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Bad and Good News for Strassen's Laser Method: Border Rank of Perm_3 and Strict Submultiplicativity

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Abstract

We determine the border ranks of tensors that could potentially advance the known upper bound for the exponent ω of matrix multiplication. The Kronecker square of the small $q = 2$ Coppersmith–Winograd tensor equals the 3×3 permanent, and could potentially be used to show $\omega = 2$. We prove the negative result for complexity theory that its border rank is 16, resolving a longstanding problem. Regarding its $q = 4$ skew cousin in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$, which could potentially be used to prove ≤ 2.11, we show the border rank of its Kronecker square is at most 42, a remarkable sub-multiplicativity result, as the square of its border rank is 64. We also determine moduli spaces VSP for the small Coppersmith–Winograd tensors.

Keywords Tensor rank · Matrix multiplication complexity · Border rank

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1 Introduction

This paper advances both upper and lower bound techniques in the study of the complexity of tensors and applies these advances to tensors that may be used to upper bound the exponent ω of matrix multiplication.

The exponent ω of matrix multiplication is defined as

$$\omega := \inf\{\tau \mid \text{ two } \mathbf{n}\times\mathbf{n} \text{ matrices may be multiplied using } O(\mathbf{n}^{\tau}) \\ \text{arithmetic operations}\}.$$

It is a fundamental constant governing the complexity of the basic operations in linear algebra. It is generally conjectured that $\omega = 2$. It has been known since 1988 that $\leq$ 2.38 [22] which was slightly improved upon 2011–2014 [36, 48, 55], and again in 2021 [3]. All new upper bounds on ω since 1987 have been obtained using Strassen's laser method, which bounds ω via auxiliary tensors, see any of [7, 22, 30] for a discussion. The bounds of 2.38 and below were obtained using the "big Coppersmith–Winograd tensor" as the auxiliary tensor. In [5] it was shown the big Coppersmith–Winograd tensor could not be used to prove $\omega < 2.3$ in the usual laser method.

In this paper we examine six tensors that potentially could be used to prove $\omega < 2.3$ with the laser method. Our approach is via algebraic geometry and representation theory, building on the recent advances in [11, 20]. We solve the longstanding problem (e.g., [7, Problem 9.8], [13, Rem. 15.44]) of determining the border rank of the Kronecker square of the only Coppersmith–Winograd tensor that could potentially prove $\omega = 2$ (the $q = 2$ small Coppersmith–Winograd tensor). The answer is a negative result for the advance of upper bounds, as it is 16, the maximum possible value. On the positive side, we show that a tensor that could potentially be used to prove $\omega < 2.11$ has border rank of its Kronecker square significantly smaller than the square of its border rank. While this result alone does not give a new upper bound on the exponent, it opens a promising new direction for upper bounds. We also develop new lower and upper bound techniques, and present directions for future research.

The tensors we study are the small Coppersmith–Winograd tensor [21] $T_{\mathrm{cw},q}$ for $q = 2$ and its skew cousin [19] $T_{\mathrm{skewcw},q}$ for even $q \leq 10$ (five such). (These tensors are defined for even $q > 10$ but they are only useful for the laser method when $q \leq 10$.) The tensors $T_{\mathrm{cw},2}$ and $T_{\mathrm{skewcw},2}$ potentially could be used to prove $\omega = 2$. Explicitly, the small Coppersmith–Winograd tensors [22] are

$$T_{\mathrm{cw},q} = \sum_{j=1}^{q} a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0$$

and, for $q = 2p$ even, its skew cousins [19] are

$$T_{\mathrm{skewcw},q} = \sum_{\xi=1}^{p} a_0 \otimes b_\xi \otimes c_{\xi+p} - a_0 \otimes b_{\xi+p} \otimes c_\xi - a_\xi \otimes b_0 \otimes c_{\xi+p} \\ + a_{\xi+p} \otimes b_0 \otimes c_\xi + a_\xi \otimes b_{\xi+p} \otimes c_0 - a_{\xi+p} \otimes b_\xi \otimes c_0.$$

The small Coppersmith–Winograd tensors are symmetric tensors and their skew cousins are skew-symmetric tensors. When $q = 2$, after a change of basis $T_{cw,2}$ is just a monomial written as a tensor, $T_{cw,2} = \sum_{\sigma\in\mathfrak{S}_3} a_{\sigma(1)}\otimes b_{\sigma(2)}\otimes c_{\sigma(3)}$ and $T_{skewcw,2} = \sum_{\sigma\in\mathfrak{S}_3} \mathrm{sgn}(\sigma) a_{\sigma(1)}\otimes b_{\sigma(2)}\otimes c_{\sigma(3)}$. Here $\mathfrak{S}_3$ denotes the permutation group on three elements.

We need the following definitions to state our results:

A tensor $T \in A\otimes B\otimes C = \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ has *rank one* if $T = a\otimes b\otimes c$ for some $a \in A$, $b \in B$, $c \in C$, and the *rank* of T, denoted $\boldsymbol{R}(T)$, is the smallest r such that T may be written as a sum of r rank one tensors. The *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, is the smallest r such that T may be written as a limit of rank r tensors. In geometric language, the border rank is smallest r such that $[T] \in \sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$, where $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ denotes the r-th secant variety of the Segre variety of rank one tensors.

For symmetric tensors $T \in S^3A \subset A\otimes A\otimes A$ we may also consider the *Waring* or *symmetric rank* of T, $\boldsymbol{R}_S(T)$, the smallest r such that $T = \sum_{s=1}^{r} v_s\otimes v_s\otimes v_s$ for some $v_s \in A$, and the Waring border rank $\underline{\mathbf{R}}_S(T)$, the smallest r such that T may be written as a limit of Waring rank r symmetric tensors. Note that $\boldsymbol{R}(T) \le \boldsymbol{R}_S(T)$ and $\underline{\mathbf{R}}(T) \le \underline{\mathbf{R}}_S(T)$.

For tensors $T \in A\otimes B\otimes C$ and $T' \in A'\otimes B'\otimes C'$, the *Kronecker product* of T and T' is the tensor $T \boxtimes T' := T\otimes T' \in (A\otimes A')\otimes(B\otimes B')\otimes(C\otimes C')$, regarded as 3-way tensor. Given $T \in A \otimes B \otimes C$, the *Kronecker powers* of T are $T^{\boxtimes N} \in A^{\otimes N} \otimes B^{\otimes N} \otimes C^{\otimes N}$, defined iteratively. Rank and border rank are submultiplicative under Kronecker product: $\mathbf{R}(T \boxtimes T') \le \mathbf{R}(T)\mathbf{R}(T')$, $\underline{\mathbf{R}}(T \boxtimes T') \le \underline{\mathbf{R}}(T)\underline{\mathbf{R}}(T')$, and both inequalities may be strict.

Strassen's *laser method* [21, 50] obtains upper bounds on ω by showing an explicit degeneration of a large Kronecker power of a "simple" tensor admits a further degeneration to a sum of disjoint matrix multiplication tensors, and then applies Schönhage's asymptotic sum inequality [42]. The relevant results for this paper are:

For all k and q, [22]

$$\omega \le \log_q\left(\frac{4}{27}(\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes k}))^{\frac{3}{k}}\right). \tag{1}$$

For all k and even q, [19]

$$\omega \le \log_q\left(\frac{4}{27}(\underline{\mathbf{R}}(T_{skewcw,q}^{\boxtimes k}))^{\frac{3}{k}}\right). \tag{2}$$

Coppersmith–Winograd [22] showed $\underline{\mathbf{R}}(T_{cw,q}) = q + 2$. Applied to (1) with $k = 1$ and $q = 8$ gives $\omega \le 2.41$, which was the previous record before 2.38.

The most natural way to upper bound the exponent of matrix multiplication would be to upper bound the border rank of the matrix multiplication tensor directly. There are very few results in this direction: work of Strassen [52], Bini [6], Pan (see, e.g., [40]), Smirnov (see, e.g., [46]) and Sedoglavic (see, e.g., [43]) are what we are aware of. In order to lower the exponent further with the matrix multiplication tensor the first opportunity to do so would be to show the border rank of the 6×6 matrix multiplication tensor equaled its known lower bound of 69 from [34].

The only still viable proposed path to prove $\omega < 2.3$ using known tensors that we are aware of would be to obtain border rank upper bounds for a Kronecker power of a small ($q \leq 10$) Coppersmith–Winograd tensor (this path has been proposed since 1989) or its skew cousin (more recently proposed in [19]). The results in this paper take a few steps further on these two paths. There is no proposed path that we are aware of to prove $\omega > 2.3$ other than by proving border rank lower bounds for the matrix multiplication tensor (or its symmetrized or skew-symmetrized versions [14]) for all $\mathbf{n}$.

1.1 Main Results

After the barriers of [5], the auxiliary tensor viewed as most promising for upper bounding the exponent, or even proving it is two, is the small Coppersmith–Winograd tensor, or more precisely its Kronecker powers. In [19] bad news in this direction was shown for the square of most of these tensors and even the cube. Left open was the square of $T_{\mathrm{cw},2}$ as it was unaccessible by the technology available at the time (Koszul flattenings and the border substitution method), although it was shown that $15 \leq \underline{\mathbf{R}}(T_{\mathrm{cw},2}^{\boxtimes 2}) \leq 16$. With the advent of border apolarity [11, 20] and the Flag Condition for border apolarity introduced in this paper (Proposition 2.5) that strengthens it, we are able to resolve this last open case. See Remark 5.4 for an explanation why this result was previously unaccessible, even with the techniques of [11, 20]. The result for the exponent is negative:

Theorem 1.1 $\underline{\mathbf{R}}(T_{\mathrm{cw},2}^{\boxtimes 2}) = 16$.

For a detailed discussion of the relation of border rank bounds to the exponent for Kronecker powers of the small Coppersmith–Winograd tensor and its skew cousin, see Section 1 of [19].

In [19] it was observed that $T_{\mathrm{cw},2}^{\boxtimes 2} = \mathrm{perm}_3$, the 3×3 permanent considered as a tensor. Previously Y. Shitov [45] showed that the Waring rank of perm_3 is at least 16, which matches the N. Ilten–Z. Teitler upper bound of [27].

Remark 1.2 P. Comon [9] had conjectured that for symmetric tensors their Waring rank equals their tensor rank and it has similarly been conjectured that their Waring border rank equals their tensor border rank. While Comon's conjecture was shown to be false in general by Shitov [44], we see both versions hold for perm_3.

Theorem 1.1 is proved in Sect. 5.

We determine the border rank of $T_{\mathrm{skewcw},q}$ in the range relevant for the laser method:

Theorem 1.3 $\underline{\mathbf{R}}(T_{\mathrm{skewcw},q}) \leq \frac{3}{2}q + 2$ *and equality holds for* $q \leq 10$.

While this is less promising than the equality $\underline{\mathbf{R}}(T_{\mathrm{cw},q}) = q+2$, in [19] a significant drop in the border rank of $T_{\mathrm{skewcw},2}^{\boxtimes 2} = \det_3$ was shown, namely that it is 17 rather than $25 = \underline{\mathbf{R}}(T_{\mathrm{skewcw},2})^2$. (The upper bound was shown in [19] and the lower bound in [20].) Theorem 1.3 implies $\underline{\mathbf{R}}(T_{\mathrm{skewcw},4})^2 \leq 64$. The following theorem is the largest drop in border rank under a Kronecker square that we are aware of:

Theorem 1.4 $(*)$ $\underline{\mathbf{R}}(T_{\text{skewcw},4}^{\boxtimes 2}) \leq \underline{\mathbf{R}}_S(T_{\text{skewcw},4}^{\boxtimes 2}) \leq 42$.

The Theorem is marked with a $(*)$ because the result is only shown to hold numerically. The expression we give has largest error 4.4×10^{-15}. We could have presented a solution to higher accuracy, but we were unable to find an algebraic expression. The new numerical techniques used to obtain this decomposition are described in Sect. 9. We also give a much simpler Waring border rank 17 expression for $\det_3 = T_{\text{skewcw},2}^{\boxtimes 2}$ than the one in [19], see Sect. 8.

Using Koszul flattenings (see Sect. 4) we show $\underline{\mathbf{R}}(T_{\text{skewcw},4}^{\boxtimes 2}) \geq 39$. For the cube we show $\underline{\mathbf{R}}(T_{\text{skewcw},4}^{\boxtimes 3}) \geq 219$ whereas for its cousin we have $180 \leq \underline{\mathbf{R}}(T_{\text{cw},4}^{\boxtimes 3}) \leq 216$. We also prove, using Koszul flattenings, lower bounds for $\underline{\mathbf{R}}(T_{\text{skewcw},q}^{\boxtimes 2})$ and $\underline{\mathbf{R}}(T_{\text{skewcw},q}^{\boxtimes 3})$ for $q \leq 10$. These results are all part of Theorem 4.1.

Remark 1.5 Starting with the fourth Kronecker power it is possible the border rank of $T_{\text{skewcw},q}^{\boxtimes 4}$ is less than that of $T_{\text{cw},q}^{\boxtimes 4}$, for $q \in \{2, 6, 8\}$. The best possible upper bound on ω obtained from some $T_{\text{skewcw},q}^{\boxtimes 4}$ would be $\omega \leq 2.39001322$ which could potentially be attained with $q = 6$. Starting with the fifth Kronecker power it is potentially possible to beat the current world record for ω with $T_{\text{skewcw},q}$ and for $T_{\text{cw},q}$ it is already possible with the fourth power.

Strassen's asymptotic rank conjecture [51] posits that for all concise tensors $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ (see Definition 1.8) with regular positive dimensional symmetry group (called *tight tensors*), $\lim_{k\to\infty}[\underline{\mathbf{R}}(T^{\boxtimes k})]^{\frac{1}{k}} = m$. As a first step towards this conjecture it is an important problem to determine which tensors T satisfy $\underline{\mathbf{R}}(T^{\boxtimes 2}) < \underline{\mathbf{R}}(T)^2$. We discuss what we understand about this problem in Sect. 3.2.

A variety that parametrizes all possible border rank decompositions of a given tensor T, denoted $\mathrm{VSP}(T)$, is defined in [11]. This variety naturally sits in a product of Grassmannians, see Sect. 3.1 for the definition. We observe that in many examples $\mathrm{VSP}(T)$ often has a large dimension when $\underline{\mathbf{R}}(T^{\boxtimes 2}) < \underline{\mathbf{R}}(T)^2$ (although not always), and in all examples we know of, when $\mathrm{VSP}(T)$ is zero-dimensional one also has $\underline{\mathbf{R}}(T^{\boxtimes 2}) = \underline{\mathbf{R}}(T)^2$. This is reflected in the following results:

Theorem 1.6 *For $q > 2$, $VSP(T_{\text{cw},q})$ is a single point.*

Theorem 1.7 *$VSP(T_{\text{cw},2})$ consists of three points.*

More precise versions of these results and their proofs are given in Sect. 6.

In contrast $\mathrm{VSP}(T_{\text{skewcw},q})$ is positive dimensional, at least for all q relevant for complexity theory ($q \leq 10$). Explicitly, $\mathrm{VSP}(T_{\text{skewcw},2})$ is at least 8-dimensional, see Corollary 3.2, and for $4 \leq q \leq 10$, $\dim \mathrm{VSP}(T_{\text{skewcw},q}) \geq \binom{q/2}{2}$, see Corollary 7.1.

Border apolarity is just in its infancy. In Sect. 2.1 we give a history leading up to it. In Sect. 2.2 we explain results from border apolarity needed in this paper. In Sect. 2.3 we discuss challenges to getting better results with the method and take first steps to overcome them in Sect. 2.4. In particular, Proposition 2.5 was critical to the proof of Theorem 1.1 as it enables one to substantially reduce the border apolarity search space in certain situations (weights occurring with multiplicities).

1.2 Previous Border Rank Bounds on $T_{cw,q}^{\boxtimes k}$ and $T_{skewcw,q}^{\boxtimes k}$

- $\underline{\mathbf{R}}(T_{\text{cw},q}^{\boxtimes 2}) = (q+2)^2$ for $q > 2$ and $15 \leq \underline{\mathbf{R}}(T_{\text{cw},2}^{\boxtimes 2}) \leq 16$. [19]
- $\underline{\mathbf{R}}(T_{\text{cw},q}^{\boxtimes 3}) = (q+2)^3$ for $q > 4$. [19]
- $\underline{\mathbf{R}}(T_{\text{skewcw},2}^{\boxtimes 2}) = 17$. [20]
- $\underline{\mathbf{R}}(T_{\text{skewcw},q}) \geq q+3$. [19]
- For all $q > 4$ and all k, $\underline{\mathbf{R}}(T_{\text{cw},q}^{\boxtimes k}) \geq (q+2)^3(q+1)^{k-3}$ and $\underline{\mathbf{R}}(T_{\text{cw},4}^{\boxtimes k}) \geq 36 \cdot 5^{k-2}$. [19]
- $\underline{\mathbf{R}}(T_{\text{cw},2}^{\boxtimes k}) \geq 15 \cdot 3^{k-2}$. [19]

With the exception of the proof $\underline{\mathbf{R}}(T_{\text{skewcw},2}^{\boxtimes 2}) \geq 17$, which was obtained via border apolarity, these lower bounds were obtained using Koszul flattenings.

Previous to these it was shown that $\underline{\mathbf{R}}(T_{\text{cw},q}^{\boxtimes k}) \geq (q+1)^k + 2^k - 1$ using the border substitution method [8].

1.3 Definitions/Notation

Throughout, A, B, C will denote complex vector spaces of dimension m. We let $\{a_i\}$ denote a basis of A, with either $0 \leq i \leq m-1$ or $1 \leq i \leq m$ and similarly for $\{b_j\}$ and $\{c_k\}$. The dual space to A is denoted A^*. Since our vector spaces have names, we re-order them freely without danger of confusion. The $\mathbb{Z}$-graded algebra of symmetric tensors is denoted $Sym(A) = \oplus_d S^d A$, it is also the algebra of homogeneous polynomials on A^*. For $X \subset A$, $X^\perp := \{\alpha \in A^* \mid \alpha(x) = 0 \forall x \in X\}$ is its annihilator, and $\langle X \rangle \subset A$ denotes the span of X. Projective space is $\mathbb{P}A = (A \backslash \{0\})/\mathbb{C}^*$, and if $x \in A\backslash\{0\}$, we let $[x] \in \mathbb{P}A$ denote the associated point in projective space (the line through x). The general linear group of invertible linear maps $A \to A$ is denoted $GL(A)$ and the special linear group of determinant one linear maps is denoted $SL(A)$. The permutation group on r elements is denoted $\mathfrak{S}_r$.

The Young diagram associated to a partition $(p_1, \dots, p_d)$ is an array of left-aligned boxes with p_j boxes in the j-th row.

The Grassmannian of r-planes through the origin is denoted $G(r, A)$, which we will view in its Plücker embedding $G(r, A) \subset \mathbb{P}\Lambda^r A$. We let $Gr(r, A)$ denote the Grassmannian of *codimension* r planes.

For a set $Z \subset \mathbb{P}A$, $\overline{Z} \subset \mathbb{P}A$ denotes its Zariski closure, $\widehat{Z} \subset A$ denotes the cone over Z union the origin, $I(Z) = I(\widehat{Z}) \subset Sym(A^*)$ denotes the ideal of Z, and $\mathbb{C}[\widehat{Z}] = Sym(A^*)/I(Z)$, denotes the homogeneous coordinate ring of $\widehat{Z}$. Both $I(Z)$, $\mathbb{C}[\widehat{Z}]$ are $\mathbb{Z}$-graded by degree.

We will be dealing with ideals on products of three projective spaces, that is, we will be dealing with polynomials that are homogeneous in three sets of variables, so our ideals with be $\mathbb{Z}^{\oplus 3}$-graded. More precisely, we will study ideals $I \subset Sym(A^*) \otimes Sym(B^*) \otimes Sym(C^*)$, and I_{stu} denotes the component in $S^s A^* \otimes S^t B^* \otimes S^u C^*$.

For $T \in A \otimes B \otimes C$, define the *symmetry group* of T, $G_T := \{g = (g_1, g_2, g_3) \in GL(A) \times GL(B) \times GL(C) \mid g \cdot T = T\}$.

Given $T, T' \in A \otimes B \otimes C$, we say that T *degenerates* to T' if $T' \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$, the closure of the orbit of T, the closures are the same in the Euclidean and Zariski topologies.

Definition 1.8 Given $T \in A\otimes B\otimes C$, we may consider it as a linear map $T_C : C^* \to A\otimes B$, and we let $T(C^*) \subset A\otimes B$ denote its image, and similarly for permuted statements. A tensor T is A-*concise* if the map T_A is injective, i.e., if it requires all basis vectors in A to write down in any basis, and T is *concise* if it is A, B, and C concise. A tensor is 1_A-*generic* if $T(A^*) \subset B\otimes C$ contains an element of maximal rank m.

2 Border Apolarity and the Challenges It Faces

2.1 History

Until very recently, essentially the only way to prove border rank lower bounds for a tensor T was to find a polynomial P in the ideal of $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ such that $P(T) \neq 0$. (See [38] for an exception.) The first nontrivial equations for tensors were found by Strassen in 1983 [49], although the equations essentially date back to E. Toeplitz [54] in the partially symmetric case. No further equations were found until 2013 [33, 35], and these are the state of the art. The equations (and a much broader class of equations) are known to have limits (see, e.g., [24]), essentially one could not prove border rank lower bounds better than $2m - 3$ for tensors in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$. A small way to improve upon this was developed in [8, 32]: this *border substitution method*, which generalizes the classical substitution method to prove rank lower bounds, is only applicable in practice to tensors with positive dimensional symmetry groups: Let $T \in A\otimes B\otimes C$ be A-concise. Let G_T be the symmetry group of T and let $\mathbb{B}_T \subset G_T$ be a Borel subgroup. Let $Gr(t, A^*)$ denote the Grassmannian of *codimension* t-planes in A^*. Note that $\mathbb{B}_T$ acts on $Gr(t, A^*)$ so it makes sense to discuss its Borel fixed elements. Then

$$\underline{\mathbf{R}}(T) \geq \min_{A' \in Gr(t,A^*),\textbf{Borel fixed}} \underline{\mathbf{R}}(T|_{A'\otimes B^*\otimes C^*}) + t. \tag{3}$$

This enables one to prove border rank lower bounds on T by proving border rank lower bounds via known equations on the restrictions of T to all Borel fixed elements of the Grassmannian $Gr(t, A^*)$. In [11] W. Buczyńska and J. Buczyński introduced *Border apolarity*, which generalizes the classical apolarity for rank to border rank, and $\overline{VSP}$ which generalizes the Variety of Sums of Powers (VSP, see, e.g., [41]) for rank decompositions to border rank decompositions.

2.2 Border Apolarity

If one has a border rank decomposition $T = \lim_{\epsilon\to 0} \sum_{j=1}^{r} T_j(\epsilon)$, for each $\epsilon > 0$, one obtains an ideal of polynomials in the coordinate ring of the Segre $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ vanishing on the r points $[T_1(\epsilon)] \sqcup \cdots \sqcup [T_r(\epsilon)]$. These are ideals in three sets of variables (those of A, B, C), and since border rank decompositions only utilize a finite number of terms in the Taylor expansion of the $T_j(\epsilon)$, one may assume that for all $\epsilon > 0$, the r points are in general position by modifying the higher order terms in the series. This has the effect that in each multi-degree $I_{stu,\epsilon} \subset S^s A^* \otimes S^t B^* \otimes S^u C^*$ has codimension r for all $s+t+u > 1$. Thus for each s, t, u there is a limiting $I_{stu} \in Gr(r, S^s A^* \otimes S^t B^* \otimes S^u C^*)$. Moreover, generalizing (3), one may assume that each of the I_{stu} is Borel fixed. By results from [26] these limiting spaces fit together to form an ideal. In particular the ideal annihilates T, which in practice means $I_{110} \subseteq T(C^*)^{\perp}$, $I_{101} \subseteq T(B^*)^{\perp}$, $I_{011} \subseteq T(A^*)^{\perp}$ and $I_{111} \subset T^{\perp}$. Moreover, since ideals are closed under multiplication, the image of the direct sum of the three multiplication maps

$$I_{s-1,t,u} \otimes A^* \oplus I_{s,t-1,u} \otimes B^* \oplus I_{s,t,u-1} \otimes C^* \to S^s A^* \otimes S^t B^* \otimes S^u C^*,$$

must be contained in I_{stu}. In particular the image must have codimension at least r, which translates to rank conditions on the map. Call the map the (stu)-map and the rank condition the (stu)-test.

Write $E_{stu} = I_{stu}^{\perp}$. It will be convenient to phrase the codimension tests dually:

Proposition 2.1 [20, Prop. 3.1] *The* (210)*-test is passed if and only if skew-symmetrization map*

$$A \otimes E_{110} \to \Lambda^2 A \otimes B \tag{4}$$

has kernel of dimension at least r. The kernel is $(A \otimes E_{110}) \cap (S^2 A \otimes B)$.

The (stu)-test is passed if and only if the triple intersection

$$(E_{s,t,u-1} \otimes C) \cap (E_{s,t-1,u} \otimes B) \cap (E_{s-1,t,u} \otimes A) \tag{5}$$

has dimension at least r.

We will make repeated use of the following lemma:

Lemma 2.2 (Fixed ideal Lemma [11]) *If T has symmetry group G_T and there exists an ideal as above, then there exists an ideal as above that is fixed under the action of a Borel subgroup of G_T which we will denote $\mathbb{B}_T$. In particular, if G_T contains a torus and there exists such an ideal, then there exists one fixed under the action of the torus.*

Border apolarity provides both lower bounds and a guide to proving upper bounds. For example, the (111) space for $T_{\text{skewcw},q}$ described in the proof of Theorem 1.3 hints at the formula (57), where the terms linear in t appear in the (111) space.

2.3 Challenges Facing Border Apolarity

In modern algebraic geometry the study of geometric objects (algebraic varieties) is replaced by the study of the ideal of polynomials that vanish on a variety. The study of a set of r points $\{z_1, \ldots, z_r\}$ in affine space $\mathbb{C}^N$ is replaced by the study of its ideal, more precisely the quotient $\mathbb{C}[x_1, \ldots, x_N]/I_{z_1 \sqcup \cdots \sqcup z_r}$ where $\mathbb{C}[x_1, \ldots, x_N]$ is the ring of all polynomials on $\mathbb{C}^N$ and $I_{z_1\sqcup\cdots\sqcup z_r}$ is the ideal. Note that ring $\mathbb{C}[x_1,\ldots,x_N]/I_{z_1\sqcup\cdots\sqcup z_r}$ is a vector space of dimension r, called the coordinate ring of the variety. (In our case we will be concerned with r points on the Segre variety $Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$ but the issues about to be discussed are local and there is no danger working in affine space.) The study becomes one of such rings, and one no longer requires them to correspond to ideals of points, only that the vector space has dimension r and that the ideal is saturated. Such ideals are called *zero dimensional schemes of length* r. If the ideal corresponds to r distinct points one says the scheme is *smooth*. A central challenge of border apolarity as a tool in the study of border rank, is that applied naïvely, it only determines necessary conditions for a not necessarily saturated ideal to be the limit of a sequence of such ideals. One could split the problem of detecting non-border rank ideals into two: first, just get rid of the ideals that are not limits of ideals of zero dimensional schemes, then, given an ideal that is a limit of ideals of zero dimensional schemes of length r, determine if it is a limit of ideals of smooth schemes (*smoothability* conditions). In this paper we address the first problem and the new additional necessary conditions we obtain (Proposition 2.5) are enough to enable us to determine $\underline{\mathbf{R}}(T_{\mathrm{cw},2}^{\boxtimes 2})$ via border apolarity. In Sect. 2.6 we show that ideals that fail to deform to saturated ideals occur already for quite low border rank. The second problem is ongoing work with J. Buczyński and his group in Warsaw.

The second problem is a serious issue: The *cactus rank* [10, 11] of a tensor T is the smallest r such that T lies in the span of a zero dimensional scheme of length r supported on the Segre variety. The cactus border rank of T, $\underline{\boldsymbol{CR}}(T)$ is the smallest r such that T is a limit of tensors of cactus rank r. One has $\underline{\mathbf{R}}(T)\geq\underline{\boldsymbol{CR}}(T)$ and for almost all tensors the inequality is strict. The (stu) tests are tests for cactus border rank. Cactus border rank is not known to be relevant for complexity theory, thus the failure of current border apolarity technology to distinguish between them is a barrier to future progress. Moreover, the cactus variety fills the ambient space of $\mathbb{P}(\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m)$ at latest border rank $6m-4$, see [25, Ex. 6.2 case $k=3$].

2.4 Viability and the Flag Conditions

We begin in the general context of secant varieties with a preliminary observation:

For a projective variety $X\subset\mathbb{P}^N$, define its variety of secant $\mathbb{P}^{r-1}$'s,

$$\sigma_r(X):=\overline{\bigcup_{x_1,\ldots,x_r\in X}\langle x_1,\ldots,x_r\rangle}.$$

Proposition 2.3 *Let $X\subset\mathbb{P}V$ be a projective variety and let $\mathbb{P}E\subset\sigma_r(X)$ be a $\mathbb{P}^{r-1}$. Then there exists a complete flag $E_1\subset E_2\subset\cdots\subset E_r=E$ such that for all $1\leq j\leq r$, $\mathbb{P}E_j\subset\sigma_j(X)$.*

Proof We may write $E=\lim_{t\to0}\langle x_1(t),\ldots,x_r(t)\rangle$ where $x_j(t)\in X$ and the limit is taken in the Grassmannian $G(r,V)$ (in particular, for all $t\neq0$ we may assume $x_1(t),\ldots,x_r(t)$ are linearly independent). Then take $E_j=\lim_{t\to0}\langle x_1(t),\ldots,x_j(t)\rangle$ where the limit is taken in the Grassmannian $G(j,V)$. □

Let $T \in A\otimes B\otimes C$ and let E_{stu} be an r-dimensional space that is $I_{stu}^{\perp}$ for a multi-graded ideal that passes all border apolarity tests up to total degree $s+t+u+1$.

Definition 2.4 A multi-graded ideal, or an E_{stu}, associated to a potential border rank decomposition of T is *viable* if it arises from an actual border rank decomposition.

Viability implies $\mathbb{P}E_{stu} \subset \sigma_r(Seg(v_s(\mathbb{P}A)\times v_t(\mathbb{P}B)\times v_u(\mathbb{P}C)))$. Here $v_s : \mathbb{P}A \to \mathbb{P}(S^sA)$ is the Veronese re-embedding, $v_s([a]) = [a^s]$.

To a $\boldsymbol{c}$-dimensional subspace $E \subset A\otimes B$, one may associate a tensor $T \in A\otimes B\otimes \mathbb{C}^{\boldsymbol{c}}$, well-defined up to isomorphism, such that $T(\mathbb{C}^{\boldsymbol{c}*}) = E$. Much of the lower bound literature exploits this correspondence to reduce questions about tensors to questions about linear subspaces of spaces of matrices. (This idea appears already in [49].) The following proposition exploits this dictionary to obtain new conditions for viability of candidate E_{stu}'s:

Proposition 2.5 (Flag conditions) *If E_{110} is viable, then there exists a $\mathbb{B}_T$-fixed filtration of E_{110}, $F_1 \subset F_2 \subset \cdots \subset F_r = E_{110}$, such that $F_j \subset \sigma_j(Seg(\mathbb{P}A\times\mathbb{P}B))$. Let $T_j \in A\otimes B\otimes\mathbb{C}^j$ be a tensor equivalent to the subspace F_j. Then $\underline{\mathbf{R}}(T_j) \le j$.*

Similarly, if E_{stu} is viable, there are complete flags in E_{stu}, A, B, C such that for all $j < m$, $E_{stu,j} \subset S^sA_j\otimes S^tB_j\otimes S^uC_j$ and for all $j \le r$, $\mathbb{P}E_{stu,j} \subset \sigma_j(Seg(v_s(\mathbb{P}A)\times v_t(\mathbb{P}B)\times v_u(\mathbb{P}C)))$.

Proof Set $\widehat{C} = C\oplus\mathbb{C}^{r-m}$. Then there exists $\widehat{T} \in A\otimes B\otimes\widehat{C}$ such that $\widehat{T}(\widehat{C}^*) = E_{110}$ and $\underline{\mathbf{R}}(\widehat{T}) \le r$. In this case the flag condition [31, Cor. 2.3] implies that since $\widehat{T} \in A\otimes B\otimes\widehat{C} = \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^r$ with $r \ge m$ is concise of minimal border rank r, there exists a complete flag $C_1 \subset C_2 \subset \cdots \subset C_r = \widehat{C}^*$ such that $\widehat{T}(C_k) \subset \sigma_k(Seg(\mathbb{P}A\times\mathbb{P}B))$. Take $F_k = \widehat{T}(C_k)$. The proof that the flag may be taken to be Borel fixed is the same as in the Fixed ideal lemma.

The second assertion follows from the preceding discussion. □

Proposition 2.5 provides additional conditions E_{stu} must satisfy for viability beyond the border apolarity tests. It allows one to utilize the known conditions for minimal border rank in a non-minimal border rank setting.

When T_j is concise, Proposition 2.5 is quite useful as there are many known conditions for concise tensors to be of minimal border rank. In particular it must have symmetry Lie algebra of dimension at least $2j-2$ and if it is $1_{\mathbb{C}^j}$-generic (for any of the factors), it must satisfy the End-closed condition (see [31]).

Remark 2.6 Proposition 2.5 also applies to cactus border rank decompositions, so it is a “non-deformable to saturated” removal condition rather than a smoothability one.

By the classification of tensors of border rank at most three [12, Thm. 1.2(iv)] the possibilities for the first two filtrands of E_{110} are $F_1 = \langle a\otimes b\rangle$, $F_{2a} = \langle a\otimes b, a'\otimes b'\rangle$ or $F_{2b} = \langle a\otimes b, a\otimes b' + a'\otimes b\rangle$ corresponding to either two distinct rank one points or a rank one point and a tangent vector, and there are five possibilities for F_3:

(1) $F_{3aa} = \langle a\otimes b, a'\otimes b', a''\otimes b''\rangle$ (three distinct points)
(2) $F_{3aab} = \langle a\otimes b, a\otimes b' + a'\otimes b, a''\otimes b''\rangle$ (two points plus a tangent vector to one of them)

(3) $F_{3bc} = \langle a\otimes b, a\otimes b' + a'\otimes b, a''\otimes b + a'\otimes b' + a\otimes b''\rangle$ (points of the form $x(0), x'(0), x''(0)$ for a curve $x(t) \subset Seg(\mathbb{P}A \times \mathbb{P}B)$)
(4) $F_{3abb} = \langle a\otimes b, a\otimes b' + a'\otimes b, a\otimes b'' + a''\otimes b\rangle$ (point plus two tangent vectors)
(5) $F_{3bd} = \langle a\otimes b, a\otimes b', a'\otimes b + a''\otimes b' + a\otimes b''\rangle$ (sum of tangent vectors to two colinear points $x' + y'$) or its mirror $F_{3bd} = \langle a\otimes b, a'\otimes b, a\otimes b' + a'\otimes b'' + a''\otimes b\rangle$.

The space E_{110} contains a distinguished subspace $T(C^*)$. Write E'_{110} for a choice of a complement to $T(C^*)$ in E_{110}.

Corollary 2.7 *If E_{110} is viable and $\mathbb{P}T(C^*) \cap \sigma_k(Seg(\mathbb{P}A \times \mathbb{P}B)) =$, then there exists a choice of E'_{110} such that $F_k \subset E'_{110}$.*

Proof Say otherwise, then there exists $M \in F_k \cap T(C^*)$. This contradicts $T(C^*) \cap \sigma_k(Seg(\mathbb{P}A \times \mathbb{P}B)) =$. □

The following Corollary originally appeared in [8, Cor. 4.2]:

Corollary 2.8 *If $\mathbb{P}T(C^*) \cap \sigma_q(Seg(\mathbb{P}A \times \mathbb{P}B)) =$, then $\underline{\mathbf{R}}(T) \geq m + q$.*

Although we have stronger lower bounds, Corollary 2.8 provides the following "for free":

Corollary 2.9 *For all k, $\underline{\mathbf{R}}(T_{\mathrm{cw},2}^{\boxtimes k}) \geq 3^k + 2^k - 1$ and $\underline{\mathbf{R}}(T_{\mathrm{skewcw},2}^{\boxtimes k}) \geq 3^k + 2^k - 1$.*

The first assertion originally appeared in [8].

Proof Let $i_\alpha, j_\beta \in \{1, 2, 3\}$. Then

$$T_{\mathrm{cw},2}^{\boxtimes k}(C^*) = \langle \sum_{\sigma \in \mathbb{Z}_2^k} \sigma \cdot (a_{i_1,\ldots,i_k} \otimes b_{j_1,\ldots,j_k}) \mid i_\alpha \neq j_\alpha \forall 1 \leq \alpha \leq k\rangle$$

and the action of σ is by swapping indices. This transparently is of rank bounded below by 2^k. The case of $T_{\mathrm{skewcw},2}^{\boxtimes k}$ is the same except that the coefficients appear with signs. □

2.5 Free, Pure and Mixed Kernels

Define three types of contribution to the kernel of the (210)-map for a given choice of E'_{110}: the *free* kernel

$$k'_f := \dim[(T(C^*)\otimes A) \cap (S^2A\otimes B)],$$

the *pure* kernel

$$k'_p = \dim[(A\otimes E'_{110}) \cap (S^2A\otimes B)],$$

and the *mixed* kernel

$$k'_m = \dim[(A\otimes E_{110}) \cap (S^2A\otimes B)] - k'_p - k'_f,$$

corresponding to elements of the kernel arising from linear combinations of elements of $A\otimes E'_{110}$ and $A\otimes T(C^*)$. In this language, E'_{110} passes the (210) test if and only if ${k'}_p + {k'}_m \geq r - {k'}_f$. Define corresponding ${k''}_p, {k''}_m$ for the (120)-test.

Conjecture 2.10 *If $r > m$ and E_{110} is such that ${k'}_m, {k''}_m = 0$, then it is not viable.*

Intuitively, if E'_{110} never "sees" the tensor, it should not be viable.

2.6 Limitations of the Total Degree 3 Border Apolarity Tests

Proposition 2.11 *Let $m \geq 9$, but $m \neq 10, 15$. Then for any tensor in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$, there are candidate ideals passing all degree three tests for border rank at most r when $r \geq 2m$.*

More generally, setting $r = m + k^2$, there are candidate ideals in total degree two passing all degree three tests once $m \leq \frac{k^3}{2} - \frac{k^2}{2}$. In particular, for all $\epsilon > 0$, $r \geq m + m^{\frac{1}{3}+\epsilon}$, and m sufficiently large, there are such candidate ideals.

Proof For the first assertion, it suffices to prove the case $r = 2m$ and the tensor T is concise. Set $k = \lfloor\sqrt{m}\rfloor$, $t = k + \lceil\frac{m-k^2}{2}\rceil$, and $t' = k + \lfloor\frac{m-k^2}{2}\rfloor$. Take $E'_{110} = \langle a_1, \ldots, a_k\rangle\otimes\langle b_1, \ldots, b_k\rangle + \langle a_{k+1}, \ldots, a_{t'}\rangle\otimes b_1 + a_1\otimes\langle b_{k+1}, \ldots, b_t\rangle$ and similarly for the other spaces. Then

$$\begin{aligned}&(E_{110}\otimes A) \cap (S^2A\otimes B) \supseteq \\ &S^2\langle a_1, \ldots, a_k\rangle\otimes\langle b_1, \ldots, b_k\rangle \oplus \langle a_{k+1}, \ldots, a_{t'}\rangle \cdot \langle a_1, \ldots, a_k\rangle\otimes b_1 \\ &\oplus S^2\langle a_{k+1}, \ldots, a_{t'}\rangle\otimes b_1 + a_1^{\otimes 2}\otimes\langle b_{k+1}, \ldots, b_t\rangle.\end{aligned}$$

This has dimension $\binom{k+1}{2}k + (t' - k)k + \binom{t'-k+1}{2} + (t - k)$ which is at least $2m$ in the specified range. (The only value greater than 8 the inequality fails for is $m = 10$.) Similarly the (120) test is passed at least as easily. Finally

$$\begin{aligned}(E_{110}\otimes C) \cap (E_{101}\otimes B) \cap (E_{011}\otimes A) \supseteq &\langle a_1, \ldots, a_k\rangle \\ &\otimes\langle b_1, \ldots, b_k\rangle\otimes\langle c_1, \ldots, c_k\rangle \oplus \langle T\rangle\end{aligned}$$

which has dimension $k^3 + 1$ which is at least $2m$ in the range of the proposition. (The only value greater than 8 the inequality fails for is $m = 15$.)

The second assertion follows with the same E'_{110}, taking $r = m + k^2$ and $t, t' = 0$. $\square$

Example 2.12 For $T_{cw,2}^{\boxtimes 3}$ it is easy to get E'_{110} of dimension 21 (so for border rank $48 < 63$) that pass the (210) and (120) tests. Take E'_{110} spanned by rank one basis vectors such that the associated Young diagram is a staircase. Then ${k'}_p = {k''}_p = 1(6) + 2(5) + 3(4) + 4(3) + 5(2) + 6(1) = 56 > 48$.

3 Moduli and Submultiplicativity

3.1 Moduli Spaces VSP

Following [11], define VSP(T) to be the set of ideals as in Sect. 2.2 arising from a border rank $\underline{\mathbf{R}}(T)$ decomposition of T. (In the notation of [11] this is VSP$(T, \underline{\mathbf{R}}(T))$.) Since for zero dimensional schemes of a fixed length there is a uniform bound on degrees of generators of their ideals, this is a finite dimensional variety which naturally embedds in a product of Grassmannians.

A more classical object also of interest is $\mathrm{VSP}_{A\otimes B\otimes C}(T) \subset G(\underline{\mathbf{R}}(T), A\otimes B\otimes C)$, which just records the $\underline{\mathbf{R}}(T)$-planes giving rise to a border rank decomposition, i.e., the annihilator of the (111)-component of the ideals in VSP(T). In particular $\dim(\mathrm{VSP}_{A\otimes B\otimes C}(T)) \leq \dim(\mathrm{VSP}(T))$.

It will be useful to state the following result in a more general context: Let $X \subset \mathbb{P}V$ be a variety not contained in a hyperplane, assume $\sigma_{r-1}(X) \neq \mathbb{P}V$ and write $\dim \sigma_r(X) = r \dim(X) + r - 1 - \delta$. Consider the incidence correspondence

$$S_r(X) = \overline{\{((x_1, \ldots, x_r), y, V) \in X^{\times r} \times \mathbb{P}V \times G(r, V) \mid y \in \langle x_1, \ldots, x_r \rangle \subseteq V\}},$$

and its projection maps

$$\begin{array}{ccc} & S_r(X) & \\ \swarrow & & \searrow \\ G(r, V) & & \sigma_r(X). \end{array}$$

Call the projections π_G, π_σ. We have $\dim S_r(X) = r \dim(X) + r - 1$ so for $y \in \sigma_r(X)_{general}$, $\dim(\pi_\sigma^{-1}(y)) = \delta$.

Define $\mathrm{VSP}_{X,\mathbb{P}V}(y) := \pi_G \pi_\sigma^{-1}(y)$. When $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, $y = T$, and $V = A\otimes B\otimes C$, this is $\mathrm{VSP}_{A\otimes B\otimes C}(T)$.

Proposition 3.1 *For all $y \in \sigma_r(X)$,* $\dim$ *$VSP_{X,\mathbb{P}V}(y) \geq \delta$.*

Proof By [17] $\dim \pi_G(S_r(X)) = r \dim(X)$, so π_G generically has $(r-1)$-dimensional fibers, which correspond to the choice of a point in the r-plane. This implies that $\pi_G|_{\pi_\sigma{}^{-1}(y)}$ is finite to one. Since $\dim \pi_\sigma^{-1}(y) \geq \delta$ we conclude. □

Corollary 3.2 *A border rank five tensor $T \in \mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ has* $\dim$ *$VSP_{A\otimes B\otimes C}(T) \geq 8$.*

Proof $\dim \sigma_5(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)) = 26$ [49]. □

Remark 3.3 In this case, by [53] T also has rank five and thus $\dim VSP_{A\otimes B\otimes C}(T) \geq 8$, where $VSP_{A\otimes B\otimes C}(T)$ is the variety of rank decompositions.

A similar argument shows:

Proposition 3.4 *Let $\mathcal{O}_{s,t,u}$ be a smallest dimensional G_T-orbit in $\mathbb{P}(S^s A\otimes S^t B\otimes S^u C)$. Then for all (s, t, u),* $\dim$ *$VSP(T) \geq$* $\dim \mathcal{O}_{s,t,u}$.

3.2 How to Find Good Tensors for the Laser Method?

The utility of a tensor $T \in A\otimes B\otimes C$ for the laser method is bounded above by the ratio of its *cost*, which is the asymptotic rank $\underset{\sim}{\mathbf{R}}(T) := \lim_{N\to\infty}[\underline{\mathbf{R}}(T^{\boxtimes N})]^{1/N}$, and its *value*, which is its asymptotic subrank $\underset{\sim}{\mathbf{Q}}(T) := \lim_{N\to\infty}[\underline{\mathbf{Q}}(T^{\boxtimes N})]^{1/N}$. See [16] for a discussion, where the ratio of their logs is called the *irreversibility* of T. Here $\underline{\mathbf{Q}}(T)$ is the maximum q such that $M_{\langle 1\rangle}^{\oplus q} \in \overline{GL(A)\times GL(B)\times GL(C)\cdot T}$, where $M_{\langle 1\rangle}^{\oplus q}$ is the so-called unit tensor. In bases, $M_{\langle 1\rangle}^{\oplus q} = \sum_{j=1}^{q} a_j\otimes b_j\otimes c_j$. Unless a tensor is of minimal border rank, we only can estimate the asymptotic rank of a tensor by computing its border rank and the border rank of its small Kronecker powers.

There are several papers regarding the search for tensors that give good upper bounds on ω in the laser method:

Papers on *barriers* may be interpreted as describing where *not* to look for good tensors: [1, 2, 4, 16] discuss limits of the laser method for various types of tensors and various types of implementations.

A program to utilize algebraic geometry and representation theory to find good tensors for the laser method was initiated in [18, 31].

Here we describe a more modest goal: determine criteria that indicate (or even guarantee) that border rank is strictly sub-multiplicative under the Kronecker square.

To our knowledge, the first example of a non-minimal border rank tensor that satisfied $\underline{\mathbf{R}}(T^{\boxtimes 2}) = \underline{\mathbf{R}}(T)^2$ was given in [19]: the small Coppersmith–Winograd tensor $T_{\mathrm{cw},q}$ for $q > 2$ and in this paper we show equality also holds when $q = 2$. This shows that tight tensors need not exhibit strict submultiplicativity. Several examples of strict submultiplicativity were known previous to this paper: the 2×2 matrix multiplication tensor $M_{\langle 2\rangle} \in \mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4$, $\underline{\mathbf{R}}(M_{\langle 2\rangle}) = 7$ [29] while $\underline{\mathbf{R}}(M_{\langle 2\rangle}^{\boxtimes 2}) \leq 46$ [47]. The tensors of [15] have a drop of one, a generic tensor $T \in \mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ satisfies $\underline{\mathbf{R}}(T) = 5$ while $\underline{\mathbf{R}}(T^{\boxtimes 2}) \leq 22$ [19], and $\underline{\mathbf{R}}(T_{\mathrm{skewcw},2}) = 5$ while $\underline{\mathbf{R}}(T_{\mathrm{skewcw},2}^{\boxtimes 2}) = 17$ [19, 20].

3.3 VSP and Strict Submultiplicativity

All the strict submultiplicativity examples have positive dimensional VSP. This is attributable to the degeneracy of $\sigma_4(Seg(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2))$ for the generic tensors in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$, and to the large symmetry groups for the other cases: If a tensor $T \in A\otimes B\otimes C$ has a positive dimensional symmetry group G_T and G_T does not have a one-dimensional submodule in each of $A\otimes B$, $A\otimes C$, $B\otimes C$, $A\otimes B\otimes C$, then $\dim(\mathrm{VSP}(T)) > 0$ because any ideal in the G_T-orbit closure of an ideal of a border rank decomposition for T will give another border rank decomposition.

It would be too much to hope that a concise tensor T not of minimal border rank satisfying $\dim \mathrm{VSP}(T) > 0$ also satisfies $\underline{\mathbf{R}}(T^{\boxtimes 2}) < \underline{\mathbf{R}}(T)^2$. Consider the following example: Let $T = T_1\oplus T_2$ with the T_j in disjoint spaces, where T_1 has non-minimal border rank and $\dim \mathrm{VSP}(T_1) = 0$ and T_2 has minimal border rank with $\dim \mathrm{VSP}(T_2) > 0$. Then there is no reason to believe $T^{\boxtimes 2}$ should have strict submultiplicativity.

It is possible that the converse holds: that strict submultiplicativity under the Kronecker square implies a positive dimensional VSP.

It might be useful, following [15] to split the submultiplicativity question into two questions: first to determine if the usual tensor square is submultiplicative and then if the border rank of the Kronecker square is less than the border rank of the tensor square. Note that in general, assuming non-defectivity, for a projective variety $X \subset \mathbb{P}V$ of dimension N, $\sigma_{R-1}(X)$ has codimension $N+1$ in $\sigma_R(X)$. In our case $R = r^2$ and in the tensor square case $N = 6m - 6$, and in the Kronecker square case $N = 3m^2 - 3$. A priori, for $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of border rank r, $T^{\otimes 2} \in \sigma_{r^2}(Seg(\mathbb{P}^{(m-1)\times 6}))$ and submultiplicativity is a codimension $6m-5$ condition, whereas $T^{\boxtimes 2} \in \sigma_{r^2}(Seg(\mathbb{P}^{(m^2-1)\times 3}))$ and submultiplicativity is a codimension $3m^2-2$ condition. Despite this, the second condition is weaker than the first.

4 Koszul Flattening Lower Bounds

The best general technique available for border rank lower bounds are Koszul flattenings [33, 35].

Fix an integer p. Given a tensor $T = \sum_{ijk} T^{ijk} a_i \otimes b_j \otimes c_k \in A \otimes B \otimes C$, the p-th *Koszul flattening* of T on the space A is the linear map

$$\begin{aligned} T_A^{\wedge p} : \Lambda^p A \otimes B^* &\to \Lambda^{p+1} A \otimes C \\ X \otimes \beta &\mapsto \textstyle\sum_{ijk} T^{ijk} \beta(b_j)(a_i \wedge X) \otimes c_k. \end{aligned}$$

Then [33, Proposition 4.1.1] states

$$\underline{\mathbf{R}}(T) \geq \frac{\operatorname{rank}(T_A^{\wedge p})}{\binom{\dim(A)-1}{p}}. \tag{6}$$

The best lower bounds for any given p are obtained by restricting T to a generic $2p+1$ dimensional subspace of A^* so the denominator becomes $\binom{2p}{p}$.

Theorem 4.1 *The following border rank lower bounds are obtained by applying Koszul flattenings to a restriction of the tensor to a sufficiently generic $\mathbb{C}^{2p+1} \otimes B \otimes C \subset A \otimes B \otimes C$. Values of p that give the bound are in parentheses.*

(1) $\underline{\mathbf{R}}(T_{\text{skewcw},4}^{\boxtimes 2}) \geq 39$ *($p = 2, 3, 4$)*
(2) $\underline{\mathbf{R}}(T_{\text{skewcw},6}^{\boxtimes 2}) \geq 70$ *($p = 2, 3, 4$)*
(3) $\underline{\mathbf{R}}(T_{\text{skewcw},8}^{\boxtimes 2}) \geq 110$ *($p = 4$)*
(4) $\underline{\mathbf{R}}(T_{\text{skewcw},10}^{\boxtimes 2}) \geq 157$ *($p = 4$)*
(5) $\underline{\mathbf{R}}(T_{\text{skewcw},2}^{\boxtimes 3}) \geq 49$ *($p = 4$)*
(6) $\underline{\mathbf{R}}(T_{\text{skewcw},4}^{\boxtimes 3}) \geq 219$ *($p = 3$)*
(7) $\underline{\mathbf{R}}(T_{\text{skewcw},6}^{\boxtimes 3}) \geq 550$ *($p = 3$)*
(8) $\underline{\mathbf{R}}(T_{\text{skewcw},8}^{\boxtimes 3}) \geq 1089$ *($p = 3$)*
(9) $\underline{\mathbf{R}}(T_{\text{skewcw},10}^{\boxtimes 3}) \geq 1886$ *($p = 3$)*.

Better lower bounds for the larger cases are potentially possible, if not easily accessible, using larger values of p.

Compare these with the values for the small Coppersmith–Winograd tensor from [19]:

(1) $\underline{\mathbf{R}}(T_{cw,4}^{\boxtimes 2}) = 36$
(2) $\underline{\mathbf{R}}(T_{cw,6}^{\boxtimes 2}) = 64$
(3) $\underline{\mathbf{R}}(T_{cw,8}^{\boxtimes 2}) = 100$
(4) $\underline{\mathbf{R}}(T_{cw,10}^{\boxtimes 2}) = 144$
(5) $\underline{\mathbf{R}}(T_{cw,4}^{\boxtimes 3}) \geq 180$
(6) $\underline{\mathbf{R}}(T_{cw,6}^{\boxtimes 3}) = 512$
(7) $\underline{\mathbf{R}}(T_{cw,8}^{\boxtimes 3}) = 1000$
(8) $\underline{\mathbf{R}}(T_{cw,10}^{\boxtimes 3}) = 1728$.

Note that $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 4}) \leq (q+2)^4$ and that $\underline{\mathbf{R}}(T_{skewcw,q}^{\boxtimes 4})$ is at least the estimate in Proposition 4.1 times $q+1$ by [19, Prop. 4.2]. Based on this, it is possible as of this writing that $\underline{\mathbf{R}}(T_{skewcw,q}^{\boxtimes 4}) \leq \underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 4})$ for $q = 2, 6, 8$.

5 Proof of Theorem 1.1 that $\underline{\mathbf{R}}(\mathrm{perm}_3) = 16$

The upper bound follows as $\underline{\mathbf{R}}(T_{cw,2}) = 4$.

For the lower bound, we prove there is no $E_{110} \subset A \otimes B$ of dimension 15 spanned by weight vectors (i.e., fixed by the torus action $\mathbb{B}_{\mathrm{perm}_3}$) that satisfies the flag condition and passes the (210) and (120) tests.

Our argument proceeds by first proving general results about linear combinations of weight vectors from $A \otimes \mathrm{perm}(C^*)$ with other weight vectors in $A \otimes (A \otimes B)$ lying in the $\mathbb{B}_{\mathrm{perm}_3}$ complement of $A \otimes \mathrm{perm}(C^*)$. We then list all such combinations that could potentially arise in some viable E_{110}. We conclude by showing that no choice of E_{110} will pass both the (210) and (120) tests. Our argument is facilitated by assuming some type of element is in the kernel, then observing what kind of flag would be needed to have such an element. Often the first few steps of the flag give enough information to eliminate the element from consideration.

5.1 General Results About the Kernel

In what follows, $\{i, j, k\} = \{1, 2, 3\}$, $\{i', j', k'\} = \{1, 2, 3\}$, and $s, t, s', t' \in \{1, 2, 3\}$ but we do not require these to be distinct from other indices.

With this notation and the terms on the right hand side running over all possible indices

$$\mathrm{perm}_3(C^*) = \langle a^i_{i'} \otimes b^j_{j'} + a^j_{i'} \otimes b^i_{j'} + a^i_{j'} \otimes b^j_{i'} + a^j_{j'} \otimes b^i_{i'} \rangle.$$

Thus $\mathrm{perm}_3(C^*) \cap \sigma_3 =$. Observe that $\kappa_f = 1$ as

$$(\mathrm{perm}_3(C^*) \otimes A) \cap (S^2 A \otimes B) = \langle \sum a^k_{k'} \otimes (a^i_{i'} \otimes b^j_{j'} + a^j_{i'} \otimes b^i_{j'} + a^i_{j'} \otimes b^j_{i'} + a^j_{j'} \otimes b^i_{i'}) \rangle.$$

Remark 5.1 In general, for any symmetric tensor T, $\kappa_f \geq 1$ due to the copy of T in $S^3 A \subset S^2 A \otimes B$.

The possible weights of elements in $A \otimes B$ are $(200)(200)$, $(110)(110)$, $(200)(110)$ and their permutations under the action of $(\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}_2$. We will say an element has *type* $(xyz)(pqr)$ if its weight is in the $(\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}_2$-orbit of $(xyz)(pqr)$.

The flag condition implies any potential E_{110} must include a flag $E_1 \subset E_2 \subset E_3 \subset \cdots \subset E_{110}$ with $\dim E_j = j$ and E_j contained in some $A_j \otimes B_j$ where $\dim A_j = \dim B_j = j$. Moreover E_1, E_2, E_3 must be in (a choice of) E'_{110}.

All weight vectors of type $(200)(200)$ have rank one, these are of the form $a^i_{i'} \otimes b^i_{i'}$. Vectors of type $(200)(110)$ have rank one or two, those of rank one are of the form $a^i_{i'} \otimes b^i_{k'}$ and vectors of type $(110)(110)$ have rank at most four, the rank one vectors among them are of the form $a^i_{i'} \otimes b^j_{j'}$.

Given the first step, we could get the second step either by adding another rank one weight vector, or taking a tangent vector to a rank one weight vector.

The rank two weight vectors tangent to a rank one element of type $(200)(200)$, which we may write as $a^j_{j'} \otimes b^j_{j'}$, are up to scale $a^j_{j'} \otimes b^j_{k'} + K a^j_{k'} \otimes b^j_{j'}$, for some $K \neq 0$, or its $\mathbb{Z}_2$-image, which are of type $(200)(110)$ or $a^j_{j'} \otimes b^k_{k'} + K a^k_{k'} \otimes b^j_{j'}$, which are of type $(110)(110)$.

No rank two vector tangent to a rank one element of type $(110)(110)$ is a weight vector.

The rank two weight vectors tangent to a rank one element of type $(200)(110)$, e.g., $a^i_{i'} \otimes b^i_{k'}$, are of the form $a^i_{i'} \otimes b^k_{k'} + K a^k_{i'} \otimes b^i_{k'}$ for some $K \neq 0$, and they are of type $(110)(110)$.

Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\Gamma \cdot a^j_{j'} \otimes b^k_{k'} = a^j_{j'} \otimes b^k_{k'} + a^j_{k'} \otimes b^k_{j'} + a^k_{j'} \otimes b^j_{k'} + a^k_{k'} \otimes b^j_{j'}$. In what follows underlined terms are elements of E'_{110}. The group G_{perm_3} allows us to unambiguously define the elements of E'_{110} except those of type $(110)(110)$.

The two ways to add to a monomial $a \otimes a' \otimes b$ to get an element in $S^2 A \otimes B$ are either to send it to zero by subtracting $a \otimes (\underline{a' \otimes b})$, which we will refer to as *cancellation* or to symmetrize it by adding $a' \otimes (\underline{a \otimes b})$, which we call *symmetrization*.

Lemma 5.2 *Only a type* $(110)(110)$ *element can be used for a cancellation of an element of* $A \otimes \mathrm{perm}_3(C^*)$. *No element can be used in more than four symmetrizations of an element of* $A \otimes \mathrm{perm}_3(C^*)$. *An element of rank greater than one can be used in at most two symmetrizations of an element of* $A \otimes \mathrm{perm}_3(C^*)$.

Proof The first assertion is obvious. For the second, if we have some $a^s_t \otimes b^i_{i'}$, then this can be used to symmetrize a term in any of $a^s_t \otimes (\Gamma \cdot a^j_{j'} \otimes b^i_{i'})$, four such, but no others. An element of rank greater than one such as $a^t_{j'} \otimes b^i_{i'} + a^t_{i'} \otimes b^i_{j'}$ can symmetrize a term in $a^t_{j'} \otimes (\Gamma \cdot a^j_{k'} \otimes b^i_{i'}) + a^t_{i'} \otimes (\Gamma \cdot a^j_{k'} \otimes b^i_{j'})$, and one such as $a^t_{j'} \otimes b^j_{i'} + a^t_{i'} \otimes b^i_{j'}$ can only symmetrize a term in $a^t_{j'} \otimes (\Gamma \cdot a^k_{k'} \otimes b^i_{i'}) + a^t_{i'} \otimes (\Gamma \cdot a^k_{k'} \otimes b^i_{j'})$. □

We use a basic fact from exterior differential systems (the easy part of Cartan's test) [28, Prop. 4.5.3]: Here we work with general $A, B = \mathbb{C}^m$. Let $A_1 \subset A_2 \subset \cdots \subset A_m$ be a generic flag in A (generic in the sense that s_1 below is maximized, and having maximized s_1, s_2 is maximized etc..). Let s_1 be the dimension of the projection of E'_{110} to $A_1 \otimes B$. Define s_2 by $s_1 + s_2$ is dimension of the projection of E'_{110} to $A_2 \otimes B$, set $s_1 + s_2 + s_3$ to be the dimension of E'_{110} projected to $A_3 \otimes B$ etc.. Then

$$\dim(S^2 A \otimes B) \cap (A \otimes E'_{110}) \leq s_1 + 2s_2 + \cdots + ms_m. \tag{7}$$

In particular, $k'_p \leq s_1 + 2s_2 + \cdots + ms_m$. If equality holds in (7), we will say E'_{110} is A-involutive.

Define the *mixed price* of a space E'_{110} to be $15 - k'_p$. In particular,

- The mixed price is at least 8 if $s_1 \geq 6$,
- The mixed price is at least 7 if $s_1 \geq 5$,
- The mixed price is at least 5 if $s_1 \geq 4$.

A necessary condition for an E'_{110} to be a candidate is that k'_m must be at least the mixed price.

Define the *flag cost* of a weight vector in E'_{110} to be the length of the smallest admissible flag that contains the weight vector.

The flag cost of a weight vector of type $(110)(110)$ is its rank, but it need not be realized via rank one $(110)(110)$ elements.

5.2 Elements in the Kernel with Weight of Type $(210)(210)$

Lemma 5.3 *If a weight vector in the kernel of the* 210*-map is of type* $(210)(210)$*, then without loss of generality it involves only one element of* $A^* \otimes \operatorname{perm}(C^*)$.

Proof Assume we have such an element $a^j_{j'} \otimes (\Gamma \cdot a^i_{i'} \otimes b^j_{j'})$. Then the three terms that need to be canceled or symmetrized cannot be dispensed with by another element of $A^* \otimes \operatorname{perm}_3(C^*)$, because it would have to have the same weight, namely there would have to appear superscripts j, j, i and subscripts j', j', i', but this is the only element of $A^* \otimes \operatorname{perm}_3(C^*)$ with that weight. Similarly, any term of E'_{110} that is used to cancel or symmetrize a term of $a^j_{j'} \otimes (\Gamma \cdot a^i_{i'} \otimes b^j_{j'})$ cannot cancel or symmetrize any other monomial in any other element of $A^* \otimes \operatorname{perm}_3(C^*)$ again by weight considerations. $\square$

Now we enumerate all such potential elements of the kernel:

To obtain an element of the kernel from $a^j_{j'} \otimes (\Gamma \cdot a^i_{i'} \otimes b^j_{j'})$ we may add any of the following terms:

$$-a^j_{j'} \otimes (\underline{a^i_{i'} \otimes b^j_{j'} + a^i_{j'} \otimes b^j_{i'} + a^j_{i'} \otimes b^i_{j'} + L a^j_{j'} \otimes b^i_{i'}}), \tag{8}$$

$$-a^j_{j'} \otimes (\underline{a^i_{i'} \otimes b^j_{j'} + a^i_{j'} \otimes b^j_{i'} + L a^j_{j'} \otimes b^i_{i'}}) + a^j_{i'} \otimes (\underline{a^j_{j'} \otimes b^i_{j'}}), \tag{9}$$

$$-a^j_{j'} \otimes (\underline{a^j_{i'} \otimes b^i_{j'} + a^i_{j'} \otimes b^j_{i'} + L a^j_{j'} \otimes b^i_{i'}}) + a^i_{i'} \otimes (\underline{a^j_{j'} \otimes b^j_{j'}}), \tag{10}$$

$$+ a^i_{i'}\otimes\underline{(a^j_{j'}\otimes b^j_{j'})} + a^i_{j'}\otimes\underline{(a^j_{j'}\otimes b^j_{i'})} - a^j_{j'}\otimes\underline{(a^j_{i'}\otimes b^i_{j'} + La^j_{j'}\otimes b^i_{i'})}, \tag{11}$$

$$- a^j_{j'}\otimes\underline{(a^i_{i'}\otimes b^j_{j'})} + a^i_{j'}\otimes\underline{(a^j_{j'}\otimes b^j_{i'} + Ka^j_{i'}\otimes b^j_{j'})} + a^j_{i'}\otimes\underline{(a^j_{j'}\otimes b^i_{j'} + Ka^i_{j'}\otimes b^j_{j'})}, \tag{12}$$

$$+ a^i_{i'}\otimes\underline{(a^j_{j'}\otimes b^j_{j'})} + a^i_{j'}\otimes\underline{(a^j_{j'}\otimes b^j_{i'} + Ka^j_{i'}\otimes b^j_{j'})} + a^j_{i'}\otimes\underline{(a^j_{j'}\otimes b^i_{j'} + Ka^i_{j'}\otimes b^j_{j'})}. \tag{13}$$

The first term consists of pure cancellations, the second and third have one symmetrization, respectively using an element of type $(200)(110)$ and $(200)(200)$ for the symmetrization, the next two use two symmetrizations. The last symmetrizes all three terms, two elements of type $(200)(110)$ and one of type $(200)(200)$ appear in the symmetrizations.

Note that if we use any one of these to obtain an element of the mixed kernel, we cannot use a second, as the difference of two such terms is an element of the pure kernel.

5.3 Elements in the Kernel with Weight of Type $(210)(111)$

If an element of the kernel of the 210-map is of type $(210)(111)$, say it involves $a^i_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{j'}) \in A\otimes \mathrm{perm}_3(C^*)$, then, by weight considerations, there are two additional elements that could efficiently appear in the same element, namely $a^i_{j'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{k'})$ and $a^i_{i'}\otimes(\Gamma\cdot a^i_{k'}\otimes b^j_{j'})$.

5.3.1 Case: One Basis Element of $A\otimes \mathrm{perm}_3(C^*)$ Appears

We obtain four terms to be canceled or symmetrized, and at least one must be symmetrized. The possibilities for the kernel element by adding to $a^i_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{j'})$ are any of the following:

$$- a^i_{k'}\otimes\underline{(a^i_{i'}\otimes b^j_{j'} + a^i_{j'}\otimes b^j_{i'} + a^j_{i'}\otimes b^i_{j'})} + a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^i_{i'}}, \tag{14}$$

$$- a^i_{k'}\otimes\underline{(a^i_{i'}\otimes b^j_{j'} + a^j_{j'}\otimes b^i_{i'} + a^j_{i'}\otimes b^i_{j'})} + a^i_{j'}\otimes\underline{a^i_{k'}\otimes b^j_{i'}}, \tag{15}$$

$$- a^i_{k'}\otimes\underline{(a^i_{i'}\otimes b^j_{j'} + a^i_{j'}\otimes b^j_{i'})} + a^j_{i'}\otimes\underline{a^i_{k'}\otimes b^i_{j'}} + a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^i_{i'}}, \tag{16}$$

$$- a^i_{k'}\otimes\underline{a^i_{i'}\otimes b^j_{j'}} + a^i_{j'}\otimes\underline{a^i_{k'}\otimes b^j_{i'}} + a^j_{i'}\otimes\underline{a^i_{k'}\otimes b^i_{j'}} + a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^i_{i'}}, \tag{17}$$

$$+ a^i_{i'}\otimes\underline{a^i_{k'}\otimes b^j_{j'}} + a^i_{j'}\otimes\underline{a^i_{k'}\otimes b^j_{i'}} + a^j_{i'}\otimes\underline{a^i_{k'}\otimes b^i_{j'}} + a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^i_{i'}}. \tag{18}$$

5.3.2 Two Elements Appear

Without loss of generality, we take them to be $a^i_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{j'})$ and $a^i_{j'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{k'})$. Then one symmetrization occurs among the 8 basis vectors in the expression, leaving six, and to send the element to the kernel we have the following possibilities to add to $a^i_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{j'}) + a^i_{j'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{k'})$:

$$-a^i_{k'}\otimes(\underline{a^j_{j'}\otimes b^i_{i'} + a^i_{i'}\otimes b^j_{j'} + a^j_{i'}\otimes b^i_{j'}}) - a^i_{j'}\otimes(\underline{a^j_{k'}\otimes b^i_{i'} + a^i_{i'}\otimes b^j_{k'} + a^j_{i'}\otimes b^i_{k'}}), \tag{19}$$

$$-a^i_{k'}\otimes(\underline{a^i_{i'}\otimes b^j_{j'} + a^j_{i'}\otimes b^i_{j'}}) - a^i_{j'}\otimes(\underline{a^j_{k'}\otimes b^i_{i'} + a^i_{i'}\otimes b^j_{k'} + a^j_{i'}\otimes b^i_{k'}}) + a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^i_{i'}}, \tag{20}$$

$$-a^i_{k'}\otimes\underline{a^j_{i'}\otimes b^i_{j'}} - a^i_{j'}\otimes(\underline{a^j_{k'}\otimes b^i_{i'} + a^i_{i'}\otimes b^j_{k'} + a^j_{i'}\otimes b^i_{k'}}) + a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^i_{i'}} + a^i_{i'}\otimes\underline{a^i_{k'}\otimes b^j_{j'}}, \tag{21}$$

$$-a^i_{k'}\otimes(\underline{a^i_{i'}\otimes b^j_{j'} + a^j_{i'}\otimes b^i_{j'}}) - a^i_{j'}\otimes(\underline{a^i_{i'}\otimes b^j_{k'} + a^j_{i'}\otimes b^i_{k'}}) + a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^i_{i'}} + a^j_{k'}\otimes\underline{a^i_{j'}\otimes b^i_{i'}}, \tag{22}$$

$$+a^i_{i'}\otimes(\underline{a^i_{k'}\otimes b^j_{j'} + a^i_{j'}\otimes b^j_{k'}}) + a^j_{i'}\otimes(\underline{a^i_{k'}\otimes b^i_{j'} + a^i_{j'}\otimes b^i_{k'}}) - a^i_{k'}\otimes\underline{a^j_{j'}\otimes b^i_{i'}} - a^i_{j'}\otimes\underline{a^j_{k'}\otimes b^i_{i'}}, \tag{23}$$

$$+a^i_{i'}\otimes(\underline{a^i_{k'}\otimes b^j_{j'} + a^i_{j'}\otimes b^j_{k'}}) + a^j_{i'}\otimes(\underline{a^i_{k'}\otimes b^i_{j'} + a^i_{j'}\otimes b^i_{k'}}) + a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^i_{i'}} - a^i_{j'}\otimes\underline{a^j_{k'}\otimes b^i_{i'}}, \tag{24}$$

$$+a^i_{i'}\otimes(\underline{a^i_{k'}\otimes b^j_{j'} + a^i_{j'}\otimes b^j_{k'}}) + a^j_{i'}\otimes(\underline{a^i_{k'}\otimes b^i_{j'} + a^i_{j'}\otimes b^i_{k'}}) - a^i_{k'}\otimes\underline{a^j_{j'}\otimes b^i_{i'}} + a^j_{k'}\otimes\underline{a^i_{j'}\otimes b^i_{i'}}, \tag{25}$$

$$+a^i_{i'}\otimes(\underline{a^i_{k'}\otimes b^j_{j'} + a^i_{j'}\otimes b^j_{k'}}) + a^j_{i'}\otimes(\underline{a^i_{k'}\otimes b^i_{j'} + a^i_{j'}\otimes b^i_{k'}}) + a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^i_{i'}} + a^j_{k'}\otimes\underline{a^i_{j'}\otimes b^i_{i'}}. \tag{26}$$

5.3.3 All Three Terms Appear

$a^i_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{j'}) + a^i_{j'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{k'}) + a^i_{i'}\otimes(\Gamma\cdot a^i_{k'}\otimes b^j_{j'})$ may be sent to the kernel by adding one of:

$$-a^i_{k'}\otimes(\underline{a^j_{i'}\otimes b^i_{j'} + a^j_{j'}\otimes b^i_{i'}}) - a^i_{i'}\otimes(\underline{a^j_{k'}\otimes b^i_{j'} + a^j_{j'}\otimes b^i_{k'}}) - a^i_{j'}\otimes(\underline{a^j_{k'}\otimes b^i_{i'} + a^j_{i'}\otimes b^i_{k'}}), \tag{27}$$

$$+a^j_{j'}\otimes(\underline{a^i_{k'}\otimes b^i_{i'} + a^i_{i'}\otimes b^i_{k'}}) + a^j_{k'}\otimes(\underline{a^i_{i'}\otimes b^i_{j'} + a^i_{j'}\otimes b^i_{i'}}) + a^j_{i'}\otimes(\underline{a^i_{k'}\otimes b^i_{j'} + a^i_{j'}\otimes b^i_{k'}}). \tag{28}$$

If we use rank one elements we get larger expressions which are easily dispensed with.

5.4 Elements in the Kernel with Weight of Type (111)(111)

5.4.1 Cases with One Element

The possible terms to add to $a^k_{k'}\otimes(\Gamma a^i_{i'}\otimes b^j_{j'})$ to send it to the kernel are:

$$+a^i_{i'}\otimes(\underline{a^k_{k'}\otimes b^j_{j'}}) + a^j_{i'}\otimes(\underline{a^k_{k'}\otimes b^i_{j'}}) + a^i_{j'}\otimes(\underline{a^k_{k'}\otimes b^j_{i'}}) + a^j_{j'}\otimes(\underline{a^k_{k'}\otimes b^i_{i'}}), \tag{29}$$

$$-a^k_{k'}\otimes(\underline{a^i_{i'}\otimes b^j_{j'}}) + a^j_{i'}\otimes(\underline{a^k_{k'}\otimes b^i_{j'}}) + a^i_{j'}\otimes(\underline{a^k_{k'}\otimes b^j_{i'}}) + a^j_{j'}\otimes(\underline{a^k_{k'}\otimes b^i_{i'}}), \tag{30}$$

$$-a^k_{k'}\otimes(\underline{a^i_{i'}\otimes b^j_{j'} + a^j_{j'}\otimes b^i_{i'}}) + a^j_{i'}\otimes(\underline{a^k_{k'}\otimes b^i_{j'}}) + a^i_{j'}\otimes(\underline{a^k_{k'}\otimes b^j_{i'}}), \tag{31}$$

$$-a^k_{k'}\otimes(\underline{a^i_{i'}\otimes b^j_{j'} + a^j_{i'}\otimes b^i_{j'}}) + a^i_{j'}\otimes(\underline{a^k_{k'}\otimes b^j_{i'}}) + a^j_{j'}\otimes(\underline{a^k_{k'}\otimes b^i_{i'}}), \tag{32}$$

$$-a^k_{k'}\otimes(\underline{a^i_{i'}\otimes b^j_{j'} + a^j_{i'}\otimes b^i_{j'} + a^i_{j'}\otimes b^j_{i'}}) + a^j_{j'}\otimes(\underline{a^k_{k'}\otimes b^i_{i'}}). \tag{33}$$

5.4.2 Cases with Two Elements

One possible term to add to $a^k_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{j'}) + a^k_{i'}\otimes(\Gamma\cdot a^i_{k'}\otimes b^j_{j'})$ to send it to the kernel is:

$$\begin{aligned} &a^i_{j'}\otimes(\underline{a^k_{k'}\otimes b^j_{i'} + a^k_{i'}\otimes b^j_{k'}}) + a^j_{j'}\otimes(\underline{a^k_{k'}\otimes b^i_{i'} + a^i_{i'}\otimes b^k_{k'}}) \\ &\quad - a^k_{k'}\otimes(\underline{a^i_{i'}\otimes b^j_{j'} + a^j_{i'}\otimes b^i_{j'}}) - a^k_{i'}\otimes(\underline{a^j_{k'}\otimes b^i_{j'} + a^k_{k'}\otimes b^j_{j'}}) \end{aligned} \tag{34}$$

All other possible terms arise by exchanging symmetrizations and cancellations, but we will see such cannot be used.

Similarly, $a^k_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{j'}) + a^i_{i'}\otimes(\Gamma\cdot a^k_{k'}\otimes b^j_{j'})$ may be sent to the kernel by adding one of

$$-a^k_{k'}\otimes(\underline{a^i_{j'}\otimes b^j_{i'} + a^j_{i'}\otimes b^i_{j'} + a^j_{j'}\otimes b^i_{i'}}) - a^i_{i'}\otimes(\underline{a^k_{j'}\otimes b^j_{k'} + a^j_{k'}\otimes b^k_{j'} + a^j_{j'}\otimes b^k_{k'}}), \tag{35}$$

$$-a^k_{k'}\otimes(\underline{a^i_{j'}\otimes b^j_{i'} + a^j_{i'}\otimes b^i_{j'}}) - a^i_{i'}\otimes(\underline{a^k_{j'}\otimes b^j_{k'} + a^j_{k'}\otimes b^k_{j'}}) + a^j_{j'}\otimes(\underline{a^k_{k'}\otimes b^i_{i'} + a^i_{i'}\otimes b^k_{k'}}). \tag{36}$$

We will see in Sect. 5.5 that elements with more symmetrizations cannot be used.

5.4.3 Cases with Three Elements

The term $a^i_{i'}\otimes(\Gamma\cdot a^j_{j'}\otimes b^k_{k'}) + a^j_{j'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^k_{k'}) + a^k_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{j'})$ may be sent to the kernel by adding

$$-a^i_{i'}\otimes(\underline{a^j_{k'}\otimes b^k_{j'} + a^k_{j'}\otimes b^j_{k'}}) - a^j_{j'}\otimes(\underline{a^i_{k'}\otimes b^k_{i'} + a^k_{i'}\otimes b^i_{k'}}) - a^k_{k'}\otimes(\underline{a^i_{j'}\otimes b^j_{i'} + a^j_{i'}\otimes b^i_{j'}}), \tag{37}$$

where all have type $(110)(110)$. One can also have symmetrizations, such will have the same mixed price but a larger flag cost so it is sufficient to eliminate (37) to eliminate all cases.

Using $a^i_{i'}\otimes(\Gamma\cdot a^j_{k'}\otimes b^k_{j'}) + a^i_{j'}\otimes(\Gamma\cdot a^j_{k'}\otimes b^k_{i'}) + a^j_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^k_{j'})$, a kernel element can be created by adding the following terms:

$$-a^i_{i'}\otimes(\underline{a^j_{j'}\otimes b^k_{k'} + a^k_{k'}\otimes b^j_{j'} + a^k_{j'}\otimes b^j_{k'}}) - a^j_{k'}\otimes\underline{a^k_{j'}\otimes b^i_{i'}} + a^k_{i'}\otimes(\underline{a^i_{j'}\otimes b^j_{k'} + a^j_{k'}\otimes b^i_{j'}})$$

$$
\begin{aligned}
&- a^i_{j'} \otimes (\underline{a^j_{i'} \otimes b^k_{k'} + a^k_{k'} \otimes b^j_{i'}}),
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
&- a^i_{i'} \otimes (\underline{a^j_{j'} \otimes b^k_{k'} + a^k_{k'} \otimes b^j_{j'}}) + a^k_{j'} \otimes (\underline{a^i_{i'} \otimes b^j_{k'} + a^j_{k'} \otimes b^i_{i'}}) + a^k_{i'} \otimes (\underline{a^i_{j'} \otimes b^j_{k'} + a^j_{k'} \otimes b^i_{j'}}) \\
&\quad - a^i_{j'} \otimes (\underline{a^j_{i'} \otimes b^k_{k'} + a^k_{k'} \otimes b^j_{i'}}),
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&- a^i_{i'} \otimes (\underline{a^j_{j'} \otimes b^k_{k'} + a^k_{k'} \otimes b^j_{j'}}) + a^k_{j'} \otimes (\underline{a^i_{i'} \otimes b^j_{k'} + a^j_{k'} \otimes b^i_{i'}}) + a^k_{i'} \otimes (\underline{a^i_{j'} \otimes b^j_{k'} + a^j_{k'} \otimes b^i_{j'}}) \\
&\quad + a^j_{i'} \otimes \underline{a^i_{j'} \otimes b^k_{k'}} + a^k_{k'} \otimes \underline{a^i_{j'} \otimes b^j_{i'}},
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
&a^k_{k'} \otimes (\underline{a^i_{i'} \otimes b^j_{j'} + a^i_{j'} \otimes b^j_{i'}}) + a^k_{j'} \otimes (\underline{a^i_{i'} \otimes b^j_{k'} + a^j_{k'} \otimes b^i_{i'}}) + a^k_{i'} \otimes (\underline{a^i_{j'} \otimes b^j_{k'} + a^j_{k'} \otimes b^i_{j'}}) \\
&\quad + a^j_{j'} \otimes \underline{a^i_{i'} \otimes b^k_{k'}} + a^j_{i'} \otimes \underline{a^i_{j'} \otimes b^k_{k'}},
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&- a^i_{i'} \otimes (\underline{a^j_{j'} \otimes b^k_{k'} + a^k_{k'} \otimes b^j_{j'}}) + a^k_{j'} \otimes (\underline{a^i_{i'} \otimes b^j_{k'} + a^j_{k'} \otimes b^i_{i'}}) - a^i_{j'} \otimes (\underline{a^j_{i'} \otimes b^k_{k'} + a^k_{i'} \otimes b^j_{k'}}) \\
&\quad + a^k_{k'} \otimes \underline{a^i_{j'} \otimes b^j_{i'}} + a^k_{i'} \otimes \underline{a^j_{k'} \otimes b^i_{j'}},
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
&- a^i_{i'} \otimes (\underline{a^j_{j'} \otimes b^k_{k'} + a^k_{k'} \otimes b^j_{j'} + a^k_{j'} \otimes b^j_{k'}}) - a^i_{j'} \otimes (\underline{a^j_{i'} \otimes b^k_{k'} + a^k_{i'} \otimes b^j_{k'} + a^k_{k'} \otimes b^j_{i'}}) \\
&\quad - a^j_{k'} \otimes (\underline{a^k_{i'} \otimes b^i_{j'} + a^k_{j'} \otimes b^i_{i'}}).
\end{aligned}
\tag{43}
$$

There are more since we can swap symmetrization and cancellation for the rank one elements. But none of them provides a larger mixed kernel.

Any other term with three basis vectors of $A^* \otimes \mathrm{perm}_3(C^*)$ will have less automatic symmetrization and thus flag cost > 6 so they need not be considered.

5.4.4 Cases with Four Elements

There are up to symmetry two types of cases with flag cost at most six: in each case there is exactly one repeated index appearing above and one below. They are distinguished by whether or not a pair of repeated indices overlap.

First $a^k_{k'} \otimes (\Gamma \cdot a^i_{i'} \otimes b^j_{j'}) + a^k_{i'} \otimes (\Gamma \cdot a^i_{k'} \otimes b^j_{j'}) + a^i_{j'} \otimes (\Gamma \cdot a^k_{k'} \otimes b^j_{i'}) + a^j_{j'} \otimes (\Gamma \cdot a^i_{i'} \otimes b^k_{k'})$ may be sent to the kernel with:

$$
\begin{aligned}
&a^i_{i'} \otimes (\underline{a^k_{k'} \otimes b^j_{j'} + a^j_{j'} \otimes b^k_{k'}}) + a^j_{i'} \otimes (\underline{a^k_{k'} \otimes b^i_{j'} + a^i_{j'} \otimes b^k_{k'}}) + a^i_{k'} \otimes (\underline{a^k_{i'} \otimes b^j_{j'} + a^j_{j'} \otimes b^k_{i'}}) \\
&\quad + a^j_{k'} \otimes (\underline{a^k_{i'} \otimes b^i_{j'} + a^i_{j'} \otimes b^k_{i'}}),
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
&a^i_{i'} \otimes (\underline{a^k_{k'} \otimes b^j_{j'} + a^j_{j'} \otimes b^k_{k'}}) + a^j_{i'} \otimes (\underline{a^k_{k'} \otimes b^j_{j'} + a^i_{j'} \otimes b^k_{k'}}) + a^i_{k'} \otimes (\underline{a^k_{i'} \otimes b^j_{j'} + a^j_{j'} \otimes b^k_{i'}}) \\
&\quad - a^k_{i'} \otimes \underline{a^j_{k'} \otimes b^i_{j'}} - a^i_{j'} \otimes \underline{a^j_{k'} \otimes b^k_{i'}},
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&- a^i_{j'} \otimes (\underline{a^j_{k'} \otimes b^k_{i'} + a^j_{i'} \otimes b^k_{k'}}) + a^i_{i'} \otimes (\underline{a^k_{k'} \otimes b^j_{j'} + a^j_{j'} \otimes b^k_{k'}}) - a^k_{i'} \otimes (\underline{a^i_{k'} \otimes b^j_{j'} + a^j_{k'} \otimes b^i_{j'}}) \\
&\quad - a^j_{j'} \otimes \underline{a^i_{k'} \otimes b^k_{i'}} - a^k_{k'} \otimes \underline{a^j_{i'} \otimes b^i_{j'}},
\end{aligned}
\tag{46}
$$

$$
- a^i_{j'} \otimes (\underline{a^j_{k'} \otimes b^k_{i'} + a^j_{i'} \otimes b^k_{k'}}) - a^j_{j'} \otimes (\underline{a^i_{k'} \otimes b^k_{i'} + a^i_{i'} \otimes b^k_{k'}}) - a^k_{k'} \otimes \underline{a^i_{i'} \otimes b^j_{j'}}
$$

$$\begin{aligned}&-a^k_{k'}\otimes\underline{a^j_{i'}\otimes b^i_{j'}} - a^k_{i'}\otimes\underline{a^i_{k'}\otimes b^j_{j'}} - a^k_{i'}\otimes\underline{a^j_{k'}\otimes b^i_{j'}},\end{aligned}\tag{47}$$

$$\begin{aligned}&-a^i_{j'}\otimes\underline{(a^j_{k'}\otimes b^k_{i'} + a^j_{i'}\otimes b^k_{k'})} - a^j_{j'}\otimes\underline{(a^i_{k'}\otimes b^k_{i'} + a^i_{i'}\otimes b^k_{k'})} + a^i_{i'}\otimes\underline{a^k_{k'}\otimes b^j_{j'}} + a^j_{i'}\otimes\underline{a^k_{k'}\otimes b^i_{j'}}\\&+ a^i_{k'}\otimes\underline{a^k_{i'}\otimes b^j_{j'}} + a^j_{k'}\otimes\underline{a^k_{i'}\otimes b^i_{j'}},\end{aligned}\tag{48}$$

$$\begin{aligned}&-a^i_{j'}\otimes\underline{(a^j_{k'}\otimes b^k_{i'} + a^j_{i'}\otimes b^k_{k'})} + a^i_{i'}\otimes\underline{(a^k_{k'}\otimes b^j_{j'} + a^j_{j'}\otimes b^k_{k'})} - a^k_{i'}\otimes\underline{a^i_{k'}\otimes b^j_{j'}}\\&-a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^k_{i'}} + a^j_{i'}\otimes\underline{a^k_{k'}\otimes b^i_{j'}} + a^j_{k'}\otimes\underline{a^k_{i'}\otimes b^i_{j'}},\end{aligned}\tag{49}$$

$$\begin{aligned}&a^i_{i'}\otimes\underline{(a^k_{k'}\otimes b^j_{j'} + a^j_{j'}\otimes b^k_{k'})} + a^j_{i'}\otimes\underline{(a^k_{k'}\otimes b^j_{j'} + a^i_{j'}\otimes b^k_{k'})} + a^i_{k'}\otimes\underline{a^k_{i'}\otimes b^j_{j'}}\\&-a^j_{j'}\otimes\underline{a^i_{k'}\otimes b^k_{i'}} + a^j_{k'}\otimes\underline{a^k_{i'}\otimes b^i_{j'}} - a^i_{j'}\otimes\underline{a^j_{k'}\otimes b^k_{i'}}.\end{aligned}\tag{50}$$

Using $a^i_{i'}\otimes(\Gamma\cdot a^j_{j'}\otimes b^k_{k'})+a^j_{j'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^k_{k'})+a^k_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{j'})+a^k_{j'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{k'})$, may be sent to the kernel using any of:

$$\begin{aligned}&a^j_{k'}\otimes\underline{a^i_{i'}\otimes b^k_{j'}} + a^k_{i'}\otimes\underline{a^j_{j'}\otimes b^i_{k'}} + a^i_{k'}\otimes\underline{a^j_{j'}\otimes b^k_{i'}} + a^j_{i'}\otimes\underline{a^k_{k'}\otimes b^i_{j'}} + a^i_{j'}\otimes\underline{a^k_{k'}\otimes b^j_{i'}}\\&-a^k_{j'}\otimes\underline{(a^j_{i'}\otimes b^i_{k'} + a^i_{k'}\otimes b^j_{i'} + a^j_{k'}\otimes b^i_{i'})},\end{aligned}\tag{51}$$

$$\begin{aligned}&-a^i_{i'}\otimes\underline{a^j_{k'}\otimes b^k_{j'}} + a^k_{i'}\otimes\underline{a^j_{j'}\otimes b^i_{k'}} + a^i_{k'}\otimes\underline{a^j_{j'}\otimes b^k_{i'}} + a^j_{i'}\otimes\underline{a^k_{k'}\otimes b^i_{j'}} + a^i_{j'}\otimes\underline{a^k_{k'}\otimes b^j_{i'}}\\&-a^k_{j'}\otimes\underline{(a^j_{i'}\otimes b^i_{k'} + a^i_{k'}\otimes b^j_{i'} + a^j_{k'}\otimes b^i_{i'})},\end{aligned}\tag{52}$$

$$\begin{aligned}&a^j_{k'}\otimes\underline{(a^i_{i'}\otimes b^k_{j'} + a^k_{j'}\otimes b^i_{i'})} - a^j_{j'}\otimes\underline{(a^k_{i'}\otimes b^i_{k'} + a^i_{k'}\otimes b^k_{i'})} - a^k_{k'}\otimes\underline{(a^j_{i'}\otimes b^i_{j'} + a^i_{j'}\otimes b^j_{i'})}\\&+a^j_{i'}\otimes\underline{a^k_{j'}\otimes b^i_{k'}} + a^i_{k'}\otimes\underline{a^k_{j'}\otimes b^j_{i'}},\end{aligned}\tag{53}$$

$$\begin{aligned}&-a^i_{i'}\otimes\underline{a^j_{k'}\otimes b^k_{j'}} + a^j_{k'}\otimes\underline{a^k_{j'}\otimes b^i_{i'}} - a^j_{j'}\otimes\underline{(a^k_{i'}\otimes b^i_{k'} + a^i_{k'}\otimes b^k_{i'})}\\&-a^k_{k'}\otimes\underline{(a^j_{i'}\otimes b^i_{j'} + a^i_{j'}\otimes b^j_{i'})} + a^j_{i'}\otimes\underline{a^k_{j'}\otimes b^i_{k'}} + a^i_{k'}\otimes\underline{a^k_{j'}\otimes b^j_{i'}},\end{aligned}\tag{54}$$

$$\begin{aligned}&a^j_{k'}\otimes\underline{(a^i_{i'}\otimes b^k_{j'} + a^k_{j'}\otimes b^i_{i'})} + a^k_{i'}\otimes\underline{a^j_{j'}\otimes b^i_{k'}} + a^i_{k'}\otimes\underline{a^j_{j'}\otimes b^k_{i'}}\\&-a^k_{k'}\otimes\underline{(a^j_{i'}\otimes b^i_{j'} + a^i_{j'}\otimes b^j_{i'})} - a^k_{j'}\otimes\underline{(a^j_{i'}\otimes b^i_{k'} + a^i_{k'}\otimes b^j_{i'})},\end{aligned}\tag{55}$$

$$\begin{aligned}&-a^i_{i'}\otimes\underline{a^j_{k'}\otimes b^k_{j'}} + a^j_{k'}\otimes\underline{a^k_{j'}\otimes b^i_{i'}} + a^k_{i'}\otimes\underline{a^j_{j'}\otimes b^i_{k'}} + a^i_{k'}\otimes\underline{a^j_{j'}\otimes b^k_{i'}}\\&-a^k_{k'}\otimes\underline{(a^j_{i'}\otimes b^i_{j'} + a^i_{j'}\otimes b^j_{i'})} - a^k_{j'}\otimes\underline{(a^j_{i'}\otimes b^i_{k'} + a^i_{k'}\otimes b^j_{i'})}.\end{aligned}\tag{56}$$

5.5 Elimination of Kernel Elements Containing an Element of Type (111)(111)

We may assume that at most four elements of $A^*\otimes\operatorname{perm}(C^*)$ appear in an element of the kernel of the 210-map of type (111)(111), because, assuming all the coefficients are $+1$, by adding a multiple of the 9-term element in the free kernel, which is of type (111)(111), we can reduce any using $k>4$ terms to one using $9-k$ terms, and the reader may verify that having coefficients other than $+1$ will only make the situation worse.

5.5.1 Case Four Basis Elements of $A \otimes \mathrm{perm}_3(C^*)$ are Used

Cases that already use six elements are easily eliminated. In the first set that leaves (44)–(46). These all have flag cost five and have $s_1 \geq 4$. But now examining the appearance of $(110)(110)$ terms in relations of $(210)(111)$, those with only one element not of type $(110)(110)$ are paired with rank three $(110)(110)$ elements. At most one term of type $(210)(210)$ may be used, so $k'_m \leq 2$ for any choice and we eliminate these cases.

Cases (51)–(56) are all easily eliminated either immediately or by a similar argument to cases (44)–(46).

5.5.2 Case Three Basis Elements of $A \otimes \mathrm{perm}_3(C^*)$ are Used

In case (37) the flag cost is at least five as the three rank two elements are in disjoint spaces. In the first step of the flag we could have a rank one element appearing in one of the two terms, or a rank one element such that one of the rank two elements is tangent, e.g. $a^j_{k'} \otimes b^k_{j'}$ or $a^j_{k'} \otimes b^j_{k'}$. At step three we need to add another, so up to symmetry there are three cases. Again since the rank two elements are in disjoint spaces, we need a third such at step five, for a total of four cases up to symmetry. This forces $s_1 \geq 5$ or $s'_1 \geq 5$ (where s'_1 is the s_1 for the (120)-test) so one of the mixed prices is at least 7. The only choice that adds to the mixed kernel is when one takes three elements of type $(200)(200)$, i.e., taking the second type in each choice. Then one can add up to three terms of type $(210)(210)$ to the mixed kernel but there is no way to have an 7-dimensional mixed kernel.

Cases (38)–(43) have flag cost 5 and mixed price at least seven, so are easily eliminated.

Case (43) also has flag cost 5 but it needs more attention as the two rank three elements may be used in (8) with no additional cost. To obtain a flag, either one will have $s_1 \geq 5$ or $s'_1 \geq 5$, so for one of the two the mixed price is at least 7 and one concludes as there is no way to enlarge the mixed kernel by four just adding in a single rank one element.

5.5.3 Case Two Basis Elements of $A \otimes \mathrm{perm}_3(C^*)$ are Used

Case (34) has a flag cost of 5 and after adding in a rank one element one has $s_1 \geq 5$, so the mixed price is at least 7 and one concludes as above.

Case (35): here we get two elements of type (8) for free, but the flag cost is 5 and any choice will either make $s_1 \geq 5$ or $s'_1 \geq 5$, and there is no way to expand it to have a 7 dimensional mixed kernel. (In fact if $s'_1 \geq 5$ the situation is even worse as this space does not give rise to a B-analog of (35).)

Case (36) has flag cost 5 as the three terms are in disjoint spaces and once the needed two rank one terms are added, one obtains $s_1 \geq 5$, so the mixed price is at least 7 and one concludes as above.

5.5.4 One Basis Element of $A\otimes$ perm$_3(C^*)$ is Used

In all cases, there is a flag cost of four.

In cases (29), (30) all elements appearing lie in different weight spaces and cannot be used together in a kernel element of type (210)(110) or (210)(210).

In cases (31) and (32), rank two elements may only be used effectively in (9), (10), and (16). Each of these requires adding in a different rank one element filling E'_{110}. Moreover, once the flag is filled in, we will have $s_1 \geq 4$ so mixed price at least five. We conclude the (210)-test cannot be passed in this case.

Consider case (33): the first step of the flag is just $a^k_{k'}\otimes b^i_{i'}$. For the second step, we could add another rank one element or a tangent vector to $a^k_{k'}\otimes b^i_{i'}$. Since eventually we need to account for the rank three element, the second choice turns out not to be as useful. For the rank one element, the naïve choice would be one of the three summands in the rank three term, but this turns out not to be as good as choosing the next three steps to be $a^i_{j'}\otimes b^i_{i'}$, $a^j_{i'}\otimes b^i_{i'}$, $a^i_{i'}\otimes b^i_{i'}$ so one gets the rank three element via the tangent to line construction. The resulting five dimensional space already passes the (210)-test. We get (8) for free using $a^j_{j'}\otimes(\Gamma a^i_{i'}\otimes b^j_{j'})$, then (9) two times, using $a^i_{j'}\otimes(\Gamma a^i_{i'}\otimes b^j_{j'})$ and $a^j_{i'}\otimes(\Gamma a^i_{i'}\otimes b^j_{j'})$ and (10) one time, using $a^i_{i'}\otimes(\Gamma a^i_{i'}\otimes b^j_{j'})$. On the other hand when one considers the (120) test, we only get a term analogous to (8) and moreover $s'_1 \geq 4$ so the mixed price is at least five. If we just add two of the three elements so the flag condition is satisfied, and then their reflections, the reflections are not useful for producing elements of type (210)(111) in the kernel and both tests are failed by one. If we add a term like $a^i_{k'}\otimes b^i_{i'}$ to enable an element of the kernel of type (14), the situation is the same. We conclude case (33) is not viable.

5.6 Kernels Consisting of Elements of Type (210)(111) and (210)(210)

5.6.1 Three Element Cases

(27) and (28) have flag cost five and are easily dispensed with using arguments similar to above.

5.6.2 Two Element Terms

All have flag cost at least 4, satisfy $s_1 \geq 4$ so the mixed price is at least 5. All cases without a rank three element are easily eliminated. Those with single a rank three element get an element of type (8) in the kernel for no extra cost, and for a cost of one each, get an element of types (9), (10), and potentially (14) and (15), but there is no way to reach five with just two more elements so these cases are eliminated.

Term (19) has flag cost four, as the first two terms in each parenthesis are tangent vectors to $a^i_{i'}\otimes b^i_{i'}$. These two terms may be used in two terms of type (8), but no others alone, and even though we may add two more vectors and utilize $a^i_{i'}\otimes b^i_{i'}$ to gain an addition term of type (13), here one of the mixed prices (once one satisfies the flag condition) is at least eight as $s_1 \geq 5$ or $s'_1 \geq 5$, so this case is also eliminated.

5.6.3 One Element Terms

Cases (16), (17), (18) are easily eliminated.

Cases (14), (15) can be made to pass the (210)-test with similar E'_{110} to case (33), but one has the identical problem with the (120)-test, and the argument to eliminate these cases is the same as that for (33).

5.7 Kernels Consisting Only of Elements of Type (210)(210)

Consider case (13). Note that even if $K \neq 0$ the flag cost of such an element is 3, and if we take one, e.g.. (13), we may obtain three more for an additional flag cost of two by considering $a^j_{j'}\otimes(\Gamma \cdot a^k_{i'}\otimes b^j_{j'})$, $a^j_{j'}\otimes(\Gamma \cdot a^i_{k'}\otimes b^j_{j'})$, $a^j_{j'}\otimes(\Gamma \cdot a^k_{k'}\otimes b^j_{j'})$. We still may add an additional element to E'_{110}, but no addition will increase k'_m to be larger than four. On the other hand we see the mixed price if 5, so such a case is not viable.

The other cases here are similar and easier. □

5.8 Remarks

Remark 5.4 The reason perm_3 was previously unaccessible was that already to choose E'_{110}, without the flag condition one needed to introduce numerous parameters due to the high weight multiplicities that made the calculation infeasible. The flag condition guaranteed the presence of low rank elements in E'_{110} which significantly reduced the search space.

Remark 5.5 It is interesting to see what happens when $\dim E'_{110} = 7$, to obtain a border rank 16 ideal fixed by the torus in G_{perm_3}. One may take for example

$$\begin{aligned} E'_{110} = \langle & a^1_1\otimes b^1_1, a^1_2\otimes b^1_1 + a^1_1\otimes b^1_2, a^1_3\otimes b^1_1 + a^1_1\otimes b^1_3, a^2_1\otimes b^1_1 \\ & + a^1_1\otimes b^2_1, a^3_1\otimes b^1_1 + a^1_1\otimes b^3_1, a^1_2\otimes b^1_2, a^1_3\otimes b^1_2 + a^1_2\otimes b^1_3\rangle. \end{aligned}$$

Then we obtain the four (200)(200) contributions to k'_m from expressions of type (18) as well as three additional contributions from expressions of type (28). Here $s_1 = s'_1 = 4$, $s_2 = s'_2 = 3$ and

$$\begin{aligned} & (A\otimes E'_{110}) \cap (S^2 A\otimes B) \\ & = \langle a^1_1\otimes a^1_1\otimes b^1_1, a^1_2\otimes a^1_1\otimes b^1_1 + a^1_1\otimes(a^1_2\otimes b^1_1 + a^1_1\otimes b^1_2), a^1_3\otimes a^1_1\otimes b^1_1 + a^1_1\otimes(a^1_3\otimes b^1_1 + a^1_1\otimes b^1_3), \\ & a^2_1\otimes a^1_1\otimes b^1_1 + a^1_1\otimes(a^2_1\otimes b^1_1 + a^1_1\otimes b^2_1), a^3_1\otimes a^1_1\otimes b^1_1 + a^1_1\otimes(a^3_1\otimes b^1_1 + a^1_1\otimes b^3_1), \\ & a^1_2\otimes a^1_2\otimes b^1_2, a^1_1\otimes a^1_2\otimes b^1_2 + a^1_2\otimes(a^1_2\otimes b^1_1 + a^1_1\otimes b^1_2), a^1_3\otimes a^1_2\otimes b^1_2 + a^1_2\otimes(a^1_3\otimes b^1_2 + a^1_2\otimes b^1_3)\rangle \end{aligned}$$

so $k'_p = k''_p = 8$ and both the (210) and (120) tests are passed.

6 Descriptions of VSP($T_{\mathrm{cw},q}$)

In this section we adopt the index range $1 \leq \alpha, \beta \leq q$. The small Coppersmith–Winograd tensor has a well-known border rank decomposition, which is also a Waring border rank decomposition.

$$\begin{aligned}
&T_{cw,q} = \lim_{t\to 0} \\
&\frac{1}{t^2}\sum_{\alpha}[(a_0+ta_\alpha)\otimes(b_0+tb_\alpha)\otimes(c_0+tc_\alpha)] \\
&\quad -\frac{1}{t^3}\left[(a_0+t^2\sum_{\alpha}a_\alpha)\otimes(b_0+t^2\sum_{\alpha}b_\alpha)\otimes(c_0+t^2\sum_{\alpha}c_\alpha)\right] \\
&\quad -(q\frac{1}{t^2}-\frac{1}{t^3})a_0\otimes b_0\otimes c_0.
\end{aligned}$$

Let $q > 2$. Write $A = B = C = L \oplus M$, where $L = \langle a_0\rangle$ and $M = \langle a_\alpha\rangle$. Set $Q = \sum_\alpha a_\alpha\otimes a_\alpha$. A straight-forward Lie algebra calculation (see, e.g., [18]) shows $G_{T_{\mathrm{cw},q}} \supseteq SO(M, Q) \times GL(L) = SO(q) \times \mathbb{C}^*$. Then

$$A\otimes B = L^{\otimes 2} \oplus L \wedge M \oplus S^2_0 M \oplus \Lambda^2 M \oplus (L \cdot M \oplus Q),$$

where the term in parenthesis is $T_{\mathrm{cw},q}(C^*)$. Here $S^2_0 M = M_{2_1}$ is the complement to the trivial $SO(M, Q)$-representation in $S^2 M$. In what follows we write L^k for $L^{\otimes k} = S^k L$.

Theorem 6.1 *For $q > 2$, VSP($T_{\mathrm{cw},q}$) is a point. The unique ideal is as follows: for all s, t, u with $s + t + u = d$, the annihilator of the ideal in degree (s, t, u) is*

$$L^d \oplus L^{d-1} \cdot M \oplus L^{d-2} \cdot Q.$$

Here

$$\begin{aligned}
L^{d-1}\cdot M = \langle & a_0^{s-1}\cdot a_\alpha\otimes b_0^t\otimes c_0^u + a_0^s\otimes b_0^{t-1}\cdot b_\alpha\otimes c_0^u + a_0^s\otimes b_0^t\otimes c_0^{u-1} \\
&\cdot c_\alpha \mid \alpha = 1,\ldots,q\rangle
\end{aligned}$$

and

$$\begin{aligned}
L^{d-2}\cdot Q = \Big\langle & \sum_\alpha a_0^{s-1}\cdot a_\alpha\otimes b_0^{t-1}\cdot b_\alpha\otimes c_0^u + a_0^{s-1}\cdot a_\alpha\otimes b_0^t\otimes c_0^{u-1}\cdot c_\alpha + a_0^s\otimes b_0^{t-1} \\
&\cdot b_\alpha\otimes c_0^{u-1}\cdot c_\alpha + a_0^{s-2}\cdot a_\alpha^2\otimes b_0^t\otimes c_0^u + a_0^s\otimes b_0^{t-2}\cdot b_\alpha^2\otimes c_0^{u-1}\cdot c_\alpha \\
&+ a_0^s\otimes b_0^t\otimes c_0^{u-2}\cdot c_\alpha^2\Big\rangle.
\end{aligned}$$

Proof We must have $\mathbb{P}E_{110} \cap Seg(\mathbb{P}A \times \mathbb{P}B) \neq$. This may be achieved by adding some

$$(u_0 a_0 + \sum_{\alpha} u_\alpha a_\alpha) \otimes (v_0 b_0 + \sum_{\beta} v_\beta b_\beta)$$

for $u_0, u_\alpha, v_0, v_\beta \in \mathbb{C}$. We also must have a flag as in Observation 2.5. Taking anything other than $a_0 \otimes b_0$, $(u_0 a_0 + a) \otimes b_0$ with $u_0 \in \mathbb{C}$, $a \in M$, or $x a_0 \otimes b_\alpha + y a_\alpha \otimes b_0$ (i.e., some $a_0 \otimes b_\alpha$ or $a_\alpha \otimes b_0$ since we are working modulo $T(C^*)$) makes the flag condition $\mathbb{P}F_2 \subset \sigma_2(Seg(\mathbb{P}A \times \mathbb{P}B))$ fail. (Here we use that $q > 2$.) Taking anything other than $a_0 \otimes b_0$ makes the flag condition $\mathbb{P}F_3 \subset \sigma_3(Seg(\mathbb{P}A \times \mathbb{P}B))$ fail. Thus there is a unique E_{110}, and by symmetry unique E_{101} and E_{011}. This triple exactly passes all degree three tests.

To see E_{200} must be as asserted, it must be such that $(E_{200} \otimes B) \supseteq E_{210}$. In order to have $L^{\otimes 3}$ in this intersection, we need $L^{\otimes 2} \subset E_{200}$. In order to have $L^2 \cdot M = \langle a_0 \otimes a_0 \otimes b_\alpha + a_0 \otimes a_\alpha \otimes b_0 + a_\alpha \otimes a_0 \otimes b_0 \rangle$ in the intersection, we see it must also contain $\langle a_0 \otimes a_\alpha + a_\alpha \otimes a_0 \rangle = L \cdot M$. In order to have $L \cdot Q = \langle \sum_\alpha (a_0 \otimes a_\alpha \otimes b_\alpha + a_\alpha \otimes a_0 \otimes b_\alpha + a_\alpha \otimes a_\alpha \otimes b_0) \rangle$ in the intersection, we see it must also contain $\langle \sum_\alpha a_\alpha \otimes a_\alpha \rangle = \langle Q \rangle$.

For the general case, assume by induction $E_{s-1,t,u}, E_{s,t-1,u}, E_{s,t,u-1}$ are as asserted and isomorphic as a module to $L^{\otimes d-1} \oplus L^{d-2} \cdot M \oplus L^{d-3} \cdot Q$. Arguing as we did for E_{200}, first obtaining $L^{\otimes d}$, then $L^{d-1} \cdot M$, then $L^{d-2} \cdot Q$ we conclude. □

Note that the ideal is $G_{T_{\mathrm{cw},q}}$-fixed as indeed it has to be if VSP is a point.

Now let $q = 2$, in this case it is more convenient to write $T_{\mathrm{cw},2}$ as

$$T_{\mathrm{cw},2} = \sum_{\sigma \in \mathfrak{S}_3} a_{\sigma(1)} \otimes b_{\sigma(2)} \otimes c_{\sigma(3)}.$$

Write $A = B = C = L_1 \oplus L_2 \oplus L_3$ where, e.g., for A, $L_j = \langle a_j \rangle$. A straight-forward Lie algebra calculation shows $G_{T_{\mathrm{cw},2}} \supseteq (\mathbb{C}^*)^{\times 3}$.

Theorem 6.2 *$VSP(T_{\mathrm{cw},2})$ and $VSP_{v_3(\mathbb{P}^2), \mathbb{P}S^3\mathbb{C}^3}(T_{\mathrm{cw},2})$ each consists of three points. One choice has for all s, t, u with $s + t + u = d$, the annihilator in degree (s, t, u) equal to*

$$L_1^s \otimes L_1^t \otimes L_1^u \oplus \phi(L_1^{d-1} \otimes L_2) \oplus \phi(L_1^{d-1} \otimes L_3) \oplus \phi(L_1^{s-2} \otimes L_2 \otimes L_3)$$

where $\phi : (L_1^{d-1} \otimes L_x) \to S^s A \otimes S^t B \otimes S^u C$ is the symmetric embedding. The other two choices arise from exchanging the role of L_1 with L_2, L_3.

Proof We have $T_{\mathrm{cw},2}(C^*) = \langle a_i \otimes b_j + b_j \otimes a_i \mid i \neq j \rangle$. The only possibilities for E_{110} for $r = 4$ that pass the (210)-test arise by adding $a_k \otimes b_k$ to this for some $k \in \{1, 2, 3\}$. Take $k = 1$. Then

$$(E_{110} \otimes A) \cap (S^2 A \otimes B) = \left\langle a_1^2 \otimes b_1, a_1 a_2 \otimes b_1 + a_1^2 \otimes b_2, a_1 a_3 \otimes b_1 + a_1^2 \otimes b_3, \right.$$
$$\left. \sum_{\sigma \in \mathfrak{S}_3} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes b_{\sigma(e)} \right\rangle$$

The only compatible choice of E_{200} is $\langle a_1^2, a_1 a_2, a_1 a_3, a_2 a_3 \rangle$. The situation for higher multi-degrees is similar. □

Remark 6.3 In contrast to $T_{\mathrm{cw},2}$, by Corollary 3.2, $\dim \mathrm{VSP}(T_{\mathrm{skewcw},2}) \geq 8$. From [23] (slightly changing notation) we have the rank five decomposition:

$$\begin{aligned} T_{\mathrm{skewcw},2} = \frac{1}{2} & [2a_1 \otimes (b_2 - b_3) \otimes (c_2 + c_3) \\ & - (a_1 + a_2) \otimes (b_1 - b_3) \otimes (c_1 + c_3) - (a_1 - a_2) \otimes (b_1 + b_3) \otimes (c_1 - c_3) \\ & + (a_1 + a_3) \otimes (b_1 - b_2) \otimes (c_1 + c_2) - (a_1 - a_3) \otimes (b_1 + b_2) \otimes (c_1 - c_2)] \end{aligned}$$

and the orbit of this decomposition already has dimension 8. (This can be seen by noting that more than four distinct vectors in $\mathbb{C}^3$ appear in the decomposition.)

7 $T_{\mathrm{skewcw},q}$, $q > 2$

Proof of Theorem 1.3 For the upper bound, we have

$$\begin{aligned} & T_{\mathrm{skewcw},q} \\ & = \lim_{t \to 0} \frac{1}{t^3} \left[\sum_{\xi} [(a_0 + t^2 a_\xi) \otimes (b_0 - t^2 b_\xi) \otimes (c_0 - t c_{\xi+p}) + (a_0 - t^2 a_\xi) \right. \\ & \otimes (b_0 - t b_{\xi+p}) \otimes (c_0 + t^2 c_\xi) + (a_0 - t a_{\xi+p}) \otimes (b_0 + t^2 b_\xi) \otimes (c_0 - t^2 c_\xi)] \\ & + \frac{1}{t^5} \left(a_0 + t^3 \sum_{\xi} a_{\xi+p} \right) \otimes \left(b_0 + t^3 \sum_{\xi} b_{\xi+p} \right) \otimes \left(c_0 + t^3 \sum_{\xi} c_{\xi+p} \right) \\ & \left. - \left(\frac{3q}{2t^2} + \frac{1}{t^5} \right) a_0 \otimes b_0 \otimes c_0 \right]. \end{aligned} \tag{57}$$

For the lower bounds, write $A = B = C = L \oplus M$ with $\dim L = 1$, $\dim M = q$ and M is equipped with a symplectic form Ω. A straight-forward Lie algebra calculation shows $G_{T_{\mathrm{skewcw},q}} \supseteq Sp(M) \times GL(L) \times M^* \otimes L$. Then

$$A \otimes B = L^{\otimes 2} \oplus L \cdot M \oplus S^2 M \oplus \Lambda^2 M_0 \oplus (L \wedge M \oplus \Omega)$$

where the term in parentheses equals $T_{\mathrm{skewcw},q}(C^*)$. Here $\Lambda^2 M_0 = M_2$, the complement to the $Sp(M)$-trivial representation in $\Lambda^2 M$.

We have the following weight diagram for the $G_{T_{\text{skewcw},q}}$-complement of $T(C^*)$ in $A\otimes B$:

$$\begin{tikzcd}
L^{\otimes 2} & L\cdot M & & S^2M & & & \Lambda^2 M & \\
a_0\otimes b_0 & & & & & & & \\
& a_0\otimes b_1+a_1\otimes b_0 \arrow[ul] & & & & & & \\
& a_0\otimes b_2+a_2\otimes b_0 \arrow[u] & & a_1\otimes b_1 \arrow[ull] & & & & \\
& a_0\otimes b_3+a_3\otimes b_0 \arrow[u] & & a_1\otimes b_2+a_2\otimes b_1 \arrow[u] \arrow[ull] & & & a_1\otimes b_2-a_2\otimes b_1 & \\
& \vdots & & a_1\otimes b_3+a_3\otimes b_1 \arrow[u] \arrow[ull] & & & a_1\otimes b_3-a_3\otimes b_1 \arrow[u] & \\
& & a_2\otimes b_3+a_3\otimes b_2 \arrow[ur] & & a_1\otimes b_4+a_4\otimes b_1 \arrow[ul] & a_2\otimes b_3-a_3\otimes b_2 \arrow[ur] & & a_1\otimes b_4-a_4\otimes b_1 \arrow[ul] \\
& & & \vdots & & & \vdots &
\end{tikzcd}$$

We will show that for $q \leq 10$, there is no choice of E'_{110} satisfying all degree three tests when $r = \frac{3}{2}q + 1$. We focus on the case $q = 10$ as that is the most difficult, the other cases are easier.

Note that elements of M may be raised to L, so an element of S^2M cannot be placed in E'_{110} unless its raising to $L\cdot M$ is also there. On the other hand, since $L\wedge M \subset E_{110}$, there is no similar restriction on elements of $\Lambda^2 M$.

We now restrict to $q = 10$. We split the types of (110) spaces into 10 types of cases depending on the dimension of E'_{110} intersected with the various irreducible modules:

case	$L^{\otimes 2}$	$L\cdot M$	S^2M	$\Lambda^2_0 M$
1	1	4	0	0
2	1	3	1	0
3	1	2	2	0
$4x$	1	2	$1+\frac{1}{2}$	$\frac{1}{2}$
5	0	0	0	5
6	1	0	0	4
7	1	1	0	3
8	1	2	0	2
9	1	1	1	2
10	1	2	1	1

Types 1, 2, 3, 8, 9, 10 are all single cases Types 5, 6, 7 each involve a choice of subset of weight vectors in $\Lambda^2_0 M$ (so they are each a collection of a finite number of cases) and case 4 involves a parameter, where we use $\frac{1}{2}$ to indicate the parameter, as the weight vector is a sum of a vector in the two indicated spaces. Explicitly, case $4x$ may be written

$$E'_{110} = \langle a_0\otimes b_0, a_0\otimes b_1 + a_1\otimes b_0, a_0\otimes b_2 + a_2\otimes b_0, a_1\otimes b_1, x(a_1\otimes b_2 + a_2\otimes b_1) \\ + a_1\otimes b_2 - a_2\otimes b\rangle.$$

Of these cases 1, 2, 3, $4x$, 8, 10 pass the (210) and (120) tests. No triple passes the (111) test. □

We remark that the decomposition (57) is $\mathbb{Z}_3$-invariant.

Corollary 7.1 *For $10 \geq q > 2$, and $q = 2p$ even, $VSP(T_{\text{skewcw},q})$ contains the isotropic Grassmannian $G_{\Omega}(\frac{q}{2}, M)$. In particular it has dimension at least $\binom{p}{2}$.*

Proof Using $Sp(M) \subset G_{T_{\text{skewcw},q}}$ we may replace $\langle a_{\xi} \rangle$ in (57) with any isotropic subspace as long as we replace $\langle a_{\xi+p} \rangle$ with the corresponding dual subspace and the same changes in B, C. □

8 A Simpler Waring Border Rank 17 Expression for $\det_3$

In this section and the next, we present explicit decompositions. The method used to obtain the decompositions is discussed after the second decomposition at the end of Sect. 9.

Set $i = \sqrt{-1}$ and $\zeta = e^{2\pi i/12}$. Then $\det_3 = \sum_{s=1}^{17} m_s^{\otimes 3}(t) + O(t)$, where the m_s are the following matrices

$$
\begin{pmatrix} \frac{\zeta^6}{t^5} & 0 & 0 \\ 0 & \zeta^6 & 0 \\ 0 & 0 & t^5 \end{pmatrix}
\begin{pmatrix} \frac{1}{t^5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\begin{pmatrix} \frac{\zeta^6}{t^5} & 0 & 0 \\ 0 & 0 & t\zeta^8 \\ 0 & t^4\zeta^4 & 0 \end{pmatrix}
\begin{pmatrix} \frac{\zeta^4}{t^5} & 0 & 0 \\ 0 & 0 & t\zeta^6 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} \frac{\zeta^5}{t^5} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t^4 & 0 \end{pmatrix}
\begin{pmatrix} \frac{\zeta^3}{t^5} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t^5 \end{pmatrix}
\begin{pmatrix} 0 & \frac{\zeta^{10}}{t^4} & \frac{\zeta^8}{t^3} \\ 0 & 0 & t\zeta^8 \\ t^3\zeta^6 & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & \frac{\zeta^8}{t^4} & \frac{\zeta^6}{t^3} \\ 0 & 0 & t\zeta^6 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & \frac{1}{t^4} & \frac{1}{t^3} \\ 0 & 0 & 0 \\ t^3\zeta^6 & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & \frac{\zeta^6}{t^4} & 0 \\ \frac{\zeta^6}{t} & 0 & 0 \\ 0 & 0 & t^5\zeta^6 \end{pmatrix}
\begin{pmatrix} 0 & \frac{\zeta^{11}}{t^4} & 0 \\ 0 & 0 & 0 \\ t^3 & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & \frac{\zeta^9}{t^4} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t^5\zeta^6 \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & 0 & \frac{1}{t^3} \\ \frac{\zeta^6}{t} & 0 & 0 \\ 0 & t^4\zeta^6 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 0 & \frac{1}{t^3} \\ 0 & \zeta^4 & 0 \\ t^3\zeta^2 & t^4 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 0 & \frac{\zeta^6}{t^3} \\ 0 & \zeta^{10} & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & \frac{5^{\frac{5}{6}}\zeta^2}{5t^4} & \frac{\left(1+\frac{2}{5}\sqrt{5}\right)^{\frac{1}{3}}\zeta^2}{t^3} \\ \frac{\left(1-\frac{2}{5}\sqrt{5}\right)^{\frac{1}{3}}\zeta^8}{t} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & \frac{5^{\frac{5}{6}}\zeta^8}{5t^4} & \frac{\left(1-\frac{2}{5}\sqrt{5}\right)^{\frac{1}{3}}\zeta^2}{t^3} \\ \frac{\left(1+\frac{2}{5}\sqrt{5}\right)^{\frac{1}{3}}\zeta^8}{t} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

9 A Numerical Border Rank 42 Expression for $T_{\text{skewcw},4}^{\boxtimes 2}$

What follows is an expression for $T_{skewcw,4}^{\boxtimes 2}$ as $\sum_{s=1}^{42} m_s(t)^{\otimes 3} + O(t)$ that is satisfied to an error of at most 4.4×10^{-15} in each entry. It consists of 42 matrices whose entries are rational expressions in the following 36 complex numbers: Let $i = \sqrt{-1}$ and let $\zeta = e^{2\pi i/12}$. Set

$$
\begin{aligned}
z_0 &= -0.8660155098072051 + 0.9452855522785384i & z_1 &= -1.2981710770246242 + 0.0008968724089185688i\\
z_2 &= 2.9260271139931078 + 0.1853833642730014i & z_3 &= 0.2542517122150322 + 0.30793819438378284i\\
z_4 &= 0.6964375578992822 + 0.2772662627986198i & z_5 &= 0.5507020325318998 - 0.0493931308002328i\\
z_6 &= 1.149228383831849 - 1.1683147648642283i & z_7 &= 0.6586404058476252 - 0.16578044112199047i\\
z_8 &= 0.7654345273805864 - 0.06877274843008892i & z_9 &= 0.544690883860558 + 0.09720573163212605i\\
z_{10} &= 0.6932236636741451 + 0.14980159446358277i & z_{11} &= 0.5862637032385472 - 0.12844523449559558i\\
z_{12} &= 2.384363992555291 - 0.08927102369428247i & z_{13} &= 0.9664252976479286 + 0.08480470055107503i\\
z_{14} &= 0.6190926897383283 + 0.15631000400545272i & z_{15} &= 0.6283592253932955 - 0.5626050553495663i\\
z_{16} &= 1.8190778570602204 - 0.22163457440913656i & z_{17} &= 1.153187286528645 - 0.07977233251120702i\\
z_{18} &= 1.4498877801613976 - 0.22515738202335905i & z_{19} &= 0.7262464450114047 + 0.7050051641972112i\\
z_{20} &= 1.1195537528292199 - 0.26381000320340176i & z_{21} &= 0.4400325048210471 + 0.6593492930106759i\\
z_{22} &= 0.3476654993676339 + 0.4095417606798612i & z_{23} &= 0.9459769225333798 + 0.24589162882727128i\\
z_{24} &= 0.7637135867709066 - 0.10529269213820387i & z_{25} &= 0.7409392923310902 - 0.10474756303325146i\\
z_{26} &= 1.0112068238001992 - 0.12695675940574122i & z_{27} &= 1.5005677845016696 - 0.24533651960180036i\\
z_{28} &= 0.6134145054919202 + 0.08121891266185506i & z_{29} &= 1.145625294745251 - 0.3813562005184122i\\
z_{30} &= 1.0607612533915372 - 0.016294891090460426i & z_{31} &= 0.941339345482511 + 0.20413704882122435i\\
z_{32} &= 0.622575977639622 + 0.2555810563389569i & z_{33} &= 0.951746321194872 - 0.2894768358835511i\\
z_{34} &= 1.0532801812660977 - 0.2502246606675517i & z_{35} &= 1.0207644184200035 - 0.2106937666100475i.
\end{aligned}
$$

The 42 matrices are:

$$
\begin{pmatrix}
\frac{\zeta^3 z_{25}z_{26}z_{31}z_{35}}{t^{269}} & \frac{\zeta^8 z_{28}t^{61}}{z_{23}z_{25}z_{26}z_{31}z_{33}z_{34}} & \frac{\zeta^{10}z_{24}z_{35}t^{13}}{z_{23}z_{25}z_{26}z_{31}z_{33}z_{34}} & 0 & 0\\
\frac{z_{25}z_{26}z_{31}z_{33}z_{35}}{t^{104}} & 0 & 0 & 0 & \frac{\zeta^9 z_{18}z_{26}z_{32}t^{148}}{z_{21}z_{23}z_{33}z_{35}^2}\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\zeta^7 z_{35}t^{121}}{z_{23}z_{25}z_{26}^2 z_{31}z_{33}z_{34}} & 0 & \frac{\zeta^8 z_{23}z_{34}t^{91}}{z_{24}z_{35}^2}\\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{\zeta^{10}z_{30}}{t^{269}} & 0 & \frac{\zeta^{10}z_{24}z_{35}t^{13}}{z_{27}z_{28}z_{30}z_{33}} & 0 & 0\\
0 & 0 & \frac{\zeta^7 z_{24}z_{35}t^{178}}{z_{27}z_{28}z_{30}} & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\zeta^6 z_{27}z_{28}z_{35}^2 t^{121}}{z_{23}^2 z_{25}z_{26}^2 z_{31}z_{33}z_{34}^2} & 0 & 0\\
0 & \frac{\zeta t^{184}}{z_{27}z_{30}z_{34}} & 0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & \frac{z_{18}z_{24}z_{28}z_{32}t^{211}}{z_{21}z_{23}^2 z_{25}z_{33}^3 z_{35}} & z_0 t^{163} & 0 & \frac{\zeta z_{34}t^{133}}{z_{25}}\\
0 & 0 & 0 & 0 & 0\\
\frac{\zeta z_{23}z_{25}z_{33}}{z_{24}z_{31}t^{146}} & \frac{\zeta^5 z_{31}z_{35}t^{184}}{z_{23}z_{27}^2 z_{28}z_{30}z_{34}^2} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\zeta^9 z_{30}}{z_{27}t^{269}} & 0 & 0 & \frac{\zeta^7}{t^{65}} & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & \frac{\zeta^5 z_{27}t^{169}}{z_{30}z_{33}} & 0 & 0 & 0\\
0 & \frac{\zeta^9 z_{27}t^{184}}{z_{30}z_{34}} & \frac{z_{28}t^{136}}{z_{26}z_{34}z_{35}} & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{\zeta^4 z_{19}z_{35}}{z_3 z_{12}z_{13}z_{18}z_{32}t^{269}} & 0 & 0 & 0 & \frac{\zeta^4 z_{18}z_{21}z_{34}}{z_2 z_{12}z_{25}z_{31}z_{32}z_{35}t^{17}}\\
\frac{\zeta z_{19}z_{33}z_{35}}{z_3 z_{12}z_{13}z_{18}z_{32}t^{104}} & 0 & 0 & 0 & \frac{\zeta^9 z_{19}z_{26}z_{32}^2 z_{33}z_{35}t^{148}}{z_3 z_{12}z_{13}z_{18}}\\
0 & 0 & 0 & 0 & 0\\
\frac{\zeta^4 z_2 z_3 z_{13}z_{31}}{z_{12}z_{19}z_{20}z_{21}z_{32}z_{34}t^{161}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{z_{34}t^{269}} & 0 & 0 & \frac{\zeta z_{24}z_{26}z_{35}}{z_{28}z_{34}t^{65}} & 0\\
0 & 0 & 0 & 0 & 0\\
\frac{\zeta^5}{t^{119}} & 0 & 0 & \frac{z_{24}z_{26}z_{35}t^{85}}{z_{28}} & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$\begin{pmatrix} 0 & \frac{\zeta^2 z_{15} z_{18} z_{19} z_{28} z_{32} t^{61}}{z_{21} z_{23} z_{25} z_{31} z_{33}^2 z_{34}} & \frac{\zeta^{11} z_{15} z_{19} z_{35} t^{13}}{z_{34}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11} z_{15} z_{21} z_{23} z_{25} z_{31} z_{35}}{z_{18} z_{19}^2 z_{26} z_{32} z_{34} t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^9 z_{21} z_{23} z_{25} z_{33} z_{34}}{z_{15}^2 z_{18} z_{19}^2 z_{26} z_{31}^2 z_{32} z_{35}^2 t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^5 z_{25} z_{31}}{z_{23} z_{34} t^{269}} & 0 & 0 & 0 & \frac{\zeta^4}{t^{17}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10} z_{25} z_{31}}{z_{23} t^{119}} & 0 & \frac{\zeta^7 z_{23} z_{24} z_{33} t^{163}}{z_{20}^2 z_{31}} & 0 & \zeta^3 z_{34} t^{133} \\ 0 & \frac{\zeta^2 z_{23}^2 z_{28} z_{34}^2 t^{169}}{z_{25}^2 z_{31}^2 z_{33} z_{35}} & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^{10} z_{23} t^{136}}{z_{25} z_{31}} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{11} z_3 z_{19}}{z_{12} z_{21} z_{23} z_{25} z_{32} t^{269}} & 0 & 0 & 0 & \frac{\zeta^{10} z_2 z_{18} z_{21} z_{23} z_{24} z_{34}}{z_{12} z_{13} z_{19} z_{20} z_{31} z_{32} z_{35} t^{17}} \\ \frac{\zeta^8 z_3 z_{19} z_{33}}{z_{12} z_{21} z_{23} z_{25} z_{32} t^{104}} & 0 & 0 & 0 & \frac{\zeta^4 z_3 z_{19} z_{26} z_{32}^2 z_{33} t^{148}}{z_{12} z_{21} z_{23} z_{25}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^9 z_{13} z_{31} z_{35}}{z_2 z_3 z_{12} z_{18} z_{24} z_{32} z_{34} t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^9 z_{28}}{t^{269}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^2 z_{28} z_{34}}{t^{119}} & \frac{\zeta^{10} z_{34} t^{211}}{z_{28} z_{33} z_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\zeta^5 z_{33} z_{35} t^{58}}{z_{34}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^5 z_{24} z_{26} z_{29}^3 z_{30}}{z_{27} t^{269}} & 0 & \frac{\zeta^{11} z_{22}^3 z_{27} z_{33} z_{35} t^{13}}{z_{20} z_{23} z_{28} z_{30} z_{31}^2} & \frac{\zeta^2}{t^{65}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10} z_{24} z_{26} z_{29}^3 z_{30} z_{34}}{z_{27} t^{119}} & 0 & 0 & \zeta z_{34} t^{85} & \frac{\zeta^{10} z_{18} z_{22}^3 z_{26} z_{27} z_{32} z_{34} z_{35} t^{133}}{z_{20} z_{21} z_{23}^2 z_{25} z_{28} z_{30}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10} z_{18} z_{22}^3 z_{25} z_{26} z_{27} z_{32} z_{33} z_{35}}{z_{20} z_{21} z_{23} z_{24} z_{28} z_{30} z_{31} t^{146}} & \frac{z_{27} t^{184}}{z_{24} z_{26} z_{29}^3 z_{30} z_{34}} & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^2 z_{25} z_{26} z_{31} z_{32} z_{35}}{z_{18} z_{21} z_{34} t^{269}} & \frac{\zeta^{11} z_{21} z_{24} z_{32}^2 z_{35} t^{61}}{z_{25} z_{31}} & 0 & 0 & 0 \\ \frac{\zeta^{11} z_{25} z_{26} z_{31} z_{32} z_{33} z_{35}}{z_{18} z_{21} z_{34} t^{104}} & 0 & 0 & 0 & \frac{\zeta^{11} z_{28} t^{148}}{z_{21}^2 z_{23} z_{24} z_{32} z_{33} z_{34} z_{35}^3} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11} z_{25} z_{31} z_{32} z_{35}}{z_{18} z_{21} z_{24} z_{34} t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\zeta^6 z_{18} z_{26} z_{31} z_{32}}{z_{20} z_{21} z_{34} t^{17}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta z_{21}^2 z_{23}^2 z_{24} z_{25} z_{33}}{z_{18}^2 z_{19}^3 z_{20} z_{26}^2 z_{31} z_{32}^2 t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11}}{t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^3 z_9 z_{10} z_{11} z_{18} z_{19} z_{30} z_{32} z_{34}}{z_5 z_7 z_8 z_{16} z_{21} z_{23} z_{25} z_{33} t^{269}} & 0 & 0 & 0 & \frac{\zeta z_5 z_{11} z_{16} z_{18} z_{24} z_{26} z_{31} z_{32}}{z_4 z_{20} z_{21} z_{25} t^{17}} \\ 0 & 0 & \frac{\zeta^{11} z_5 z_{11} z_{16} z_{20} z_{33} t^{178}}{z_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10} z_4}{z_{19} z_{20} z_{27} z_{30} z_{33} t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^2 z_4}{z_{19} z_{20} z_{27} z_{30} z_{34} t^{146}} & 0 & \frac{\zeta^6 z_9 z_{10} z_{11} z_{18} z_{19} z_{27} z_{28} z_{32} t^{136}}{z_5 z_7 z_8 z_{16} z_{21} z_{23} z_{25} z_{26} z_{33} z_{35}} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^5 z_{28} z_{35}^2}{t^{269}} & 0 & 0 & \frac{\zeta^5}{t^{65}} & 0 \\ \frac{\zeta^2 z_{28} z_{33} z_{35}^2}{t^{104}} & 0 & 0 & \zeta^8 z_{33} t^{100} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^{43} & 0 \\ 0 & 0 & 0 & \frac{\zeta^{10} z_{33} t^{58}}{z_{34}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{\zeta^9 z_{17}^2 z_{21} z_{24} z_{28} z_{29}^2 t^{61}}{z_{18} z_{22}^2 z_{25} z_{27}^2 z_{32} z_{33} z_{34} z_{35}} & \frac{\zeta^4 z_{22} z_{27} t^{13}}{z_{17} z_{26} z_{29} z_{31}^2} & \frac{\zeta z_{22} z_{27} z_{31} z_{33} z_{35}}{z_{17} z_{28} z_{29} t^{65}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta^2 z_{17}^2 z_{21} z_{24} z_{28} z_{29}^2 t^{211}}{z_{18} z_{22}^2 z_{25} z_{27}^2 z_{32} z_{33} z_{35}} & 0 & \frac{z_{22} z_{27} z_{31} z_{33} z_{34} z_{35} t^{85}}{z_{17} z_{28} z_{29}} & \frac{\zeta^3 z_{18} z_{22} z_{27} z_{32} z_{34} t^{133}}{z_{17} z_{21} z_{23} z_{25} z_{29} z_{33}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^3 z_{18} z_{22} z_{25} z_{27} z_{32}}{z_{17} z_{21} z_{24} z_{29} z_{31} t^{146}} & \frac{\zeta^2 z_{17}^2 z_{22} t^{184}}{z_{20} z_{23} z_{24} z_{29} z_{30}^2 z_{31} z_{34}} & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{11} z_{27} z_{28} z_{30}^2}{z_{26} z_{35} t^{269}} & 0 & 0 & 0 & \frac{\zeta^3}{t^{17}} \\ 0 & 0 & \frac{\zeta z_{20}^2 z_{21} z_{25} z_{33} t^{178}}{z_{18} z_{24} z_{26} z_{31} z_{32}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^7 z_{26} z_{35} t^{121}}{z_{27} z_{28} z_{30}^2 z_{33}} & 0 & 0 \\ 0 & 0 & \frac{z_{25} z_{31} t^{136}}{z_{23} z_{26} z_{34}^2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^8 z_{25} z_{26} z_{31} z_{32} z_{35}}{z_{18} z_{21} z_{34} t^{269}} & 0 & \frac{\zeta^4 z_{21} z_{24} z_{35} t^{13}}{z_{25} z_{26} z_{31} z_{32}} & 0 & 0 \\ \frac{\zeta^5 z_{25} z_{26} z_{31} z_{32} z_{33} z_{35}}{z_{18} z_{21} z_{34} t^{104}} & 0 & 0 & 0 & \frac{\zeta^5 z_{28} t^{148}}{z_{21}^2 z_{23} z_{24} z_{32} z_{33} z_{34} z_{35}^3} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^5 z_{25} z_{31} z_{32} z_{35}}{z_{18} z_{21} z_{24} z_{34} t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{10} z_5 z_{18} z_{19} z_{28} z_{30} z_{32}}{z_{21} z_{23} z_{25} z_{33} z_{35} t^{269}} & 0 & 0 & 0 & \frac{z_4 z_{11} z_{16} z_{31}}{z_5 z_{19} z_{20} t^{17}} \\ 0 & 0 & \frac{\zeta^{10} z_4 z_{11} z_{16} z_{20} z_{21} z_{25} z_{33} t^{178}}{z_5 z_{18} z_{19} z_{24} z_{26} z_{32}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10} z_9 z_{10} z_{11} z_{18} z_{24} z_{26} z_{32} z_{34} z_{35}}{z_4 z_7 z_8 z_{16} z_{20} z_{21} z_{25} z_{27} z_{28} z_{30} z_{33} t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^2 z_9 z_{10} z_{11} z_{18} z_{24} z_{26} z_{32} z_{35}}{z_4 z_7 z_8 z_{16} z_{20} z_{21} z_{25} z_{27} z_{28} z_{30} t^{146}} & 0 & \frac{\zeta z_5 z_{18} z_{19} z_{27} z_{28}^2 z_{32} t^{136}}{z_{21} z_{23} z_{25} z_{26} z_{33} z_{34} z_{35}^2} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^4 z_{17}^2 z_{22} z_{26} z_{29}}{z_{20} z_{23} z_{30} z_{34} t^{269}} & 0 & \frac{z_{22} z_{27} z_{28} z_{29} z_{30} z_{34}^2 t^{13}}{z_{17} z_{26} z_{31} z_{35}} & \frac{\zeta^8 z_{22} z_{24} z_{26} z_{29} z_{33} z_{35}^2}{z_{17} z_{27} z_{28}^2 z_{31} z_{34} t^{65}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^9 z_{17}^2 z_{22} z_{26} z_{29}}{z_{20} z_{23} z_{30} t^{119}} & 0 & 0 & \frac{\zeta^7 z_{22} z_{24} z_{26} z_{29} z_{33} z_{35}^2 t^{85}}{z_{17} z_{27} z_{28}^2 z_{31}} & \frac{\zeta^{11} z_{18} z_{22} z_{27} z_{28} z_{29} z_{30} z_{31} z_{32} z_{34}^3 t^{133}}{z_{17} z_{21} z_{23} z_{25} z_{33} z_{35}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11} z_{18} z_{22} z_{25} z_{27} z_{28} z_{29} z_{30} z_{32} z_{34}^2}{z_{17} z_{21} z_{24} z_{35} t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \zeta^5 z_{19} z_{31} t^{13} & 0 & 0 \\ 0 & 0 & \zeta^8 z_{19} z_{31} z_{33} t^{178} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^8 z_{21} z_{23} z_{25}}{z_{18} z_{19}^2 z_{26} z_{31} z_{32} t^{161}} & 0 & 0 & 0 & 0 \\ \frac{z_{21} z_{23} z_{25} z_{33}}{z_{18} z_{19}^2 z_{26} z_{31} z_{32} z_{34} t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^3}{t^{269}} & \frac{\zeta^7 t^{61}}{z_{33}} & 0 & \frac{\zeta^{10}}{t^{65}} & 0 \\ \frac{z_{33}}{t^{104}} & 0 & 0 & \zeta z_{33} t^{100} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^4}{t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{\zeta^{10} z_{16}^2 z_{21} z_{23} z_{25} z_{28} t^{61}}{z_{18} z_{19} z_{32} z_{35}} & 0 & \frac{\zeta^4 z_{18} z_{19}^2 z_{26} z_{31} z_{32} z_{35}}{z_{16} z_{21} z_{23} z_{25} z_{28} t^{65}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta}{z_{16} z_{19} z_{26} z_{31} z_{33} t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^5}{z_{16} z_{19} z_{26} z_{31} z_{34} t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^7 z_{28}}{t^{269}} & \frac{\zeta^3 t^{61}}{z_{28} z_{33} z_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{z_{28} z_{34}}{t^{119}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta^4 t^{169}}{z_{28} z_{33} z_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\zeta^3 z_{33} z_{35} t^{58}}{z_{34}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{z_{14}}{t^{269}} & \frac{\zeta^4 z_{30} t^{61}}{z_{14}^2 z_{27} z_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\zeta^{11} z_{14} z_{27} t^{43}}{z_{30}} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^{10} z_{30}}{t^{269}} & \frac{\zeta^2 t^{61}}{z_{27} z_{30} z_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta z_{35} t^{121}}{z_{26} z_{27} z_{28} z_{30} z_{33}} & 0 & \frac{\zeta z_{27} z_{28} t^{91}}{z_{24} z_{35}} \\ 0 & 0 & 0 & 0 & \frac{\zeta^5 z_{27} z_{28} z_{33} t^{106}}{z_{24} z_{34} z_{35}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^4}{t^{269}} & \frac{\zeta^3 z_{28} t^{61}}{z_{33} z_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^9 z_{34}}{t^{119}} & \frac{\zeta^2 z_{28} z_{34} t^{211}}{z_{33} z_{35}} & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^2 t^{121}}{z_{26} z_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\zeta^4 z_{33} t^{106}}{z_{24} z_{34}} \end{pmatrix} \begin{pmatrix} \frac{\zeta^2 z_{14}}{t^{269}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta^9 z_{30} t^{226}}{z_{14}^2 z_{27}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta z_{30} t^{169}}{z_{14}^2 z_{27} z_{33}} & 0 & \frac{\zeta z_{14} z_{27} t^{43}}{z_{30}} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^4 z_{30}}{t^{269}} & 0 & \frac{\zeta^4 z_{24} z_{35} t^{13}}{z_{27} z_{28} z_{30} z_{33}} & 0 & 0 \\ 0 & 0 & \frac{\zeta z_{24} z_{35} t^{178}}{z_{27} z_{28} z_{30}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\zeta^7 z_{27} z_{28} t^{91}}{z_{24} z_{35}} \\ 0 & 0 & 0 & 0 & \frac{\zeta^{11} z_{27} z_{28} z_{33} t^{106}}{z_{24} z_{34} z_{35}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^8 z_{18} z_{22} z_{24} z_{26} z_{27} z_{32}}{z_{20} z_{21} z_{23} z_{31} z_{33} t^{269}} & 0 & \frac{\zeta^2 z_{20} z_{22} z_{27} z_{34} t^{13}}{z_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta z_{18} z_{22} z_{24} z_{26} z_{27} z_{32} z_{34}}{z_{20} z_{21} z_{23} z_{31} z_{33} t^{119}} & 0 & \frac{\zeta^3 z_{21} z_{23} z_{24} z_{31} t^{163}}{z_{18} z_{22}^2 z_{26} z_{27}^2 z_{32}} & 0 & \frac{\zeta^{10} z_{22} z_{27} z_{31} z_{33} t^{133}}{z_{20} z_{23} z_{25}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10} z_{22} z_{25} z_{27} z_{33}^2}{z_{20} z_{24} z_{34} t^{146}} & \frac{\zeta^3 z_{20} z_{22} z_{35} t^{184}}{z_{24} z_{27} z_{28} z_{30}} & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{10}}{t^{269}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^3 z_{34}}{t^{119}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^8 t^{121}}{z_{26} z_{33}} & 0 & 0 \\ 0 & z_1 t^{184} & \frac{\zeta^6 z_{25} z_{31} t^{136}}{z_{23} z_{34}^2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^5 z_{24} z_{25} z_{33}}{z_{20}^2 z_{34} t^{269}} & 0 & 0 & 0 & \frac{\zeta^8}{t^{17}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10} z_{24} z_{25} z_{33}}{z_{20}^2 t^{119}} & 0 & \zeta^9 t^{163} & 0 & \frac{\zeta^3 z_{23} z_{24} z_{33} z_{34} t^{133}}{z_{20}^2 z_{31}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta}{t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^6}{t^{269}} & 0 & \frac{\zeta z_{24} t^{13}}{z_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11} z_{34}}{t^{119}} & 0 & \frac{z_{24} z_{34} t^{163}}{z_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\zeta^6 z_{33} t^{106}}{z_{24} z_{34}} \end{pmatrix} \begin{pmatrix} \frac{\zeta^{10} z_9 z_{34}}{t^{269}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta z_7 z_8 z_{11} z_{18} z_{31} z_{32} z_{35} t^{226}}{z_{10} z_{21} z_{23} z_{25} z_{27} z_{28} z_{30}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^5 z_9 z_{34}}{t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^9 z_9 z_{33}}{t^{146}} & 0 & \frac{\zeta z_9 z_{27} z_{28} t^{136}}{z_{26} z_{30} z_{35}} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^7 z_{21} z_{24}}{z_{18} z_{23} z_{26} z_{32} t^{269}} & 0 & 0 & 0 & \frac{\zeta^5}{t^{17}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{z_{21} z_{24} z_{34}}{z_{18} z_{23} z_{26} z_{32} t^{119}} & 0 & 0 & 0 & \frac{\zeta^5 z_{21} z_{24} z_{34}^2 t^{133}}{z_{18} z_{25} z_{26} z_{31} z_{32}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^8 z_{18} z_{23} z_{25} z_{26} z_{32} z_{33}}{z_{20}^2 z_{21} z_{34} t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^9}{t^{269}} & 0 & 0 & \frac{\zeta^4}{t^{65}} & 0 \\ \frac{\zeta^6 z_{33}}{t^{104}} & 0 & 0 & \zeta^7 z_{33} t^{100} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10}}{t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{11} z_{25} z_{35}}{z_{13} t^{269}} & 0 & 0 & 0 & \frac{\zeta^7 z_{23} z_{34}}{z_{13} z_{31} z_{35}^2 t^{17}} \\ \frac{\zeta^8 z_{25} z_{33} z_{35}}{z_{13} t^{104}} & 0 & 0 & 0 & \frac{\zeta^4 z_{25} z_{26} z_{32}^3 z_{33} z_{35} t^{148}}{z_{13}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta^{11} z_{13}^2 z_{28} t^{169}}{z_{25}^2 z_{33}} & \frac{\zeta^9 z_{13}^2 z_{31} z_{35} t^{121}}{z_{23} z_{25} z_{33} z_{34}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^9 z_{25} z_{26} z_{31} z_{35}}{t^{269}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^6 z_{25} z_{26} z_{31} z_{33} z_{35}}{t^{104}} & 0 & 0 & 0 & \frac{\zeta^3 z_{18} z_{26} z_{32} t^{148}}{z_{21} z_{23} z_{33} z_{35}^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\zeta^2 z_{23} z_{34} t^{91}}{z_{24} z_{35}^2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^9 z_7 z_{11} z_{16} z_{30} z_{34}}{z_6 z_{27} t^{269}} & 0 & 0 & \frac{\zeta^{11} z_6 z_{10} z_{21} z_{23} z_{25} z_{26} z_{30}}{z_9 z_{18} z_{32} t^{65}} & 0 \\ 0 & \frac{\zeta^2 z_8 z_{10} z_{16}^2 z_{26} z_{28} z_{33} z_{34} t^{226}}{z_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^8 z_8 z_{10} z_{18} z_{32} z_{34}}{z_{16} z_{21} z_{23} z_{25} z_{31} z_{33} t^{161}} & 0 & 0 & 0 & 0 \\ \frac{z_8 z_{10} z_{18} z_{32}}{z_{16} z_{21} z_{23} z_{25} z_{31} t^{146}} & 0 & \frac{z_7 z_{11} z_{16} z_{28} t^{136}}{z_6 z_{26} z_{35}} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{10} z_6 z_8 z_{10} z_{26} z_{30} z_{34}}{z_{27} t^{269}} & 0 & 0 & \frac{\zeta^2 z_{10} z_{16} z_{28} z_{30}}{z_6 z_9 z_{31} z_{35} t^{65}} & 0 \\ 0 & \frac{\zeta^4 z_7 z_{11} z_{16}^2 z_{21} z_{23} z_{25} z_{26} z_{31} z_{33} z_{34} t^{226}}{z_{18} z_{32}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10} z_7 z_{11} z_{34} z_{35}}{z_{16} z_{28} z_{33} t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^2 z_7 z_{11} z_{35}}{z_{16} z_{28} t^{146}} & 0 & \frac{\zeta z_6 z_8 z_{10} z_{28} t^{136}}{z_{35}} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^8}{z_{15}^2 z_{18} z_{21} z_{23} t^{269}} & \frac{\zeta^3 z_{15} z_{18} z_{28} t^{61}}{z_{23} z_{25} z_{26} z_{31} z_{32} z_{33}^2 z_{34} z_{35}} & \frac{\zeta^4 z_{15} z_{21} z_{24} z_{32} z_{35}^3 t^{13}}{z_{34}} & 0 & 0 \\ \frac{\zeta^5 z_{33}}{z_{15}^2 z_{18} z_{21} z_{23} t^{104}} & 0 & 0 & 0 & \frac{\zeta z_{26} z_{32}^3 z_{33} t^{148}}{z_{15}^2 z_{18} z_{21} z_{23}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11} z_{15} z_{23} z_{25} z_{31}}{z_{18} z_{21} z_{24} z_{34} t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^7 z_{28} z_{35}^2}{t^{269}} & \frac{t^{61}}{z_{28} z_{33} z_{35}^2} & 0 & \frac{\zeta^7}{t^{65}} & 0 \\ \frac{\zeta^4 z_{28} z_{33} z_{35}^2}{t^{104}} & 0 & 0 & \zeta^{10} z_{33} t^{100} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta t^{169}}{z_{28} z_{33} z_{35}^2} & 0 & \zeta^2 t^{43} & 0 \\ 0 & 0 & 0 & \frac{z_{33} t^{58}}{z_{34}} & 0 \end{pmatrix}.$$

We now give an overview of the method used to obtain the expressions for $\det_3$ and $T_{\text{skewcw},4}^{\boxtimes 2}$:

Fix bases $a_i \in A$, $b_i \in B$, and $c_i \in C$. A tensor $T \in A \otimes B \otimes C$ has an expression $T = \sum_{i,j,k} T^{ijk} a_i \otimes b_j \otimes c_k$ and is *standard tight* in this basis if there exist injective functions $\omega_A : [m] \to \mathbb{Z}$, $\omega_B : [m] \to \mathbb{Z}$, $\omega_C : [m] \to \mathbb{Z}$ so that $T^{ijk} \neq 0$ implies $\omega_A(i) + \omega_B(j) + \omega_C(k) = 0$. In this case, we will call a choice of $(\omega_A, \omega_B, \omega_C)$ satisfying the constraints a set of *tight weights*. Given a set of tight weights for T, we consider border rank decompositions of the form:

$$T = \sum_{s=1}^r \mathcal{A}_s(t) \otimes \mathcal{B}_s(t) \otimes \mathcal{C}_s(t) + O(t), \tag{58}$$

where $\mathcal{A}_s(t) = \sum_{i=1}^m \mathcal{A}_{si} t^{\omega_A(i)} a_i$, $\mathcal{B}_s(t) = \sum_{j=1}^m \mathcal{B}_{sj} t^{\omega_B(j)} b_j$, and $\mathcal{C}_s(t) = \sum_{k=1}^m \mathcal{C}_{sk} t^{\omega_C(k)} c_k$. Note that when the tight weights are trivial, this is an ordinary rank decomposition. In our situation, the equations correspond to a strict subset of the equations describing a rank decomposition, namely those corresponding to triples (i, j, k) where $\omega_A(i) + \omega_B(j) + \omega_C(k) \leq 0$. In the case of $T_{\text{skewcw},4}^{\boxtimes 2}$ this reduces the number of equations down from $\binom{25+2}{3} = 2925$ to 692 and just as with a rank decomposition, there are $3rm = 3150$ unknowns.

We pick a choice of tight weights which minimizes the number of equations to be solved. The problem of obtaining a border rank decomposition is then split into two questions: first, to compute a set of tight weights $(\omega_A, \omega_B, \omega_C)$ so that $\#\{(i, j, k) \mid \omega_A(i) + \omega_B(j) + \omega_C(k) \leq 0\}$ is minimal, and second, to solve the resulting equations (58) in the $\mathcal{A}_{si}, \mathcal{B}_{sj}, \mathcal{C}_{sk}$.

Consider the first question. Given sets $S_{\leq}, S_{>} \subset [m] \times [m] \times [m]$, consider the problem of deciding if there are tight weights $(\omega_A, \omega_B, \omega_C)$ satisfying the additional constraints that $\omega_A(i) + \omega_B(j) + \omega_C(k) \leq 0$ for $(i, j, k) \in S_{\leq}$ and $\omega_A(i) + \omega_B(j) + \omega_C(k) \geq 1$ for $(i, j, k) \in S_{>}$. These conditions along with the original equality conditions form a linear program on the images of $(\omega_A, \omega_B, \omega_C)$ which may be efficiently solved. There is no harm in letting the linear program be defined over the rationals, as we may clear denominators to obtain a solution in integers. One can use this fact to prune an exhaustive search of choices of $S_{\leq}, S_{>}$ to find one for which $S_{\leq} \cup S_{>} = [m] \times [m] \times [m]$, there exists a corresponding set of tight weights, and $\#S_{\leq}$ is minimal. While this is an exponential procedure, this optimization was sufficient to solve the problem for this decomposition.

The second problem, solving the associated system, is solved with the Levenberg–Marquardt nonlinear least squares algorithm [37, 39]. The sparse structure of the answer is obtained by speculatively zeroing (or setting to simple values) coefficients until all freedom with respect to the equations is lost. In other words, we impose additional simple equations on the solution and solve again until we obtain an isolated point, which can be verified by checking that the Jacobian has full rank numerically. This procedure is repeated many times in order to find a simple solution. Ideally, we would prove the resulting parameters indeed approximate an exact solution to the equations by searching for additional relations between the parameters and then using such relations to make symbolic methods tractable. In this case, all such attempts failed. See [19] for further discussion of these techniques.

The border rank decomposition in this section is also a Waring border rank decomposition, that is, $A = B = C$, and $\mathcal{A}_s(t) = \mathcal{B}_s(t) = \mathcal{C}_s(t)$; in particular, $\omega_A = \omega_B = \omega_C$. This condition was imposed to make the nonlinear search more tractable, and it also has independent interest. The techniques presented are equally applicable in the symmetric case as well as the asymmetric.

We remark that numerous relaxations of this method are possible. It was inspired by the improved expression for $\det_3$, which had the structure we assume. It remains to determine how useful it will be for more general types of tensors.

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