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A proof of the Theta Operator Conjecture

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Abstract

In the context of the (generalized) Delta Conjecture and its compositional form, D'Adderio, Iraci, and Vanden Wyngaerd recently stated a conjecture relating two symmetric function operators, D_k and Θ_k. We prove this Theta Operator Conjecture, finding it as a consequence of the five-term relation of Mellit and Garsia. As a result, we find surprising ways of writing the D_k operators. Even though we deal specifically with the relation between the D_k and Θ_k operators, our work introduces a method for finding relations between Θ_k and other plethystic operators which are important in this area of study.

1. Introduction

In what follows, we will assume the reader is familiar with symmetric functions and plethystic substitution. For a standard symmetric function reference, there is Macdonald's book [4]. For some of the plethystic identities shown here, we will mostly reference [2] and [3]. We will also adopt the notation and conventions in [1]. Garsia, Haiman, and Tesler defined in [2] a family of plethystic operators $\{D_k\}_{k\in\mathbb{Z}}$ by setting

$$D_k F[X] = F\left[X + \frac{M}{z}\right] \operatorname{Exp}[-zX]\Big|_{z^k}$$

where

$$\operatorname{Exp}[X] = \sum_{n\geq 0} h_n[X] = \exp\left(\sum_{k\geq 1} \frac{p_k}{k}\right)$$

is the plethystic exponential and $M = (1-q)(1-t)$. In the definition of D_k, we would have

$$\operatorname{Exp}[-zX] = \sum_{k\geq 0} (-z)^k e_k[X].$$

For every partition μ, set

$$\Pi_\mu = \prod_{(i,j)\in\mu/(1)} (1 - q^i t^j)$$

and define the linear operator Π by setting

$$\Pi\, \widetilde{\mathrm{H}}_\mu = \Pi_\mu\, \widetilde{\mathrm{H}}_\mu\,.$$

To get a compositional refinement of the (generalized) Delta Conjecture [5], D'Adderio, Iraci, and Vanden Wyngaerd [6] define

$$\Theta_k G = \begin{cases} G & \text{if } G \text{ is constant and } k = 0, \\ 0 & \text{if } G \text{ is constant but } k > 0, \text{ and} \\ \Pi\, \underline{e}_k^*\, \Pi^{-1} G & \text{otherwise} \end{cases}$$

where $\underline{f^*}$ denotes multiplication by $f^* = f[X/M]$.

We will prove Conjecture 10.3 in [6], which asserts that

Theorem 1.1. *For $k \geq 0$,*

$$[\Theta_k, D_1] = \sum_{i=1}^{k} (-1)^i D_{i+1} \Theta_{k-i}.$$

Multiplying both sides of the equation from Theorem 1.1 by $(-1)^k$ and expanding $[\Theta_k, D_1]$ as $\Theta_k D_1 - D_1 \Theta_k$ gives

$$(-1)^k \Theta_k D_1 = D_1 (-1)^k \Theta_k + \sum_{i=1}^{k} D_{i+1} (-1)^{k-i} \Theta_{k-i}.$$

Let us set $\overline{\Theta}_k = (-1)^k \Theta_k$,

$$\overline{\Theta}(z) = \sum_{k \geq 0} z^k \, \overline{\Theta}_k \qquad \text{and} \qquad D_+(z) = \sum_{k \geq 1} z^k D_k.$$

Then note that

$$D_+(z) \, \overline{\Theta}(z)|_{z^{k+1}} = D_1 \, \overline{\Theta}_k + D_2 \, \overline{\Theta}_{k-1} + \cdots + D_{k+1} \, \overline{\Theta}_0 \, .$$

Therefore, Theorem 1.1 is equivalent to

Proposition 1.2.

$$z \, \overline{\Theta}(z) D_1 = D_+(z) \, \overline{\Theta}(z).$$

This is what we will prove in the third section. One of the most important aspects of our proof is the realization of $\overline{\Theta}(z)$ through Garsia and Mellit's five-term relation. This gives a general method for producing identities involving Theta operators.

2. The D_k operators

Another way to write the D_k operators is using the translation and multiplication operators $\mathcal{T}_Y$ and $\mathcal{P}_Z$, defined for any two expressions Y and Z by setting

$$\mathcal{T}_Y \, F[X] = F[X+Y] \qquad \text{and} \qquad \mathcal{P}_Z \, F[X] = \mathrm{Exp}[ZX] F[X].$$

Then following the definition of D_k, we have as in Equation 1.10 of [2],

$$\sum_{-\infty < k < \infty} z^k D_k = \mathcal{P}_{-z} \, \mathcal{T}_{\frac{M}{z}} \, .$$

3. The proof

For any symmetric function F, define the linear operator Δ_F by setting

$$\Delta_F \, \widetilde{\mathrm{H}}_\mu = F[B_\mu] \, \widetilde{\mathrm{H}}_\mu, \qquad \text{where} \qquad B_\mu = \sum_{(i,j) \in \mu} q^i t^j.$$

This is the Delta eigenoperator for the modified Macdonald basis, defined in [3]. We will now follow the notation from Garsia and Mellit's five-term relation [1]. First note that if we set

$$\Delta_u = \sum_{n \geq 0} (-u)^n \Delta_{e_n},$$

then we have

$$\Delta_u \widetilde{\mathrm{H}}_\mu = \widetilde{\mathrm{H}}_\mu \prod_{(i,j)\in\mu} (1 - uq^i t^j) \qquad \text{and} \qquad \Delta_u^{-1} = \sum_{n\geq 0} u^n \Delta_{h_n}.$$

Furthermore, $\Pi = \Delta_u/(1-u)|_{u=1}$ on non-constants, where we can write

$$\overline{\Theta}_k = \Delta_u (-1)^k \underline{e}_k^* \Delta_u^{-1} \Big|_{u=1}.$$

We will not need this evaluation since the u's will vanish in our calculations. For this reason we will now, instead, consider the unspecialized operator

$$\widetilde{\Theta}_k = \Delta_v (-1)^k \underline{e}_k^* \Delta_v^{-1}$$

and let

$$\widetilde{\Theta}(z) = \sum_{k\geq 0} z^k \, \widetilde{\Theta}_k = \Delta_v \, \mathcal{P}_{-\frac{z}{M}} \, \Delta_v^{-1}$$

for some monomial v. We will show that

$$z\, \widetilde{\Theta}(z) D_1 = D_+(z)\, \widetilde{\Theta}(z),$$

or rather

$$z\, \widetilde{\Theta}(z) D_1 \, \widetilde{\Theta}(z)^{-1} = D_+(z).$$

In other words, we want to show that

$$D_+(z) = z \Delta_v \, \mathcal{P}_{-\frac{z}{M}} \, \Delta_v^{-1} D_1 \Delta_v \, \mathcal{P}_{\frac{z}{M}} \, \Delta_v^{-1}.$$

Next, for $\mu \vdash n$, we set $\nabla \widetilde{\mathrm{H}}_\mu = \Delta_u \, \widetilde{\mathrm{H}}_\mu \, |_{u^n} = (-1)^n q^{n(\mu')} t^{n(\mu)} \, \widetilde{\mathrm{H}}_\mu$, with

$$n(\mu) = \sum_{i=1}^{\ell(\mu)} \mu_i (i-1) = \sum_{(i,j)\in\mu} j.$$

With this convention, we have $D_1 = \nabla \underline{e}_1 \nabla^{-1}$ ([2], Equation 2.11). We can then rewrite the conjecture by substituting for D_1, giving

$$D_+(z) = z \Delta_v \, \mathcal{P}_{-\frac{z}{M}} \, \Delta_v^{-1} \nabla \underline{e}_1 \nabla^{-1} \Delta_v \, \mathcal{P}_{\frac{z}{M}} \, \Delta_v^{-1}.$$

This gives an expression for D_k in terms of Delta operators and multiplication by symmetric functions evaluated at X/M.

One of the main results from the five-term relation is the following identity:

Theorem 3.1 (Garsia-Mellit [1],Theorem 2.8)**.** *For any two monomials u and v, we have*

$$\nabla^{-1} \, \mathcal{T}_{uv} \, \nabla = \Delta_v^{-1} \, \mathcal{T}_u \, \Delta_v \, \mathcal{T}_u^{-1}.$$

The dual version is given by translating $\mathcal{T}_z$ to $\mathcal{P}_{-z/M}$ and reversing the order:

$$\nabla \, \mathcal{P}_{-\frac{uv}{M}} \, \nabla^{-1} = \mathcal{P}_{\frac{u}{M}} \, \Delta_v \, \mathcal{P}_{-\frac{u}{M}} \, \Delta_v^{-1}.$$

We get

$$\Delta_v \, \mathcal{P}_{-\frac{z}{M}} \, \Delta_v^{-1} = \mathcal{P}_{-\frac{z}{M}} \, \nabla \, \mathcal{P}_{-\frac{zv}{M}} \, \nabla^{-1}$$

and the inverse formula

$$\Delta_v \, \mathcal{P}_{\frac{z}{M}} \, \Delta_v^{-1} = \nabla \, \mathcal{P}_{\frac{zv}{M}} \, \nabla^{-1} \, \mathcal{P}_{\frac{z}{M}}.$$

Substituting this in our conjectured formula, we get

$$\begin{aligned}\Delta_v \mathcal{P}_{-\frac{z}{M}} \Delta_v^{-1}\nabla\underline{e}_1\nabla^{-1}\Delta_v \mathcal{P}_{\frac{z}{M}} \Delta_v^{-1} &= \mathcal{P}_{-\frac{z}{M}} \nabla \mathcal{P}_{-\frac{zv}{M}} \nabla^{-1}\nabla\underline{e}_1\nabla^{-1}\nabla \mathcal{P}_{\frac{zv}{M}} \nabla^{-1} \mathcal{P}_{\frac{z}{M}}\\ &= \mathcal{P}_{-\frac{z}{M}} \nabla \mathcal{P}_{-\frac{zv}{M}} \underline{e}_1 \mathcal{P}_{\frac{zv}{M}} \nabla^{-1} \mathcal{P}_{\frac{z}{M}}\\ &= \mathcal{P}_{-\frac{z}{M}} \nabla\underline{e}_1\nabla^{-1} \mathcal{P}_{\frac{z}{M}}\\ &= \mathcal{P}_{-\frac{z}{M}} D_1 \mathcal{P}_{\frac{z}{M}} .\end{aligned}$$

We can now write the final form of the conjecture. It is an extension of Proposition 1.6 in [2].

Theorem 3.2.

$$D_+(z) = \sum_{k\geq 1} z^k D_k = z\, \mathcal{P}_{-\frac{z}{M}} D_1 \mathcal{P}_{\frac{z}{M}}$$

or

$$D_+(z) = z\, \mathcal{P}_{-\frac{z}{M}} \nabla\underline{e}_1\nabla^{-1} \mathcal{P}_{\frac{z}{M}} .$$

Proof. We will show that for any r

$$\mathcal{P}_{-\frac{z}{M}} D_r \mathcal{P}_{\frac{z}{m}} = \sum_{k\geq 0} z^k D_{k+r}.$$

In particular for $r = 1$, this proves the theorem. Following Proposition 1.2 in [2], for any two expressions Y and Z, we have

$$\begin{aligned}\mathcal{T}_Y \mathcal{P}_Z \; F[X] &= \mathcal{T}_Y \; \mathrm{Exp}[XZ]F[X]\\ &= \mathrm{Exp}[(X+Y)Z]F[X+Y]\\ &= \mathrm{Exp}[YZ]\,\mathrm{Exp}[XZ]F[X+Y]\\ &= \mathrm{Exp}[YZ]\,\mathcal{P}_Z \mathcal{T}_Y \; F[X],\end{aligned}$$

meaning

$$\mathcal{T}_Y \mathcal{P}_Z = \mathrm{Exp}[YZ]\,\mathcal{P}_Z \mathcal{T}_Y .$$

Therefore, we have

$$\begin{aligned}\mathcal{P}_{-\frac{z}{M}} \mathcal{P}_{-u} \mathcal{T}_{\frac{M}{u}} \mathcal{P}_{\frac{z}{M}} &= \mathcal{P}_{-\frac{z}{M}} \mathcal{P}_{-u}\,\mathrm{Exp}\left[\frac{z}{u}\right] \mathcal{P}_{\frac{z}{M}} \mathcal{T}_{\frac{M}{u}}\\ &= \frac{1}{1-z/u} \mathcal{P}_{-u} \mathcal{T}_{\frac{M}{u}} .\end{aligned}$$

The coefficient of $z^k u^r$ (with $k \geq 0$) on the right hand side of the last equality is

$$\mathcal{P}_{-u} \mathcal{T}_{\frac{M}{u}} \Big|_{u^{k+r}} = D_{k+r}.$$

Equating coefficients on both sides, we have

$$\mathcal{P}_{-\frac{z}{M}} D_r \mathcal{P}_{\frac{z}{M}} \Big|_{z^k} = D_{k+r}.$$

□

4. Specializing to the main conjecture

In the process described above, we took Π and replaced it by Δ_v. This jump is not well established, but it is valid since for any homogeneous symmetric functions F and G we have

$$\Delta_v\underline{F}\Delta_v^{-1}G\Big|_{v=1} = \begin{cases} FG & \text{if both } F \text{ and } G \text{ are constant,}\\ 0 & \text{if } G \text{ is constant but } F \text{ is not,}\\ \Pi\underline{F}\Pi^{-1}G & \text{otherwise.}\end{cases}$$

When $F = e_k^*$, this is precisely the definition of Θ_k, meaning that the identity

$$z\,\widetilde{\Theta}(z) D_1 = D_+(z)\,\widetilde{\Theta}(z)$$

specializes to Proposition 1.2 when $v = 1$. On constants, we get another interesting identity. Specializing the left-hand side at $v = 1$ gives

$$z\,\widetilde{\Theta}(z)(D_1 \circ 1)\Big|_{v=1} = z\Pi\,\mathcal{P}_{-\frac{z}{M}}\,\Pi^{-1}(-\underline{e}_1) = \sum_{m\geq 0}(-z)^{m+1}\Pi e_m^* e_1.$$

On the other hand,

$$D_+(z)\,\widetilde{\Theta}(z)\circ 1 \qquad = D_+(z)\Delta_v\,\mathcal{P}_{-\frac{z}{M}}\,\Delta_v^{-1}\circ 1 \qquad = D_+(z)\Delta_v\,\mathcal{P}_{-\frac{z}{M}}\circ 1.$$

Specializing at $v = 1$ gives

$$D_+(z)\Delta_v\,\mathcal{P}_{-\frac{z}{M}}\circ 1\Big|_{v=1} \qquad = D_+(z)\circ 1 \qquad = \sum_{m\geq 0}(-z)^{m+1}e_{m+1}.$$

This means

$$\Theta_m D_1 \circ 1 = D_{m+1}\circ 1,$$

or rather

Corollary 4.1. *We have*

$$\Pi e_m^* e_1 = e_{m+1}.$$

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