THE VIRTUAL K-THEORY OF QUOT SCHEMES OF SURFACES

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Abstract. We study virtual invariants of Quot schemes parametrizing quotients of dimension at most 1 of the trivial sheaf of rank N on nonsingular projective surfaces. We conjecture that the generating series of virtual K-theoretic invariants are given by rational functions. We prove rationality for several geometries including punctual quotients for all surfaces and dimension 1 quotients for surfaces X with $p_g > 0$. We also show that the generating series of virtual cobordism classes can be irrational.

Given a K-theory class on X of rank r, we associate natural series of virtual Segre and Verlinde numbers. We show that the Segre and Verlinde series match in the following cases:

(i) Quot schemes of dimension 0 quotients,
(ii) Hilbert schemes of points and curves over surfaces with $p_g > 0$,
(iii) Quot schemes of minimal elliptic surfaces for quotients supported on fiber classes.

Moreover, for punctual quotients of the trivial sheaf of rank N, we prove a new symmetry of the Segre/Verlinde series exchanging r and N. The Segre/Verlinde statements have analogues for punctual Quot schemes over curves.

Contents

1. Introduction

1.1. **Overview.** While moduli spaces of sheaves on higher dimensional varieties rarely carry 2-term perfect obstructions theories, moduli spaces of sheaves on varieties of dimension at most 3 often have well-defined virtual fundamental classes. In many cases, the resulting virtual invariants have a rich structure, reflecting the underlying geometry of the moduli

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spaces used in their definition. We refer the reader to [PT3] for an introduction to sheaf counting methods in enumerative geometry.

An example whose virtual geometry can be studied effectively is the Quot scheme of 1-dimensional quotients over surfaces. If X is a nonsingular projective surface and $\beta \in H_2(X,\mathbb{Z})$ is an effective curve class, we let $\mathsf{Quot}_X(\mathbb{C}^N,\beta,n)$ parametrize short exact sequences

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0\,, \tag{1}$$

where

$$\text{rank } Q = 0\,,\ \ c_1(Q) = \beta\,,\ \ \chi(Q) = n\,.$$

By [MOP1], $\mathsf{Quot}_X(\mathbb{C}^N,\beta,n)$ carries a canonical 2-term perfect obstruction theory and a virtual fundamental class of dimension

$$\text{vdim} = \chi(S,Q) = Nn + \beta^2\,.$$

The virtual fundamental class of Quot schemes over curves had been constructed previously in [MO1]. Due to connections with Seiberg-Witten theory, the virtual fundamental class of Quot schemes over surfaces with $p_g > 0$ is typically more accessible, while the case $p_g = 0$ is less understood.

For the Quot schemes of 1-dimensional quotients over surfaces, the generating series of the following invariants have been considered in [OP, L, JOP] respectively:

(i) virtual Euler characteristics,
(ii) virtual χ_y-genera,
(iii) descendent invariants.

Conjecturally, series (i) – (iii) are always given by rational functions. Rationality was shown for arbitrary surfaces when $\beta = 0$, see [JOP, L, OP]. For non-zero curve classes, rationality was confirmed for (i) and (ii) for all surfaces with $p_g > 0$, see [L, OP]. Furthermore, series (i) and (iii) were also proven to be rational for simply connected surfaces with $p_g = 0$ when $N = 1$ in [JOP].

We refine here the techniques of [JOP, L, OP] to study the virtual K-theory of the Quot scheme $\mathsf{Quot}_X(\mathbb{C}^N,\beta,n)$. A central result is the rationality of natural series of virtual K-theoretic invariants for many geometries. The methods also allow us to prove several new identities and symmetries satisfied by the virtual Segre and Verlinde series. Along the way, we take up the series of cobordism classes, and study the virtual structure sheaf for punctual quotients.

1.2. **Seiberg-Witten invariants.** Let X be a nonsingular projective surface. Curve classes β of Seiberg-Witten length N were introduced in [L]. By definition, $\beta \in H_2(X,\mathbb{Z})$ is of Seiberg-Witten length N if for all effective decompositions

$$\beta = \beta_1 + \ldots + \beta_N$$

such that $\mathsf{SW}(\beta_i) \neq 0$ for all i, we have

$$\beta_i \cdot (K_X - \beta_i) = 0 \text{ for all } i\,.$$

Here, $\mathsf{SW}(\beta_i) \in H_\star(\mathrm{Pic}(X)) = \wedge^\star H^1(X)$ denotes the Seiberg-Witten invariant of [DKO] constructed via the Hilbert scheme of curves $\mathsf{Hilb}_{\beta_i}(X)$ of class β_i and the associated Abel-Jacobi map

$$\mathsf{AJ} : \mathsf{Hilb}_{\beta_i}(X) \to \mathrm{Pic}^{\beta_i}(X).$$

When the condition $\beta_i \cdot (K_X - \beta_i) = 0$ is satisfied, the Hilbert scheme of curves admits a virtual fundamental class of dimension zero whose length is

$$\mathsf{SW}(\beta_i) = \deg\, [\mathsf{Hilb}_{\beta_i}(X)]^{\mathrm{vir}} \in \mathbb{Z}\,.$$

As noted in [DKO, L], examples of curve classes of Seiberg-Witten length N, for all N, include:

(i) $\beta = 0$ for all surfaces,
(ii) arbitrary curve classes β for surfaces with $p_g > 0$,
(iii) curve classes supported on fibers for relatively minimal elliptic surfaces.

1.3. **K-theory.** For a scheme S with a 2-term perfect obstruction theory, let $\mathcal{O}_S^{\mathrm{vir}}$ denote the virtual structure sheaf, see [BF, CFK, Lee]. Given a K-theory class $V \to S$, write

$$\chi^{\mathrm{vir}}(S,V) = \chi(S, V \otimes \mathcal{O}_S^{\mathrm{vir}})\,.$$

For $\alpha \in K^0(X)$, we define the tautological classes

$$\alpha^{[n]} = \mathrm{R}\pi_{1*}(\mathcal{Q} \otimes \pi_2^*\alpha) \in K^0(\mathsf{Quot}_X(\mathbb{C}^N, \beta, n))\,.$$

Here,

$$\mathcal{Q} \to \mathsf{Quot}_X(\mathbb{C}^N, \beta, n) \times X$$

is the universal quotient, and π_1, π_2 are the two projections.

The Riemann-Roch numbers of exterior and symmetric powers of tautological sheaves over the Hilbert schemes of points $X^{[n]}$ (with its usual geometry) were studied in [A, D, EGL, K, Sc, Z], among others. Closed-form expressions are difficult to write down. By

working with the virtual class, we cover more general Quot scheme geometries and obtain answers that satisfy simpler structural results. We first prove

Theorem 1. *Fix integers $k_1, \ldots, k_\ell \geq 0$, and let $\alpha_1, \ldots, \alpha_\ell$ be K-theory classes on a nonsingular projective surface X. If $\beta \in H_2(X, \mathbb{Z})$ is a curve class of Seiberg-Witten length N, the series*

$$\mathsf{Z}^K_{X,N,\beta}(\alpha_1, \ldots, \alpha_\ell \,|\, k_1, \ldots, k_\ell) = \sum_{n \in \mathbb{Z}} q^n \chi^{\mathrm{vir}} \left(\mathsf{Quot}_X(\mathbb{C}^N, \beta, n), \wedge^{k_1} \alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_\ell} \alpha_\ell^{[n]} \right)$$

is the Laurent expansion of a rational function in q.

By taking α of negative rank, we also access the symmetric powers of the tautological sheaves via the identities

$$\wedge_t \alpha = \sum_k t^k \wedge^k \alpha\,, \quad \mathsf{S}_t(\alpha) = \sum_k t^k \operatorname{Sym}^k \alpha\,, \quad \wedge_{-t}(-\alpha) = \mathsf{S}_t(\alpha)\,.$$

Theorem 1 applies to all curve classes on surfaces with $p_g > 0$. The case of surfaces with $p_g = 0$ is more complicated, and our results are not as general. Nonetheless, when $N = 1$, we show

Theorem 2. *For all nonsingular simply connected surfaces X with $p_g = 0$ and all $\alpha \in K^0(X)$, the series*

$$\mathsf{Z}^K_{X,1,\beta} = \sum_{n \in \mathbb{Z}} q^n \chi^{\mathrm{vir}} \left(\mathsf{Quot}_X(\mathbb{C}^1, \beta, n),\, \alpha^{[n]} \right)$$

is the Laurent expansion of a rational function in q.

Based on Theorems 1 and 2, we formulate the following

Conjecture 3. *Let $k_1, \ldots, k_\ell \geq 0$ be integers, and let $\alpha_1, \ldots, \alpha_\ell$ be K-theory classes on a nonsingular projective surface X. For every $\beta \in H_2(X, \mathbb{Z})$, the series*

$$\mathsf{Z}^K_{X,N,\beta}(\alpha_1, \ldots, \alpha_\ell \,|\, k_1, \ldots, k_\ell) = \sum_{n \in \mathbb{Z}} q^n \chi^{\mathrm{vir}} \left(\mathsf{Quot}_X(\mathbb{C}^N, \beta, n), \wedge^{k_1} \alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_\ell} \alpha_\ell^{[n]} \right)$$

is the Laurent expansion of a rational function in q.

Several other variations are possible. We can dualize some of the factors of the tensor product. We could also consider twists by the virtual cotangent bundle

$$\widetilde{\mathsf{Z}}^K_{X,N,\beta}(\alpha_1, \ldots, \alpha_\ell \,|\, k_1, \ldots, k_\ell) = \\ = \sum_{n \in \mathbb{Z}} q^n \chi^{\mathrm{vir}} \left(\mathsf{Quot}_X(\mathbb{C}^N, \beta, n), \wedge^{k_1} \alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_\ell} \alpha_\ell^{[n]} \otimes \wedge_y \Omega^{\mathrm{vir}} \right)\,.$$

These variations can be studied by the same methods, yielding rational functions in q, though we do not explicitly record the answers.

It is natural to ask about the locations and orders of the poles of the K-theoretic generating series. In case $N = 1$, we have a complete answer.

Theorem 4. *If $\beta \in H_2(X, \mathbb{Z})$ is a curve class of Seiberg-Witten length $N = 1$, the shifted series*

$$\overline{\mathsf{Z}}^K_{X,1,\beta}(\alpha_1, \ldots, \alpha_\ell \,|\, k_1, \ldots, k_\ell) = q^{\beta \cdot K_X} \sum_{n \in \mathbb{Z}} q^n \chi^{\mathrm{vir}} \left(\mathsf{Quot}_X(\mathbb{C}^1, \beta, n), \wedge^{k_1} \alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_\ell} \alpha_\ell^{[n]} \right)$$

is the Laurent expansion of a rational function with a pole only at $q = 1$ of order at most

$$2(k_1 + \ldots + k_\ell)\,.$$

We formulate the following non-virtual analogue of Theorem 4.

Question 5. *For integers $k_1, \ldots, k_\ell \geq 0$ and K-theory classes $\alpha_1, \ldots, \alpha_\ell$, is the series*

$$\sum_{n=0}^{\infty} q^n \chi \left(X^{[n]}, \wedge^{k_1} \alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_\ell} \alpha_\ell^{[n]} \right)$$

the Laurent expansion of a rational function in q?

Explicit calculations in [A, Section 6], [EGL, Section 5], [K, Section 8] , [Sc, Section 5], [Z, Section 7] answer Question 5 in the affirmative for several values of the parameters ℓ, k_i and rank α_i. We will investigate the general case in future work.

Example 6. The simplest case of Theorem 1 occurs when $k_i = 0$ for all i. By the results of [L],

$$\mathsf{Z}^K_{X,N} = \sum_{n \in \mathbb{Z}} q^n \chi^{\mathrm{vir}} \left(\mathsf{Quot}_X(\mathbb{C}^N, n), \mathcal{O} \right) = 1\,, \tag{2}$$

and, in general,

$$\mathsf{Z}^K_{X,N,\beta} = \sum_{n \in \mathbb{Z}} q^n \chi^{\mathrm{vir}} \left(\mathsf{Quot}_X(\mathbb{C}^N, \beta, n), \mathcal{O} \right)$$

is rational whenever β is of Seiberg-Witten length N.

Example 7. Let $\beta = 0$, and take $\ell = 1$, $k_1 = 1$. To illustrate Theorems 1 and 4, we compute[1]

$$\sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(X^{[n]}, \alpha^{[n]}) = -\mathrm{rk}\ \alpha \cdot K_X^2 \cdot \frac{q^2}{(1-q)^2} - \langle K_X, c_1(\alpha)\rangle \cdot \frac{q}{1-q}\,. \tag{3}$$

Here, $\langle , \rangle$ is the intersection pairing on X. For $N > 1$, we find the surprising identity

$$\sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, n), \alpha^{[n]}) = N \sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(X^{[n]}, \alpha^{[n]})\,. \tag{4}$$

Since the above series vanish for K-trivial surfaces, the following question is natural.

Question 8. *What are the corresponding series for the reduced tangent-obstruction theory of Quot schemes of $K3$ surfaces?*

Example 9. Perhaps the simplest example with $\beta \neq 0$ is the case of relatively minimal elliptic surfaces $X \to C$ with β supported on fibers. We have

$$\sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, \beta, n), \alpha^{[n]}) = \mathsf{sw}_\beta \cdot \left(-N\langle K_X, c_1(\alpha)\rangle \frac{q}{1-q} + \langle \beta, c_1(\alpha)\rangle \frac{1}{1-q} \right) \tag{5}$$

where

$$\mathsf{sw}_\beta = \sum_{\beta_1 + \ldots + \beta_N = \beta} \mathsf{SW}(\beta_1) \cdots \mathsf{SW}(\beta_N)\,.$$

The Seiberg-Witten invariants of fiber classes are known by [DKO, Proposition 5.8].

Example 10. For a more complicated example, let X be a minimal surface of general type with $p_g > 0$, and let $\beta = K_X$. We find

$$\sum_{n \in \mathbb{Z}} q^n \chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, K_X, n), \alpha^{[n]}) = \mathsf{SW}(K_X) \cdot \left(-\frac{N}{1-q} \right)^{K_X^2} \left(\sum_{i=1}^{N} z_i^{-K_X^2} \cdot \mathsf{p}_\alpha(z_i) \right). \tag{6}$$

Here, $z_1, \ldots, z_N$ are the roots of the equation

$$z^N - q(z-1)^N = 0.$$

In general, p_α is an explicit rational function, but the simplest expressions occur when $\mathrm{rank}\ \alpha = 0$:

$$\mathsf{p}_\alpha(z) = \langle K_X, c_1(\alpha)\rangle \left(\frac{1 - (N+1)q}{1-q} - z \right).$$

[1] Non-virtually, we can compare (3) with Proposition 5.6 in [EGL]:

$$\sum_{n=1}^{\infty} q^{n-1} \chi(X^{[n]}, \alpha^{[n]}) = \frac{\chi(\alpha)}{(1-q)^{\chi(\mathcal{O}_X)}}\,.$$

1.4. **Stable pairs.** The Quot schemes studied here can be compared to the moduli space of higher rank stable pairs on surfaces X,

$$\mathbb{C}^N \otimes \mathcal{O}_X \xrightarrow{s} F, \quad [F] = \beta\,, \quad \chi(F) = n\,,$$

whose virtual fundamental class was constructed in [Lin]. The tautological classes $\alpha^{[n]}$ can be defined analogously. It is natural to inquire whether the above rationality results still hold.

Increasing the dimension, let X be a smooth projective threefold, and let $P_n(X,\beta)$ denote the moduli space of stable pairs

$$\mathcal{O}_X \xrightarrow{s} F, \quad [F] = \beta, \quad \chi(F) = n,$$

as defined in [PT1]. In cohomology, the series of descendant invariants[2]

$$\sum_{n\in\mathbb{Z}} q^n \cdot \int_{[P_n(X,\beta)]^{\mathrm{vir}}} \mathsf{ch}_{k_1}(\alpha_1^{[n]}) \ldots \mathsf{ch}_{k_\ell}(\alpha_\ell^{[n]}) \tag{7}$$

is conjectured to be rational [MNOP, P, PP1, PP2, PT1, PT2]. Rationality is proven for toric threefolds in [PP1, PP2] via localization of the virtual class [GP]. The analogous conjecture in K-theory reads

Conjecture 11. *For all integers $k_1,\ldots,k_\ell \geq 0$, and K-theory classes $\alpha_1,\ldots,\alpha_\ell$ on a smooth projective threefold X, the series*

$$\mathsf{Z}^K_{X,\beta}(\alpha_1,\ldots,\alpha_\ell \,|\, k_1,\ldots,k_\ell) = \sum_{n\in\mathbb{Z}} q^n \chi^{\mathrm{vir}}\left(P_n(X,\beta), \wedge^{k_1}\alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_\ell}\alpha_\ell^{[n]}\right)$$

is the Laurent expansion of a rational function in q.

For stable pairs on threefolds, rationality of the descendent series in cohomology *implies*[3] the rationality of Conjecture 11 for K-theory via the virtual Hirzebruch-Riemann-Roch theorem [FG]. Indeed, the K-theoretic invariants can be expressed solely in terms of the Chern characters $\mathrm{ch}_k(\alpha^{[n]})$ appearing in the descendent series. The Chern classes of the virtual tangent bundle appearing in the calculation can be written in terms of descendent invariants in a form which does not depend on n, see [Sh, Section 3.1]. The result crucially relies on the independence of the virtual dimension of $P_n(X,\beta)$ on n.

[2]The descendent series (7) is in a slightly different form than the descendent series in the stable pairs references, but the rationality of (7) follows from the descendent study in the references by an application of Grothendieck-Riemann-Roch.

[3]See also [Sm] for a direct approach to the rationality of the 1-leg descendent vertex in K-theory via quasimaps.

For Quot schemes of surfaces, the descendent rationality of [JOP] does *not* obviously imply Conjecture 3. The virtual dimension of the Quot scheme on a surface grows with n. Therefore, we cannot bound the degree of the descendent classes appearing in Hirzerbruch-Riemann-Roch independently of n, and hence the K-theoretic series are not expressed in terms of finitely many descendent series

$$\sum_{n\in\mathbb{Z}} q^n \cdot \int_{[\mathsf{Quot}_X(\mathbb{C}^N,\beta,n)]^{\mathrm{vir}}} \mathsf{ch}_{k_1}(\alpha_1^{[n]}) \ldots \mathsf{ch}_{k_\ell}(\alpha_\ell^{[n]})\, c(T^{\mathrm{vir}}\mathsf{Quot}) \tag{8}$$

of [JOP]. Nevertheless, we will remark in Section 3 that the methods used in the proof of Theorem 1 also establish rationality of the Quot scheme descendent series for all surfaces with $p_g > 0$, partially answering Conjecture 2 of [JOP].

1.5. **Cobordism.** It is natural to inquire whether the rationality results proven in cohomology or K-theory also hold at the level of virtual cobordism.[4] We show that this is not the case:

Theorem 12. *The series*

$$\mathsf{Z}^{\mathsf{cob}}_{X,\beta} = \sum_{n\in\mathbb{Z}} q^n \left[\mathsf{Quot}_X(\mathbb{C}^1,\beta,n)\right]^{\mathrm{vir}}_{\mathrm{cob}} \in \Omega^\star((q))$$

is not given by a rational function, in general.

In particular, we show that the virtual Pontryagin series is given by an explicit algebraic irrational function when $N = 1$ for surfaces with $p_g > 0$. Under the same assumptions, Theorem 18 of Section 2 provides an explicit expression for the cobordism series.

1.6. **A virtual Segre/Verlinde correspondence.** Let X be a nonsingular projective surface, and let $\alpha \in K^0(X)$. Using the usual (non-virtual) geometry of the Hilbert scheme of points, we define Segre and Verlinde series as follows:

$$\begin{aligned} \mathsf{S}^{\mathsf{Hilb}}_\alpha &= \sum_{n=0}^{\infty} q^n \int_{X^{[n]}} s(\alpha^{[n]})\,, \\ \mathsf{V}^{\mathsf{Hilb}}_\alpha &= \sum_{n=0}^{\infty} q^n\, \chi(X^{[n]}, \det \alpha^{[n]})\,. \end{aligned}$$

Precise closed-form expressions for $\mathsf{S}^{\mathsf{Hilb}}_\alpha$ and $\mathsf{V}^{\mathsf{Hilb}}_\alpha$ are in general difficult to write down, but see [EGL, Le, MOP1, MOP3, V] for results and conjectures when α has small rank.

[4] For the parallel theory of stable pairs over 3-folds, both the series of invariants and the series of cobordism classes [Sh] are conjectured to be rational.

Nevertheless, in the absence of explicit expressions, a connection

$$\mathsf{S}_{\alpha}^{\mathsf{Hilb}} \;\leftrightarrow\; \mathsf{V}_{\widetilde{\alpha}}^{\mathsf{Hilb}}$$

after an explicit change of variables, for pairs $(\alpha, \widetilde{\alpha})$ related in an explicit fashion, was proposed in [J, MOP2]. The resulting *Segre/Verlinde correspondence* is aligned with the larger conjectural framework of strange duality. An extension to higher rank moduli spaces of sheaves was announced in [GK].

In the virtual context, the Segre/Verlinde correspondence takes a simpler form. We define the virtual (shifted)[5] Segre and Verlinde series [6] by

$$\begin{aligned}
\mathsf{S}_\alpha(q) &= q^{\beta.K_X} \sum_{n\in\mathbb{Z}} q^n \int_{[\mathsf{Quot}_X(\mathbb{C}^N,\beta,n)]^{\mathrm{vir}}} s(\alpha^{[n]})\,, \\
\mathsf{V}_\alpha(q) &= q^{\beta.K_X} \sum_{n\in\mathbb{Z}} q^n\, \chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N,\beta,n),\, \det \alpha^{[n]})\,.
\end{aligned}$$

The more precise notation $\mathsf{S}_{N,\alpha}$ and $\mathsf{V}_{N,\alpha}$ also keeps track of the dimension of $\mathbb{C}^N$.

Theorem 13. *The virtual Segre and Verlinde series match*

$$\mathsf{S}_{N,\alpha}\left((-1)^N q\right) = \mathsf{V}_{N,\alpha}(q)$$

in the following three cases:

(i) *X is a nonsingular projective surface, $\beta = 0$, N is arbitrary,*

(ii) *X is a nonsingular projective surface with $p_g > 0$, $N = 1$ and β is arbitrary,*

(iii) *X is a relatively minimal elliptic surface, β is supported on fibers, N is arbitrary.*

However, the virtual Segre/Verlinde correspondence does not always hold. The simplest counterexample occurs for minimal surfaces of general type, $N = 2$ and $\beta = K_X$, with discrepancy even at the first few terms.

1.7. **Symmetry.** For $\beta = 0$, we show that the expressions for the Segre series $\mathsf{S}_{N,\,\alpha}$ are symmetric when the rank $r = \mathrm{rk}\ \alpha$ and the dimension of $\mathbb{C}^N$ are interchanged.

Theorem 14. *Let X be a nonsingular projective surface, and let $\alpha, \widetilde{\alpha} \in K^0(X)$ satisfy*

$$\mathsf{rk}\,\alpha = r, \quad \mathsf{rk}\,\widetilde{\alpha} = N\,, \qquad \frac{\langle K_X, c_1(\alpha)\rangle}{\mathsf{rk}\,\alpha} = \frac{\langle K_X, c_1(\widetilde{\alpha})\rangle}{\mathsf{rk}\,\widetilde{\alpha}}\,.$$

Then, the following symmetry holds:

$$\mathsf{S}_{N,\,\alpha}\left((-1)^N q\right) = \mathsf{S}_{r,\,\widetilde{\alpha}}\left((-1)^r q\right)\,.$$

[5] The shift is needed to ensure a uniform statement in Theorem 13.

[6] By contrast with the series considered in Section 1.3, the Segre/Verlinde series are given by algebraic functions; see Section 5 for details.

1.8. **The virtual structure sheaf.** Our calculations crucially rely on the simpler form of the virtual fundamental class of Quot schemes for curve classes of Seiberg-Witten length N. When $\beta = 0$, the following result was proven in [OP, Section 4.2.3]. Let $C \in |K_X|$ be a nonsingular canonical curve, and let

$$\iota : \mathsf{Quot}_C(\mathbb{C}^N, n) \hookrightarrow \mathsf{Quot}_X(\mathbb{C}^N, n)$$

denote the natural inclusion. Then, the virtual class localizes on the canonical curve:

$$\big[\mathsf{Quot}_X(\mathbb{C}^N, n)\big]^{\mathrm{vir}} = (-1)^n \iota_\star \left[\mathsf{Quot}_C(\mathbb{C}^N, n)\right] . \tag{9}$$

We provide a similar expression for the virtual structure sheaf. Let $\Theta = N_{C/X}$ denote the theta characteristic. Write

$$\mathcal{D}_n = p^\star \det\left(\Theta^{[n]}\right)$$

for the pullback via the support morphism

$$p : \mathsf{Quot}_C(\mathbb{C}^N, n) \to C^{[n]}, \;\; Q \mapsto \mathrm{supp}\; Q.$$

Theorem 15. *Assume $C \subset X$ is a nonsingular canonical curve. In rational K-theory, the virtual structure sheaf localizes on the canonical curve*

$$\mathcal{O}^{\mathrm{vir}}_{\mathsf{Quot}_X(\mathbb{C}^N,n)} = (-1)^n \, \iota_\star \, \mathcal{D}_n \, .$$

Furthermore, letting $\mathcal{N}^{\mathrm{vir}}$ denote the virtual normal bundle of the embedding ι, the sheaf $\mathcal{D}_n$ is a square root of $\det \mathcal{N}^{\mathrm{vir}}$.

1.9. **The 8-fold equivalence.** Let X be a nonsingular projective surface with a nonsingular canonical curve $C \subset X$. Putting together Theorems 13 and 14, we match the following 4 virtual invariants of Quot schemes of X in the $\beta = 0$ case:

(i) the Segre integrals $(-1)^{Nn} \int_{[\mathsf{Quot}_X(\mathbb{C}^N,n)]^{\mathrm{vir}}} s(\alpha^{[n]})$
(ii) the Segre integrals $(-1)^{rn} \int_{[\mathsf{Quot}_X(\mathbb{C}^r,n)]^{\mathrm{vir}}} s(\widetilde{\alpha}^{[n]})$
(iii) the virtual Verlinde numbers $\chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, n), \;\; \det \alpha^{[n]})$
(iv) the virtual Verlinde numbers $\chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^r, n), \;\; \det \widetilde{\alpha}^{[n]})$.

With the aid of Theorem 15 and equation (9), the same results can be stated over the canonical curve. For simplicity, let $\alpha = L^{\oplus r}, \widetilde{\alpha} = L^{\oplus N}$, where $L \to C$ is a line bundle. We match:

(i)′ the Segre integrals $(-1)^{Nn} \int_{\mathsf{Quot}_C(\mathbb{C}^N,n)} s(L^{[n]})^r$
(ii)′ the Segre integrals $(-1)^{rn} \int_{\mathsf{Quot}_C(\mathbb{C}^r,n)} s(L^{[n]})^N$
(iii)′ the twisted Verlinde numbers $\chi\left(\mathsf{Quot}_C(\mathbb{C}^N, n), \;\; \left(\det L^{[n]}\right)^r \otimes \mathcal{D}_n\right)$

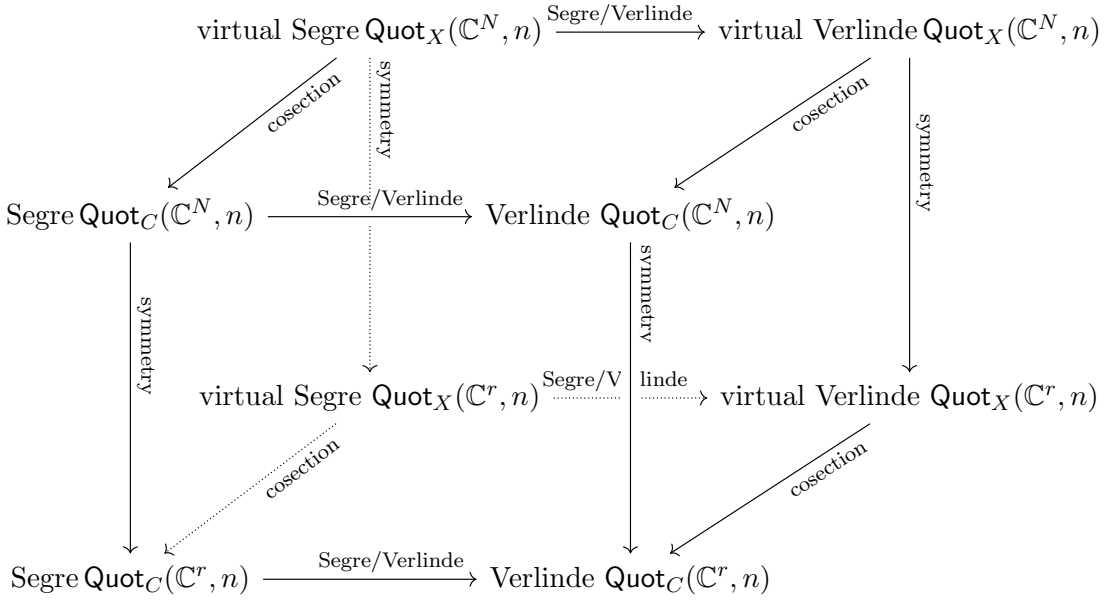

FIGURE 1. The Segre/Verlinde series for curves/surfaces and their symmetry

(iv)$'$ the twisted Verlinde numbers $\chi\left(\mathsf{Quot}_C(\mathbb{C}^r,n),\ \left(\det L^{[n]}\right)^N\otimes\mathcal{D}_n\right)$.

The diagram in Figure 1 summarizes the above 8-fold equivalence.

Question 16. *Find a geometric interpretation of the above equalities.*

It would be satisfying to see (i)$'$-(iv)$'$ as solutions to the same enumerative problem. The match between (iii)$'$ and (iv)$'$:

$$\chi\left(\mathsf{Quot}_C(\mathbb{C}^N,n),\ \left(\det L^{[n]}\right)^r\otimes\mathcal{D}_n\right)=\chi\left(\mathsf{Quot}_C(\mathbb{C}^r,n),\ \left(\det L^{[n]}\right)^N\otimes\mathcal{D}_n\right)$$

is reminiscent of strange duality for bundles over curves [Be, MO2]. We will investigate these matters elsewhere.

1.10. **Plan of the paper.** Section 2 is the most demanding computationally, but it plays a central role in our arguments. It generalizes the techniques of [JOP, L, OP] and records structural results for the series of invariants we consider. It also contains a proof of Theorem 12 regarding the cobordism series. Section 3 shows the rationality of the K-theoretic series in several contexts: punctual quotients, curve classes of Seiberg-Witten length N, surfaces with $p_g=0$, and certain rank 1 quotients. Section 4 establishes Theorem 15 regarding cosection localization for punctual quotients. Finally, Section 5 discusses the Segre/Verlinde

correspondence and the symmetry of the Segre/Verlinde series, proving Theorems 13 and 14.

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2. A general calculation and cobordism

2.1. **Setup.** While our interest here lies foremost in the series which appear in Theorems 1, 4 and 12 and 13, it is useful to work more generally. Let

$$f(x) \in F[\![x]\!]$$

be an invertible power series with coefficients in a field F. For our purposes, F will be either the field of complex numbers or the field of rational functions in one or several variables. For a vector bundle V over a complex projective scheme S, we consider the corresponding genus

$$f(V) = \prod_i f(x_i) \in H^\star(S, F),$$

where x_i are the Chern roots of V. This definition extends multiplicatively to K-theory.

Fix invertible power series

$$f_1, \ldots, f_\ell, g \in F[\![x]\!], \quad g(0) = 1,$$

and classes

$$\alpha_1, \ldots, \alpha_\ell \in K^0(X) \text{ with rank } \alpha_s = r_s, \ \ 1 \leq s \leq \ell.$$

As explained in Section 1.3, this yields

$$\alpha_1^{[n]}, \ldots, \alpha_\ell^{[n]} \to \mathsf{Quot}_X(\mathbb{C}^N, \beta, n),$$

and we may consider the associated characteristic classes

$$f_1(\alpha_1^{[n]}), \ldots, f_\ell(\alpha_\ell^{[n]})$$

in the cohomology of $\mathsf{Quot}_X(\mathbb{C}^N, \beta, n)$. Recall furthermore that the standard deformation-obstruction theory of the Quot scheme is governed by

$$\mathrm{Ext}^0(S,Q), \quad \mathrm{Ext}^1(S,Q), \quad \mathrm{Ext}^{\geq 2}(S,Q) = 0.$$

Here, we keep the notation in (1), with S, Q denoting the subsheaf and the quotient respectively. The last Ext vanishes by Serre duality and the assumption that rank $Q = 0$, see [MOP1]. In particular, the virtual tangent bundle is given by the alternating sum

$$T^{\mathrm{vir}}\mathsf{Quot} = \mathrm{Ext}^\bullet(S,Q) = \mathrm{Ext}^0(S,Q) - \mathrm{Ext}^1(S,Q).$$

We consider the following general expression mixing the tautological classes and the virtual tangent bundle

$$\mathsf{Z}^{f_1,\ldots,f_\ell,g}_{X,N,\beta}(q \,|\, \alpha_1, \ldots, \alpha_\ell) = \sum_{n\in\mathbb{Z}} q^n \int_{[\mathsf{Quot}_X(\mathbb{C}^N,\beta,n)]^{\mathrm{vir}}} f_1(\alpha_1^{[n]}) \cdots f_\ell(\alpha_\ell^{[n]}) \cdot g(T^{\mathrm{vir}}\mathsf{Quot}). \tag{10}$$

As we will see below, the examples in Theorems 1, 4, 12, 13 correspond to

(i) the virtual tautological series: $f_s(x) = 1 + y_s e^x$ and $g(x) = \frac{x}{1-e^{-x}}$,
(ii) the virtual Verlinde series: $f(x) = e^x$ and $g(x) = \frac{x}{1-e^{-x}}$,
(iii) the virtual Segre series: $f(x) = \frac{1}{1+x}$ and $g(x) = 1$,
(iv) the virtual cobordism class: $f(x) = 1$ and g arbitrary.

An example covered by (iv), but not considered in further detail here is given by

(v) the virtual elliptic genus: $f(x) = 1$ and $g(x) = x \cdot \frac{\theta\left(\frac{x}{2\pi i} - z, \tau\right)}{\theta\left(\frac{x}{2\pi i}, \tau\right)}$, with θ denoting the Jacobi theta function.[7]

[7]The virtual elliptic genus has interesting automorphic properties for virtual Calabi-Yau manifolds. However, for $N = 1$, $\beta = 0$, the virtual canonical bundle

$$K^{\mathrm{vir}}_{X^{[n]}} = (\det TX^{[n]})^\vee \otimes \det \mathrm{Obs} = (K_X)_{(n)} \otimes \det\left((K_X)^{[n]}\right)^\vee = E^\vee$$

where E is $-\frac{1}{2}$ of the exceptional divisor. To get weak Jacobi forms, the modified elliptic genus can be used:

$$g(x) = x \cdot \frac{\theta\left(\frac{x}{2\pi i} - z, \tau\right)}{\theta\left(\frac{x}{2\pi i}, \tau\right)} \cdot \exp\left(-\frac{x}{2\pi i} \cdot \frac{\theta'}{\theta}(z,\tau)\right).$$

Alternatively, weak Jacobi forms can be obtained by twisting with pure classes coming from the symmetric product; these pair trivially with $K^{\mathrm{vir}}_{X^{[n]}}$ and Theorem 5.4 of [FG] applies.

2.2. **Calculation.** We compute (10) explicitly for curve classes β of Seiberg-Witten length N following the approach in [L, OP, JOP]. The strategy is to combine torus localization with universality arguments over the Hilbert scheme of points. Specifically, we let $\mathbb{C}^\star$ act with distinct weights

$$w_1, \ldots, w_N$$

on the middle term of the exact sequence

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_X \to Q \to 0,$$

thus inducing an action on $\mathsf{Quot}_X(\mathbb{C}^N, \beta, n)$. The series (10) admits a natural equivariant lift. Writing K for the canonical bundle of X to ease notation, we show

Theorem 17. *For fixed N, $r_1, \ldots, r_\ell$, there are universal series*

$$\mathsf{A}, \quad \mathsf{B}_s, \quad \mathsf{U}_i, \quad \mathsf{V}_{i,s}, \quad \mathsf{W}_{i,j}$$

such that for all nonsingular projective surfaces X, all curve classes β of Seiberg-Witten length N, and all classes $\alpha_1, \ldots, \alpha_\ell$ of ranks $r_1, \ldots, r_\ell$, the equivariant series above takes the form

$$\mathsf{Z}^{f_1,\ldots,f_\ell,g}_{X,N,\beta}(q \,|\, \alpha_1, \ldots, \alpha_\ell) = q^{-\beta.K} \sum_{\beta=\beta_1+\cdots+\beta_N} \mathsf{SW}(\beta_1) \cdot \ldots \cdot \mathsf{SW}(\beta_N) \cdot \left(\mathsf{A}^{K^2} \cdot \prod_{s=1}^{\ell} \mathsf{B}_s^{c_1(\alpha_s).K} \cdot \prod_{i=1}^{N} \mathsf{U}_i^{\beta_i.K} \cdot \prod_{\substack{1\le i\le N \\ 1\le s\le \ell}} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_i.\beta_j} \right). \tag{11}$$

The universal expressions appearing in the answer are explicitly recorded in equations (15) – (20) below.

Proof. Under the above torus action, the $\mathbb{C}^\star$-fixed subsheaves split

$$S = \bigoplus_{i=1}^{N} I_{Z_i}(-D_i), \quad [D_i] = \beta_i, \quad \text{length}(Z_i) = m_i$$

where

$$\beta = \beta_1 + \ldots + \beta_N, \quad m = m_1 + \ldots + m_N, \quad m = n + \frac{1}{2}\sum_{i=1}^{N} \beta_i(\beta_i + K).$$

Thus, the torus fixed loci of $\mathsf{Quot}_X(\mathbb{C}^N, \beta, n)$ are isomorphic to products of Hilbert schemes of points and curves

$$\mathsf{F}[\underline{m}, \underline{\beta}] = X^{[\underline{m}]} \times \mathsf{Hilb}_{\underline{\beta}}(X).$$

The underline notation

$$\underline{m} = (m_1, \ldots, m_N), \quad \underline{\beta} = (\beta_1, \ldots, \beta_N)$$

is used to denote vectors of integers, curve classes etc. We set

$$X^{[\underline{m}]} = X^{[m_1]} \times \ldots \times X^{[m_N]}, \quad \mathsf{Hilb}_{\underline{\beta}}(X) = \mathsf{Hilb}_{\beta_1}(X) \times \ldots \times \mathsf{Hilb}_{\beta_N}(X).$$

Crucially, the arguments of Section 2 in [L], originally written for the integrals computing the virtual χ_y-genus, apply verbatim to the more general expressions considered here. Indeed, via the virtual localization theorem in [GP], the integrals (11) can be written as sums of contributions from the fixed loci $\mathsf{F}[\underline{m}, \underline{\beta}]$. Whenever β is a curve class of Seiberg-Witten length N, nonvanishing contributions only arise from splittings satisfying

$$\beta_i \cdot (K - \beta_i) = 0. \tag{12}$$

In all other cases, the Seiberg-Witten invariants in the class β_i vanish. As explained in [L], this in turn forces the vanishing of the corresponding localization terms. Furthermore, when (12) is satisfied, the Hilbert schemes of curves $\mathsf{Hilb}_{\beta_i}(X)$ carry virtual fundamental classes of dimension zero and of length $\mathsf{SW}(\beta_i)$. Thus

$$\big[\mathsf{F}[\underline{m}, \underline{\beta}]\big]^{\mathrm{vir}} = \mathsf{SW}(\beta_1) \cdots \mathsf{SW}(\beta_N) \cdot \Big[X^{[\underline{m}]}\Big]^{\mathrm{vir}} \times [\mathrm{point}]\,. \tag{13}$$

The virtual class of $X^{[\underline{m}]}$ is determined by the obstruction bundle

$$\mathrm{Obs} = \sum_i \mathrm{Obs}_i, \quad \mathrm{Obs}_i = \Big((K - \beta_i)^{[m_i]}\Big)^{\vee}.$$

Note that $m = n + \beta \cdot K_X$. The equivariant series $\mathsf{Z}_{X,N,\beta}$ takes the following structure

$$\mathsf{Z}^{f,g}_{X,N,\beta}(q \,|\, \underline{\alpha}) = q^{-\beta.K_X} \sum_{\beta = \beta_1 + \cdots + \beta_N} \mathsf{SW}(\beta_1) \cdots \mathsf{SW}(\beta_N) \cdot \widehat{\mathsf{Z}}^{f,g}_{X,N,(\beta_1,\ldots,\beta_N)}(q, \underline{w} \,|\, \underline{\alpha}),$$

where we used the above convention to underline quantities understood as vectors. The last term in the expression above arises via equivariant localization [GP] and is given by

$$\widehat{\mathsf{Z}}^{f,g}_{X,N,\underline{\beta}}(q, \underline{w} \,|\, \underline{\alpha}) = \sum_{m_i \geq 0} q^{\sum m_i} \int_{X^{[\underline{m}]}} \prod_{s=1}^{\ell} f_s\Big(\iota^\star \alpha_s^{[n]}\Big) \cdot \frac{\mathsf{e}}{g}\Big(\sum_{i=1}^{N} \mathrm{Obs}_i\Big) \cdot \frac{g}{\mathsf{e}}(N^{\mathrm{vir}}) \cdot g\Big(\sum_{i=1}^{N} T_{X^{[m_i]}}\Big)$$

where e is the equivariant Euler class and N^{vir} denotes the virtual normal bundle. Furthermore,

(i) $\mathrm{Obs}_i = \big((K - D_i)^{[m_i]}\big)^{\vee}$,

(ii) $N^{\mathrm{vir}} = \sum_{i \neq j} N_{ij}[w_j - w_i]$, where $N_{ij} = \mathrm{RHom}_\pi(I_{\mathcal{Z}_i}(-D_i), \mathcal{O} - I_{\mathcal{Z}_j}(-D_j))$,

(iii) $\iota^\star \alpha^{[n]} = \sum_{i=1}^{N} \alpha_{\beta_i}^{[m_i]}[w_i]$, with $\alpha_{\beta_i}^{[m_i]} = \mathrm{R}\pi_*(\mathcal{Q}_i \otimes q^* \alpha) = \alpha(-D_i)^{[m_i]} + \mathbb{C}^{\chi(\alpha|_{D_i})}$.

Here $\mathcal{Z}_i$ denotes the universal subscheme of $X^{[m_i]} \times X$, and the equivariant weights are recorded within brackets. Furthermore, $D_i \in \mathsf{Hilb}_{\beta_i}(X)$ are any representatives for the point classes appearing in (13). The integral does not depend on this choice.

The obstruction and normal bundles in (i) and (ii) are lifted from [L, OP]. Item (iii) concerns the splitting of the tautological sheaves $\alpha^{[n]}$ to the product $X^{[\underline{m}]}$. This uses the splitting of the universal quotient

$$\mathcal{Q} = \sum_{i=1}^{N} \mathcal{Q}_i[w_i], \qquad \mathcal{Q}_i = \mathcal{O} - \mathcal{O}(-D_i) \otimes \mathcal{I}_{\mathcal{Z}_i}.$$

We point out the following notational ambiguity in (iii): the tautogical sheaf $\alpha^{[n]}$ on the left is restricted from the Quot scheme to the fixed locus, but the tautological sheaves on the right are over the Hilbert scheme of m_i points. The notation $\alpha_{\beta_i}^{[m_i]}$ records the twist by the curve class β_i in the universal quotient.

The expresssion $\widehat{Z}$ is a tautological integral over products of Hilbert schemes of points. Using the arguments of [EGL], it follows that $\widehat{Z}$ admits a multiplicative universal formula

$$\widehat{Z}^{\underline{f},g}_{X,N,(\beta_1,\ldots,\beta_N)}(q,\underline{w}\,|\,\underline{\alpha}) = \mathsf{A}^{K^2} \cdot \prod_{s=1}^{\ell} \mathsf{B}_s^{K.c_1(\alpha_s)} \cdot \prod_{1\le s,t\le \ell} \mathsf{C}_{s,t}^{c_1(\alpha_s).c_1(\alpha_t)} \cdot \prod_{s=1}^{\ell} \mathsf{D}_s^{c_2(\alpha_s)} \cdot \mathsf{E}^{\chi(\mathcal{O}_X)}$$
$$\cdot \prod_{i=1}^{N} \mathsf{U}_i^{\beta_i.K} \cdot \prod_{\substack{1\le i\le N\\ 1\le s\le \ell}} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{1\le i<j\le N} \mathsf{W}_{i,j}^{\beta_i.\beta_j}.$$

The above universal series are independent of the choice of surface and curve classes β_i, though they do depend on $N, r_s, \underline{f}, g$. We omitted the Chern numbers β_i^2 because we will only use the formula for $\beta_i^2 = \beta_i.K$.

We next determine relations between the universal series by specializing to simpler geometries. When $\beta_1 = \cdots = \beta_N = K_X$, the obstruction bundle

$$\mathrm{Obs}_i = \left(\mathcal{O}_X^{[m_i]}\right)^{\vee}$$

admits a trivial summand, hence its Euler class vanishes, for all i with $m_i > 0$. The case $m_i = 0$ corresponds to the Hilbert scheme isomorphic to a point, and the only equivariant contributions come from (iii). Since for $D_i \in |K_X|$ we have

$$\chi(\alpha_s|_{D_i}) = -r_s \cdot K^2 + K \cdot c_1(\alpha_s),$$

the series $\widehat{Z}$ becomes

$$\prod_{s=1}^{\ell} \prod_{i=1}^{N} \left(\frac{1}{f_s(w_i)^{r_s}}\right)^{K^2} \cdot f_s(w_i)^{K.c_1(\alpha_s)}.$$

This also equals

$$\mathsf{A}^{K^2}\prod_{s=1}^{\ell}\mathsf{B}_s^{K.c_1(\alpha_s)}\prod_{1\le s,t\le \ell}\mathsf{C}_{s,t}^{c_1(\alpha_s).c_1(\alpha_t)}\prod_{s=1}^{\ell}\mathsf{D}_s^{c_2(\alpha_s)}\mathsf{E}^{\chi(\mathcal{O}_X)}\prod_{i=1}^{N}\mathsf{U}_i^{K^2}\prod_{\substack{1\le i\le N\\ 1\le s\le \ell}}\mathsf{V}_{i,s}^{K.c_1(\alpha_s)}\prod_{1\le i<j\le N}\mathsf{W}_{i,j}^{K^2}.$$

By the independence of the Chern numbers, we conclude that

$$\mathsf{C}_{s,t} \;=\; \mathsf{D}_s \;=\; \mathsf{E} \;=\; 1.$$

Therefore we have

$$\widehat{Z}^{f,g}_{\underline{X},N,(\beta_1,\ldots,\beta_N)}(q,\underline{w}\,|\,\underline{\alpha}) = \mathsf{A}^{K^2}\cdot\prod_{s=1}^{\ell}\mathsf{B}_s^{K.c_1(\alpha_s)}\cdot\prod_{i=1}^{N}\mathsf{U}_i^{\beta_i.K}\cdot\prod_{\substack{1\le i\le N\\ 1\le s\le \ell}}\mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)}\cdot\prod_{1\le i<j\le N}\mathsf{W}_{i,j}^{\beta_i.\beta_j}. \tag{14}$$

Next we specialize X to a $K3$ surface blown up at a point, with exceptional divisor $E = K_X$. Let $I\sqcup J = [N]$ be a partition of the set $[N] = \{1,\ldots,N\}$ into two disjoint parts. Define a vector of curve classes

$$\underline{E}_{I\sqcup J} = (\beta_1,\ldots,\beta_N) \quad \text{where} \quad \begin{cases}\beta_i = E, & \text{if } i\in I\\ \beta_j = 0, & \text{if } j\in J.\end{cases}$$

By (14), we have

$$\widehat{Z}^{f,g}_{\underline{X},N,\underline{E}_{I\sqcup J}}(q,\underline{w}\,|\,\underline{\alpha}) = \mathsf{A}^{-1}\cdot\prod_{s=1}^{\ell}\mathsf{B}_s^{E.c_1(\alpha_s)}\cdot\prod_{i\in I}\mathsf{U}_i^{-1}\cdot\prod_{\substack{i\in I\\ 1\le s\le \ell}}\mathsf{V}_{i,s}^{E.c_1(\alpha_s)}\cdot\prod_{\substack{i_1<i_2\\ i_1,i_2\in I}}\mathsf{W}_{i_1,i_2}^{-1}.$$

By varying the degree $d_s = E.c_1(\alpha_s)$ and the partition $I\sqcup J = [N]$, we determine all universal series. In fact, it will be enough to consider the cases when $|I| = 0, 1$ or 2.

From now on, to ease the notation, we assume $\ell = 1$ so that

$$\widehat{\mathsf{Z}}^{f,g}_{X,N,\underline{\beta}}(q,\underline{w}\,|\,\alpha) = \sum_{m_i\ge 0} q^{\sum m_i}\int_{X^{[\underline{m}]}} f\Big(\sum_{i=1}^{N}\alpha_{\beta_i}^{[m_i]}[w_i]\Big)\cdot\frac{\mathsf{e}}{g}\Big(\sum_{i=1}^{N}\mathrm{Obs}_i\Big)\cdot\frac{g}{\mathsf{e}}(N^{\mathrm{vir}})\cdot g\Big(\sum_{i=1}^{N}T_{X^{[m_i]}}\Big).$$

We will make the convention to temporarily denote indices in I, J and $[N]$ by i, j, and k, respectively. Since

$$\mathrm{Obs}_i = \big((K_X - E)^{[m_i]}\big)^\vee = \big(\mathcal{O}_X^{[m_i]}\big)^\vee \text{ if } i\in I$$
$$\mathrm{Obs}_j = \big((K_X - 0)^{[m_i]}\big)^\vee = \big(E^{[m_j]}\big)^\vee \text{ if } j\in J,$$

all contributions with $m_i > 0$, for some $i\in I$, must vanish. Therefore we assume that $m_i = 0$ for all $i\in I$. For $j\in J$, the Euler class of

$$\mathrm{Obs}_j = \big(E^{[m_j]}\big)^\vee$$

can be represented up to sign by the Hilbert scheme of m_j points on $E = \mathbb{P}^1$:

$$\mathsf{e}(\mathrm{Obs}_j) = (-1)^{m_j} \left[E^{[m_j]}\right] = (-1)^{m_j} \left[\mathbb{P}^{m_j}\right].$$

We obtain $\widehat{Z}^{f,g}_{X,N,\underline{E}_{I\sqcup J}}(q, \underline{w} \,|\, \alpha)$ equals

$$\sum_{\forall j\in J,\ m_j\geq 0} (-q)^{\sum m_j} \int_{\prod_j \mathbb{P}^{m_j}} \iota^* \left(f\Big(\sum_{k=1}^N \alpha_{\beta_k}^{[m_k]}[w_k]\Big) \cdot \frac{g}{\mathsf{e}}(N^{\mathrm{vir}}) \cdot g\Big(\sum_j \Big(T_{X^{[m_j]}} - \mathrm{Obs}_j\Big)\Big)\right)$$

where

$$\iota : \textstyle\prod_j \mathbb{P}^{m_j} \hookrightarrow \prod_j X^{[m_j]}.$$

Let $d = E.c_1(\alpha)$. From the calculations of [L, Section 3] and [OP, Section 5] respectively, we have

$$\begin{aligned}
&\text{(i) } \iota^*\alpha_{\beta_i}^{[m_i=0]}[w_i] && = \mathbb{C}^{\chi(\alpha|_E)}[w_i] = \mathbb{C}^{r+d}[w_i],\\
&\text{(ii) } \iota^*\alpha_{\beta_j=0}^{[m_j]}[w_j] && = \mathcal{O}(-h_j)^{\oplus r m_j}[w_j] + \big(\mathcal{O} - \mathcal{O}(-h_j)\big)^{\oplus(r+d)}[w_j],\\
&\text{(iii) } \iota^*(T_{X^{[m_j]}} - \mathrm{Obs}_j) && = \mathcal{O}(-h_j)^{\oplus m_j} + \mathcal{O}(h_j) - \mathcal{O}\\
&\text{(iv) } \iota^* N^{\mathrm{vir}} && = \sum_{j_1\neq j_2} \Big(\mathcal{O}(-h_{j_2})^{\oplus m_{j_2}} + \mathcal{O}(h_{j_1}) - \mathcal{O}(h_{j_1} - h_{j_2})\Big)[w_{j_2} - w_{j_1}]\\
& && \quad + \sum_{i,j} \Big(\mathcal{O}(-h_j)^{\oplus m_j}[w_j - w_i] + \mathcal{O}(h_j)[w_i - w_j]\Big).
\end{aligned}$$

Here the h_j's denote the pullbacks of the hyperplane classes from each of the factors. Integrating over the projective space means extracting the suitable coefficient of h_j; we denote this operation by square brackets. After carefully collecting all terms, we obtain that

$$\widehat{Z}^{f,g}_{X,N,\underline{E}_{I\sqcup J}}(q, \underline{w} \,|\, \alpha) = \sum_{m_j\geq 0} (-q)^{\sum m_j} \left[\textstyle\prod_{j\in J} h_j^{m_j}\right] \textstyle\prod_{j\in J} \Phi_j(h_j)^{m_j} \cdot \Psi_{I\sqcup J}(\{h_j\}_{j\in J})$$

where

$$\begin{aligned}
\Phi_j(h_j) &= (-h_j) \cdot f(-h_j + w_j)^r \cdot \prod_{k=1}^N \frac{g(-h_j + w_j - w_k)}{-h_j + w_j - w_k}\\
\Psi_{I\sqcup J} &= \prod_{j\in J} h_j \cdot \prod_{k=1}^N f(w_k)^{r+d} \cdot \prod_{j\in J} \frac{1}{f(-h_j + w_j)^{r+d}} \cdot \prod_{\substack{j\in J\\ 1\leq k\leq N}} \frac{g(h_j - w_j + w_k)}{h_j - w_j + w_k}\\
&\quad \cdot \prod_{j_1\neq j_2} \frac{h_{j_1} - w_{j_1} - h_{j_2} + w_{j_2}}{g(h_{j_1} - w_{j_1} - h_{j_2} + w_{j_2})}.
\end{aligned}$$

We evaluate the above expression via the multivariable Lagrange-Bürmann formula [G]. Specifically, fix $\ell \geq 1$, and for each $1 \leq j \leq \ell$, consider power series ϕ_j in the variable x_j,

with nonzero constant terms. Set

$$K = \prod_{j=1}^{\ell} \left(1 - x_j \frac{d}{dx_j} \log \phi_j(x_j)\right).$$

Then, for all power series ψ in $x_1, \ldots, x_\ell$, we have

$$\sum_{m_j \geq 0} q^{m_1+\ldots+m_\ell} \left[x_1^{m_1} \cdots x_\ell^{m_\ell}\right] \ \phi_1(x_1)^{m_1} \cdot \ldots \cdot \phi_\ell(x_\ell)^{m_\ell} \cdot \psi(x_1, \ldots, x_\ell) = \frac{\psi}{K}(x_1, \ldots, x_\ell).$$

On the right hand side, the x_j's are the unique solutions to

$$q = \frac{x_j}{\phi_j(x_j)}, \quad x_j(q=0) = 0.$$

Applying this to our situation, we obtain

$$\widehat{Z}^{f,g}_{X,N,\underline{E}_{I\sqcup J}}(q, \underline{w} \,|\, \alpha) = \frac{\Psi_{I\sqcup J}}{K_{I\sqcup J}}(\{h_j\}_{j\in J})$$

where

$$K_{I\sqcup J} = \prod_{j\in J} \left(1 - h_j \frac{d}{dh_j} \log \Phi_j(h_j)\right),$$

and for variables related by

$$q = \frac{-h_j}{\Phi_j(h_j)}, \quad h_j(q=0) = 0.$$

In our case, we obtain

$$q = \frac{1}{f(-h_j+w_j)^r} \cdot \prod_{k=1}^{N} \frac{-h_j+w_j-w_k}{g(-h_j+w_j-w_k)} \quad \text{with} \quad h_j(q=0) = 0$$

and

$$K_{I\sqcup J} = \prod_{j\in J} h_j \cdot \prod_{j\in J} \left(r \cdot \frac{f'}{f}(-h_j+w_j) + \sum_{k=1}^{N} \left(\frac{g'}{g}(-h_j+w_j-w_k) + \frac{1}{h_j-w_j+w_k}\right)\right).$$

Using the additional change of variables

$$H_j = h_j - w_j,$$

we can simplify this to

$$\widehat{Z}^{f,g}_{X,N,\underline{E}_{I\sqcup J}}(q, \underline{w} \,|\, \alpha) = \frac{\prod_{k=1}^{N} f(w_k)^{r+d} \cdot \prod_{j\in J} \frac{1}{f(-H_j)^{r+d}} \cdot \prod_{\substack{j\in J \\ 1\leq k\leq N}} \frac{g(H_j+w_k)}{H_j+w_k} \cdot \prod_{\substack{j_1\neq j_2 \\ j_1,j_2\in J}} \frac{H_{j_1}-H_{j_2}}{g(H_{j_1}-H_{j_2})}}{\prod_{j\in J} \left(r \cdot \frac{f'}{f}(-H_j) + \sum_{k=1}^{N} \left(\frac{g'}{g}(-H_j-w_k) + \frac{1}{H_j+w_k}\right)\right)},$$

where

$$q = \frac{1}{f(-H_j)^r} \cdot \prod_{k=1}^{N} \frac{-H_j-w_k}{g(-H_j-w_k)} \quad \text{with} \quad H_j(q=0) = -w_j. \tag{15}$$

This determines all the universal series by comparing with

$$\widehat{Z}^{f,g}_{X,N,\underline{E}_{I\sqcup J}}(q,\underline{w}\,|\,\alpha) = \mathsf{A}^{-1}\cdot\mathsf{B}^{\,d}\cdot\prod_{i\in I}\mathsf{U}_i^{-1}\cdot\prod_{i\in I}\mathsf{V}_i^{d}\cdot\prod_{\substack{i_1<i_2\\ i_1,i_2\in I}}\mathsf{W}_{i_1,i_2}^{-1}.$$

First, we find

$$\mathsf{B}=\prod_{k=1}^{N}\frac{f(w_k)}{f(-H_k)},\quad \mathsf{V}_i=f(-H_i), \tag{16}$$

by considering the terms with exponent d and letting $I=\emptyset$ or $I=\{i\}$. Similarly, we see that

$$\begin{aligned}\mathsf{A} \;&=\prod_{i=1}^{N}\tfrac{f(-H_i)^r}{f(w_i)^r}\cdot\prod_{1\le k,i\le N}\tfrac{H_i+w_k}{g(H_i+w_k)}\cdot\prod_{i_1\neq i_2}\tfrac{g(H_{i_1}-H_{i_2})}{H_{i_1}-H_{i_2}}\cdot\\ &\cdot\prod_{i=1}^{N}\left(r\cdot\tfrac{f'}{f}(-H_i)+\sum_{k=1}^{N}\left(\tfrac{g'}{g}(-H_i-w_k)+\tfrac{1}{H_i+w_k}\right)\right)\end{aligned} \tag{17}$$

and

$$\begin{aligned}\mathsf{U}_i \;&=\tfrac{1}{f(-H_i)^r}\cdot\prod_{k=1}^{N}\tfrac{g(H_i+w_k)}{H_i+w_k}\cdot\prod_{i'\neq i}\tfrac{H_{i'}-H_i}{g(H_{i'}-H_i)}\cdot\tfrac{H_i-H_{i'}}{g(H_i-H_{i'})}\\ &\cdot\left(r\cdot\tfrac{f'}{f}(-H_i)+\sum_{k=1}^{N}\left(\tfrac{g'}{g}(-H_i-w_k)+\tfrac{1}{H_i+w_k}\right)\right)^{-1}.\end{aligned} \tag{18}$$

by comparing the terms with exponent (-1) and letting $I=\emptyset$ or $I=\{i\}$. In the above expressions, the indices i,i' now range over $[N]$ because the partition $I\sqcup J=[N]$ is not involved. Finally,

$$\mathsf{W}_{i_1,i_2}=\frac{g(H_{i_1}-H_{i_2})}{H_{i_1}-H_{i_2}}\cdot\frac{g(H_{i_2}-H_{i_1})}{H_{i_2}-H_{i_1}}, \tag{19}$$

by considering $I=\{i_1,i_2\}$ for $i_1<i_2$.

When several functions $f_1,\ldots,f_\ell$ are involved, each instance of f gets replaced by product contributions from each of the f_s's. For instance, the change of variables becomes

$$q=\prod_{s=1}^{\ell}\frac{1}{f_s(-H_j)^{r_s}}\cdot\prod_{k=1}^{N}\frac{-H_j-w_k}{g(-H_j-w_k)}. \tag{20}$$

This also affects the logarithmic derivatives $\frac{f'}{f}$ accordingly, turning them into sums of ℓ terms. □

2.3. **Cobordism and the virtual Pontryagin class.** As an application of the calculations in the previous subsection, we consider the virtual cobordism series of the Hilbert

scheme of curves and points:

$$\mathsf{Z}^{\mathsf{cob}}_{X,\beta} = \sum_{n\in\mathbb{Z}} q^n \left[\mathsf{Quot}_X(\mathbb{C}^1,\beta,n)\right]^{\mathrm{vir}}_{\mathrm{cob}} \in \Omega^\star((q)).$$

For a proper scheme S with a 2-term perfect obstruction theory, virtual cobordism [Sh] is encoded by all virtual Chern numbers

$$\int_{[S]^{\mathrm{vir}}} c_{i_1}(T^{\mathrm{vir}}S)\cdots c_{i_k}(T^{\mathrm{vir}}S).$$

For bookkeeping, we set

$$g(x) = \sum_{k=0}^{\infty} y_k x^k,\ \ y_0 = 1,$$

and note that the virtual cobordism class of S is determined by

$$[S]^{\mathrm{vir}}_{\mathrm{cob}} = \int_{[S]^{\mathrm{vir}}} g(T^{\mathrm{vir}}S) \in \mathbb{C}[y_0, y_1, \ldots].$$

This is consistent with the fact that $\Omega_\star$ is a polynomial ring in infinitely many variables.

For surfaces with $p_g > 0$, the series

$$\mathsf{Z}^{\mathsf{cob}}_{X,\beta} = \sum_{n\in\mathbb{Z}} q^n \int_{[\mathsf{Quot}_X(\mathbb{C}^1,\beta,n)]^{\mathrm{vir}}} g(T^{\mathrm{vir}}\mathsf{Quot}_X(\mathbb{C}^1,\beta,n))$$

can be calculated by letting $N = 1$ and $f = 1$ in the formulas of Section 2.2, also setting $w = 0$ in the answer. Then,

$$\mathsf{A} = \mathsf{U}^{-1} = \frac{H}{g(H)}\cdot\left(\frac{g'}{g}(-H) + \frac{1}{H}\right),\ \ \mathsf{B} = \mathsf{V} = 1$$

for the change of variables

$$q = \frac{-H}{g(-H)}.$$

Thus

$$\mathsf{Z}^{\mathsf{cob}}_{X,\beta}(q) = q^{-K\cdot\beta}\cdot\mathsf{SW}(\beta)\cdot\mathsf{A}^{K^2-K\cdot\beta}.$$

It is easy to see that $\mathsf{Z}^{\mathsf{cob}}_{X,\beta}$ is not given by a rational function. This fails for instance for the specialization

$$y_1 = 0,\ y_2 = y,\ \ y_k = 0 \text{ for } k \geq 3 \implies g(x) = 1 + yx^2$$

corresponding to the virtual Pontryagin class. In this case

$$q = \frac{-H}{1+yH^2},\quad \mathsf{A} = \frac{1-yH^2}{(1+yH^2)^2} \implies \mathsf{A} = \frac{1}{2}\left(1+\sqrt{1-4q^2y}\right)\cdot\sqrt{1-4q^2y}.$$

We can however write down a different expression for the cobordism series. The cobordism ring $\Omega_\star$ is generated by the classes of projective spaces $\mathbb{P}^0, \mathbb{P}^1, \ldots$. Consider the generating series

$$\mathsf{P} = \sum_{n=0}^{\infty} q^n \, [\mathbb{P}^n] \in \Omega_\star((q)).$$

More generally, for each ℓ, we consider the projective spaces $\mathbb{P}^n$ endowed with the nontrivial obstruction theory $\mathcal{O}(\ell)$. When $\ell \geq 1$ this corresponds to the class of a smooth degree ℓ hypersurface. We denote by $[\mathbb{P}^n]_{(\ell)}$ the associated virtual cobordism class and define

$$\mathsf{P}_\ell = \sum_{n=0}^{\infty} q^n \, [\mathbb{P}^n]_{(\ell)} \in \Omega_\star((q)).$$

Theorem 18. *For surfaces with $p_g > 0$, we have*

$$\mathsf{Z}^{\mathsf{cob}}_{X,\beta} = \mathsf{SW}(\beta) \cdot q^{-K^2} \cdot \left(-\frac{\mathsf{P}_{-1}}{\mathsf{P}^2} \right)^{K^2 - K\cdot\beta}.$$

Proof. The series P corresponds under our conventions to

$$\mathsf{P} = \sum_{n=0}^{\infty} q^n \int_{\mathbb{P}^n} g(T\mathbb{P}^n) = \sum_{n=0}^{\infty} q^n \int_{\mathbb{P}^n} g(h)^{n+1} = \sum_{n=0}^{\infty} q^n \cdot \left([h^n] \, g(h)^{n+1} \right) = \frac{dh}{dq}$$

by Lagrange-Bürmann formula for $q = \frac{h}{g(h)}$. Note that $h = -H$ in the notation used above. Similarly,

$$\begin{aligned} \mathsf{P}_\ell &= \sum_{n=0}^{\infty} q^n \int_{[\mathbb{P}^n]_{(\ell)}} g(T^{\mathrm{vir}} \, [\mathbb{P}^n]_{(\ell)}) = \sum_{n=0}^{\infty} q^n \int_{\mathbb{P}^n} \mathsf{e}(\mathrm{Obs}) \cdot g(T\mathbb{P}^n - \mathrm{Obs}) \\ &= \sum_{n=0}^{\infty} q^n \int_{\mathbb{P}^n} \ell h \cdot \frac{g(h)^{n+1}}{g(\ell h)} = \sum_{n=0}^{\infty} q^n \cdot \left([h^n] \, \frac{\ell h \cdot g(h)}{g(\ell h)} \cdot g(h)^n \right) = \frac{\ell h}{g(\ell h)} \cdot \frac{dh}{dq}, \end{aligned}$$

again using Lagrange-Bürmann in the last equality. By direct calculation, we find

$$\mathsf{A} = \frac{-h}{g(-h)} \cdot \left(\frac{g'}{g}(h) - \frac{1}{h} \right) = -\frac{1}{q} \cdot \frac{\mathsf{P}_{-1}}{\mathsf{P}^2},$$

completing the proof. □

3. Rationality in K-theory and examples

3.1. **Overview.** We here establish Theorems 1 and 4 using the general calculations of Section 2.2. In addition, we justify Examples 7, 9 and 10 in the Introduction. Finally, we consider surfaces with $p_g = 0$, thereby proving Theorem 2, and we discuss a calculation involving rank 1 quotients.

3.2. **Rationality.** *Proof of Theorem 4.* The theorem concerns the shifted series

$$\overline{\mathsf{Z}} = \sum_{n\in\mathbb{Z}} q^{n+\beta\cdot K} \chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^1,\beta,n), \wedge^{k_1}\alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_\ell}\alpha_\ell^{[n]}).$$

For formal variables $y_1,\ldots,y_\ell$, define

$$\overline{\mathsf{Z}}^\dagger = \sum_{n\in\mathbb{Z}} q^{n+\beta\cdot K} \chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^1,\beta,n), \wedge_{y_1}\alpha_1^{[n]} \otimes \cdots \otimes \wedge_{y_\ell}\alpha_\ell^{[n]}).$$

Then

$$\overline{\mathsf{Z}} = \frac{1}{k_1!\ldots k_\ell!}\cdot\left(\frac{\partial}{\partial y_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial y_\ell}\right)^{k_\ell}\overline{\mathsf{Z}}^\dagger\Big|_{\underline{y}=0}.$$

The series $\overline{\mathsf{Z}}^\dagger$ are special cases of the general expressions of Section 2. This can be seen by invoking the Hirzebruch-Riemann-Roch theorem of [CFK, FG]. For a scheme S endowed with a 2-term perfect obstruction theory, and any K-theory class $V\to S$, we have

$$\chi^{\mathrm{vir}}(S,V) = \int_{[S]^{\mathrm{vir}}} \mathrm{ch}\,(V)\cdot \mathrm{Todd}\,(T^{\mathrm{vir}}S).$$

In our context, we take

$$S = \mathsf{Quot}_X(\mathbb{C}^1,\beta,n), \quad V = \wedge_{y_1}\alpha_1^{[n]}\otimes\ldots\otimes\wedge_{y_\ell}\alpha_\ell^{[n]}.$$

Thus, in the notation of Section 2, the series $\overline{\mathsf{Z}}^\dagger$ corresponds to functions

$$f_s(x) = 1+y_s e^x, \quad 1\le s\le \ell, \quad g(x) = \frac{x}{1-e^{-x}}.$$

For $p_g>0$, we write via (11)

$$\overline{\mathsf{Z}}^\dagger = \mathsf{SW}(\beta)\cdot \mathsf{A}^{K^2-\beta\cdot K}\cdot\prod_{s=1}^{\ell}\mathsf{B}_s^{c_1(\alpha_s).K}\cdot\prod_{s=1}^{\ell}\mathsf{V}_s^{c_1(\alpha_s)\cdot\beta},$$

where we used $\mathsf{A}=\mathsf{U}^{-1}$. The change of variables (15) becomes

$$q = (1-e^H)\prod_{s=1}^{\ell}(1+y_s e^{-H})^{-r_s}.$$

For convenience, set $e^H = 1-z$ so that

$$q = z\cdot\prod_{s=1}^{\ell}\left(\frac{1-z}{1-z+y_s}\right)^{r_s}.$$

We regard z as a function of $q, y_1, \ldots, y_\ell$. From formulas (16)–(18) of Section 2.2 we find

$$\mathsf{A} = \mathsf{U}^{-1} = \prod_{s=1}^{\ell} \left(1 + \frac{y_s}{1-z}\right)^{r_s} \cdot \prod_{s=1}^{\ell} (1+y_s)^{-r_s} \cdot \left(\sum_{s=1}^{\ell} r_s \frac{y_s}{1+y_s-z} \cdot \frac{-z}{1-z} + 1\right)$$

$$\mathsf{B}_s = \frac{(1+y_s)(1-z)}{1+y_s-z}$$

$$\mathsf{V}_s = 1 + \frac{y_s}{1-z}.$$

We immediately see that

$$\mathsf{A}\Big|_{y=0} = \mathsf{B}_s\Big|_{y=0} = \mathsf{V}_s\Big|_{y=0} = 1.$$

Let us first assume $\ell = 1$ for simplicity, also writing $y_1 = y$ and $r_1 = r$. Note that $z\Big|_{y=0} = q$. The function z is differentiable in y. We claim that for $k \geq 1$, the derivatives take the form

$$\frac{\partial^k z}{\partial y^k}\Big|_{y=0} = \frac{P_k(q)}{(1-q)^{2k-1}}$$

for certain polynomials P_k.[8] This can be seen by induction on k starting from the equation

$$F(y,z) = z \cdot \left(\frac{1-z}{1-z+y}\right)^r = q.$$

For instance, differentiating once we obtain

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0 \implies \frac{\partial z}{\partial y}\Big|_{y=0} = \frac{rq}{1-q}.$$

The case of arbitrary k is notationally more involved, but it is useful to spell out the details in order to bound the order of the poles. Key to the argument is the general expression for the kth derivative of $F(y,z)$ with respect to y. It takes the form

$$\sum \mathsf{C}\,(p \,|\, m_1, \ldots, m_t) \cdot \left(\frac{\partial}{\partial z}\right)^t \left(\frac{\partial}{\partial y}\right)^p F \cdot \frac{\partial^{m_1} z}{\partial y^{m_1}} \cdots \frac{\partial^{m_t} z}{\partial y^{m_t}} = 0.$$

In the above expression, the length of the tuple $(m_1, \ldots, m_t)$ must equal the number of z derivatives, namely t, and of course the total number of y derivatives equals

$$p + m_1 + \ldots + m_t = k.$$

The derivative $\frac{\partial^k z}{\partial y^k}$ appears in the form

$$\frac{\partial F}{\partial z} \cdot \frac{\partial^k z}{\partial y^k} + \text{ lower order terms} = 0.$$

[8] A more precise calculation similar to that in Lemma 22 below shows that the derivative equals

$$(k-1)! \sum_{n=1}^{\infty} r \binom{rn-1}{k-1} \binom{n+k-2}{k-1} q^n.$$

The coefficient of q^n is a polynomial in n of degree $2k-2$, from where the claim follows.

(The leading coefficient $\mathsf{C}(0\,|\,k)$ can be seen to be 1 by induction.) The lower order terms involve lower derivatives of z with respect to y and derivatives of F. By direct calculation we find $\frac{\partial F}{\partial z}\Big|_{y=0} = 1$ so we can solve for the derivative $\frac{\partial^k z}{\partial y^k}\Big|_{y=0}$ in terms of the lower order terms. In fact, using the explicit expression for F, all derivatives take the form

$$\left(\frac{\partial}{\partial z}\right)^t \left(\frac{\partial}{\partial y}\right)^p F\Big|_{y=0} = \frac{F_{p,t}(q)}{(1-q)^{p+t}}$$

for certain polynomials $F_{p,t}$. Inductively, all terms $\frac{\partial^{m_1} z}{\partial y^{m_1}} \cdots \frac{\partial^{m_t} z}{\partial y^{m_t}}\Big|_{y=0}$ are rational functions in q with pole at $q=1$ of order less or equal to $(2m_1-1)+\ldots+(2m_t-1)$. Solving for $\frac{\partial^k z}{\partial y^k}\Big|_{y=0}$ we obtain a rational function with pole of order at most

$$(p+t)+(2m_1-1)+\ldots+(2m_t-1) = p+2m_1+\ldots+2m_t \leq 2k-1.$$

The only exception is $p=0$ which requires $t \geq 2$ (since $t=1$ gives the leading term), but then by direct calculation

$$\left(\frac{\partial}{\partial z}\right)^t F\Big|_{y=0} = 0.$$

This completes the proof of the claim.

Now let $G(y,z)$ be one of the factors that appears in the product expressions of A, B, V or their inverses A^{-1}, B^{-1}, V^{-1}. These terms are fractions in y and z. We claim that

$$\frac{\partial^k G}{\partial y^k}\Big|_{y=0} \text{ is a rational function in } q \text{ with pole of order at most } 2k \text{ at } q=1. \tag{21}$$

Indeed, the derivative is given by the same formula

$$\sum \mathsf{C}\,(p\,|\,m_1,\ldots,m_t) \cdot \left(\frac{\partial}{\partial z}\right)^t \left(\frac{\partial}{\partial y}\right)^p G \cdot \frac{\partial^{m_1} z}{\partial y^{m_1}} \cdots \frac{\partial^{m_t} z}{\partial y^{m_t}}.$$

In all cases at hand, we can compute

$$\left(\frac{\partial}{\partial z}\right)^t \left(\frac{\partial}{\partial y}\right)^p G\Big|_{y=0}$$

directly and observe that the answers are rational functions in q with poles of order at most $2p+t$ at $q=1$. Thus, at $y=0$, the derivative in (21) is rational with poles of order at most

$$(2p+t)+(2m_1-1)+\ldots+(2m_t-1) = 2k.$$

Property (21) is preserved under taking products. We therefore conclude that $\overline{\mathsf{Z}}^\dagger$ satisfies (21) as well. Equivalently, $\overline{\mathsf{Z}}$ is rational in q with pole of order at most $2k$ at $q=1$, as claimed.

Finally, the case $\ell > 1$ is similar. The above statements hold true, their proof requiring only more diligent bookkeeping via multiindices, but no new ideas. □

Proof of Theorem 1. For simplicity, first take $\ell = 1$. We seek to prove the rationality of the series

$$\mathsf{Z}_{X,N,\beta}(\alpha \,|\, k) = \sum_{n\in\mathbb{Z}} q^n \chi^{\mathrm{vir}}\left(\mathsf{Quot}_X(\mathbb{C}^N,\beta,n), \wedge^k \alpha^{[n]}\right).$$

The proof exploits the symmetric structure of the expression (11), and also relies on the rewriting of the change of variables (15) and of the functions (16) – (19) in terms of rational functions, in the case at hand.

Specifically, as in the proof of Theorem 4 above, set

$$\mathsf{Z}^\dagger = \sum_{n\in\mathbb{Z}} q^n \chi^{\mathrm{vir}}\left(\mathsf{Quot}_X(\mathbb{C}^N,\beta,n), \wedge_y \alpha^{[n]}\right) \implies \mathsf{Z} = \frac{1}{k!}\frac{\partial^k \mathsf{Z}^\dagger}{\partial y^k}\bigg|_{y=0}.$$

For Seiberg-Witten classes β of length N, by (11) we have

$$\mathsf{Z}^\dagger = q^{-\beta\cdot K_X} \sum_{\beta_1+\ldots+\beta_N=\beta} \mathsf{SW}(\beta_1)\cdots\mathsf{SW}(\beta_N)\cdot \mathsf{A}^{K^2}\cdot \mathsf{B}^{K.c_1(\alpha)}\cdot \prod_{i=1}^N \mathsf{U}_i^{\beta_i.K}\cdot \prod_{i=1}^N \mathsf{V}_i^{\beta_i.c_1(\alpha)}\cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_i.\beta_j}.$$

For every decomposition $\beta = \beta_1 + \ldots + \beta_N$, we consider the symmetrized expression

$$\mathsf{C} = \frac{1}{|\mathrm{Stab}|}\sum_{\sigma\in S_N} \prod_{i=1}^N \mathsf{U}_i^{\beta_{\sigma(i)}.K}\cdot \prod_{i=1}^N \mathsf{V}_i^{\beta_{\sigma(i)}.c_1(\alpha)}\cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_{\sigma(i)}.\beta_{\sigma(j)}}$$

where the sum is over the orbit of the symmetric group S_N permuting the classes β_i. We will drop the dependence of C on the classes β_i for simplicity.

We use the results of Section 2.2 for the functions

$$f(x) = 1 + ye^x, \quad g(x) = \frac{x}{1-e^{-x}}.$$

To simplify the formulas, we further set $e^{-H_j} = 1 - z_j$, $e^{w_k} = 1 + t_k$. The change of variables (15) becomes

$$q = (1+y(1-z_j))^{-r}\cdot \prod_{k=1}^N \left(\frac{-z_j - t_k}{1-z_j}\right), \quad z_j(0) = -t_j. \tag{22}$$

We record

$$\mathsf{A} = \prod_{i=1}^{N} \left(\frac{1+y(1-z_i)}{1+y(1+t_i)} \right)^r \prod_{1\leq i,k\leq N} \frac{z_i+t_k}{1+t_k} \prod_{i\neq j} \frac{1-z_j}{z_i-z_j} \prod_{i=1}^{N} \left(\frac{ry(1-z_i)}{1+y(1-z_i)} + \sum_{k=1}^{N} \frac{1+t_k}{z_i+t_k} \right)$$

$$\mathsf{B} = \prod_{i=1}^{N} \frac{1+y(1+t_i)}{1+y(1-z_i)}$$

$$\mathsf{U}_i = \frac{1}{(1+y(1-z_i))^r} \prod_{k=1}^{N} \frac{1+t_k}{z_i+t_k} \prod_{i'\neq i} \left(\frac{z_i-z_{i'}}{1-z_{i'}} \frac{z_{i'}-z_i}{1-z_i} \right) \left(\frac{ry(1-z_i)}{1+y(1-z_i)} + \sum_{k=1}^{N} \frac{1+t_k}{z_i+t_k} \right)^{-1}$$

$$\mathsf{V}_i = 1+y(1-z_i)$$

$$\mathsf{W}_{i_1,i_2} = \frac{1-z_{i_1}}{z_{i_2}-z_{i_1}} \frac{1-z_{i_2}}{z_{i_1}-z_{i_2}}.$$

To show that

$$\mathsf{Z} = \frac{1}{k!} \frac{\partial^k \mathsf{Z}^\dagger}{\partial y^k}\Big|_{y=0,\,\underline{t}=0}$$

is a rational function in q, it suffices to show that the derivatives

$$\frac{\partial^k \mathsf{A}}{\partial y^k}\Big|_{y=0,\,\underline{t}=0}, \quad \frac{\partial^k \mathsf{B}}{\partial y^k}\Big|_{y=0,\,\underline{t}=0}, \quad \frac{\partial^k \mathsf{C}}{\partial y^k}\Big|_{y=0,\,\underline{t}=0}, \tag{23}$$

have the same property. We do not have closed form expressions for these derivatives, but a qualitative argument suffices.

Property (23) is in fact valid for any rational function R in the $\{z_i\}$, $\{t_k\}$ and y, which is symmetric in the z's. The functions $\mathsf{A}, \mathsf{B}, \mathsf{C}$ are of this type. This is clear for A and B from the explicit expressions, and it also holds for C because we sum over the orbit of the S_N-action.

The assertion above was proven in [JOP], Section 2.2, and we only indicate the main points. Indeed, $\mathsf{R}\Big|_{y=0,\,\underline{t}=0}$ is a rational function in q since it can be expressed in terms of the elementary symmetric functions in the z_i's. These symmetric functions are continuous in y and $\underline{t}$. At $y=0$, $\underline{t}=0$, they are rational functions in q since the change of variables becomes

$$z^N = q(z-1)^N.$$

The derivative

$$\frac{\partial \mathsf{R}}{\partial y} = \sum_{i=1}^{N} \frac{\partial \mathsf{R}}{\partial z_i} \cdot \frac{\partial z_i}{\partial y} + \partial_y \mathsf{R}$$

is also given by a rational function in $\{z_i\}$, $\{t_k\}$ and y, symmetric in the $\{z_i\}$'s. The second term $\partial_y \mathsf{R}$ is calculated keeping z fixed, and is symmetric in the z's since R is. By implicit

differentiation

$$F(y,\, z,\, \underline{t}) = (1+y(1-z))^{-r} \cdot \prod_{k=1}^{N} \frac{-z-t_k}{1-z} = q \implies \frac{\partial z}{\partial y} = \mathsf{S}(y,\, z,\, \underline{t})$$

for an explicit rational function S. Using that R is symmetric and the formula for the implicit derivatives, it follows that transposing z_i and z_j turns

$$\frac{\partial \mathsf{R}}{\partial z_i} \text{ into } \frac{\partial \mathsf{R}}{\partial z_j} \quad \text{and} \quad \frac{\partial z_i}{\partial y} \text{ into } \frac{\partial z_j}{\partial y}.$$

Since we sum over all i's, $\frac{\partial \mathsf{R}}{\partial y}$ is symmetric in the z_i's.

Inductively, it follows that all higher derivatives of $\frac{\partial^k \mathsf{R}}{\partial y^k}$ are rational functions of $\{z_i\}$, $\{t_k\}$ and y, symmetric in the $\{z_i\}$. Their values $\frac{\partial^k \mathsf{R}}{\partial y^k}\Big|_{y=0,\, \underline{t}=0}$ are therefore rational in q.

Finally, introducing additional formal variables $y_1, \ldots, y_\ell$ and running the argument above, we obtain the rationality of the series

$$\sum_{n\in\mathbb{Z}} q^n \chi^{\mathrm{vir}}\left(\mathsf{Quot}_X(\mathbb{C}^N, \beta, n), \wedge^{k_1}\alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_\ell}\alpha_\ell^{[n]}\right). \qquad \square$$

Remark 19. The same arguments prove the rationality of the generating series of descendent invariants defined in [JOP], for all curve classes β of Seiberg-Witten length N. This partially confirms Conjecture 2 in the work cited.

3.3. **Examples.** Equations (3) and (4) in the Introduction are obtained by executing the above proof in more precise detail. We continue to use the notation

$$\mathsf{Z} = \sum_{n\in\mathbb{Z}} q^n \chi^{\mathrm{vir}}\left(\mathsf{Quot}_X(\mathbb{C}^N, n), \alpha^{[n]}\right), \quad \mathsf{Z}^\dagger = \sum_{n\in\mathbb{Z}} q^n \chi^{\mathrm{vir}}\left(\mathsf{Quot}_X(\mathbb{C}^N, n), \wedge_y \alpha^{[n]}\right).$$

We have

$$\mathsf{Z} = \frac{\partial \mathsf{Z}^\dagger}{\partial y}\Big|_{y=0}, \quad \mathsf{Z}^\dagger = \mathsf{A}^{K^2} \cdot \mathsf{B}^{c_1(\alpha).K}.$$

The functions A, B were recorded in the proof of Theorem 1. We claim that

(i) $\mathsf{A}\Big|_{y=0,\, \underline{t}=0} = \mathsf{B}\Big|_{y=0,\, \underline{t}=0} = 1$

(ii) $\frac{\partial \mathsf{A}}{\partial y}\Big|_{y=0,\, \underline{t}=0} = -Nr \cdot \frac{q^2}{(1-q)^2}, \quad \frac{\partial \mathsf{B}}{\partial y}\Big|_{y=0,\, \underline{t}=0} = -N \cdot \frac{q}{1-q}.$

From here, (3) and (4) follow immediately since

$$\mathsf{Z} = \frac{\partial \mathsf{Z}^\dagger}{\partial y}\Big|_{y=0} = \frac{\partial}{\partial y}\left(\mathsf{A}^{K^2} \cdot \mathsf{B}^{c_1(\alpha).K}\right)\Big|_{y=0} = -Nr \cdot K^2 \cdot \frac{q^2}{(1-q)^2} - N \cdot (c_1(\alpha)\cdot K) \cdot \frac{q}{1-q}.$$

Item (i) is clear due to equation (2). This can also be seen directly by setting

$$y = 0,\, \underline{t} = 0$$

in the explicit formulas. The fact that these substitutions make sense is a consequence of the arguments in Section 2.2 of [JOP]: the elementary symmetric functions in the roots $z_1, \dots, z_N$ are continuous in the parameters. When $y = 0$, $\underline{t} = 0$, the equation giving the change of variables becomes

$$z^N - q(z-1)^N = 0$$

with roots z_i. It is then clear that $\mathsf{B}\Big|_{y=0,\,\underline{t}=0} = 1$ and

$$\mathsf{A}\Big|_{y=0,\,\underline{t}=0} = \prod_{i=1}^N \mathsf{z}_i^N \cdot \prod_{i\neq j} \frac{1-\mathsf{z}_j}{\mathsf{z}_i - \mathsf{z}_j} \cdot \prod_{i=1}^N \frac{N}{\mathsf{z}_i} = 1.$$

Item (ii) is similar. It is more convenient to work with the logarithmic derivative. Set $\mathsf{B}^\dagger = \log \mathsf{B}$. Using (i), we have

$$\begin{aligned}\frac{\partial \mathsf{B}}{\partial y}\Big|_{y=0,\,\underline{t}=0} = \frac{\partial \mathsf{B}^\dagger}{\partial y}\Big|_{y=0,\,\underline{t}=0} &= \sum_{i=1}^N \frac{\partial}{\partial y}\left(\log(1+y) - \log(1+y(1-z_i))\right)\Big|_{y=0} \\ &= \sum_{i=1}^N \mathsf{z}_i = -N \cdot \frac{q}{1-q}.\end{aligned}$$

We record here that

$$\sum_{i=1}^N \mathsf{z}_i = -N\frac{q}{1-q}, \qquad \sum_{i<j} \mathsf{z}_i\mathsf{z}_j = -\binom{N}{2}\frac{q}{1-q}. \tag{24}$$

The calculation involving A is slightly more involved and requires the implicit derivatives

$$\frac{\partial z_i}{\partial y}\Big|_{y=0,\,\underline{t}=0} = \frac{r}{N}\mathsf{z}_i(1-\mathsf{z}_i)^2.$$

These are found by differentiating (22). Writing $\mathsf{A}^\dagger = \log \mathsf{A}$, we obtain

$$\frac{\partial \mathsf{A}}{\partial y}\Big|_{y=0,\,\underline{t}=0} = \frac{\partial \mathsf{A}^\dagger}{\partial y}\Big|_{y=0,\,\underline{t}=0} = \sum_{i=1}^N \frac{\partial \mathsf{A}^\dagger}{\partial z_i}\Big|_{y=0,\,\underline{t}=0} \cdot \frac{\partial z_i}{\partial y}\Big|_{y=0,\,\underline{t}=0} + \frac{\partial \mathsf{A}^\dagger}{\partial y}\Big|_{y=0,\,\underline{t}=0}.$$

The last y-derivative is understood as keeping the z's fixed. The answer is symmetric in the z's and setting $y = \underline{t} = 0$ is therefore allowed. Direct calculation gives

$$\frac{\partial \mathsf{A}^\dagger}{\partial y}\Big|_{y=0,\,\underline{t}=0} = -r\sum_{i=1}^N \mathsf{z}_i + \frac{r}{N}\sum_{i=1}^N \mathsf{z}_i(1-\mathsf{z}_i) = -Nr \cdot \frac{q^2}{(1-q)^2}.$$

Similarly, the following sum is also symmetric in the roots, and by direct computation we find

$$\sum_{i=1}^{N} \frac{\partial \mathsf{A}^\dagger}{\partial z_i}\Big|_{y=0,\,\underline{t}=0} \cdot \frac{\partial z_i}{\partial y}\Big|_{y=0,\,\underline{t}=0} = \frac{r}{N} \cdot \sum_{i=1}^{N} \frac{\partial \mathsf{A}^\dagger}{\partial z_i}\Big|_{y=0,\,\underline{t}=0} \cdot \mathsf{z}_i(1-\mathsf{z}_i)^2$$
$$= \frac{r}{N}\left((N-1)\sum_i \mathsf{z}_i - 2\sum_{i<j} \mathsf{z}_i\mathsf{z}_j\right) = 0,$$

using (24). This completes the argument.

Finally, equations (5) and (6) of the Introduction require not only the series A, B studied in (i)–(ii), but also U_i, V_i and their first derivatives. For rank $\alpha = 0$, we find

(iii) $\mathsf{U}_i\Big|_{y=0,\,\underline{t}=0} = -\frac{Nq}{1-q}\cdot\frac{1}{\mathsf{z}_i}, \quad \mathsf{V}_i\Big|_{y=0,\,\underline{t}=0} = 1$

(iv) $\frac{\partial \mathsf{U}_i}{\partial y}\Big|_{y=0,\,\underline{t}=0} = 0, \quad \frac{\partial \mathsf{V}_i}{\partial y}\Big|_{y=0,\,\underline{t}=0} = 1-\mathsf{z}_i.$

The expressions in Examples 9 and 10 in the Introduction follow by substituting (i) – (iv) in (11). Items (i) – (iv) and equation (19) suffice to treat all classes $\beta = \ell K_X$ for $0 \le \ell \le N$, but the formulas are more cumbersome.

3.4. **Surfaces with $p_g = 0$.** We next give a proof of Theorem 2. Let X be a simply connected surface with $p_g = 0$ and set $N = 1$. We show that the shifted series

$$\overline{\mathsf{Z}} = q^{\frac{1}{2}\beta(\beta+K_X)} \cdot \sum_{n\in\mathbb{Z}} q^n \chi^{\mathrm{vir}}\left(\mathsf{Quot}_X(\mathbb{C}^1,\beta,n),\, \alpha^{[n]}\right)$$

is given by a rational function with poles only at $q = 1$. This case reduces to the calculation of Riemann-Roch numbers of tautological sheaves over the (non-virtual) Hilbert scheme of points.

We have the following identification

$$\mathsf{Quot}_X(\mathbb{C}^1,\beta,n) \simeq X^{[m]} \times \mathbb{P}$$

where $\mathbb{P} = |\beta|$ and $m = n + \frac{\beta(\beta+K_X)}{2}$. When $p_g = 0$, the obstruction bundle was found in [OP]:

$$\mathrm{Obs} = H^1(\beta)\otimes\zeta + \left(M^{[m]}\right)^\vee \otimes \zeta,$$

where $\zeta = \mathcal{O}_{\mathbb{P}}(1)$ denotes the tautological bundle, and $M = K_X - \beta$. Thus

$$\mathcal{O}^{\mathrm{vir}}_{\mathsf{Quot}} = \wedge_{-1}\mathrm{Obs}^\vee = \left(\wedge_{-1}\zeta^{-1}\right)^{h^1(\beta)} \otimes \wedge_{-1}\left(M^{[m]}\otimes\zeta^{-1}\right).$$

The term

$$\wedge_{-1}\zeta^{-1} = \mathcal{O} - \zeta^{-1}$$

is supported on hyperplanes, and its powers can be used to cut down the dimension of the projective space. Hence

$$\mathcal{O}^{\mathrm{vir}}_{\mathsf{Quot}} = \iota_\star \wedge_{-1} \left(M^{[m]} \otimes \zeta^{-1} \right)$$

is supported on $X^{[m]} \times \mathbb{P}(V)$ where

$$\mathrm{dim} V = h^0(\beta) - h^1(\beta) = \chi(\beta) = \nu + 1, \quad \text{assuming } \nu \geq 0.$$

We used here that $h^2(\beta) = 0$ by Serre duality, combined with the fact that β is effective and $p_g = 0$. In consequence, for an arbitrary class γ, we have

$$\begin{aligned} \chi^{\mathrm{vir}}\left(\mathsf{Quot}_X(\mathbb{C}^1, \beta, n), \gamma\right) &= \chi\left(X^{[m]} \times \mathbb{P}(V),\ \wedge_{-1}\left(M^{[m]} \otimes \zeta^{-1}\right) \otimes \gamma\right) \\ &= \sum_{k=0}^{m} (-1)^k \chi\left(X^{[m]} \times \mathbb{P}(V),\ \wedge^k M^{[m]} \otimes \zeta^{-k} \otimes \gamma\right). \end{aligned}$$

Under the identification of the Quot scheme with $X^{[m]} \times \mathbb{P}$, the tautological classes take the form

$$\alpha^{[n]}_{\mathsf{Quot}} = \chi(\alpha) \cdot \mathcal{O} - \chi(\widetilde{\alpha}) \cdot \zeta^{-1} + \widetilde{\alpha}^{[m]}_{\mathsf{Hilb}} \otimes \zeta^{-1}$$

where $\widetilde{\alpha} = \alpha \otimes \mathcal{O}(-\beta)$; see [JOP], Section 3.2. As a result, we write

$$\overline{\mathsf{Z}} = \mathsf{Z}_1 + \mathsf{Z}_2 + \mathsf{Z}_3$$

where

$$\begin{aligned} \mathsf{Z}_1 &= \chi(\alpha) \cdot \sum_{m=0}^{\infty} q^m \sum_{k=0}^{m} (-1)^k \chi\left(X^{[m]} \times \mathbb{P}(V),\ \wedge^k M^{[m]} \otimes \zeta^{-k}\right) \\ \mathsf{Z}_2 &= -\chi(\widetilde{\alpha}) \cdot \sum_{m=0}^{\infty} q^m \sum_{k=0}^{m} (-1)^k \chi\left(X^{[m]} \times \mathbb{P}(V),\ \wedge^k M^{[m]} \otimes \zeta^{-k-1}\right) \\ \mathsf{Z}_3 &= \sum_{m=0}^{\infty} q^m \sum_{k=0}^{m} (-1)^k \chi\left(X^{[m]} \times \mathbb{P}(V),\ \wedge^k M^{[m]} \otimes \widetilde{\alpha}^{[m]} \otimes \zeta^{-k-1}\right). \end{aligned}$$

All three series yield rational functions. The first two series require Theorem 5.2.1 in [Sc] or Corollary 6.1 in [A]:

$$\chi\left(X^{[m]},\ \wedge^k M^{[m]}\right) = \binom{m-k+\chi(\mathcal{O}_X)-1}{m-k} \cdot \binom{\chi(M)}{k} = \binom{\nu+1}{k}$$

using that in our case $\chi(\mathcal{O}_X) = 1$. Thus

$$\begin{aligned} \mathsf{Z}_1 &= \chi(\alpha) \cdot \sum_{m=0}^{\infty} q^m \sum_{k=0}^{m} (-1)^k \chi\left(X^{[m]},\ \wedge^k M^{[m]}\right) \cdot \chi\left(\mathbb{P}(V), \zeta^{-k}\right) \\ &= \chi(\alpha) \cdot \sum_{m=0}^{\infty} q^m \cdot \sum_{k=0}^{m} (-1)^k \binom{\nu+1}{k} \cdot \binom{-k+\nu}{\nu} \\ &= \chi(\alpha) \cdot (1 + q + \ldots + q^\nu). \end{aligned}$$

The last evaluation follows by examining the nonzero binomial contributions $k = 0$ and $k = \nu + 1$ in the sum above. In a similar fashion, we find

$$\mathsf{Z}_2 = -\chi(\widetilde{\alpha}) \cdot (\nu + 1) \cdot q^{\nu}.$$

Regarding the third sum, we note

$$\begin{aligned}\mathsf{Z}_3 &= \sum_{m=0}^{\infty} q^m \sum_{k=0}^{m} (-1)^k \chi\left(X^{[m]} \times \mathbb{P}(V),\ \wedge^k M^{[m]} \otimes \widetilde{\alpha}^{[m]} \otimes \zeta^{-k-1}\right) \\ &= \sum_{m=0}^{\infty} q^m \sum_{k=0}^{m} (-1)^k \chi\left(X^{[m]}, \wedge^k M^{[m]} \otimes \widetilde{\alpha}^{[m]}\right) \cdot \binom{\nu - k - 1}{\nu}.\end{aligned}$$

We require the following:

Proposition 20. *For all nonsingular surfaces X, line bundles $M \to X$, and K-theory classes $\widetilde{\alpha}$ on X, we have*

$$\mathsf{W} = \sum_{m=0}^{\infty} q^m \chi\left(X^{[m]}, \wedge_y M^{[m]} \otimes \widetilde{\alpha}^{[m]}\right) = q \cdot \frac{(1+qy)^{\chi(M)}}{(1-q)^{\chi(\mathcal{O}_X)}} \cdot \chi\left(X, \frac{\wedge_y M}{\wedge_{qy} M} \otimes \widetilde{\alpha}\right). \tag{25}$$

To prove rationality of Z_3, note that

$$\mathsf{Z}_3 = \frac{1}{\nu!} \cdot \frac{\partial^{\nu} \mathsf{W}}{\partial y^{\nu}}\bigg|_{y=-1}.$$

We examine the derivatives of each term in the product (25). For instance, for $\nu \geq 1$, we have

$$\frac{\partial^{\nu}}{\partial y^{\nu}} \chi\left(X, \frac{\wedge_y M}{\wedge_{qy} M} \otimes \widetilde{\alpha}\right)\bigg|_{y=-1} = (-1)^{\nu-1}\, \nu!\, q^{\nu-1}(1-q) \cdot \chi\left(X, \frac{M^{\nu}}{(1-qM)^{\nu+1}} \otimes \widetilde{\alpha}\right).$$

Expanding using the binomial theorem, we arrive at the expression

$$(-1)^{\nu-1}\, \nu!\, q^{\nu-1}(1-q) \cdot \sum_{k=0}^{\infty} (-q)^k \cdot \binom{-\nu-1}{k} \cdot \chi(X,\, M^{\nu+k} \otimes \widetilde{\alpha}).$$

By Hirzebruch-Riemann-Roch, the Euler characteristics $\chi(X,\, M^{\nu+k} \otimes \widetilde{\alpha})$ depend polynomially on k, and the same is true about the binomial prefactors. Thus, the answer is a rational function in q with possible pole only at $q = 1$.

The interested reader can compute Z_3 explicitly by following the above method, substituting $\chi(M) = \chi(\beta) = \nu + 1$ and $\chi(\mathcal{O}_X) = 1$. For example, when $\nu \geq 2$, after cancellations, we obtain

$$\mathsf{Z}_3 = q^{\nu} \cdot (-\chi(M^{-1} \otimes \widetilde{\alpha}) + (\nu + 1) \cdot \chi(\widetilde{\alpha}))$$

so that

$$\overline{\mathsf{Z}} = \chi(\alpha) \cdot (1 + q + \ldots + q^{\nu}) - \chi(K^{-1} \otimes \alpha) \cdot q^{\nu}. \tag{26}$$

The answers for $\nu = 0$ or 1 have poles at $q = 1$ of order at most 2; we do not record them here. □

Proof[9] *of Proposition 20.* Since both sides are additive in $\widetilde{\alpha}$, it suffices to assume $\widetilde{\alpha}$ is a line bundle. Consider the more general expression

$$\mathsf{W}^\dagger = \sum_{n=0}^{\infty} q^n \chi\left(X^{[n]}, \wedge_y M^{[n]} \otimes \operatorname{Sym}_z \widetilde{\alpha}^{[n]}\right).$$

We show that

$$\mathsf{W}^\dagger = \frac{(1+qy)^{\chi(M)}}{(1-q)^{\chi(\mathcal{O}_X)}} \cdot \left(1 + qz\chi\left(X, \frac{\wedge_y M}{\wedge_{qy} M} \otimes \widetilde{\alpha}\right) + \dots\right),$$

where the omitted terms have order at least 2 in z. The Proposition follows from here by extracting the z-coefficient. Note that the z-constant term

$$\mathsf{W}^\dagger\Big|_{z=0} = \sum_{n=0}^{\infty} q^n \chi\left(X^{[n]}, \wedge_y M^{[n]}\right) = \frac{(1+qy)^{\chi(M)}}{(1-q)^{\chi(\mathcal{O}_X)}}$$

is indeed correct by [A, Sc], and was already used above. Consequently, we only need to confirm the asymptotics of the normalized expression

$$\mathsf{W}^\star = \frac{\mathsf{W}^\dagger}{\mathsf{W}^\dagger\big|_{z=0}} \stackrel{?}{=} 1 + qz\chi\left(X, \frac{\wedge_y M}{\wedge_{qy} M} \otimes \widetilde{\alpha}\right) + \dots. \tag{27}$$

By the universality results of [EGL], it suffices to establish (27) for toric surfaces X. Via equivariant localization in K-theory, we reduce to the case of equivariant $X = \mathbb{C}^2$. These steps are standard, and the interested reader can find more details in a similar context in [A, EGL].

Let the torus $\mathbb{C}^\star \times \mathbb{C}^\star$ act on $\mathbb{C}^2$ with weights t_1, t_2, which we write multiplicatively. The line bundles M and $\widetilde{\alpha}$ are trivial, but they carry possibly nontrivial equivariant weights m and a respectively. The equivariant version of (27) becomes

$$\mathsf{W}^\star = 1 + qz \cdot \frac{a}{(1-t_1^{-1})(1-t_2^{-1})} \cdot \frac{1+ym}{1+qym} + \dots.$$

The torus fixed points of the Hilbert scheme $\left(\mathbb{C}^2\right)^{[n]}$ are standardly known to correspond to monomial ideals; these are indexed by partitions λ of n. We note the equivariant restrictions of the tautological bundles to the fixed points

$$M^{[n]}\Big|_\lambda = \sum_{(b_1,b_2)\in\lambda} m\, t_1^{-b_1} t_2^{-b_2}, \quad \alpha^{[n]}\Big|_\lambda = \sum_{(b_1,b_2)\in\lambda} a\, t_1^{-b_1} t_2^{-b_2},$$

[9] This also follows from [Z], equation (194) by setting $u = -y$, $v = 0$ in the corresponding formulas. In [Z], the last term is claimed to be $\chi(X, \widetilde{\alpha})$. There is however a substitution oversight on the previous page just above equation (192). When the correct values of u, v are used, the above expression naturally appears. For the convenience of the reader, we decided to include an independent argument here.

as well as the character of the tangent space

$$T_\lambda \left(\mathbb{C}^2\right)^{[n]} = \sum_{\square \in \lambda} \left(t_1^{-\ell(\square)} t_2^{a(\square)+1} + t_1^{\ell(\square)+1} t_2^{-a(\square)} \right).$$

The equivariant sum

$$\mathsf{W}^\dagger = \sum_{n=0}^{\infty} q^n \chi\left(\left(\mathbb{C}^2\right)^{[n]}, \wedge_y M^{[n]} \otimes \operatorname{Sym}_z \widetilde{\alpha}^{[n]} \right)$$

thus takes the form

$$\mathsf{W}^\dagger = \sum_\lambda \frac{q^{|\lambda|}}{\wedge_{-1} T_\lambda^\vee} \cdot \prod_{(b_1,b_2)\in\lambda} \frac{1 + ym\, t_1^{-b_1} t_2^{-b_2}}{1 - za\, t_1^{-b_1} t_2^{-b_2}}.$$

It is more convenient to introduce the following modified expression

$$\mathsf{F}(q,u,v) = \sum_\lambda \frac{q^{|\lambda|}}{\wedge_{-1} T_\lambda^\vee} \cdot \prod_{(b_1,b_2)\in\lambda} \frac{1 - u t_1^{b_1} t_2^{b_2}}{v - t_1^{b_1} t_2^{b_2}}.$$

Then

$$\mathsf{W}^\dagger = \mathsf{F}\left(-qym, -\frac{1}{ym}, za\right), \quad \mathsf{W}^\star = \frac{\mathsf{F}\left(-qym, -\frac{1}{ym}, za\right)}{\mathsf{F}\left(-qym, -\frac{1}{ym}, 0\right)}.$$

Crucially for our arguments, F admits the following (q,v)-symmetry

$$\frac{\mathsf{F}(q,u,v)}{\mathsf{F}(q,u,0)} = \frac{\mathsf{F}(v,u,q)}{\mathsf{F}(v,u,0)}.$$

This is contained in Theorem 1.2 of [A], and is proven via localization over the Donaldson-Thomas moduli space of Calabi-Yau toric threefolds. The symmetry greatly simplifies our calculation: by switching parameters, we only need to find the contribution of the Hilbert scheme of one point. Specifically, we have

$$\mathsf{W}^\star = \frac{\mathsf{F}\left(-qym, -\frac{1}{ym}, za\right)}{\mathsf{F}\left(-qym, -\frac{1}{ym}, 0\right)} = \frac{\mathsf{F}\left(za, -\frac{1}{ym}, -qym\right)}{\mathsf{F}\left(za, -\frac{1}{ym}, 0\right)}.$$

From the definitions, we see that

$$\mathsf{F}\left(za, -\frac{1}{ym}, -qym\right) = 1 + \frac{za}{(1-t_1^{-1})(1-t_2^{-1})} \cdot \frac{1+\frac{1}{ym}}{-qym-1} + \text{h.o.t in } z,$$

$$\mathsf{F}\left(za, -\frac{1}{ym}, 0\right) = 1 - \frac{za}{(1-t_1^{-1})(1-t_2^{-1})} \cdot \left(1 + \frac{1}{ym}\right) + \text{h.o.t in } z,$$

from where the equivariant version of (27) follows at once. □

3.5. **Rank 1 quotients.** We end this section with a calculation involving rank 1 quotients. Let X be a nonsingular projective surface with $p_g = 0$. Let Quot_n parametrize short exact sequences

$$0 \to S \to \mathbb{C}^2 \otimes \mathcal{O}_X \to Q \to 0 \tag{28}$$

where

$$\text{rk } Q = 1, \quad c_1(Q) = 0, \quad \chi(Q) = n + \chi(\mathcal{O}_X).$$

It was shown in [OP, Sch] that Quot_n admits a virtual fundamental class of dimension

$$\text{vd} = \chi(\mathcal{O}_X)$$

whenever $p_g = 0$. We write $\chi = \chi(\mathcal{O}_X) = 1 - h^1(\mathcal{O}_X)$ for simplicity. We must have $0 \leq \chi \leq 1$ in order to obtain non-zero invariants. As an application of our computations in Section 3.2, we show

Proposition 21. *The series*

$$\mathsf{Z} = \sum_{n=0}^{\infty} q^n \chi^{\text{vir}}(\mathsf{Quot}_n, \mathcal{O})$$

is the Laurent expansion of a rational function in q.

The method of proof is well-suited to handle twists by tautological sheaves as well. It is natural to inquire whether the Proposition can be generalized to higher N and higher rank quotients Q, and to include curve classes $\beta \neq 0$. These questions are more difficult and we will address them elsewhere.

Proof. We let the torus $\mathbb{C}^\star$ act with weights w_1, w_2 on the middle term of the sequence (28). Set $w = w_2 - w_1$ for simplicity. There are two fixed point loci

$$\mathsf{F}^{\pm} = X^{[n]}$$

corresponding to exact sequences

$$0 \to S = I_Z \to \mathcal{O}_X[w_1] + \mathcal{O}_X[w_2] \to Q = \mathcal{O}_Z + \mathcal{O}_X \to 0$$

where $\text{length}(Z) = n$. The $\pm$ decorations correspond to the inclusion of the subsheaf S into the first or second trivial summands respectively. The fixed part of the tangent-obstruction theory is

$$T^{\text{vir}}\mathsf{F}^{\pm} = \text{Ext}^\bullet_X(I_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}}) = TX^{[n]} - \text{Obs} = TX^{[n]} - \left(K^{[n]}\right)^\vee.$$

Here $\mathcal{Z}$ is the universal subscheme. Thus, the fixed locus $\mathsf{F}^{\pm} = X^{[n]}$ comes equipped with the virtual fundamental class we studied above. Similarly, the virtual normal bundles carry weights $\pm w$ and are given by

$$\begin{aligned}\mathsf{N}^{\pm} &= \mathrm{Ext}^{\bullet}_X(I_{\mathcal{Z}}, \mathcal{O}_X) = \mathrm{Ext}^{\bullet}_X(\mathcal{O}_X - \mathcal{O}_{\mathcal{Z}}, \mathcal{O}_X) = \mathbb{C}^{\chi} \otimes \mathcal{O} - \mathrm{Ext}^{\bullet}_X(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_X) \\ &= \mathbb{C}^{\chi} \otimes \mathcal{O} - \left(K^{[n]}\right)^{\vee},\end{aligned}$$

using Serre duality in the last step. In particular, for $y = e^{-w}$, we have

$$\frac{1}{\wedge_{-1}\mathsf{N}^{+\vee}} = \frac{\wedge_{-1}\left(K^{[n]}[-w]\right)}{(\wedge_{-1}\mathbb{C}[-w])^{\chi}} = \left(\frac{1}{1-y}\right)^{\chi} \cdot \left(\sum_k (-1)^k \wedge^k K^{[n]} y^k\right) = \frac{\wedge_{-y} K^{[n]}}{(1-y)^{\chi}}$$

and similarly

$$\frac{1}{\wedge_{-1}\mathsf{N}^{-\vee}} = \frac{\wedge_{-y^{-1}} K^{[n]}}{(1-y^{-1})^{\chi}}.$$

By virtual localization in K-theory [FG, Q], we find

$$\begin{aligned}\chi^{\mathrm{vir}}(\mathsf{Quot}_n, \mathcal{O}) &= \chi^{\mathrm{vir}}\left(\mathsf{F}^{+}, \frac{1}{\wedge_{-1}\mathsf{N}^{+\vee}}\right) + \chi^{\mathrm{vir}}\left(\mathsf{F}^{-}, \frac{1}{\wedge_{-1}\mathsf{N}^{-\vee}}\right) \\ &= \frac{1}{(1-y)^{\chi}}\chi^{\mathrm{vir}}\left(X^{[n]}, \wedge_{-y} K^{[n]}\right) + \frac{1}{(1-y^{-1})^{\chi}}\chi^{\mathrm{vir}}\left(X^{[n]}, \wedge_{-y^{-1}} K^{[n]}\right)\Big|_{y=1}.\end{aligned}$$

Expressions of this form were studied in Section 3.2. In particular, from the formulas for the functions A and B immediately following equation (22), we find

$$\sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}\left(X^{[n]}, \wedge_{-y} K^{[n]}\right) = \left(\frac{1-y+yu^2}{1-y+yu}\right)^{K^2}$$

where u is the solution of the equation

$$-q = \frac{u}{1-u} \cdot \frac{1}{1-y+yu}, \quad u(q=0) = 0.$$

To evaluate the second term, we similarly solve the equation

$$-q = \frac{v}{1-v} \cdot \frac{1}{1-y^{-1}+y^{-1}v}, \quad v(q=0) = 0.$$

Therefore,

$$\begin{aligned}\mathsf{Z} &= \sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(\mathsf{Quot}_n, \mathcal{O}) \\ &= \frac{1}{(1-y)^{\chi}} \cdot \left(\frac{1-y+yu^2}{1-y+yu}\right)^{K^2} + \frac{1}{(1-y^{-1})^{\chi}} \cdot \left(\frac{1-y^{-1}+y^{-1}v^2}{1-y^{-1}+y^{-1}v}\right)^{K^2}\Bigg|_{y=1}.\end{aligned}$$

In order to cancel the poles at $y = 1$ in the above expression, we record the following:

Lemma 22. *We have*

$$u = (1-y)\cdot\frac{-q}{1+q} + (1-y)^2\cdot\frac{-q^2(2+q)}{(1+q)^3} + O\left((1-y)^3\right)$$

$$v = (1-y)\cdot\frac{q}{1+q} + (1-y)^2\cdot\frac{q}{(1+q)^3} + O\left((1-y)^3\right).$$

Proof. We only give the details of the first expansion. We could derive this directly by implicit differentiation as in Section 3.2, but a different and slightly stronger argument will be given instead. (The same ideas can be used to give a different proof of Theorem 4.) We change variables $1-y=t$, so that

$$-q = \frac{u}{1-u}\cdot\frac{1}{t+(1-t)u}.$$

We claim that the implicit solution takes the form

$$u(q,t) = \sum_{n\geq 1, k\geq 1} a_{n,k}\, q^n t^k \tag{29}$$

where

$$a_{n,k} = \frac{(-1)^{n+k+1}}{n}\binom{n+k}{k-1}\binom{n}{k}.$$

The Lemma follows since

$$a_{n,1} = (-1)^n, \quad a_{n,2} = (-1)^{n+1}\cdot\frac{(n-1)(n+2)}{2},$$

and

$$\sum_{n=1}^{\infty} a_{n,1}\, q^n = \frac{-q}{1+q}, \quad \sum_{n=1}^{\infty} a_{n,2}\, q^n = -\frac{q^2(2+q)}{(1+q)^3}.$$

Expansion (29) uses Lagrange inversion for implicit functions [So], Theorem 3.6. Specifically, let

$$\mathsf{G}(u,q,t) = -q(1-u)(t+(1-t)u)$$

so that we implicitly solve $\mathsf{G}(u,q,t)=u$. Therefore,

$$u(q,t) = \sum_{n\geq 1, k\geq 1} a_{n,k} q^n t^k$$

where

$$\begin{aligned}
a_{n,k} &= \sum_m \frac{1}{m}[z^{m-1}\, q^n\, t^k]\, \mathsf{G}(z,q,t)^m \\
&= \sum_m \frac{1}{m}[z^{m-1}\, q^n\, t^k]\, (-1)^m\, q^m\, (1-z)^m\, (t+(1-t)z)^m \\
&= \frac{(-1)^n}{n}[z^{n-1}\, t^k]\, (1-z)^n\, (t+(1-t)z)^n \\
&= \frac{(-1)^n}{n}\, [z^{n-1}\, t^k]\, (1-z)^n (z+t(1-z))^n \\
&= \frac{(-1)^n}{n} \binom{n}{k}\, [z^{n-1}]\, (1-z)^n\, z^{n-k}\, (1-z)^k \\
&= \frac{(-1)^n}{n} \binom{n}{k} [z^{k-1}](1-z)^{n+k} = \frac{(-1)^{n+k+1}}{n} \binom{n}{k} \binom{n+k}{k-1}.
\end{aligned}$$

□

Using Lemma 22, we find that

$$\frac{u}{1-y} = \frac{-q}{1+q} + (1-y)\cdot \frac{-q^2(2+q)}{(1+q)^3} + O\left((1-y)^2\right),$$

and from here we derive

$$\frac{1-y+yu^2}{1-y+yu} = \frac{1+\frac{yu^2}{1-y}}{1+\frac{yu}{1-y}} = (1+q)\left(1-(1-y)\cdot \frac{q(1-q)}{(1+q)^2} + O\left((1-y)^2\right)\right).$$

A similar argument shows

$$\frac{1-y^{-1}+y^{-1}v^2}{1-y^{-1}+y^{-1}v} = (1+q)\cdot \left(1+(1-y)\cdot \frac{q(1-q)}{(1+q)^2} + O\left((1-y)^2\right)\right).$$

The Proposition follows from here. For instance, when $\chi = 0$, corresponding to surfaces of irregularity 1, we immediately obtain

$$\mathsf{Z} = 2(1+q)^{K^2}.$$

For regular surfaces, we have $\chi = 1$. In this case, we need to consider the linear terms in $1-y$ as well, yielding

$$\begin{aligned}
\mathsf{Z} &= \frac{(1+q)^{K^2}}{1-y}\left(1-K^2(1-y)\frac{q(1-q)}{(1+q)^2}\right) + \frac{(1+q)^{K^2}}{1-y^{-1}}\left(1+K^2(1-y)\frac{q(1-q)}{(1+q)^2}\right)\Bigg|_{y=1} \\
&= (1+q)^{K^2}\cdot\left(1-2K^2\cdot\frac{q(1-q)}{(1+q)^2}\right).
\end{aligned}$$

This completes the proof. □

4. K-theoretic cosection localization

To complement the explicit calculations of Section 3, we now discuss the virtual structure sheaf of Quot schemes with $\beta = 0$, giving a proof of Theorem 15.

As in the Introduction, assume $C \subset X$ is a nonsingular canonical curve, and let $\Theta = N_{C/X} = \iota^\star K_X$ denote the theta characteristic. Let

$$\iota : \mathsf{Quot}_C(\mathbb{C}^N, n) \hookrightarrow \mathsf{Quot}_X(\mathbb{C}^N, n)$$

be the natural inclusion. We show that the virtual structure sheaf of $\mathsf{Quot}_X(\mathbb{C}^N, n)$ localizes on the curve C:

$$\mathcal{O}^{\mathrm{vir}}_{\mathsf{Quot}_X(\mathbb{C}^N,n)} = (-1)^n \, \iota_\star \, \mathcal{D}_n \tag{30}$$

where we set

$$\mathcal{D}_n \to \mathsf{Quot}_C(\mathbb{C}^N, n), \quad \mathcal{D}_n = \det \mathrm{R}\pi_\star(\Theta \otimes \det \mathcal{Q})^\vee.$$

Along the way, we also prove that

$$\mathcal{D}_n^{\otimes 2} = \det \mathcal{N}^{\mathrm{vir}} \tag{31}$$

where $\mathcal{N}^{\mathrm{vir}}$ stands for the virtual normal bundle of the embedding ι.

To align (30) with the statement of Theorem 15 given in the Introduction, consider the support map

$$p : \mathsf{Quot}_C(\mathbb{C}^N, n) \to C^{[n]}, \;\; Q \mapsto \operatorname{supp} Q.$$

We have

$$\mathcal{D}_n = p^\star \det\left(\Theta^{[n]}\right).$$

Indeed, for $\mathcal{Z} \subset C \times C^{[n]}$ denoting the universal subscheme, over $C^{[n]}$ we establish that

$$\det \mathrm{R}\pi_\star(\Theta \otimes \mathcal{O}(\mathcal{Z}))^\vee = \det \mathrm{R}\pi_\star(\Theta \otimes \mathcal{O}_{\mathcal{Z}}).$$

This follows by relative duality, the fact that Θ is a theta characteristic, and the ideal exact sequence for $\mathcal{Z}$:

$$\det \mathrm{R}\pi_\star(\Theta \otimes \mathcal{O}(\mathcal{Z}))^\vee = \det \mathrm{R}\pi_\star(\Theta \otimes \mathcal{O}(-\mathcal{Z}))^\vee = \det \mathrm{R}\pi_\star(\Theta \otimes \mathcal{O}_{\mathcal{Z}}).$$

Example 23. For $N = 1$, equation (30) states

$$\mathcal{O}^{\mathrm{vir}}_{X^{[n]}} = (-1)^n \iota_\star \det \Theta^{[n]},$$

for $\iota : C^{[n]} \hookrightarrow X^{[n]}$. This is indeed correct since $\mathrm{Obs} = \left((K_X)^{[n]}\right)^\vee$ as noted in [OP, Section 4.2.3], and we have

$$\begin{aligned} \mathcal{O}^{\mathrm{vir}}_{X^{[n]}} &= \wedge_{-1}\mathrm{Obs}^\vee = (-1)^n \wedge^n \mathrm{Obs}^\vee \otimes \wedge_{-1}\mathrm{Obs} \\ &= (-1)^n \det(K_X)^{[n]} \otimes \iota_\star \mathcal{O}_{C^{[n]}} \\ &= (-1)^n \iota_\star \det \Theta^{[n]}. \end{aligned} \tag{32}$$

Here, we used the Koszul resolution

$$\wedge^\bullet \left((K_X)^{[n]}\right)^\vee \to \mathcal{O}_{C^{[n]}}$$

of the canonical section $s^{[n]}$ cutting out $\iota : C^{[n]} \hookrightarrow X^{[n]}$, where s defines $\iota : C \hookrightarrow X$. Furthermore,

$$\begin{aligned} \mathcal{N}^{\mathrm{vir}} &= \iota^\star T^{\mathrm{vir}} X^{[n]} - TC^{[n]} \\ &= \iota^\star \left(TX^{[n]} - \mathrm{Obs}\right) - TC^{[n]} \\ &= N_{C^{[n]}/X^{[n]}} - \iota^\star \mathrm{Obs} \\ &= \Theta^{[n]} - \left(\Theta^{[n]}\right)^\vee. \end{aligned}$$

On the last line, we used the identifications

$$N_{C^{[n]}/X^{[n]}} = \Theta^{[n]}, \quad \iota^\star \mathrm{Obs} = \iota^\star \left((K_C)^{[n]}\right)^\vee = \left(\Theta^{[n]}\right)^\vee,$$

which can be found in [OP, Section 4.3.2]. This allows us to confirm equation (31):

$$\det \mathcal{N}^{\mathrm{vir}} = \left(\det \Theta^{[n]}\right)^2.$$

Proof of Theorem 15. We justify equations (30) and (31). For simplicity, we write Quot_C and Quot_X for the two Quot schemes. We let $\mathcal{S}_C, \mathcal{S}_X$ denote the two universal subsheaves, and let $\mathcal{Q}$ be universal quotient for Quot_C. We begin by noting that

$$\begin{aligned} \mathcal{N}^{\mathrm{vir}} &= \iota^\star T^{\mathrm{vir}} \mathsf{Quot}_X - T\mathsf{Quot}_C \\ &= \mathrm{Ext}^\bullet_X(\mathcal{S}_X, \mathcal{Q}) - \mathrm{Ext}^\bullet_C(\mathcal{S}_C, \mathcal{Q}) \\ &= -\mathrm{Ext}^\bullet_X(\mathcal{Q}, \mathcal{Q}) + \mathrm{Ext}^\bullet_C(\mathcal{Q}, \mathcal{Q}) \\ &= \mathrm{Ext}^\bullet_C(\mathcal{Q}, \mathcal{Q} \otimes \Theta). \end{aligned}$$

The third equality follows by expressing the universal subsheaves in terms of the universal quotient in K-theory. The last identity follows from the exact sequence

$$\ldots \to \mathrm{Ext}^i_C(\mathcal{Q}, \mathcal{Q}) \to \mathrm{Ext}^i_X(\mathcal{Q}, \mathcal{Q}) \to \mathrm{Ext}^{i-1}_C(\mathcal{Q}, \mathcal{Q} \otimes \Theta) \to \ldots$$

proven, for instance, in [T, Lemma 3.42].

For future reference, over $\mathsf{Quot}_C(\mathbb{C}^N, n)$, we set

$$\mathcal{U}_n = \det \mathcal{N}^{\mathrm{vir}} = \det \mathrm{Ext}^\bullet_C(\mathcal{Q}, \mathcal{Q} \otimes \Theta).$$

We show first that

$$\mathcal{O}^{\mathrm{vir}}_{\mathsf{Quot}_X(\mathbb{C}^N,n)} = (-1)^n \, \iota_\star \mathcal{U}_n^{1/2} \tag{33}$$

where the right hand side is only well-defined up to 2-torsion.

To this end, we apply $\mathbb{C}^\star$-equivariant localization to both Quot schemes over C and X using the same weights $w_1, \ldots, w_N$ for the two torus actions. The fixed loci are

$$\mathsf{F}_C[n_1, \ldots, n_N] = C^{[n_1]} \times \cdots \times C^{[n_N]}, \quad \mathsf{F}_X[n_1, \ldots, n_N] = X^{[n_1]} \times \cdots \times X^{[n_N]}$$

where $n_1 + \ldots + n_N = n$. There is a natural embedding

$$\iota : \mathsf{F}_C[n_1, \ldots, n_N] \hookrightarrow \mathsf{F}_X[n_1, \ldots, n_N].$$

In the localization argument below, we will match contributions coming from the two fixed loci corresponding to the same values of $n_1, \ldots, n_N$. For simplicity, we drop the n_i's from the notation, writing F_C and F_X for the fixed loci. We let

$$j_C : \mathsf{F}_C \to \mathsf{Quot}_C, \quad j_X : \mathsf{F}_X \to \mathsf{Quot}_X$$

denote the natural inclusions. We also write N_C and N_X for the virtual normal bundles.

The proof of (33) requires several steps. By the virtual localization theorem in K-theory [Q], we have

$$\mathcal{O}^{\mathrm{vir}}_{\mathsf{Quot}_X} = \sum (j_X)_\star \ \frac{\mathcal{O}^{\mathrm{vir}}_{\mathsf{F}_X}}{\wedge_{-1} \mathsf{N}_X^\vee}$$
$$\mathcal{O}_{\mathsf{Quot}_C} = \sum (j_C)_\star \ \frac{\mathcal{O}_{\mathsf{F}_C}}{\wedge_{-1} \mathsf{N}_C^\vee}.$$

In particular, from the second expression, we find

$$(-1)^n \iota_\star \mathcal{U}_n^{1/2} = \sum (-1)^n \iota_\star (j_C)_\star \ \frac{j_C^\star \, \mathcal{U}_n^{1/2}}{\wedge_{-1} \mathsf{N}_C^\vee}$$
$$= \sum (-1)^n (j_X)_\star \, \iota_\star \ \frac{j_C^\star \, \mathcal{U}_n^{1/2}}{\wedge_{-1} \mathsf{N}_C^\vee}.$$

To prove (33), it suffices to establish

$$\frac{\mathcal{O}^{\mathrm{vir}}_{\mathsf{F}_X}}{\wedge_{-1} \mathsf{N}_X^\vee} = (-1)^n \iota_\star \ \frac{j_C^\star \, \mathcal{U}_n^{1/2}}{\wedge_{-1} \mathsf{N}_C^\vee}. \tag{34}$$

We compute the virtual structure sheaf on the left. The obstruction theory of the fixed locus $\mathsf{F}_X = X^{[n_1]} \times \cdots \times X^{[n_N]}$ splits as

$$\mathrm{Obs} = \oplus_{i=1}^N \mathrm{Obs}_i, \quad \mathrm{Obs}_i = \left(K_X^{[n_i]} \right)^\vee.$$

Thus, using (32), we find

$$\mathcal{O}^{\mathrm{vir}}_{\mathsf{F}_X} = \wedge_{-1}\mathrm{Obs}^\vee = \otimes_{i=1}^N \wedge_{-1} \mathrm{Obs}_i^\vee = \iota_\star\left(\otimes_{i=1}^N (-1)^{n_i} \det \Theta^{[n_i]}\right). \tag{35}$$

To prove (34), we show

$$\frac{\otimes_{i=1}^N \det \Theta^{[n_i]}}{\iota^\star \wedge_{-1} \mathsf{N}_X^\vee} = \frac{j_C^\star\, \mathcal{U}_n^{1/2}}{\wedge_{-1}\mathsf{N}_C^\vee}. \tag{36}$$

Next, we note that over the fixed locus F_C, the quotient takes the form

$$\mathcal{Q} = \mathcal{O}_{\mathcal{Z}_1} + \ldots + \mathcal{O}_{\mathcal{Z}_N}.$$

Thus the equivariant restriction becomes

$$j_C^\star \mathcal{U}_n = \det \mathrm{Ext}^\bullet_C \left(\sum_{i=1}^N \mathcal{O}_{\mathcal{Z}_i}, \sum_{i=1}^N \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta\right). \tag{37}$$

Both sides carry weight 0.

Lemma 34 in [OP] gives the difference

$$\iota^\star \mathsf{N}_X - \mathsf{N}_C = \bigoplus_{i<j} \mathsf{V}_{ij} \tag{38}$$

where we define

$$\mathsf{V}_{ij} = \mathrm{Ext}^\bullet_C(\mathcal{O}_{\mathcal{Z}_i}, \mathcal{O}_{\mathcal{Z}_j} \otimes \Theta)[w_j - w_i] \oplus \mathrm{Ext}^\bullet_C(\mathcal{O}_{\mathcal{Z}_j}, \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta)[w_i - w_j].$$

We noted in [OP] that by Serre duality and the fact that Θ is a theta characteristic, we have

$$\mathsf{V}_{ij} = \mathsf{U}_{ij} + \mathsf{U}_{ij}^\vee[-1]$$

where

$$\mathsf{U}_{ij} = \mathrm{Ext}^\bullet_C(\mathcal{O}_{\mathcal{Z}_i}, \mathcal{O}_{\mathcal{Z}_j} \otimes \Theta)[w_j - w_i].$$

Observe that

$$\mathrm{rank}\ \mathsf{U}_{ij} = 0.$$

In general, if $\mathsf{V} = \mathsf{U} + \mathsf{U}^\vee[-1]$ we have

$$\frac{1}{\wedge_{-1}\mathsf{V}^\vee} = (-1)^{\mathrm{rk}\ \mathsf{U}} \det \mathsf{U}. \tag{39}$$

To see this, note that since both sides are multiplicative, we may assume U is an equivariant line bundle L. We then need to show

$$\frac{1}{\wedge_{-1}(L^\vee - L)} = -L \iff \wedge_{-1} L = -\wedge_{-1} L^\vee \cdot L$$

which is correct.

From (38) and (39) we find

$$\frac{1}{\iota^\star \wedge_{-1} \mathsf{N}_X^\vee} = \frac{1}{\wedge_{-1} N_C^\vee} \cdot \prod_{i<j} \frac{1}{\wedge_{-1} \mathsf{V}_{ij}^\vee} = \frac{1}{\wedge_{-1} N_C^\vee} \cdot \det\left(\sum_{i<j} \mathsf{U}_{ij}\right). \tag{40}$$

Comparing (37) and (40) with (36), we need to show

$$\otimes_i \det \Theta^{[n_i]} \otimes \det\left(\sum_{i<j} \mathsf{U}_{ij}\right) = \det\left(\mathrm{Ext}_C^\bullet\left(\sum_{i=1}^N \mathcal{O}_{\mathcal{Z}_i}, \sum_{i=1}^N \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta\right)\right)^{1/2}. \tag{41}$$

All terms above carry zero weights. A priori, the bundles U_{ij} carry weights $w_j - w_i$, but the ranks are equal to 0, so the weights cancel after taking determinants.

To establish (41) we extend the definition of U_{ij} to include the case $i = j$:

$$\mathsf{U}_{ii} = \mathrm{Ext}_C^\bullet(\mathcal{O}_{\mathcal{Z}_i}, \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta).$$

By Serre duality, we have

$$\begin{aligned}
\det \mathsf{U}_{ii} &= \det\left(\mathrm{Ext}_C^0(\mathcal{O}_{\mathcal{Z}_i}, \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta) - \mathrm{Ext}_C^1(\mathcal{O}_{\mathcal{Z}_i}, \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta)\right) \\
&= \det\left(\mathrm{Ext}_C^0(\mathcal{O}_{\mathcal{Z}_i}, \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta) - \mathrm{Ext}_C^0(\mathcal{O}_{\mathcal{Z}_i}, \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta)^\vee\right) \\
&= \det \mathrm{Ext}_C^0(\mathcal{O}_{\mathcal{Z}_i}, \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta)^{\otimes 2} = \det \mathrm{Ext}_C^0(\mathcal{O}_C, \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta)^{\otimes 2} \\
&= \det \Theta^{[n_i]^{\otimes 2}}.
\end{aligned}$$

The ideal exact sequence for $\mathcal{Z}_i$ was used on the third line. We already noted above that as a consequence of Serre duality $\mathsf{U}_{ij} \simeq \mathsf{U}_{ji}^\vee[1]$, hence

$$\det \mathsf{U}_{ij} = \det \mathsf{U}_{ji}.$$

Therefore

$$\begin{aligned}
\otimes_i \det \Theta^{[n_i]} \otimes \det\left(\sum_{i<j} \mathsf{U}_{ij}\right) &= \det\left(\sum_{1 \le i,j \le N} \mathsf{U}_{ij}\right)^{1/2} \\
&= \det\left(\mathrm{Ext}_C^\bullet\left(\sum_{i=1}^N \mathcal{O}_{\mathcal{Z}_i}, \sum_{i=1}^N \mathcal{O}_{\mathcal{Z}_i} \otimes \Theta\right)\right)^{1/2}.
\end{aligned}$$

This confirms (41), and equation (33) along with it.

Finally, recall that

$$\mathcal{D}_n,\ \mathcal{U}_n \to \mathsf{Quot}_C(\mathbb{C}^N, n), \quad \mathcal{D}_n = \det \mathrm{R}\pi_\star(\det \mathcal{Q} \otimes \Theta)^\vee, \quad \mathcal{U}_n = \det \mathrm{Ext}_C^\bullet(\mathcal{Q}, \mathcal{Q} \otimes \Theta).$$

We show that

$$\mathcal{D}_n^{\otimes 2} = \mathcal{U}_n.$$

If in the expression for $\mathcal{U}_n$ we replace $\mathcal{Q}$ by $\det \mathcal{Q} - \mathcal{O}$, we obtain

$$\begin{aligned}\widetilde{\mathcal{U}}_n &= \det \mathrm{Ext}^\bullet_C(\det \mathcal{Q} - \mathcal{O}, \det \mathcal{Q} \otimes \Theta - \Theta) \\ &= \det \left(\mathrm{R}\pi_\star(\det \mathcal{Q} \otimes \Theta)\right)^\vee \otimes \det \mathrm{R}\pi_\star(\det \mathcal{Q}^\vee \otimes \Theta)^\vee \\ &= \det \left(\mathrm{R}\pi_\star(\det \mathcal{Q} \otimes \Theta)\right)^{\vee \otimes 2} = \mathcal{D}_n^{\otimes 2},\end{aligned}$$

where Serre duality was used in the last line. It remains to show that $\mathcal{U}_n \simeq \widetilde{\mathcal{U}}_n$. Write

$$\Delta = (\det \mathcal{Q} - \mathcal{O}) - \mathcal{Q},$$

and note that

$$\widetilde{\mathcal{U}}_n = \mathcal{U}_n \otimes \det \mathrm{Ext}^\bullet_C(\Delta, \Delta \otimes \Theta) \otimes \det \mathrm{Ext}^\bullet_C(\Delta, \mathcal{Q} \otimes \Theta) \otimes \det \mathrm{Ext}^\bullet_C(\mathcal{Q}, \Delta \otimes \Theta).$$

With respect to the codimension filtration in K-theory, Δ has pieces of degree 2 or higher, and $\mathcal{Q}$ has pieces of degree 1 or higher. It follows that the last three Ext's above are supported in codimension 2, hence their determinants are trivial. This completes the argument. □

5. The Segre/Verlinde correspondence and symmetry

5.1. **Overview.** We study here the virtual shifted Verlinde and Segre series

$$\begin{aligned}\mathsf{V}_\alpha(q) &= q^{\beta.K_X} \sum_{n=0}^\infty q^n \, \chi^{\mathrm{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, \beta, n), \det \alpha^{[n]}) \\ \mathsf{S}_\alpha(q) &= q^{\beta.K_X} \sum_{n=0}^\infty q^n \int_{[\mathsf{Quot}_X(\mathbb{C}^N,\beta,n)]^{\mathrm{vir}}} s(\alpha^{[n]}).\end{aligned}$$

We establish the Segre/Verlinde correspondence of Theorem 13. For $\beta = 0$, we prove the symmetry of the Segre/Verlinde series with respect to N and $r = \mathrm{rank}\, \alpha$, as stated in Theorem 14.

5.2. **The Verlinde Series.** The virtual Verlinde series corresponds to the series $\mathsf{Z}^{f,g}_{X,N,\beta}(\alpha)$ of Section 2 for

$$f(x) = e^x, \quad g(x) = \frac{x}{1 - e^{-x}}.$$

This is a consequence of the virtual Hirzebruch-Riemann-Roch theorem [CFK, FG]. In particular,

$$\frac{f'}{f}(x) = 1, \quad \frac{g(x)}{x} = \frac{1}{1 - e^{-x}}, \quad \frac{g'}{g}(-x) + \frac{1}{x} = \frac{1}{1 - e^{-x}}.$$

We can simultaneously simplify the universal series and the change of variables by setting

$$e^{H_i} = 1 - \widetilde{H}_i \text{ and } e^{-w_k} = 1 + \widetilde{w}_k, \tag{42}$$

so that equation (15) becomes

$$q = \big(1-\widetilde{H}_i\big)^r \cdot \prod_{k=1}^N \frac{\widetilde{H}_i+\widetilde{w}_k}{1+\widetilde{w}_k} \quad \text{with} \quad \widetilde{H}_i(q=0) = -\widetilde{w}_i.$$

Using (16) – (19), we obtain the explicit expressions

$$\begin{aligned}
\mathsf{A} &= \prod_{i=1}^N \left(\frac{1-\widetilde{H}_i}{1+\widetilde{w}_i}\right)^{-r} \cdot \prod_{k,i} \frac{\widetilde{H}_i+\widetilde{w}_k}{-(1-\widetilde{H}_i)} \cdot \prod_{i_1\neq i_2} \frac{1-\widetilde{H}_{i_1}}{\widetilde{H}_{i_2}-\widetilde{H}_{i_1}} \cdot \prod_{i=1}^N \left(r - \sum_{k=1}^N \frac{1-\widetilde{H}_i}{\widetilde{H}_i+\widetilde{w}_k}\right) \\
\mathsf{B} &= \prod_{i=1}^N \frac{1-\widetilde{H}_i}{1+\widetilde{w}_i} \\
\mathsf{U}_i &= \big(1-\widetilde{H}_i\big)^r \cdot \prod_{k=1}^N \frac{-(1-\widetilde{H}_i)}{\widetilde{H}_i+\widetilde{w}_k} \cdot \prod_{i'\neq i} \left(\frac{\widetilde{H}_{i'}-\widetilde{H}_i}{1-\widetilde{H}_i} \cdot \frac{\widetilde{H}_i-\widetilde{H}_{i'}}{1-\widetilde{H}_{i'}}\right) \cdot \left(r - \sum_{k=1}^N \frac{1-\widetilde{H}_i}{\widetilde{H}_i+\widetilde{w}_k}\right)^{-1} \\
\mathsf{V}_i &= \frac{1}{1-\widetilde{H}_i} \\
\mathsf{W}_{i_1,i_2} &= \frac{1-\widetilde{H}_{i_1}}{\widetilde{H}_{i_2}-\widetilde{H}_{i_1}} \cdot \frac{1-\widetilde{H}_{i_2}}{\widetilde{H}_{i_1}-\widetilde{H}_{i_2}}.
\end{aligned}$$

5.3. **The Segre Series.** To obtain the virtual Segre series, we consider the functions

$$\widetilde{f}(x) = \frac{1}{1+x}, \quad \widetilde{g}(x) = 1.$$

In particular,

$$\frac{\widetilde{f}'}{\widetilde{f}}(x) = -\frac{1}{1+x}, \quad \frac{\widetilde{g}}{x} = \frac{1}{x}, \quad \frac{\widetilde{g}'}{\widetilde{g}}(-x) = 0.$$

We write $\widetilde{w}_1, \ldots, \widetilde{w}_N$ for the weights used in the corresponding localization computation. Via equations (15) – (19), we find the universal series

$$\begin{aligned}
\widetilde{\mathsf{A}} &= \prod_{i=1}^N \left(\frac{1-\widetilde{H}_i}{1+\widetilde{w}_i}\right)^{-r} \cdot \prod_{k,i} (\widetilde{H}_i+\widetilde{w}_k) \cdot \prod_{i_1\neq i_2} \frac{1}{\widetilde{H}_{i_1}-\widetilde{H}_{i_2}} \cdot \prod_{i=1}^N \left(\frac{-r}{1-\widetilde{H}_i} + \sum_{k=1}^N \frac{1}{\widetilde{H}_i+\widetilde{w}_k}\right) \\
\widetilde{\mathsf{B}} &= \prod_{i=1}^N \frac{1-\widetilde{H}_i}{1+\widetilde{w}_i} \\
\widetilde{\mathsf{U}}_i &= \big(1-\widetilde{H}_i\big)^r \cdot \prod_{k=1}^N \frac{1}{\widetilde{H}_i+\widetilde{w}_k} \cdot \prod_{i'\neq i} \big(\widetilde{H}_{i'}-\widetilde{H}_i\big)\big(\widetilde{H}_i-\widetilde{H}_{i'}\big) \cdot \left(\frac{-r}{1-\widetilde{H}_i} + \sum_k \frac{1}{\widetilde{H}_i+\widetilde{w}_k}\right)^{-1} \\
\widetilde{\mathsf{V}}_i &= \frac{1}{1-\widetilde{H}_i} \\
\widetilde{\mathsf{W}}_{i_1,i_2} &= \frac{1}{\widetilde{H}_{i_1}-\widetilde{H}_{i_2}} \cdot \frac{1}{\widetilde{H}_{i_2}-\widetilde{H}_{i_1}}
\end{aligned}$$

after the change of variables

$$\widetilde{q} = \big(1 - \widetilde{H}_i\big)^r \cdot \prod_{k=1}^{N} \big(- \widetilde{H}_i - \widetilde{w}_k\big) \quad \text{with} \quad \widetilde{H}_i(q=0) = -\widetilde{w}_i.$$

5.4. **Comparison.** We show the Segre and Verlinde series match[10] for the following geometries

(i) X is smooth projective, $\beta = 0$, N is arbitrary,
(ii) X is smooth projective with $p_g > 0$, $N = 1$ and β is arbitrary,
(iii) X is a relatively minimal elliptic surface, β is supported on fibers, N is arbitrary.

Proof of Theorem 13. We begin by rewriting the Segre series as

$$\mathsf{S}^{\dagger}_{\alpha}(q) = q^{\beta.K_X} \sum_{n=0}^{\infty} q^n \int_{[\mathsf{Quot}_X(\mathbb{C}^N,\beta,n)]^{\mathrm{vir}}} s\left(\alpha^{[n]} - \big(\mathbb{C}^N\big)^{\oplus n}\right).$$

This does not change the original nonequivariant expression. Equivariantly, we use the lift of the trivial bundle $\mathbb{C}^N$ with weights $\widetilde{w}_1, \ldots, \widetilde{w}_N$. Assuming that the torus weights $w_1, \ldots, w_N$ on the Verlinde side and the torus weights $\widetilde{w}_1, \ldots, \widetilde{w}_N$ on the Segre side are connected by (42), we show that equivariantly

$$\mathsf{V}_{\alpha}(q) = \mathsf{S}^{\dagger}_{\alpha}((-1)^N q).$$

Theorem 13 follows from here.

We begin by analyzing the series V_{α} and S_{α}. By the calculations in Sections 5.2 and 5.3, the following universal series match exactly on both series:

$$\mathsf{A} = \widetilde{\mathsf{A}}, \quad \mathsf{B} = \widetilde{\mathsf{B}}, \quad \mathsf{V}_i = \widetilde{\mathsf{V}}_i$$

when evaluated at the variables q and $\tilde{q}$ related by

$$\frac{q}{\widetilde{q}} = (-1)^N \cdot \prod_{k=1}^{N} \frac{1}{1+\widetilde{w}_k}. \tag{43}$$

However, the remaining universal series no longer match precisely. In fact,

$$\frac{\mathsf{U}_i}{\widetilde{\mathsf{U}}_i} = (-1)^{N-1} \cdot \prod_{i' \neq i} \frac{1}{1-\widetilde{H}_{i'}}$$

$$\frac{\mathsf{W}_{i_1,i_2}}{\widetilde{\mathsf{W}}_{i_1,i_2}} = (1-\widetilde{H}_{i_1})(1-\widetilde{H}_{i_2}).$$

[10]There are other quadruples $(f, g, \widetilde{f}, \widetilde{g})$ for which the corresponding series of invariants match as well, but their geometric interpretation is less clear.

The geometries (i)–(iii) prevent these terms from taking effect: in cases (i) and (iii) they appear with exponent 0, while in case (ii) only U makes sense and in this case we do obtain a perfect match. Thus, for (i)–(iii), we obtain via (11) the equivariant identity

$$\mathsf{V}_\alpha(q) = \mathsf{S}_\alpha(\widetilde{q})$$

where

$$\frac{q}{\widetilde{q}} = (-1)^N \cdot \prod_{k=1}^{N} \frac{1}{1+\widetilde{w}_k}.$$

We can absorb the difference in the variables by working with $\mathsf{S}^\dagger_\alpha$ instead. Recall the lift of the trivial bundle $\mathbb{C}^N$ with weights $\widetilde{w}_1, \ldots, \widetilde{w}_N$. This affects each individual summand in $\mathsf{S}^\dagger_\alpha$ by the factor

$$(1+\widetilde{w}_1)^n \cdots (1+\widetilde{w}_N)^n \;\text{ so that }\; \mathsf{S}_\alpha\left(q \cdot \prod_{k=1}^{N}(1+\widetilde{w}_k)\right) = \mathsf{S}^\dagger_\alpha(q).$$

The claimed equality follows

$$\mathsf{V}_\alpha(q) = \mathsf{S}^\dagger_\alpha((-1)^N q).$$

□

Example 24. When $N = 1$ and for rank $\alpha = r$, we have

$$\sum_{n=0}^{\infty} q^n \cdot \chi^{\mathrm{vir}}(X^{[n]}, \det \alpha^{[n]}) = (1-t)^{c_1(\alpha)\cdot K_X} \cdot \left(\frac{1-t(r+1)}{(1-t)^{r+1}}\right)^{K_X^2} \tag{44}$$

for $q = t(1-t)^r$. This is a consequence of Theorem 13 and [MOP2, Theorem 2] giving the Segre series. Thus, the theorems in Section 1.3 do not extend to the highest exterior powers and to the Verlinde series which are typically given by algebraic functions.

Rationality does occur however in the special case when rank $\alpha = 0$ (also for rank $\alpha = -1$), so that $\det \alpha^{[n]}$ agrees with the pullback from the symmetric power $(\det \alpha)_{(n)}$. Then,

$$\sum_{n=0}^{\infty} q^n \cdot \chi^{\mathrm{vir}}(X^{[n]}, \det \alpha^{[n]}) = (1-q)^{-\chi^{\mathrm{vir}}(X,\alpha)} = (1-q)^{c_1(\alpha)\cdot K_X}. \tag{45}$$

By (32), we have

$$\mathcal{O}^{\mathrm{vir}}_{X^{[n]}} = \sum_{k=0}^{n} (-1)^k \wedge^k (K_X)^{[n]}.$$

The computation (45) is thus aligned with Theorem 5.2.1 in [Sc] which gives the individual cohomology groups

$$H^\star(X^{[n]}, (\det \alpha)_{(n)} \otimes \wedge^k (K_X)^{[n]}) = \wedge^k H^\star(\det \alpha \otimes K_X) \otimes \mathsf{Sym}^{n-k} H^\star(\det \alpha).$$

5.5. **Symmetry.** We now prove Theorem 14 giving the symmetry of the Segre/Verlinde series when $\beta = 0$. We pick two classes α and $\widetilde{\alpha}$ such that

$$\mathrm{rk}\ \alpha = r, \quad \mathrm{rk}\ \widetilde{\alpha} = N, \quad \frac{c_1(\alpha)\cdot K}{\mathrm{rk}\ \alpha} = \frac{c_1(\widetilde{\alpha})\cdot K}{\mathrm{rk}\ \widetilde{\alpha}} = \mu.$$

We show

$$\mathsf{S}_{N,\alpha}((-1)^N q) = \mathsf{S}_{r,\widetilde{\alpha}}((-1)^r q).$$

For clarity, we change the notation from the previous subsections, writing

$$\mathsf{S}_{N,\alpha}((-1)^N q) = \sum_{n=0}^{\infty} \big((-1)^N q\big)^n \int_{[\mathsf{Quot}_X(\mathbb{C}^N,n)]^{\mathrm{vir}}} s(\alpha^{[n]}) = \mathsf{M}^{K^2}\cdot \mathsf{N}^{\mu}$$

where

$$\mathsf{M} = \prod_{i=1}^{N}\left(\frac{-r}{1-H_i} + \sum_{k=1}^{N}\frac{1}{H_i+w_k}\right)\prod_{i=1}^{N}\left(\frac{1-H_i}{1+w_i}\right)^{-r} \prod_{1\le i,k\le N}(H_i+w_k)\prod_{i_1\neq i_2}\frac{1}{H_{i_1}-H_{i_2}}$$

$$\mathsf{N} = \prod_{i=1}^{N}\left(\frac{1-H_i}{1+w_i}\right)^{r}$$

for

$$q = \big(1-H_i\big)^r\cdot\prod_{k=1}^{N}\big(H_i+w_k\big), \quad H_i(q=0) = -w_i. \tag{46}$$

In a similar fashion,

$$\mathsf{S}_{r,\widetilde{\alpha}}((-1)^r q) = \sum_{n=0}^{\infty}((-1)^r q)^n \int_{[\mathsf{Quot}_X(\mathbb{C}^r,n)]^{\mathrm{vir}}} s(\widetilde{\alpha}^{[n]}) = \widetilde{\mathsf{M}}^{K^2}\cdot\widetilde{\mathsf{N}}^{\mu}$$

where

$$\widetilde{\mathsf{M}} = \prod_{j=1}^{r}\left(\frac{-N}{1-\widetilde{H}_j} + \sum_{k=1}^{r}\frac{1}{\widetilde{H}_j+\widetilde{w}_k}\right)\prod_{j=1}^{r}\left(\frac{1-\widetilde{H}_j}{1+\widetilde{w}_j}\right)^{-N} \prod_{1\le j,k\le r}(\widetilde{H}_j+\widetilde{w}_k)\prod_{j_1\neq j_2}\frac{1}{\widetilde{H}_{j_1}-\widetilde{H}_{j_2}}$$

$$\widetilde{\mathsf{N}} = \prod_{i=1}^{r}\left(\frac{1-\widetilde{H}_j}{1+\widetilde{w}_j}\right)^{N}$$

for

$$q = \big(1-\widetilde{H}_j\big)^N\cdot\prod_{k=1}^{r}\big(\widetilde{H}_j+\widetilde{w}_k\big), \quad \widetilde{H}_j(q=0) = -\widetilde{w}_j.$$

Two sets of weights $w = (w_1,\ldots,w_N)$ and $\widetilde{w} = (\widetilde{w}_1,\ldots,\widetilde{w}_r)$ are used on the two different Quot schemes in the localization computation. We will show that

$$\mathsf{M}\Big|_{w=0} = \widetilde{\mathsf{M}}\Big|_{\widetilde{w}=0}, \quad \mathsf{N}\Big|_{w=0} = \widetilde{\mathsf{N}}\Big|_{\widetilde{w}=0}.$$

In fact, we argue that the limit is found by setting $w = 0$ directly in the above equations. In other words

$$\mathsf{M}\Big|_{w=0} = \mathsf{M}^\star, \quad \mathsf{N}\Big|_{w=0} = \mathsf{N}^\star$$

where

$$\mathsf{M}^\star = \prod_{i=1}^{N} \left(\frac{-r}{1-H_i} + \frac{N}{H_i} \right) \cdot \prod_{i=1}^{N} (1-H_i)^{-r} \cdot \prod_{i=1}^{N} H_i^N \cdot \prod_{i_1 \neq i_2} \frac{1}{H_{i_1} - H_{i_2}}$$

$$\mathsf{N}^\star = \prod_{i=1}^{N} (1-H_i)^r,$$

and $H_1, \ldots, H_N$ are the roots[11] of the equation

$$q = (1-H)^r \cdot H^N$$

that vanish at $q = 0$. These roots are given by formal Puiseux series in the variable $q^{\frac{1}{N}}$. To justify the limit, note that expressions M and N are symmetric in the H's, so they can be recast in terms of the elementary symmetric functions. The assertion follows from the following:

Claim: The elementary symmetric functions of the solutions $H_1, \ldots, H_N$ of (46) are continuous as functions of w at $w = 0$.

We will justify this shortly. In a similar fashion, in the limit, $\widetilde{\mathsf{M}}$ and $\widetilde{\mathsf{N}}$ become

$$\widetilde{\mathsf{M}}^\star = \prod_{j=1}^{r} \left(\frac{-N}{1-\widetilde{H}_j} + \frac{r}{\widetilde{H}_j} \right) \cdot \prod_{j=1}^{r} \left(1-\widetilde{H}_j\right)^{-N} \cdot \prod_{j=1}^{r} \widetilde{H}_j^r \cdot \prod_{j_1 \neq j_2} \frac{1}{\widetilde{H}_{j_1} - \widetilde{H}_{j_2}}$$

$$\widetilde{\mathsf{N}}^\star = \prod_{j=1}^{r} (1-\widetilde{H}_j)^N$$

where $\widetilde{H}_1, \ldots, \widetilde{H}_r$ are the roots of $q = (1-\widetilde{H})^N \cdot \widetilde{H}^r$ that vanish at $q = 0$.

We show

$$\mathsf{M}^\star = \widetilde{\mathsf{M}}^\star, \quad \mathsf{N}^\star = \widetilde{\mathsf{N}}^\star.$$

Let

$$f(z) = z^N (1-z)^r - q.$$

We note that $f(z) = 0$ has $r+N$ roots namely

$$H_1, \ldots, H_N, 1-\widetilde{H}_1, \ldots, 1-\widetilde{H}_r.$$

[11] By slight abuse, we do not introduce new notation for the roots of the equation obtained by setting $w = 0$.

Thus

$$f(z) = \prod_{i=1}^{N}(z - H_i) \cdot \prod_{j=1}^{r}(1 - z - \widetilde{H}_j).$$

Using the equations, we can rewrite

$$\widetilde{\mathsf{N}}^\star = \prod_{j=1}^{r} \frac{q}{\widetilde{H}_j^r} \implies \frac{\mathsf{N}^\star}{\widetilde{\mathsf{N}}^\star} = q^{-r} \prod_{i=1}^{N}(1 - H_i)^r \cdot \prod_{j=1}^{r} \widetilde{H}_j^r = q^{-r} f(1)^r (-1)^r = 1 \implies \mathsf{N}^* = \widetilde{\mathsf{N}}^*.$$

The computation for the remaining series is similar. We first note

$$f'(z) = z^N (1 - z)^r \cdot \left(\frac{-r}{1 - z} + \frac{N}{z} \right).$$

From here, we find

$$\begin{aligned}
\mathsf{M}^\star \mathsf{N}^{\star 2} &= \prod_{i=1}^{N} \left(\frac{-r}{1 - H_i} + \frac{N}{H_i} \right) \cdot \prod_{i=1}^{N} (1 - H_i)^r \cdot \prod_{i=1}^{N} H_i^N \cdot \prod_{i_1 \neq i_2} \frac{1}{H_{i_1} - H_{i_2}} \\
&= \prod_{i=1}^{N} f'(H_i) \cdot \prod_{i_1 \neq i_2} \frac{1}{H_{i_1} - H_{i_2}} \\
&= \prod_{1 \leq i \leq N, 1 \leq j \leq r} (1 - H_i - \widetilde{H}_j) = \widetilde{\mathsf{M}}^\star \widetilde{\mathsf{N}}^{\star 2}.
\end{aligned}$$

This shows $\mathsf{M}^\star = \widetilde{\mathsf{M}}^\star$, completing the proof of Theorem 14.

Proof of the Claim: The proof is a modification of the argument of Section 2.2 of [JOP] which does not apply directly here. We work over the ring $A = \mathbb{C}(w)[[q]]$ of formal power series whose coefficients are rational functions in w. For simplicity, set

$$\mathsf{P}(z) = (1 - z)^r, \quad \mathsf{Q}(z) = \prod_{k=1}^{N} (z + w_k),$$

and write

$$\mathsf{P}(z) = (-1)^r \left(z^r + p_1 z^{r-1} + \ldots + p_r \right), \quad \mathsf{Q}(z) = z^N + q_1 z^{N-1} + \ldots + q_N.$$

By assumption, $H_1, \ldots, H_N \in A$ are the power series solutions of

$$\mathsf{P}(H) \cdot \mathsf{Q}(H) - q = 0, \quad H_i(q = 0) = -w_i.$$

Let $e_i \in A$ be the i^{th} elementary symmetric function in $H_1, \ldots, H_N$ times $(-1)^i$, and note that from the initial conditions, we have at $q = 0$:

$$e_i(0) = q_i.$$

We factor in $A[z]$:

$$\mathsf{P}(z)\cdot\mathsf{Q}(z)-q=(-1)^r\cdot\prod_{k=1}^{N}(z-H_k)\cdot\left(z^r+f_1z^{r-1}+\ldots+f_r\right) \tag{47}$$
$$=(-1)^r\cdot(z^N+e_1z^{N-1}+\ldots+e_N)\cdot\left(z^r+f_1z^{r-1}+\ldots+f_r\right).$$

Setting $q=0$, we find

$$\mathsf{P}(z)\cdot\mathsf{Q}(z)=(-1)^r\cdot\prod_{k=1}^{N}(z-H_k(0))\cdot\left(z^r+f_1(0)z^{r-1}+\ldots+f_r(0)\right).$$

From the initial conditions $H_k(0)=-w_k$, we obtain

$$f_j(0)=p_j.$$

We use the convention $e_0=f_0=1$.

We show by induction on m that the coefficients of q^m in e_i and f_j are rational functions in w whose denominators are powers of

$$\Delta=\prod_{k=1}^{N}\mathsf{P}(-w_k).$$

Since $\mathsf{P}(0)\neq 0$, the substitution $w=0$ is therefore allowed, completing the proof of the **Claim.**

The base case $m=0$ is a consequence of the above remarks. For the inductive step, write

$$e_i=q_i+q^{m+1}e_i^{(m+1)}+\text{other terms},\quad f_j=p_j+q^{m+1}f_j^{(m+1)}+\text{ other terms}.$$

The omitted terms correspond to powers of q with exponent between 1 and m, or greater than $m+1$. From (47), we see that

$$[q^{m+1}]\sum_{i+j=k}e_if_j=[q^{m+1}z^{N+r-k}]\,(-1)^r\,(\mathsf{P}(z)\mathsf{Q}(z)-q)=0\text{ or }(-1)^{r+1}.$$

On the other hand, by direct computation

$$[q^{m+1}]\sum_{i+j=k}e_if_j=\sum_{i+j=k}p_je_i^{(m+1)}+q_if_j^{(m+1)}+\text{ other terms}.$$

By the inductive hypothesis, we obtain that

$$\sum_{i+j=k}p_je_i^{(m+1)}+q_if_j^{(m+1)}=c_k$$

where c_k is a rational function in w whose denominator is a power of Δ for $1\leq k\leq N+r$. We regard the above equations as a linear system in

$$e_1^{(m+1)},\ldots,e_N^{(m+1)},f_1^{(m+1)},\ldots,f_r^{(m+1)}$$

which we solve in terms of the c_k's by Cramer's rule. To complete the inductive step, we show that the determinant of the matrix of coefficients equals Δ. Indeed, the matrix of coefficients has entries equal to p_j or q_i, and zero elsewhere. This is in fact the Sylvester matrix of the two polynomials $(-1)^r \mathsf{P}$, Q, and its determinant is therefore the resultant

$$\Delta = \text{Res } ((-1)^r \mathsf{P}, \mathsf{Q}) = \prod_{k=1}^{N} \mathsf{P}(-w_k).$$

□

Example 25. The match between expressions (i)-(iv) and (i)′-(iv)′ in Section 1.9 is established by combining Theorems 13, 14, 15. Here, we offer an analogy with the projective space. Indeed, the following identity can be seen as the most basic instance of the Segre/Verlinde correspondence

$$(-1)^k \int_{\mathbb{P}^k} s(\mathcal{O}_{\mathbb{P}^k}(1))^{r+1} = \chi(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(r)), \tag{48}$$

while the simplest rank-section symmetry, when formulated on the Verlinde side, is

$$\chi(\mathbb{P}^k, \mathcal{O}(r)) = \chi(\mathbb{P}^r, \mathcal{O}(k)).$$

In fact the above statements correspond precisely to the case $n = 1,\ \beta = 0,\ N$ arbitrary, so that our results can be interpreted as possible extensions to arbitrary n. Note that

$$\mathsf{Quot}_X(\mathbb{C}^N, 1) = X \times \mathbb{P}^{N-1}$$

in such a fashion that $(x, \zeta) \in X \times \mathbb{P}^{N-1}$ corresponds to the quotient $\mathbb{C}^N \xrightarrow{\zeta} \mathbb{C} \to \mathbb{C}_x \to 0$. Since

$$\mathcal{Q} = \mathcal{O}_\Delta \boxtimes \mathcal{O}_{\mathbb{P}^{N-1}}(1)$$

over $X \times X \times \mathbb{P}^{N-1}$, we compute

$$\text{Ext}^\bullet(\mathcal{S}, \mathcal{Q}) = TX + T\mathbb{P}^{N-1} - K_X^\vee \implies \text{Obs} = K_X^\vee$$

and

$$\alpha^{[1]} = \alpha \boxtimes \mathcal{O}_{\mathbb{P}^{N-1}}(1) \implies \det \alpha^{[1]} = \det \alpha \boxtimes \mathcal{O}_{\mathbb{P}^{N-1}}(r).$$

Thus the Verlinde number is

$$\chi^{\text{vir}}(\mathsf{Quot}_X(\mathbb{C}^N, 1), \det \alpha^{[1]}) = \chi^{\text{vir}}(X, \det \alpha) \cdot \chi(\mathbb{P}^{N-1}, \mathcal{O}_{\mathbb{P}^{N-1}}(r)).$$

On the Segre side, we find

$$\begin{aligned}\int_{[\mathsf{Quot}_X(\mathbb{C}^N,1)]^{\mathrm{vir}}} s(\alpha^{[1]}) &= \int_{X\times\mathbb{P}^{N-1}} \mathsf{e}(K_X^\vee)\cdot\ s\left(\alpha\boxtimes\mathcal{O}_{\mathbb{P}^{N-1}}(1)\right)\\ &= -\left(\int_X \mathsf{e}(K_X^\vee)\cdot c_1(\alpha)\right)\cdot\int_{\mathbb{P}^{N-1}} s(\mathcal{O}_{\mathbb{P}^{N-1}}(1))^{r+1}\\ &= -\chi^{\mathrm{vir}}(X,\det\alpha)\cdot\int_{\mathbb{P}^{N-1}} s(\mathcal{O}_{\mathbb{P}^{N-1}}(1))^{r+1}.\end{aligned}$$

Equation (48) confirms Theorem 13 in this case. We invite the reader to check that all vertices of the cube in Section 1.9 match precisely.

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