , i.e.,

$$X = \sum_{i < j: i, j \in \mathcal{V}_u} \mathbb{1}\{i \equiv j\}.$$

We will show that $P(X = 0) \rightarrow 0$ as $n \rightarrow \infty$ if the condition (6) in Theorem 2 is satisfied.

We start by upper-bounding $P(X = 0)$ using Chebyshev's inequality

$$P(X = 0) \leq \frac{\text{Var}(X)}{E[X]^2} = \frac{E[X^2] - E[X]^2}{E[X]^2}. \quad (15)$$

For the first moment term $E[X]$, we have

$$E[X] = \sum_{i < j} P(i \equiv j) = \binom{n}{2} P(i \equiv j). \quad (16)$$

For the second moment term $E[X^2]$, we expand the sum as

$$\begin{aligned} E[X^2] &= E\left[\sum_{i < j} \mathbb{1}\{i \equiv j\} \cdot \sum_{k < l} \mathbb{1}\{k \equiv l\}\right] \\ &= E\left[\sum_{i < j} \mathbb{1}\{i \equiv j\} + \sum_{\substack{i, j, k, l: i < j, k < l \\ i, j, k, l \text{ are distinct}}} \mathbb{1}\{i \equiv j\} \cdot \mathbb{1}\{k \equiv l\} \right. \\ &\quad \left. + \sum_{\substack{i, j, k, l: i < j, k < l \\ \{i, j\} \text{ and } \{k, l\} \text{ share one element}}} \mathbb{1}\{i \equiv j \equiv k \equiv l\}\right] \\ &= \binom{n}{2} P(i \equiv j) + \binom{n}{4} \binom{4}{2} P(i \equiv j \text{ and } k \equiv l) \\ &\quad + 6 \binom{n}{3} P(i \equiv j \equiv k), \end{aligned} \quad (17)$$

where i, j, k, l are distinct in (17). With (16) and (17), the upper bound given by Chebyshev's inequality in (15) can be written as

$$\begin{aligned} \frac{\text{Var}(X)}{E[X]^2} &= \frac{2}{n(n-1)P(i \equiv j)} + \frac{4(n-2)}{n(n-1)} \frac{P(i \equiv j \equiv k)}{P(i \equiv j)^2} \\ &\quad + \frac{(n-2)(n-3)}{n(n-1)} \frac{P(i \equiv j \text{ and } k \equiv l)}{P(i \equiv j)^2} - 1. \end{aligned} \quad (18)$$

To compute $P(i \equiv j)$, we look into the event $\{i \equiv j\}$ which is the intersection of A_1 and A_2 , where $A_1 = \{\forall v \in$

$\mathcal{V}_u \setminus \{i, j\}$, $G((i, v)) = G((j, v))$, and $A_2 = \{\forall u \in \mathcal{V}_a, G((i, u)) = G((j, u))\}$. Recall that in the intersection graph $G = G_1 \wedge G_2$, the edge probability is p_{11} for user-user pairs and q_{11} for user-attribute pairs. Therefore,

$$\begin{aligned} P(A_1) &= \sum_{i=0}^{n-2} \binom{n-2}{i} p_{11}^{2i} (1-p_{11})^{2(n-2-i)} \\ &= (p_{11}^2 + (1-p_{11})^2)^{n-2}, \\ P(A_2) &= \sum_{i=0}^m \binom{m}{i} p_{11}^{2i} (1-p_{11})^{2(m-i)} \\ &= (q_{11}^2 + (1-q_{11})^2)^m. \end{aligned}$$

Since A_1 and A_2 are independent, we have

$$\begin{aligned} P(i \equiv j) &= P(A_1)P(A_2) \\ &= (p_{11}^2 + (1-p_{11})^2)^{n-2} (q_{11}^2 + (1-q_{11})^2)^m \\ &= (1-2p_{11}+2p_{11}^2)^{n-2} (1-2q_{11}+2q_{11}^2)^m. \end{aligned} \quad (19)$$

Similarly, to compute $P(i \equiv j \equiv k)$, we look into the event $\{i \equiv j \equiv k\}$ which is the intersection of events B_0 , B_1 and B_2 , where $B_0 = \{G((i, j)) = G((j, k)) = G((i, k))\}$, $B_1 = \{\forall v \in \mathcal{V}_u \setminus \{i, j, k\}, G((i, v)) = G((j, v)) = G((k, v))\}$, and $B_2 = \{\forall u \in \mathcal{V}_a, G((i, u)) = G((j, u)) = G((k, u))\}$. Then, the probabilities of those three events are

$$\begin{aligned} P(B_0) &= p_{11}^3 + (1-p_{11})^3, \\ P(B_1) &= (p_{11}^3 + (1-p_{11})^3)^{n-3}, \\ P(B_2) &= (q_{11}^3 + (1-q_{11})^3)^m. \end{aligned}$$

Since the events B_0 , B_1 and B_2 are independent, we have

$$\begin{aligned} P(i \equiv j \equiv k) &= P(B_0)P(B_1)P(B_2) \\ &= (1-3p_{11}+3p_{11}^2)^{n-2} (1-3q_{11}+3q_{11}^2)^m. \end{aligned}$$

To compute $P(i \equiv j \text{ and } k \equiv l)$, we look into the event $\{i \equiv j \text{ and } k \equiv l\}$ which is the intersection of C_0 , C_1 , C'_1 , C_2 and C'_2 , where $C_0 = \{G(i, k) = G(j, k) = G(i, l) = G(j, l)\}$, $C_1 = \{\forall v \in \mathcal{V}_u \setminus \{i, j, k, l\}, G(i, v) = G(j, v)\}$, $C'_1 = \{\forall v \in \mathcal{V}_u \setminus \{i, j, k, l\}, G(k, v) = G(l, v)\}$, $C_2 = \{\forall u \in \mathcal{V}_a, G(i, u) = G(j, u)\}$ and $C'_2 = \{\forall u \in \mathcal{V}_a, G(k, u) = G(l, u)\}$. The probabilities of those events are

$$\begin{aligned} P(C_0) &= p_{11}^6 + p_{11}^4(1-p_{11})^2 + p_{11}^2(1-p_{11})^4 + (1-p_{11})^6, \\ P(C_1) &= P(C'_1) = (p_{11}^2 + (1-p_{11})^2)^{n-4}, \\ P(C_2) &= P(C'_2) = (q_{11}^2 + (1-q_{11})^2)^m. \end{aligned}$$

Since C_0 , C_1 , C'_1 , C_2 and C'_2 are independent, we have

$$\begin{aligned} P(i \equiv j \text{ and } k \equiv l) &= P(C_0)P(C_1)P(C'_1)P(C_2)P(C'_2) \\ &= P(C_0)(p_{11}^2 + (1-p_{11})^2)^{2n-8} (q_{11}^2 + (1-q_{11})^2)^{2m}. \end{aligned}$$

Now we are ready to analyze the terms in (18). For the last two terms, note that $\frac{(n-2)(n-3)}{n(n-1)} \rightarrow 1$ and $\frac{P(i \equiv j \text{ and } k \equiv l)}{P(i \equiv j)^2} \rightarrow 1$ because $p_{11} < \frac{\log n}{n}$ from the condition (6). Therefore, we have $\frac{(n-2)(n-3)}{n(n-1)} \frac{P(i \equiv j \text{ and } k \equiv l)}{P(i \equiv j)^2} - 1 \rightarrow 0$ as $n \rightarrow \infty$. Then we

just need to bound the first two terms in (18). For the first term $\frac{2}{n(n-1)P(i \equiv j)}$, plugging in the expression in (19) gives

$$\begin{aligned} & -\log \frac{2}{n(n-1)P(i \equiv j)} \\ &= 2 \log n + (n-2) \log(1-2p_{11}+2p_{11}^2) \\ & \quad + m \log(1-2q_{11}+2q_{11}^2) + O(1) \\ & \geq 2 \log n - 2np_{11} - 2mq_{11} + O(1) \\ &= \omega(1). \end{aligned} \quad (20)$$

Here (20) follows from the inequality $\log(1-2x+2x^2) \geq -2x$ for any $x \in [0, 1]$, which can be verified by showing that function $f_1(x) = \log(1-2x+2x^2) + 2x$ is monotone increasing in $[0, 1]$ and thus $f_1(x) \geq f_1(0) = 0$. Equation (21) follows from the condition (6) in Theorem 2. Therefore, the first term in (18) $\frac{2}{n(n-1)P(i \equiv j)} \rightarrow 0$ as $n \rightarrow \infty$.

Next, for the second term $\frac{4(n-2)}{n(n-1)} \frac{P(i \equiv j \equiv k)}{P(i \equiv j)^2}$ in (18), we have

$$\begin{aligned} & -\log \left(\frac{4(n-2)}{n(n-1)} \frac{P(i \equiv j \equiv k)}{P(i \equiv j)^2} \right) \\ &= \log n - (n-2) \log \left(\frac{1-3p_{11}+3p_{11}^2}{(1-2p_{11}+2p_{11}^2)^2} \right) \\ & \quad - m \log \left(\frac{1-3q_{11}+3q_{11}^2}{(1-2q_{11}+2q_{11}^2)^2} \right) + O(1) \\ & \geq \log n - np_{11} - mq_{11} + O(1) \\ &= \omega(1). \end{aligned} \quad (22)$$

Here (22) follows from the inequality $\log \left(\frac{1-3x+3x^2}{(1-2x+2x^2)^2} \right) \leq x$ for any $x \in [0, 1]$, which can be verified by showing that the function $f_2(x) = \log \left(\frac{1-3x+3x^2}{(1-2x+2x^2)^2} \right) - x$ is monotone decreasing in $[0, 1]$ and thus $f_2(x) \leq f_2(0) = 0$. Equation (23) follows from the condition (6) in Theorem 2. Hence, the second term in (18) also converges to 0 as $n \rightarrow \infty$, which completes the proof for $P(X=0) \rightarrow 0$ as $n \rightarrow \infty$.

Now we derive an upper bound on the probability of exact alignment under the MAP estimator, which is also an upper bound for any estimator since MAP minimizes the probability of error. Note that by Lemma 4, $P(\pi_{\text{MAP}} = \Pi^* | X = x) \leq \frac{1}{x+1}$, which is at most 1/2 when $x \geq 1$. Therefore,

$$\begin{aligned} P(\pi_{\text{MAP}} = \Pi^*) &= P(\pi_{\text{MAP}} = \Pi^* | X = 0)P(X = 0) \\ & \quad + P(\pi_{\text{MAP}} = \Pi^* | X \geq 1)P(X \geq 1) \\ & \leq P(X = 0) + \frac{1}{2}P(X \geq 1) \\ &= \frac{1}{2} + \frac{1}{2}P(X = 0), \end{aligned}$$

which goes to 1/2 as $n \rightarrow \infty$ and thus is bounded away from 1. This completes the proof that no algorithm can guarantee exact alignment w.h.p. $\square$

ACKNOWLEDGMENT

This work was supported in part by the NSERC Discovery Grant No. RGPIN-2019-05448 and in part by the NSERC Collaborative Research and Development Grant CRDPJ 543676-19.

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