

Positive scalar curvature and minimal hypersurface singularities

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ABSTRACT. In this paper we develop methods to extend the minimal hypersurface approach to positive scalar curvature problems to all dimensions. This includes a proof of the positive mass theorem in all dimensions without a spin assumption. It also includes statements about the structure of compact manifolds of positive scalar curvature extending the work of [SY1] to all dimensions. The technical work in this paper is to construct minimal slicings and associated weight functions in the presence of small singular sets and to show that the singular sets do not become too large in the lower dimensional slices. It is shown that the singular set in any slice is a closed set with Hausdorff codimension at least three. In particular for arguments which involve slicing down to dimension 1 or 2 the method is successful. The arguments can be viewed as an extension of the minimal hypersurface regularity theory to this setting of minimal slicings.

1. Introduction

The study of manifolds of positive scalar curvature has a long history in both differential geometry and general relativity. The theorems involved include the positive mass theorem, the topological classification of manifolds of positive scalar curvature, and the local geometric study of metrics of positive scalar curvature. There are two methods which have been successful in this study in general situations, the Dirac operator method and the minimal hypersurface method. Both of these methods have restrictions on their applicability, the Dirac operator methods require the topological assumption that the manifold be spin, and the minimal hypersurface method has been restricted to the case of manifolds with dimension at most 8 because of the possibility of singularities which might occur in the hypersurfaces. The purpose of this paper is to extend the minimal hypersurface method to all dimensions.

The first author was partially supported by NSF grant DMS-2005431.

The Dirac operator method was pioneered by A. Lichnerowicz [Li] and M. Atiyah, I. Singer [AS] in the early 1960s. It was extended by N. Hitchin [H] and then systematically developed by M. Gromov and H. B. Lawson in [GL1], [GL2], and [GL3]. Surgery methods for manifolds of positive scalar curvature were developed in [SY1] and [GL2]. For simply connected manifolds M^n with $n \geq 5$ Gromov and Lawson conjectured necessary and conditions for M to have a metric of positive scalar curvature (related to the index of the Dirac operator in the spin case). The conjecture was solved in the affirmative by S. Stolz [St]. The Dirac operator method was used by E. Witten [W] to prove the positive mass theorem for spin manifolds (see also [PT]).

The minimal hypersurface method originated in [SY4] for the three dimensional case and was extended to higher dimensions in [SY1]. The extension to the positive mass theorem was initiated in [SY2] and in higher dimensions in [SY5] and [Sc]. In this paper we extend the minimal hypersurface argument to all dimensions at least as regards the applications to the positive mass theorem and results which can be proven by slicing down to dimension two.

The basic objects of study in this paper are called *minimal k -slicings* and we now describe them. We start with a compact oriented Riemannian manifold M which will be our top dimensional slice Σ_n . We choose an oriented volume minimizing hypersurface Σ_{n-1} . Since Σ_{n-1} is stable, the second variation form $S_{n-1}(\varphi, \varphi)$ has first eigenvalue which is non-negative. We choose a positive first eigenfunction u_{n-1} and we use it as a weight ρ_{n-1} for the volume functional on $n-2$ cycles which are contained in Σ_{n-1} . We assume we have a $\Sigma_{n-2} \subset \Sigma_{n-1}$ which minimizes the weighted volume $V_{\rho_{n-1}}(\cdot)$. The second variation $S_{n-2}(\varphi, \varphi)$ for the weighted volume on Σ_{n-2} then has non-negative first eigenvalue and we let u_{n-2} be a positive first eigenfunction. We then define $\rho_{n-2} = u_{n-2}\rho_{n-1}$ and we continue this process. That is if we have $\Sigma_{j+1} \subset \Sigma_{j+2} \subset \dots \subset \Sigma_n$ which have been constructed, we choose Σ_j to be a minimizer of the weighted volume $V_{\rho_{j+1}}(\cdot)$. Such a nested family $\Sigma_k \subset \Sigma_{k+1} \subset \dots \subset \Sigma_n$ is called a *minimal k -slicing*.

The basic geometric theorem about minimal k -slicings which is generalized in Section 2 is the statement that if Σ_n has positive scalar curvature then for any minimal k -slicing we have that Σ_k is Yamabe positive and so admits a metric of positive scalar curvature. In particular if $k = 2$ then Σ_2 must be diffeomorphic to S^2 and there can be no minimal 1-slicing.

If we start with Σ_n with $n \geq 8$, there might be a closed singular set \mathcal{S}_{n-1} of Hausdorff dimension at most $n-8$ in Σ_{n-1} . In this paper we develop methods to carry out the construction of minimal k -slicings allowing for the possibility that the Σ_j may have nonempty singular sets \mathcal{S}_j . In order to do this it is necessary to extend the existence and regularity theory for minimal hypersurfaces to this setting. To do this requires maintaining some integral control of the geometry of the Σ_j in the ambient manifold Σ_n , and also of constructing the eigenfunctions u_j which are bounded in appropriate

weighted Sobolev spaces. This control is gotten by carefully exploiting the terms which are left over in the geometry of the second variation at each stage of the slicing. This is done by modifying the second variation form S_j to a larger form Q_j . The form Q_j is more coercive and can be diagonalized with respect to the weighted L^2 norm even in the presence of small singular sets. We can then construct the next slice using the first eigenfunction for the form Q_j to modify the weight. This procedure only works if the singular sets \mathcal{S}_j do not become too large. We prove that for a minimal k -slicing the Hausdorff dimension of the singular set \mathcal{S}_k is at most $k - 3$. The regularity theorem is proven by establishing appropriate compactness theorems for minimal k - slicings and showing that at a singular point there is a homogeneous minimal k -slicing gotten by rescaling and using appropriate monotonicity theorems (volume monotonicity and monotonicity of an appropriate frequency function). A homogeneous minimal k -slicing is one in \mathbb{R}^n for which all of the Σ_j are cones and all of the u_j are homogeneous of some degree. It is then possible to show that if we had a Σ_{k+1} with singular set of codimension at least 3, but Σ_k had a singular set of Hausdorff dimension larger than $k - 3$ then there would exist a nontrivial homogeneous 2-slicing with Σ_2 having an isolated singularity at the origin. We show that no such homogeneous slicings exist to conclude that if \mathcal{S}_{k+1} has codimension at least 3 in Σ_{k+1} , then \mathcal{S}_k has codimension at least 3 in Σ_k . In particular if $k = 2$ then Σ_2 is regular.

We now state the main theorems of the paper beginning with the positive mass theorem. A manifold M^n is called asymptotically flat if there is a compact set $K \subset M$ such that $M \setminus K$ is diffeomorphic to the exterior of a ball in \mathbb{R}^n and there are coordinates near infinity x^1, \dots, x^n so that the metric components g_{ij} satisfy

$$g_{ij} = \delta_{ij} + O(|x|^{-p}), \quad |x| |\partial g_{ij}| + |x|^2 |\partial^2 g_{ij}| = O(|x|^{-p})$$

for some $p > \frac{n-2}{2}$. We also require the scalar curvature R to satisfy

$$|R| = O(|x|^{-q})$$

for some $q > n$. Under these assumptions the ADM mass is well defined by the formula (see [Sc] for the n dimensional case)

$$m = \frac{1}{4(n-1)\omega_{n-1}} \lim_{\sigma \rightarrow \infty} \int_{S_\sigma} \sum_{i,j} (g_{ij,i} - g_{ii,j}) \nu_j \, d\xi(\sigma)$$

where S_σ is the euclidean sphere in the x coordinates, $\omega_{n-1} = \text{Vol}(S^{n-1}(1))$, and the unit normal and volume integral are with respect to the euclidean metric. The positive mass theorem is as follows.

THEOREM 1.1. *Assume that M is an asymptotically flat manifold with $R \geq 0$. We then have that the ADM mass is nonnegative. Furthermore, if the mass is zero, then M is isometric to \mathbb{R}^n .*

It is shown in Section 5 using results of [SY3] to simplify the asymptotic behavior and an observation of J. Lohkamp which allows us to compactify the manifold keeping the scalar curvature positive. The result which is needed for compact manifolds follows.

THEOREM 1.2. *If M_1 is any closed manifold of dimension n , then $M_1 \# T^n$ does not have a metric of positive scalar curvature.*

Both of these theorems were known if either $n \leq 8$ or for any n assuming the manifold is a spin manifold. Actually for $n = 8$ there may be isolated singularities, but in this dimension a result of N. Smale [Sm] shows that there is a dense set of ambient metrics for which the singularities do not occur. Using this result the eight dimensional case can also be done without dealing with singularities. In this paper we remove the dimensional and spin assumptions.

Finally we prove the following more precise theorem about compact manifolds with positive scalar curvature.

THEOREM 1.3. *Assume that M is a compact oriented n -manifold with a metric of positive scalar curvature. If $\alpha_1, \dots, \alpha_{n-2}$ are classes in $H^1(M, \mathbb{Z})$ with the property that the class σ_2 given by $\sigma_2 = \alpha_{n-2} \cap \alpha_{n-3} \cap \dots \cap \alpha_1 \cap [M] \in H_2(M, \mathbb{Z})$ is nonzero, then the class σ_2 can be represented by a sum of smooth two spheres. If α_{n-1} is any class in $H^1(M, \mathbb{Z})$, then we must have $\alpha_{n-1} \cap \sigma_2 = 0$. In particular, if M has classes $\alpha_1, \dots, \alpha_{n-1}$ with $\alpha_{n-1} \cap \dots \cap \alpha_1 \cap [M] \neq 0$, then M cannot carry a metric of positive scalar curvature.*

We also point out the recent series of papers by J. Lohkamp [Lo1], [Lo2], [Lo3], and [Lo4]. These papers also present an approach to the high dimensional positive mass theorem by extending the minimal hypersurface approach to all dimensions. Our approach seems quite different both conceptually and technically, and is more in the classical spirit of the calculus of variations. In any case we feel that, for such a fundamental result, it is of value to have multiple approaches.

2. Terminology and statements of main theorems

We begin by introducing the notation involved in the construction of a *minimal k -slicing*; that is, a nested family of hypersurfaces beginning with a smooth manifold Σ_n of dimension n and going down to Σ_k of dimension $k \leq n - 1$. This consists of $\Sigma_k \subset \Sigma_{k+1} \subset \dots \subset \Sigma_n$ where each Σ_j will be constructed as a volume minimizer of a certain weighted volume in Σ_{j+1} .

Let Σ_n be a properly embedded n -dimensional submanifold in an open set Ω contained in \mathbb{R}^N . We will consider a minimal slicing of Σ_n defined in an inductive manner. First, let $u_n = 1$, and let Σ_{n-1} be a volume minimizing hypersurface in Σ_n . Of course, it may happen that Σ_{n-1} has a singular set \mathcal{S}_{n-1} which is a closed subset of Hausdorff dimension at most $n - 8$. On Σ_{n-1} we will construct a positive definite quadratic form Q_{n-1} on functions by suitably modifying the index form associated to the second variation of

volume. We will then construct a positive function u_{n-1} on Σ_{n-1} which is a least eigenfunction of Q_{n-1} . We then define $\rho_{n-1} = u_{n-1}u_n$, and we let Σ_{n-2} be a hypersurface in Σ_{n-1} which is a minimizer of the ρ_{n-1} -weighted volume $V_{\rho_{n-1}}(\Sigma) = \int_{\Sigma} \rho_{n-1} d\mu_{n-2}$ for an $n-2$ dimensional submanifold of Σ_{n-1} and we denote μ_j to be the Hausdorff j -dimensional measure. Inductively, assume that we have constructed a slicing down to dimension $k+1$; that is, we have a nested family of hypersurfaces, quadratic forms, and positive functions (Σ_j, Q_j, u_j) for $j = k+1, \dots, n$ such that Σ_j minimizes the ρ_{j+1} -weighted volume where $\rho_{j+1} = u_{j+1}u_{j+2} \dots u_n$, Q_j is a positive definite quadratic form related to the second variation of the ρ_{j+1} -weighted volume (see (2.1) below), and u_j is a lowest eigenfunction of Q_j with eigenvalue $\lambda_j \geq 0$. We will always take λ_j to be the lowest Dirichlet eigenvalue (if $\partial\Sigma_j \neq \emptyset$) of Q_j with respect to the weighted L^2 norm and we take u_j to be a corresponding eigenfunction. We will show in Section 3 that such λ_j and u_j exist. We then inductively construct (Σ_k, Q_k, u_k) by letting Σ_k be a minimizer of the ρ_{k+1} weighted volume where $\rho_{k+1} = u_{k+1}u_{k+2} \dots u_n$, Q_k a positive definite quadratic form described below, and u_k a positive eigenfunction of Q_k .

Note that if Σ_j is a leaf in a minimal k -slicing, then choosing a unit normal vector ν_j to Σ_j in Σ_{j+1} gives us an orthonormal basis $\nu_k, \nu_{k+1}, \dots, \nu_{n-1}$ for the normal bundle of Σ_k defined on the regular set \mathcal{R}_k . Thus the second fundamental form of Σ_k in Σ_n consists of the scalar forms $A_k^{\nu_j} = \langle A_k, \nu_j \rangle$ for $j = k, \dots, n-1$ and we have $|A_k|^2 = \sum_{j=k}^{n-1} |A_k^{\nu_j}|^2$.

Now if we have a minimal k -slicing, we let g_k denote the metric induced on Σ_k from Σ_n , and we let \hat{g}_k denote the metric $\hat{g}_k = g_k + \sum_{p=k}^{n-1} u_p^2 dt_p^2$ on $\Sigma_k \times (S^1)^{n-k}$ where we use S^1 to denote a circle of length 1, and we denote by t_p a coordinate on the p th factor of S^1 . We then note that the volume measure of the metric \hat{g}_k is given by $\rho_k d\mu_k$ where we have suppressed the t_p variables since we will consider only objects which do not depend on them; for example, the ρ_k -weighted volume of Σ_k is the volume of the n -dimensional manifold $\Sigma_k \times T^{n-k}$. We will need to introduce another metric \tilde{g}_k on $\Sigma_k \times (S^1)^{n-k-1}$. This is defined by $\tilde{g}_k = g_k + \sum_{p=k+1}^{n-1} u_p^2 dt_p^2$. Note that \tilde{g}_k is the metric induced on $\Sigma_k \times (S^1)^{n-k-1}$ by \hat{g}_{k+1} . We also let \tilde{A}_k denote the second fundamental form of $\Sigma_k \times (S^1)^{n-k-1}$ in $(\Sigma_{k+1} \times (S^1)^{n-k-1}, \hat{g}_{k+1})$. The following lemma computes this second fundamental form.

LEMMA 2.1. *We have $\tilde{A}_k = A_k^{\nu_k} - \sum_{p=k+1}^{n-1} u_p \nu_k(u_p) dt_p^2$, and the square length with respect to \tilde{g}_k is given by $|\tilde{A}_k|^2 = |A_k^{\nu_k}|^2 + \sum_{p=k+1}^{n-1} (\nu_k(\log u_p))^2$.*

PROOF. If we consider a hypersurface Σ in a Riemannian manifold with unit normal ν , then we can consider the parallel hypersurfaces parametrized on Σ by $F_\varepsilon(x) = \exp(\varepsilon\nu(x))$ for small ε and $x \in \Sigma$. We then have a family of induced metrics g_ε from F_ε on Σ , and the second fundamental form is given by $A = -\frac{1}{2}\dot{g}$ where \dot{g} denotes the ε derivative of g_ε at $\varepsilon = 0$.

If we let \exp denote the exponential map of Σ_k in Σ_{k+1} , then since Σ_{k+1} is totally geodesic in $\Sigma_{k+1} \times T^{n-k-1}$, we have

$$F_\varepsilon(x, t) = (\exp(\varepsilon\nu_k(x)), t)$$

for $(x, t) \in \Sigma_k \times T^{n-k-1}$, and the induced family of metrics is given by

$$\tilde{g}_\varepsilon = (g_k)_\varepsilon + \sum_{p=k+1}^{n-1} (u_p(\exp(\varepsilon\nu_k)))^2 dt_p^2.$$

Thus we have

$$\dot{\tilde{g}} = -2A_k^{\nu_k} + 2 \sum_{p=k+1}^{n-1} u_p \nu_k(u_p) dt_p^2$$

since $A_k^{\nu_k}$ is the second fundamental form of Σ_k in Σ_{k+1} . It follows that $\tilde{A}_k = A_k^{\nu_k} - \sum_{p=k+1}^{n-1} u_p \nu_k(u_p) dt_p^2$, and taking the square norm with respect to the metric \tilde{g}_k then gives the desired formula for $|\tilde{A}_k|^2$. \square

We now describe the choice we will make for Q_j . Let S_j be the second variation form for the weighted volume $V_{\rho_{j+1}}$ at Σ_j , and define

$$(2.1) \quad Q_j(\varphi, \varphi) = S_j(\varphi, \varphi) + \frac{3}{8} \int_{\Sigma_j} \left(|\tilde{A}_j|^2 + \frac{1}{3n} \sum_{p=j+1}^n (|\nabla_j \log u_p|^2 + |\tilde{A}_p|^2) \right) \varphi^2 \rho_{j+1} d\mu_j$$

where, for now, φ is a function supported in the regular set \mathcal{R}_j and we define $\tilde{A}_n = 0$, $u_n = 1$. We will discuss an extended domain for Q_j in the Section 3.

Up to this point our discussion is formal because we have not discussed issues related to the singularities of the Σ_j in a minimal slicing. We first define the *regular set*, \mathcal{R}_j of Σ_j to be the set of points x for which there is a neighborhood of x in \mathbb{R}^N in which all of $\Sigma_j, \Sigma_{j+1}, \dots, \Sigma_n$ are smooth embedded submanifolds of \mathbb{R}^N . The *singular set*, \mathcal{S}_j is then defined to be the complement of \mathcal{R}_j in Σ_j . Thus \mathcal{S}_j is a closed set by definition. The following result follows from the standard minimizing hypersurface regularity theory. In this paper $\dim(A)$ always refers to the Hausdorff dimension of a subset $A \subset \mathbb{R}^N$.

PROPOSITION 2.2. *For $j \leq n-1$ we have $\dim(\mathcal{S}_j \sim \mathcal{S}_{j+1}) \leq j-7$, and in particular we have $\dim(\mathcal{S}_{n-1}) \leq n-8$.*

In light of this result, we see that our main task in controlling singularities is to control the size of the set $\mathcal{S}_j \cap \mathcal{S}_{j+1}$. We will do this by extending the minimal hypersurface regularity theory to this slicing setting. In order to do this we need to establish the relevant compactness and tangent cone properties and this requires establishing suitable bounds on the slicings. To begin this process we make the following definition.

DEFINITION 2.1. For a constant $\Lambda > 0$, a **Λ -bounded minimal k -slicing** is a minimal k -slicing satisfying the following bounds

$$\lambda_j \leq \Lambda, \text{Vol}_{\rho_{j+1}}(\Sigma_j) \leq \Lambda, \int_{\Sigma_j} \left(1 + |A_j|^2 + \sum_{p=j+1}^n |\nabla_j \log u_p|^2 \right) u_j^2 \rho_{j+1} d\mu_j \leq \Lambda$$

for $j = k, k+1, \dots, n-1$, where μ_j is Hausdorff measure, ∇_j is taken on (the regular set of) Σ_j , and A_j is the second fundamental form of Σ_j in \mathbb{R}^N .

The minimal k -slicings we will consider in this paper will always be Λ -bounded for some Λ . We have the following regularity theorem.

THEOREM 2.3. *Given any Λ -bounded minimal k -slicing, we have for each $j = k, k+1, \dots, n-1$ the bound on the singular set $\dim(\mathcal{S}_j) \leq j-3$.*

We now formulate an existence theorem for minimal k -slicings in Σ_n . We consider the case in which Σ_n is a closed oriented manifold. We assume that there is closed oriented k -dimensional manifold X^k and a smooth map $F : \Sigma_n \rightarrow X \times T^{n-k}$ of non-zero degree s . We let Ω denote a k -form of X with $\int_X \Omega = 1$, and we denote by dt^{k+1}, \dots, dt^n the basic one forms on T^{n-k} where we assume the periods are equal to one. We introduce the notation $\Theta = F^*\Omega$ and $\omega^p = F^*(dt^p)$ for $p = k+1, \dots, n$.

We can now state our first existence theorem. A more refined existence theorem is given by Theorem 4.6 which we will not state here.

THEOREM 2.4. *For a manifold $M = \Sigma_n$ as described above, there is a Λ -bounded, partially regular, minimal k -slicing. Moreover, if $k \leq j \leq n-1$ and Σ_j is regular, then $\int_{\Sigma_j} \Theta \wedge \omega^{k+1} \wedge \dots \wedge \omega^j = s$.*

The proofs of Theorems 2.3 and 2.4 will be given in Sections 3 and 4. In the remainder of this section we discuss the quadratic forms Q_j in more detail and derive important geometric consequences for minimal 1-slicings and 2-slicings under the assumption that Σ_n has positive scalar curvature. Consequences of these results, which are the main geometric theorems of the paper, will be given in Section 5.

Recall that in general if Σ is a stable two-sided (trivial normal bundle) minimal hypersurface in a Riemannian manifold M , then we may choose a globally defined unit normal vector ν , and we may parametrize normal deformations by functions $\varphi \cdot \nu$. The second variation of volume then becomes the quadratic form

$$(2.2) \quad S(\varphi, \varphi) = \int_{\Sigma} \left[|\nabla \varphi|^2 - \frac{1}{2} (R_M - R_{\Sigma} + |A|^2) \varphi^2 \right] d\mu$$

where R_M and R_{Σ} are the scalar curvature functions of M and Σ and A denotes the second fundamental form of Σ in M .

We have the following result which computes the scalar curvature \tilde{R}_k of \tilde{g}_k .

LEMMA 2.5. *The scalar curvature of the metric \tilde{g}_k is given by*

$$\tilde{R}_k = R_k - 2 \sum_{p=k+1}^{n-1} u_p^{-1} \Delta_k u_p - 2 \sum_{k+1 \leq p < q \leq n-1} \langle \nabla_k \log u_p, \nabla_k \log u_q \rangle$$

where Δ_k and ∇_k denote the Laplace and gradient operators with respect to g_k .

PROOF. The calculation is a finite induction using the formula

$$\tilde{R} = R - 2u^{-1} \Delta u$$

for the scalar curvature of the metric $\tilde{g} = g + u^2 dt^2$.

For $j = k, \dots, n-1$ Let $\bar{g}_j = g_k + \sum_{p=j}^{n-1} u_p^2 dt_p^2$. Note that $\bar{g}_k = \hat{g}_k$ and $\bar{g}_{k+1} = \tilde{g}_k$. We prove the formula

$$\bar{R}_j = R_k - 2 \sum_{p=j}^{n-1} u_p^{-1} \Delta_k u_p - 2 \sum_{j \leq p < q \leq n-1} \langle \nabla_k \log u_p, \nabla_k \log u_q \rangle$$

by a finite reverse induction on j . First note that for $j = n-1$ the formula follows from the one above. Now assume the formula is correct for \bar{g}_{j+1} . We then apply the formula above to obtain

$$\bar{R}_j = \bar{R}_{j+1} - 2u_j^{-1} \bar{\Delta}_j u_j.$$

Since u_j does not depend on the extra variables t_p , we have

$$u_j^{-1} \bar{\Delta}_j u_j = u_j^{-1} \rho_j^{-1} \operatorname{div}_k(\rho_j \nabla_k u_j) = u_j^{-1} \Delta_k u_j + \sum_{p=j+1}^{n-1} \langle \nabla_k \log u_p, \nabla_k \log u_j \rangle$$

where as above $\rho_j = u_{j+1} \cdots u_{n-1}$. The statement now follows from the inductive assumption. Since $\bar{g}_{k+1} = \tilde{g}_k$, we have proven the required statement. \square

We now consider consequences of having a minimal k -slicing of a manifold of positive scalar curvature.

THEOREM 2.6. *Assume that the scalar curvature of Σ_n is bounded below by a constant κ . If Σ_k is a leaf in a minimal k -slicing, then we have the following scalar curvature formula and eigenvalue estimate*

$$\hat{R}_k = R_n + 2 \sum_{p=k}^{n-1} \lambda_p + \frac{1}{4} \sum_{p=k}^{n-1} \left(|\tilde{A}_p|^2 - \frac{1}{n} \sum_{q=p+1}^n (|\nabla_p \log u_q|^2 + |\tilde{A}_q|^2) \right)$$

$$\int_{\Sigma_k} \left(\kappa + \frac{3}{4} \sum_{j=k+1}^n |\nabla_k \log u_j|^2 - R_k \right) \varphi^2 d\mu_k \leq 4 \int_{\Sigma_k} |\nabla_k \varphi|^2 d\mu_k$$

where φ is any smooth function with compact support in \mathcal{R}_k .

PROOF. First note that from (2.1) and (2.2) we have

$$Q_j(\varphi, \varphi) = \int_{\Sigma_j} \left[|\nabla_j \varphi|^2 - \frac{1}{2}(\hat{R}_{j+1} - \tilde{R}_j)\varphi^2 - \frac{1}{8} \left(|\tilde{A}_j|^2 - \frac{1}{n} \sum_{p=j+1}^n (|\nabla_j \log u_p|^2 + |\tilde{A}_p|^2) \right) \varphi^2 \right] \rho_{j+1} d\mu_j,$$

and therefore u_j satisfies the equation $L_j u_j = -\lambda_j u_j$ where

(2.3)

$$L_j = \tilde{\Delta}_j + \frac{1}{2}(\hat{R}_{j+1} - \tilde{R}_j) + \frac{1}{8} \left(|\tilde{A}_j|^2 - \frac{1}{n} \sum_{p=j+1}^n (|\nabla_j \log u_p|^2 + |\tilde{A}_p|^2) \right).$$

We derive the scalar curvature formula by a finite downward induction beginning with $k = n - 1$. In this case the eigenvalue estimates follow from the standard stability inequality (2.2) since $\rho_n = u_n = 1$ and $\tilde{R}_{n-1} = R_{n-1}$. We also have from Lemma 2.5 that $\hat{R}_{n-1} = R_{n-1} - 2u_{n-1}^{-1}\Delta_{n-1}u_{n-1}$. The equation satisfied by u_{n-1} is

$$\Delta_{n-1}u_{n-1} + \frac{1}{2}(R_n - R_{n-1})u_{n-1} + \frac{1}{8}|\tilde{A}_{n-1}|^2u_{n-1} = -\lambda_{n-1}u_{n-1}$$

and so we have $\hat{R}_{n-1} = R_n + 2\lambda_{n-1} + \frac{1}{4}|\tilde{A}_{n-1}|^2$. This proves the result for $k = n - 1$.

Now we assume the conclusions are true for integers k and larger, and we will derive them for $k - 1$. We first observe that $\hat{g}_{k-1} = \tilde{g}_{k-1} + u_{k-1}^2 dt_{k-1}^2$ and so $\hat{R}_{k-1} = \tilde{R}_{k-1} - 2u_{k-1}^{-1}\tilde{\Delta}_{k-1}u_{k-1}$. On the other hand from (2.3) applied with $j = k - 1$ we see that u_{k-1} satisfies the equation

$$\begin{aligned} \tilde{\Delta}_{k-1}u_{k-1} + \frac{1}{2}(\hat{R}_k - \tilde{R}_{k-1})u_{k-1} + \frac{1}{8} \left(|\tilde{A}_{k-1}|^2 - \frac{1}{n} \sum_{p=k}^n (|\nabla_{k-1} \log u_p|^2 + |\tilde{A}_p|^2) \right) u_{k-1} = -\lambda_{k-1}u_{k-1}. \end{aligned}$$

Substituting this above we have

$$\begin{aligned} \hat{R}_{k-1} = \tilde{R}_{k-1} + 2 \left[\lambda_{k-1} + \frac{1}{2}(\hat{R}_k - \tilde{R}_{k-1}) + \frac{1}{8} \left(|\tilde{A}_{k-1}|^2 - \frac{1}{n} \sum_{q=k}^n (|\nabla_{k-1} \log u_q|^2 + |\tilde{A}_q|^2) \right) \right], \end{aligned}$$

so we have

$$\hat{R}_{k-1} = 2\lambda_{k-1} + \hat{R}_k + \frac{1}{4} \left(|\tilde{A}_{k-1}|^2 - \frac{1}{n} \sum_{q=k}^n (|\nabla_{k-1} \log u_q|^2 + |\tilde{A}_q|^2) \right).$$

Using the inductive hypothesis we get the desired formula

$$\hat{R}_{k-1} = R_n + 2 \sum_{p=k-1}^{n-1} \lambda_p + \frac{1}{4} \sum_{p=k-1}^{n-1} \left(|\tilde{A}_p|^2 - \frac{1}{n} \sum_{q=p+1}^n (|\nabla_p \log u_q|^2 + |\tilde{A}_q|^2) \right).$$

Now observe that by an easy rewrite and estimate on the first term

$$\begin{aligned} & \sum_{p=k}^{n-1} \left(n |\tilde{A}_p|^2 - \sum_{q=p+1}^{n-1} (|\nabla_p \log u_q|^2 + |\tilde{A}_q|^2) \right) \\ & \geq \sum_{p=k}^{n-1} \left(\sum_{r=k}^{n-1} |\tilde{A}_r|^2 - \sum_{q=p+1}^{n-1} (|\nabla_p \log u_q|^2 + |\tilde{A}_q|^2) \right) \\ & = \sum_{p=k}^{n-1} \left(\sum_{r=k}^p |\tilde{A}_r|^2 - \sum_{q=p+1}^{n-1} |\nabla_p \log u_q|^2 \right). \end{aligned}$$

From Lemma 2.1 we have the bound

$$\begin{aligned} \sum_{r=k}^p |\tilde{A}_r|^2 & \geq \sum_{r=k}^p \sum_{q=r+1}^{n-1} (\nu_r \log u_q)^2 \geq \sum_{r=k}^p \sum_{q=p+1}^{n-1} (\nu_r \log u_q)^2 \\ & = \sum_{q=p+1}^{n-1} \sum_{r=k}^p (\nu_r \log u_q)^2 \end{aligned}$$

since $r \leq p$. Combining these we have

$$\begin{aligned} & \sum_{p=k}^{n-1} \left(n |\tilde{A}_p|^2 - \sum_{q=p+1}^{n-1} (|\nabla_p \log u_q|^2 + |\tilde{A}_q|^2) \right) \\ & \geq \sum_{p=k}^{n-1} \sum_{q=p+1}^{n-1} \left(\sum_{r=k}^{p-1} (\nu_r \log u_q)^2 - |\nabla_p \log u_q|^2 \right) \\ & = - \sum_{p=k}^{n-1} \sum_{q=p+1}^{n-1} |\nabla_k \log u_q|^2 \geq -n \sum_{q=k}^{n-1} |\nabla_k \log u_q|^2. \end{aligned}$$

This formula implies that for each k we have

$$\hat{R}_k \geq \kappa - 1/4 \sum_{j=k}^n |\nabla_k \log u_j|^2.$$

It then follows from Lemma 2.1 that

$$(2.4) \quad |\tilde{A}_k|^2 + \hat{R}_{k+1} \geq \kappa - 1/4 \sum_{j=k+1}^n |\nabla_k \log u_j|^2$$

and so the following eigenvalue estimate follows from (2.2)

$$\int_{\Sigma_k} \left(\kappa - \frac{1}{4} \sum_{j=k+1}^{n-1} n |\nabla_k \log u_j|^2 - \tilde{R}_k \right) \varphi^2 \rho_{k+1} d\mu_k \leq 2 \int_{\Sigma_k} |\nabla_k \varphi|^2 \rho_{k+1} d\mu_k$$

The remainder of the proof derives the eigenvalue estimate from this one. Since φ is arbitrary we may replace φ by $\varphi(\rho_{k+1})^{1/2}$ to obtain

$$\begin{aligned} & \int_{\Sigma_k} \left(\kappa - \frac{1}{4} \sum_{j=k+1}^n |\nabla_k \log u_j|^2 - \tilde{R}_k \right) \varphi^2 d\mu_k \\ & \leq 2 \int_{\Sigma_k} |\nabla_k(\varphi/\sqrt{\rho_{k+1}})|^2 \rho_{k+1} d\mu_k \\ & \leq 4 \int_{\Sigma_k} |\nabla_k(\varphi/\sqrt{\rho_{k+1}})|^2 \rho_{k+1} d\mu_k \end{aligned}$$

where we used the inequality $2 \leq 4$. After expanding, the term on the right becomes

$$4 \int_{\Sigma_k} (|\nabla_k \varphi|^2 - \varphi \langle \nabla_k \varphi, \nabla_k \log \rho_{k+1} \rangle + 1/4 \varphi^2 |\nabla_k \log \rho_{k+1}|^2) d\mu_k.$$

Rewriting the middle term in terms of $\nabla_k(\varphi)^2$ and integrating by parts the term becomes

$$4 \int_{\Sigma_k} \left(|\nabla_k \varphi|^2 + 1/2 \varphi^2 \left[\sum_{p=k+1}^{n-1} (u_p^{-1} \Delta_k u_p - |\nabla_k \log u_p|^2) + 1/2 |\nabla_k \log \rho_{k+1}|^2 \right] \right) d\mu_k.$$

Now recall from Lemma 2.5 that

$$\tilde{R}_k = R_k - 2 \sum_{p=k+1}^{n-1} u_p^{-1} \Delta_k u_p - 2 \sum_{k+1 \leq p < q \leq n-1} \langle \nabla_k \log u_p, \nabla_k \log u_q \rangle.$$

Thus we see that the terms involving $\Delta_k u_p$ cancel out, and note also that

$$|\nabla_k \log \rho_{k+1}|^2 = \sum_{p=k+1}^{n-1} |\nabla_k \log u_p|^2 + 2 \sum_{k+1 \leq p < q \leq n-1} \langle \nabla_k \log u_p, \nabla_k \log u_q \rangle$$

so the second term also cancels. Thus we are left with

$$\begin{aligned} & \int_{\Sigma_k} \left(\kappa - \frac{1}{4} \sum_{j=k+1}^n |\nabla_k \log u_j|^2 - R_k \right) \varphi^2 d\mu_k \\ & \leq 4 \int_{\Sigma_k} \left(|\nabla_k \varphi|^2 - \frac{1}{4} \sum_{j=k+1}^n |\nabla_k \log u_j|^2 \right) d\mu_k. \end{aligned}$$

This gives the desired eigenvalue estimate. \square

This theorem will be central to the regularity proof in the next section and it also has an important geometric consequence which is the main tool in the applications of Section 5.

THEOREM 2.7. *Assume that $R_n \geq \kappa > 0$. If Σ_k is regular, then (Σ_k, g_k) is a Yamabe positive conformal manifold. If Σ_2 lies in a minimal 2-slicing, Σ_2 is regular, and $\partial\Sigma_2 = 0$, then each connected component of Σ_2 is homeomorphic to the two sphere. If Σ_1 lies in a minimal 1-slicing and Σ_1 is regular, then each component of Σ_1 is an arc of length at most $2\pi/\sqrt{\kappa}$.*

PROOF. Recall that the condition that g_k be Yamabe positive is that the lowest eigenvalue of the conformal Laplacian $-\Delta_k + c(k)R_k$ be positive where $c(k) = \frac{k-2}{4(k-1)}$. In variational form this condition says

$$-\int_{\Sigma_k} R_k \varphi^2 d\mu_k < c(k)^{-1} \int_{\Sigma_k} |\nabla_k \varphi|^2 d\mu_k$$

for all nonzero functions φ which vanish on $\partial\Sigma_k$ (if Σ_k has a boundary). Since $4 < c(k)^{-1}$ we see that this follows from the eigenvalue estimate of Theorem 2.6.

Now consider Σ_2 , and apply the eigenvalue estimate of Theorem 2.6 with $\varphi = 1$ to a component S of Σ_2 to see that $\int_S R_2 d\mu_2 > 0$. It then follows from the Gauss-Bonnet Theorem that S is homeomorphic to the two sphere (note that S is orientable).

Finally, if γ is a connected component of Σ_1 of length l , then the eigenvalue estimate of Theorem 2.6 implies that the lowest Dirichlet eigenvalue of γ is at least $\kappa/4$. Thus $\kappa/4 \leq \pi^2/l^2$ and $l \leq 2\pi/\sqrt{\kappa}$ as claimed. \square

3. Compactness and regularity of minimal k - slicings

The main goal of this section is to prove Theorem 2.3. In order to do this we first must clarify some analytic issues concerning the domain of the quadratic form Q_j . We let $L^2(\Sigma_j)$ denote the space of square integrable functions on Σ_j with respect to the measure $\rho_{j+1}\mu_j$. We let

$$\|\varphi\|_{0,j}^2 = \int_{\Sigma_j} \varphi^2 \rho_{j+1} d\mu_j$$

denote the square norm on $L^2_{\Sigma_j}$. We introduce some notation, defining P_j to be the function defined on Σ_j

$$P_j = |A_j|^2 + \sum_{p=j+1}^n |\nabla_j \log u_p|^2.$$

We will say that a minimal k -slicing in an open set Ω is *partially regular* if $\dim(\mathcal{S}_j) \leq j-3$ for $j = k, \dots, n-1$. It follows from Proposition 2.2 that if the $(k+1)$ -slicing associated to a minimal k -slicing is partially regular, then $\dim(\mathcal{S}_k) \leq \min\{\dim(\mathcal{S}_{k+1}), k-7\} \leq k-2$.

For functions φ which are Lipschitz (with respect to ambient distance) on Σ_j with compact support in $\mathcal{R}_j \cap \bar{\Omega}$, we define a square norm by

$$\|\varphi\|_{1,j}^2 = \|\varphi\|_{0,j}^2 + \int_{\Sigma_j} (|\nabla_j \varphi|^2 + P_j \varphi^2) \rho_{j+1} d\mu_j.$$

We let \mathcal{H}_j denote the Hilbert space which is the completion with respect to this norm. Note that functions in \mathcal{H}_j are clearly locally in $W_{1,2}$ on \mathcal{R}_j . We will assume from now on that $u_j \in \mathcal{H}_j$ for $j \geq k$; in fact, we take this as part of the definition of a bounded minimal k -slicing. We define $\mathcal{H}_{j,0}$ to be the closed subspace of \mathcal{H}_j consisting of the completion of the Lipschitz functions with compact support in $\mathcal{R}_j \cap \Omega$. In order to handle boundary effects we also assume that there is a larger domain Ω_1 which contains $\bar{\Omega}$ as a compact subset and that the k -slicing is defined and boundaryless in Ω_1 . Note that this is automatic if $\partial\Sigma_j = \phi$. Thus $\mathcal{H}_{j,0}$ consists of those functions in \mathcal{H}_j with 0 boundary data on $\Sigma_j \cap \partial\Omega$. The existence of eigenfunctions u_j in this space will be discussed in the next section. The following estimate of the $L^2(\Sigma_k)$ norm near the singular set will be used both in this section and the next. The result may be thought of as a non-concentration result for the weighted L^2 norm near the singular set in case the \mathcal{H}_k norm is bounded.

PROPOSITION 3.1. *Let \mathcal{S} be a closed subset of Ω_1 with zero $(k-1)$ -dimensional Hausdorff measure. Let Σ_k be a member of a bounded minimal k -slicing such that Σ_{k+1} is partially regular in Ω_1 . For any $\eta > 0$ there exists an open set $V \subset \Omega_1$ containing $\mathcal{S} \cap \bar{\Omega}$ such that whenever $\mathcal{S}_k \cap \bar{\Omega} \subset V$ we have the following estimate*

$$\int_{\Sigma_k \cap V} \varphi^2 \rho_{k+1} d\mu_k \leq \eta \int_{\Sigma_k \cap \Omega} [|\nabla_k \varphi|^2 + (1 + P_k) \varphi^2] \rho_{k+1} d\mu_k$$

for all $\varphi \in \mathcal{H}_{k,0}$.

PROOF. Let $\varepsilon > 0$, $\delta > 0$ be given. We may choose a finite covering of the compact set $\mathcal{S} \cap \bar{\Omega}$ by balls $B_{r_\alpha}(x_\alpha)$ with $r_\alpha \leq \delta/5$ such

$$\sum_{\alpha} r_\alpha^{k-1} \leq \varepsilon.$$

We let V denote the union of the balls, $V = \cup_{\alpha} B_{r_\alpha}(x_\alpha)$.

Assume that $\mathcal{S}_k \cap \bar{\Omega} \subset V$ and let $\varphi \in \mathcal{H}_{k,0}$. We may extend φ to $\Sigma_k \cap \Omega_1$ by taking $\varphi = 0$ in $\Omega_1 \setminus \Omega$. By a standard first variation argument for submanifolds of \mathbb{R}^N , for a nonnegative function we have

$$\begin{aligned} k \int_{\Sigma_k \cap B_r} \varphi^2 \rho_{k+1} d\mu_k &\leq r \int_{\Sigma_k \cap B_r} (|\nabla_k \varphi|^2 \rho_{k+1} + |H_k| \varphi^2 \rho_{k+1}) d\mu_k \\ &\quad + r \int_{\Sigma_k \cap \partial B_r} \varphi^2 \rho_{k+1} d\mu_{k-1}. \end{aligned}$$

Let $L_\alpha(r) = \int_{\Sigma_k \cap B_r(x_\alpha)} \varphi^2 \rho_{k+1} d\mu_k$ and

$$M_\alpha(r) = \int_{\Sigma_k \cap B_r(x_\alpha)} (|\nabla_k(\varphi^2 \rho_{k+1})| + |H_k| \varphi^2 \rho_{k+1}) d\mu_k.$$

The above inequality then implies

$$kL_\alpha(r) \leq rM_\alpha(r) + r \frac{d}{dr}(L_\alpha(r)).$$

Now for any α and a small constant ε_0 we consider two cases: (1) There exists r with $r_\alpha \leq r \leq \delta/5$ such that the inequality

$$\varepsilon_0 L_\alpha(5r) \leq r M_\alpha(r).$$

We denote such a choice of r by r'_α . Secondly, we have case (2) For all r with $r_\alpha \leq r \leq \delta/5$ we have

$$r M_\alpha(r) < \varepsilon_0 L_\alpha(5r).$$

The collection of α for which the first case holds will be labeled A_1 , and that for which the second holds A_2 . We will handle the two cases separately.

For the collection of balls with radius r'_α indexed by A_1 we may apply the five times covering lemma to extract a subset $A'_1 \subseteq A_1$ for which the balls in A'_1 are disjoint and such that

$$V_1 \equiv \cup_{\alpha \in A_1} B_{r_\alpha}(x_\alpha) \subseteq \cup_{\alpha \in A_1} B_{r'_\alpha}(x_\alpha) \subseteq \cup_{\alpha \in A'_1} B_{5r'_\alpha}(x_\alpha).$$

From the inequality of case (1) above applied for $\alpha \in A'_1$ we have

$$L_\alpha(r_\alpha) \leq L_\alpha(5r'_\alpha) \leq \varepsilon_0^{-1} r'_\alpha M_\alpha(r'_\alpha) \leq \varepsilon_0^{-1} \delta M_\alpha(r'_\alpha).$$

Summing over $\alpha \in A_1$ and using disjointness of the balls we have

$$(3.1) \quad \int_{\Sigma_k \cap V_1} \varphi^2 \rho_{k+1} d\mu_k \leq \varepsilon_0^{-1} \delta \int_{\Sigma_k \cap \Omega} (|\nabla_k \varphi^2 \rho_{k+1}| + |H_k| \varphi^2 \rho_{k+1}) d\mu_k.$$

Now for $\alpha \in A_2$ we have

$$k L_\alpha(r) \leq \varepsilon_0 L_\alpha(5r) + r \frac{d}{dr} (L_\alpha(r))$$

for $r_\alpha \leq r \leq \delta/5$. For $j = 0, 1, 2, \dots$ define $\sigma_j = 5^j r_\alpha$ and let p be the positive integer such that $\sigma_{p-1} < \delta/5 \leq \sigma_p$. We define Λ_j by $\Lambda_j = L_\alpha(\sigma_j)$ for $j = 0, 1, \dots, p$. For $\sigma_j \leq r \leq \sigma_{j+1}$ we then have

$$k L_\alpha(r) \leq \varepsilon_0 \Lambda_{j+2} \Lambda_j^{-1} L_\alpha(r) + r \frac{d}{dr} (L_\alpha(r)).$$

Integrating we find

$$\Lambda_{j+1} \Lambda_j^{-1} \geq 5^{k-\varepsilon_0 \Lambda_{j+2} \Lambda_j^{-1}}.$$

Setting $R_j = \Lambda_{j+1} \Lambda_j^{-1}$ we have shown

$$R_j \geq 5^{k-\varepsilon_0 R_j R_{j+1}}.$$

Now if $R_j \leq 5^{k-1}$ then we would have $5^{k-1} \geq 5^{k-\varepsilon_0 R_j R_{j+1}}$ which in turn implies $\varepsilon_0 5^{k-1} R_{j+1} \geq \varepsilon_0 R_j R_{j+1} \geq 1$. Thus if we choose $\varepsilon_0 = 5^{-3k+3}$ we find $R_{j+1} \geq 5^{2(k-1)}$ and hence it follows that $R_j R_{j+1} \geq 5^{2(k-1)}$. Thus we have shown that for any $j = 0, 1, \dots, p-1$ we either have $R_j \geq 5^{k-1}$ or $R_j R_{j+1} \geq 5^{2(k-1)}$. This implies that $\Lambda_p \Lambda_0^{-1} \geq 5^{(p-1)(k-1)} \geq 5^{1-k} (\delta/r_\alpha)^{k-1}$ and therefore we have $L_\alpha(r_\alpha) \leq c(r_\alpha/\delta)^{k-1} L_\alpha(\sigma_p)$ for each $\alpha \in A_2$. Summing this over these α and using the choice of the covering we have

$$\int_{\Sigma_k \cap V_2} \varphi^2 \rho_{k+1} d\mu_k \leq c \varepsilon \delta^{1-k} \int_{\Sigma_k \cap \Omega} \varphi^2 \rho_{k+1} d\mu_k.$$

Combining this with (3.1) we finally obtain

$$\begin{aligned} & \int_{\Sigma_k \cap V} \varphi^2 \rho_{k+1} d\mu_k \\ & \leq c\varepsilon\delta^{1-k} \int_{\Sigma_k \cap \Omega} \varphi^2 \rho_{k+1} d\mu_k + c\delta \int_{\Sigma_k \cap \Omega} (|\nabla_k \varphi^2 \rho_{k+1}| + |H_k| \varphi^2 \rho_{k+1}) d\mu_k, \end{aligned}$$

since we have now fixed ε_0 . We can estimate the second term on the right using

$$\begin{aligned} & |\nabla_k \varphi^2 \rho_{k+1}| + |H_k| \varphi^2 \rho_{k+1} \\ & \leq (\varphi^2 + |\nabla_k \varphi|^2) \rho_{k+1} + \frac{1}{2} \varphi^2 (2 + |\nabla_k \log \rho_{k+1}|^2 + |H_k|^2) \rho_{k+1}. \end{aligned}$$

This implies the bound

$$\begin{aligned} & \int_{\Sigma_k \cap V} \varphi^2 \rho_{k+1} d\mu_k \\ & \leq c(\varepsilon\delta^{1-k} + \delta) \int_{\Sigma_k \cap \Omega} \varphi^2 \rho_{k+1} d\mu_k + c\delta \int_{\Sigma_k \cap \Omega} [|\nabla_k \varphi|^2 + P_k \varphi^2] \rho_{k+1} d\mu_k. \end{aligned}$$

The desired conclusion now follows by choosing δ so that $c\delta = \eta/2$ and then choosing ε so that $c\varepsilon\delta^{1-k} = \eta$. This completes the proof. \square

The following coercivity bound will be useful both in this section and in the next. We assume here that we have a partially regular minimal k -slicing.

PROPOSITION 3.2. *Assume that our k -slicing is bounded. There is a constant c such that for $\varphi \in \mathcal{H}_{k,0}$ we have*

$$c^{-1} \int_{\Sigma_k} [|\nabla_k \varphi|^2 + (P_k + |\nabla_k \log u_k|^2) \varphi^2] \rho_{k+1} d\mu_k \leq Q_k(\varphi, \varphi) + \int_{\Sigma_k} \varphi^2 \rho_{k+1} d\mu_k.$$

Moreover we have the bound

$$c^{-1} \int_{\Sigma_k} (|\nabla_k(\varphi \sqrt{\rho_{k+1}})|^2 + |A_k|^2 \varphi^2 \rho_{k+1}) d\mu_k \leq Q_k(\varphi, \varphi) + \int_{\Sigma_k} \varphi^2 \rho_{k+1} d\mu_k.$$

PROOF. We can see from (2.1) that

$$Q_k(\varphi, \varphi) \geq S_k(\varphi, \varphi) + \frac{1}{8n} \int_{\Sigma_k} \left(\sum_{p=k}^n |\tilde{A}_p|^2 + \sum_{p=k+1}^n |\nabla_k \log u_p|^2 \right) \varphi^2 \rho_{k+1} d\mu_k.$$

Using the stability of Σ_k we have

$$(3.2) \quad Q_k(\varphi, \varphi) \geq \frac{1}{8n} \int_{\Sigma_k} \left(\sum_{p=k}^n |\tilde{A}_p|^2 + \sum_{p=k+1}^n |\nabla_k \log u_p|^2 \right) \varphi^2 \rho_{k+1} d\mu_k.$$

Finally we use Lemma 2.1 to conclude that (note that $\tilde{A}_n = 0$)

$$\sum_{p=k}^n |\tilde{A}_p|^2 \geq \sum_{p=k}^{n-1} |A_p^{\nu_p}|^2 \geq \sum_{p=k}^{n-1} |A_k^{\nu_p}|^2 = |A_k|^2,$$

and thus we have

$$Q_k(\varphi, \varphi) \geq \frac{1}{8n} \int_{\Sigma_k} P_k \varphi^2 \rho_{k+1} d\mu_k.$$

Recall that $S_k(\varphi, \varphi) = \int_{\Sigma_k} (|\nabla_k \varphi|^2 - q_k \varphi^2) \rho_{k+1} d\mu_k$ where

$$q_k = \frac{1}{2}(|\tilde{A}_k|^2 + \hat{R}_{k+1} - \tilde{R}_k)$$

where \hat{R}_{k+1} is given in Theorem 2.6 and \tilde{R}_k is given in Lemma 2.5. We will need an upper bound on q_k , so we first see from Theorem 2.6 with k replace by $k+1$

$$q_k \leq c + \frac{1}{2} \sum_{p=k}^{n-1} |\tilde{A}_p|^2 - \frac{1}{2} \tilde{R}_k$$

where the constant bounds the curvature of Σ_n and the eigenvalues. Now from Lemma 2.5 we can obtain the bound

$$-\frac{1}{2} \tilde{R}_k \leq \frac{1}{2} |R_k| + \sum_{p=k+1}^{n-1} |\nabla_k \log u_p|^2 + \operatorname{div}_k(\mathcal{X}_k)$$

where $\mathcal{X}_k = \sum_{p=k+1}^{n-1} \nabla_k \log u_p$. We observe that the Gauss equation implies that $|R_k| \leq c(1 + |A_k|^2)$, and so we have

$$q_k \leq c + c \sum_{p=k}^{n-1} |\tilde{A}_p|^2 + \sum_{p=k+1}^{n-1} |\nabla_k \log u_p|^2 + \operatorname{div}_k(\mathcal{X}_k)$$

Now observe that $Q_k \geq S_k$ and so we have

$$\int_{\Sigma_k} \left(|\nabla_k \varphi|^2 + \frac{1}{8n} P_k \varphi^2 \right) \rho_{k+1} d\mu_k \leq 2Q_k(\varphi, \varphi) + \int_{\Sigma_k} q_k \varphi^2 \rho_{k+1} d\mu_k.$$

We want to bound the second term on the right by a constant times the first plus up to the square of the L^2 norm of φ , so we use the bound for q_k to obtain

$$\begin{aligned} \int_{\Sigma_k} q_k \varphi^2 \rho_{k+1} d\mu_k &\leq c \int_{\Sigma_k} \left(1 + \sum_{p=k}^{n-1} |\tilde{A}_p|^2 + \sum_{p=k+1}^{n-1} |\nabla_k \log u_p|^2 \right) \varphi^2 \rho_{k+1} d\mu_k \\ &\quad + \int_{\Sigma_k} \operatorname{div}_k(\mathcal{X}_k) \varphi^2 \rho_{k+1} d\mu_k. \end{aligned}$$

Now since φ has compact support we have

$$\int_{\Sigma_k} \operatorname{div}_k(\mathcal{X}_k) \varphi^2 \rho_{k+1} d\mu_k = - \int_{\Sigma_k} \langle \mathcal{X}_k, \nabla(\varphi^2 \rho_{k+1}) \rangle d\mu_k.$$

Easy estimates then imply the bound

$$\begin{aligned} & \left| \int_{\Sigma_k} \operatorname{div}_k(\mathcal{X}_k) \varphi^2 \rho_{k+1} d\mu_k \right| \\ & \leq \frac{1}{2} \int_{\Sigma_k} |\nabla_k \varphi|^2 \rho_{k+1} d\mu_k + c \int_{\Sigma_k} \left(\sum_{p=k+1}^{n-1} |\nabla_k \log u_p|^2 \right) \varphi^2 \rho_{k+1} d\mu_k. \end{aligned}$$

We may now absorb the first term back to the left and use (3.2) to obtain the bound

$$\int_{\Sigma_k} (|\nabla_k \varphi|^2 + P_k \varphi^2) \rho_{k+1} d\mu_k \leq cQ_k(\varphi, \varphi) + \int_{\Sigma_k} \varphi^2 \rho_{k+1} d\mu_k.$$

To bound the term involving $|\nabla_k \log u_k|^2$ we recall that on the regular set we have

$$\tilde{\Delta}_k u_k + q_k u_k = -\lambda_k u_k$$

where $\lambda_k \geq 0$. This implies by direct calculation

$$\tilde{\Delta} \log u_k = -q_k - \lambda_k - |\nabla_k \log u_k|^2.$$

(Note that $\tilde{\nabla}_k = \nabla_k$ on functions which do not depend on the extra variables t_p .) Now if φ has compact support in \mathcal{R}_k , we multiply by φ^2 , integrate by parts to obtain

$$\int_{\Sigma_k} (|\nabla_k \log u_k|^2 + q_k) \varphi^2 \rho_{k+1} d\mu_k \leq 2 \int_{\Sigma_k} \varphi \langle \nabla_k \varphi, \nabla_k \log u_k \rangle \rho_{k+1} d\mu_k.$$

By the arithmetic-geometric mean inequality

$$\begin{aligned} \int_{\Sigma_k} (|\nabla_k \log u_k|^2 + q_k) \varphi^2 \rho_{k+1} d\mu_k & \leq \frac{1}{2} \int_{\Sigma_k} (|\nabla_k \log u_k|^2 + q_k) \varphi^2 \rho_{k+1} d\mu_k \\ & + 2 \int_{\Sigma_k} |\nabla_k \varphi|^2 \rho_{k+1} d\mu_k. \end{aligned}$$

This implies

$$\frac{1}{2} \int_{\Sigma_k} |\nabla_k \log u_k|^2 \varphi^2 \rho_{k+1} d\mu_k \leq \frac{1}{2} Q_k(\varphi, \varphi) + \frac{3}{2} \int_{\Sigma_k} |\nabla_k \varphi|^2 \rho_{k+1} d\mu_k.$$

The first inequality then follows from this and our previous estimate.

The second conclusion follows since $|\nabla_k \log \rho_{k+1}|^2 \leq cP_k$, and so the integrand on the left $|\nabla_k(\varphi \sqrt{\rho_{k+1}})|^2 + |A_k|^2 \varphi^2 \rho_{k+1}$ is bounded pointwise by a constant times $(|\nabla_k \varphi|^2 + P_k \varphi^2) \rho_{k+1}$. \square

Recall that an important analytic step in the minimal hypersurface regularity theory is the local reduction to the case in which the hypersurface is the boundary of a set. This makes comparisons particularly simple and reduces consideration to a multiplicity one setting. We will need an analogous reduction in our situation. Since the leaves of a k -slicing can be singular, we must consider the possibility that local topology comes into play and prohibits such a reduction to boundaries of sets. What saves us here is the

fact that k -slicings come with a natural trivialization of the normal bundle (on the regular set). We have the following result.

PROPOSITION 3.3. *Assume that U is compactly contained in Ω , and that $U \cap \Sigma_n$ is diffeomorphic to a ball. Assume that we have a minimal k -slicing in Ω such that the associated $(k+1)$ -slicing is partially regular. Let $\hat{\Sigma}_k$ denote the closure of any connected component of $\Sigma_k \cap U \cap \mathcal{R}_{k+1}$. Then it follows that $\hat{\Sigma}_k$ divides the corresponding connected component (denoted $\hat{\Sigma}_{k+1}$) of Σ_{k+1} into a union of two relatively open subsets, and choosing the one, denoted U_{k+1} , for which the unit normal of $\hat{\Sigma}_k$ points outward, we have $\hat{\Sigma}_k = \partial U_{k+1}$ as a point set boundary in $\hat{\Sigma}_{k+1}$, and as an oriented boundary in \mathcal{R}_{k+1} .*

PROOF. Since $\hat{\Sigma}_k \cap \mathcal{R}_{k+1}$ and $\hat{\Sigma}_{k+1} \cap \mathcal{R}_{k+1}$ are connected, it follows that the complement of $\hat{\Sigma}_k \cap \mathcal{R}_{k+1}$ in $\hat{\Sigma}_{k+1} \cap \mathcal{R}_{k+1}$ has either 1 or 2 connected components. These consist of the connected components of points lying near $\hat{\Sigma}_k$ on either side. Locally these are separate components, but they may reduce globally to a single connected component. If this were to happen, then since $\dim(\mathcal{S}_{k+1}) \leq k-2$, we could find a smooth embedded closed curve $\gamma(t)$ parametrized by a periodic variable $t \in [0, 1]$ with $\gamma(0) \in \hat{\Sigma}_k \cap \mathcal{R}_{k+1}$ and $\gamma(t) \in \mathcal{R}_{k+1} \sim \hat{\Sigma}_k$ for $t \neq 0$. We may also assume that $\gamma'(0)$ is transverse to $\hat{\Sigma}_k$. We choose local coordinates x^1, \dots, x^k for $\hat{\Sigma}_k$ in a neighborhood V of $\gamma(0)$ and we may find an embedding F of $V \times S^1$ in \mathcal{R}_{k+1} with the property that $F(0, t) = \gamma(t)$, $F(x, 0) \in \hat{\Sigma}_k$, $F(x, t) \notin \hat{\Sigma}_k$ for $t \neq 0$, and $\frac{\partial F}{\partial t}(x, 0)$ is transverse to $\hat{\Sigma}_k$. The k -form $\omega = \zeta(x) dx^1 \wedge \dots \wedge dx^k$, where ζ is a nonnegative and nonzero function with compact support in V , is a closed form which has positive integral over $\hat{\Sigma}_k$. Since the image $V_1 = F(V \times S^1)$ is compactly contained in \mathcal{R}_{k+1} and the normal bundle of $\hat{\Sigma}_{k+1}$ is trivial, we may choose coordinates x^{k+2}, \dots, x^n for a normal disk, and the coordinates $x^1, \dots, x^k, t, x^{k+2}, \dots, x^n$ are then coordinates on a neighborhood of V_1 in Σ_n . We may then extend ω to an $(n-1)$ -form on this neighborhood by setting

$$\omega_1 = \omega \wedge \zeta_1(x^{k+2}, \dots, x^n) dx^{k+2} \wedge \dots \wedge dx^n$$

where ζ_1 is a nonzero, nonnegative function with compact support in the domain of x^{k+2}, \dots, x^n . Thus ω_1 is a closed $(n-1)$ -form with compact support in $U \cap \Sigma_n$ which has positive integral on $\hat{\Sigma}_{n-1}$, the connected component of Σ_{n-1} containing $\gamma(0)$. This contradicts the condition that each connected component of Σ_{n-1} must divide the ball $U \cap \Sigma_n$ into 2 connected components and is the oriented boundary of one of them, say $\hat{\Sigma}_{n-1} = \partial U_n$, since Stokes theorem would imply that $\int_{\hat{\Sigma}_{n-1}} \omega_1 = \int_{U_n} d\omega_1 = 0$ (note that ω_1 has compact support in $U \cap \Sigma_n$). \square

We will prove a boundedness theorem which will be needed in the proof of the compactness theorem. Note that we will obtain the partial regularity theorem by finite induction down from dimension $n-1$, so we may assume in the following theorems that we have already established partial regularity for

$(k+1)$ -slicings. In the following result we will consider the restriction of a k -slicing to a small ball $B_\sigma(x)$ where $x \in \mathbb{R}^N$. We consider the rescaled k -slicing of the unit ball given by $\Sigma_{j,\sigma} = \sigma^{-1}(\Sigma_j - x)$ with $u_{j,\sigma}(y) = a_j u_j(x + \sigma y)$ with a_j chosen so that $\int_{\Sigma_{j,\sigma}} (u_{j,\sigma})^2 \rho_{j+1,\sigma} d\mu_j = 1$. We note that by Proposition 3.3 we may assume that each Σ_j in $B_\sigma(x)$ is the oriented boundary of a relatively open set $O_{j+1} \subseteq \Sigma_{j+1}$. We take $O_{j+1,\sigma}$ to be the rescaled open set. The following result implies that the rescaled k -slicing remains Λ -bounded for a suitably chosen Λ .

THEOREM 3.4. *Assume that all bounded $(k+1)$ -slicings are partially regular. If we take any bounded minimal k -slicing (Σ_j, u_j) in Ω and a ball $B_\sigma(x)$ compactly contained in Ω , then there is a Λ depending only on Σ_n such that $(\Sigma_{j,\sigma}, u_{j,\sigma})$, $j = k, \dots, n-1$ is Λ -bounded in $B_{1/2}(0)$.*

PROOF. The proof is by a finite induction beginning with $k = n-1$. The boundedness of $\mu_{n-1}(\Sigma_{n-1,\sigma})$ follows by comparison with a portion of the sphere of radius 1 in a standard way (see a similar argument below). We normalize $\int_{\Sigma_{n-1,\sigma}} (u_{n-1,\sigma})^2 d\mu_{n-1} = 1$, so it remains to show

$$\int_{\Sigma_{n-1,\sigma} \cap B_{1/2}(0)} |A_{n-1,\sigma}|^2 u_{n-1,\sigma}^2 d\mu_{n-1} \leq \Lambda.$$

To see this, we use stability with the variation $\zeta u_{n-1,\sigma}$ to obtain

$$\frac{1}{4} \int_{\Sigma_{n-1,\sigma}} |A_{n-1,\sigma}|^2 \zeta^2 u_{n-1,\sigma}^2 d\mu_{n-1} \leq Q_{n-1,\sigma}(\zeta u_{n-1,\sigma}, \zeta u_{n-1,\sigma}).$$

Now we have by direct calculation for any $W_{1,2}(\Sigma_{n-1,\sigma})$ function v

$$Q_{n-1,\sigma}(\zeta v, \zeta v) = Q_{n-1,\sigma}(\zeta^2 v, v) + \int_{\Sigma_{n-1,\sigma}} v^2 |\nabla_{n-1,\sigma} \zeta|^2 d\mu_{n-1}.$$

Taking $v = u_{n-1,\sigma}$ and choosing ζ to be a function which is 1 on $B_{1/2}(0)$ with support in $B_1(0)$ and with bounded gradient we find

$$\int_{\Sigma_{n-1,\sigma}} |A_{n-1,\sigma}|^2 u_{n-1,\sigma}^2 d\mu_{n-1} \leq 4\lambda_{n-1,\sigma} + c \leq \Lambda$$

for a constant Λ where we have used the eigenvalue condition

$$Q_{n-1,\sigma}(\zeta^2 u_{n-1,\sigma}, u_{n-1,\sigma}) = \lambda_{n-1,\sigma} \int_{\Sigma_{n-1,\sigma}} \zeta^2 u_{n-1,\sigma}^2 d\mu_{n-1}$$

and the obvious relation $\lambda_{n-1,\sigma} = \sigma^2 \lambda_{n-1}$. This proves Λ -boundedness for $k = n-1$.

Now assume that we have Λ -boundedness for $j \geq k+1$ in $B_{3/4}(0)$. Thus it follows that $\int_{\Sigma_{k+1,\sigma} \cap B_{3/4}(0)} (1 + (u_{k+1,\sigma})^2) \rho_{k+2,\sigma} d\mu_{k+1}$ is bounded and hence $\int_{\Sigma_{k+1,\sigma} \cap B_{3/4}(0)} \rho_{k+1,\sigma} d\mu_{k+1}$ is bounded. We may then use the coarea

formula to find a radius $r \in (1/2, 3/4)$ so that

$$\int_{\Sigma_{k+1,\sigma} \cap \partial B_r(0)} \rho_{k+1,\sigma} d\mu_k \leq \Lambda.$$

Using the portion of $\Sigma_{k+1,\sigma} \cap \partial B_r(0)$ lying outside $O_{k,\sigma}$ as a comparison surface we find

$$\text{Vol}_{\rho_{k+1,\sigma}}(\Sigma_{k,\sigma} \cap B_{1/2}(0)) \leq \text{Vol}_{\rho_{k+1,\sigma}}(\Sigma_{k+1,\sigma} \cap \partial B_r(0)) \leq \Lambda.$$

Finally we prove the bound

$$\int_{\Sigma_{k,\sigma} \cap B_{1/2}(0)} \left(|A_{k,\sigma}|^2 + \sum_{p=k+1}^n |\nabla_{k,\sigma} \log u_{p,\sigma}|^2 \right) u_{k,\sigma}^2 \rho_{k+1,\sigma} d\mu_k \leq \Lambda$$

by the use of stability as we did above for the case $k = n - 1$. \square

We will now formulate and prove a compactness theorem for minimal k -slicings under the assumption that the associated $(k+1)$ -slicings for the sequence are partially regular. We will say that a Λ -bounded sequence of k -slicings $(\Sigma_j^{(i)}, u_j^{(i)})$, $j = k, \dots, n-1$ converges to a minimal k -slicing (Σ_j, u_j) in an open set U if $\Sigma_j^{(i)}$ converges in C^2 norm to Σ_j in \bar{U} locally on the complement of the singular set (of the limit) \mathcal{S}_j , and such that for $j = k, \dots, n-1$

$$(3.3) \quad \lim_{i \rightarrow \infty} V_{\rho_{j+1}}^{(i)}(\Sigma_j^{(i)} \cap U_i) = V_{\rho_{j+1}}(\Sigma_j \cap U),$$

$$(3.4) \quad \begin{aligned} \lim_{i \rightarrow \infty} \|u_j^{(i)}\|_{0,j,U_i}^2 &= \|u_j\|_{0,j,U}^2 \\ \lim_{i \rightarrow \infty} \int_{\Sigma_j^{(i)} \cap U_i} (|\nabla_j u_j^{(i)}|^2 + P_j^{(i)}(u_j^{(i)})^2) \rho_{j+1}^{(i)} d\mu_j \\ &= \int_{\Sigma_j \cap U} (|\nabla_j u_j|^2 + P_j u_j^2) \rho_{j+1} d\mu_j \end{aligned}$$

where U_i is a sequence of compact subdomains of U with $U_i \subseteq U_{i+1} \subseteq U$ and $U = \cup_i U_i$.

To make precise the meaning of convergence on compact subsets for this problem involves some subtlety since changing the u_p , $p \geq j+1$ by multiplication by a positive constant has no effect on the Σ_j , so in order to get nontrivial limits for the u_p we must normalize them appropriately. In case $\Sigma_j \cap U$ has multiple components this normalization must be done on each component. If (Σ_j, u_j) is a minimal k -slicing with Σ_j being partially regular for $j \geq k+1$, then we call a compact subdomain U of Ω *admissible* for (Σ_j, u_j) if U is a smooth domain which meets $\partial \Sigma_j$ transversally and $\dim(\partial U \cap \mathcal{S}_j) \leq j-3$. It follows from the coarea formula that any smooth domain can be perturbed to be admissible. We make the following definition.

DEFINITION 3.1. We say that a sequence of k -slicings $(\Sigma_j^{(i)}, u_j^{(i)})$ converges on compact subsets to a k -slicing (Σ_j, u_j) if for any compact subdomain U of Ω which is admissible for (Σ_j, u_j) and for any admissible domains U_i for $(\Sigma_j^{(i)}, u_j^{(i)})$ with $U_i \subseteq U_{i+1} \subseteq U$ compactly contained in U it is true that each connected component of $\Sigma_j \cap \mathcal{R}_{j+1} \cap U$ is a limit of connected components of $\Sigma_j^{(i)} \cap \mathcal{R}_{j+1}^{(i)} \cap U_i$ in the sense of (3.3) and (3.4) with u_j appropriately normalized on each connected component.

REMARK 3.1. Because of the connectedness of the regular set and the Harnack inequality, we may normalize the u_j to be equal to 1 at a point of $x_0 \in \mathcal{R}_k$ about which we have a uniform ball on which the Σ_j have bounded curvature, and this normalization suffices for the connected component of $\Sigma_k \cap U$ for any compact admissible domain for (Σ_j, u_j) . A consequence of the compactness theorem below implies that this normalization suffices.

The following compactness and regularity theorem includes Theorem 2.3 as a special case.

THEOREM 3.5. Assume that all bounded minimal $(k+1)$ -slicings are partially regular. Given a Λ -bounded sequence of k -slicings, there is a subsequence which converges to a Λ -bounded k -slicing on compact open subsets of Ω . Furthermore Σ_k is partially regular.

PROOF. We will proceed as usual by downward induction beginning with $k = n - 1$. We will break the proof into two separate steps, the first establishing the first statement of (3.3) for convergence of the Σ_k and the second showing the other two statements (3.4) involving convergence of the u_k . For $k = n - 1$ the first step follows from the usual compactness theorem for volume minimizing hypersurfaces (see [Si]). To complete the proof we will need to develop some monotonicity ideas both for the Σ_j and for the u_j . We digress on this topic and return to the proof below.

We now prove a version of the monotonicity of the frequency-type function. This idea is due to F. Almgren [A], and it gives a method to prove that solutions of variationally defined elliptic equations are approximately homogeneous on a small scale. The importance of this method for us is that it works in the presence of singularities provided certain integrals are defined. We will apply this to show that the u_k become homogeneous upon rescaling at a given singular point. Assume that C is a k dimensional cone in \mathbb{R}^n which is regular except for a set \mathcal{S} with $\dim(\mathcal{S}) \leq k - 3$. Assume that Q is a quadratic form on C of the form

$$Q(\varphi, \varphi) = \int_C (|\nabla \varphi|^2 - q(x)\varphi^2) \rho \, d\mu$$

where ρ is a homogeneous weight function on C of degree p ; i.e. assume that $\rho(\lambda x) = \lambda^p \rho(x)$ for $x \in C$ and $\lambda > 0$. Assume also that ρ is smooth and positive on the regular set \mathcal{R} of C and that ρ is locally L^1 on C . Assume also that q is smooth on \mathcal{R} and is homogeneous of degree -2 ; i.e. assume

that $q(\lambda x) = \lambda^{-2}q(x)$ for $x \in C$ and $\lambda > 0$. Finally assume that u is a minimizer for Q in a neighborhood of 0 and in particular that u is smooth and positive on \mathcal{R} . Assume also that $q = \operatorname{div}(\mathcal{X}) + \bar{q}$ where $|\mathcal{X}|^2 + |\bar{q}| \leq P$ for some positive function P and that the following integral bound holds

$$\int_C [|\nabla u|^2 + (1 + |\nabla \log \rho|^2 + P)u^2] \rho \, d\mu < \infty.$$

Under these conditions we may define the frequency function $N(\sigma)$ which is a function of a radius $\sigma > 0$ such that $B_\sigma(0)$ is contained in the domain of definition of u . It is defined by

$$(3.5) \quad N(\sigma) = \frac{\sigma Q_\sigma(u)}{I_\sigma(u)}$$

where $Q_\sigma(u)$ and $I_\sigma(u)$ are defined by

$$Q_\sigma(u) = \int_{C \cap B_\sigma(0)} (|\nabla u|^2 - q(x)u^2) \rho \, d\mu_k, \quad I_\sigma(u) = \int_{C \cap \partial B_\sigma(0)} u^2 \rho \, d\mu_{k-1}$$

where the last integral is taken with respect to $k-1$ dimensional Hausdorff measure. We may now prove the following monotonicity result for $N(\sigma)$.

THEOREM 3.6. *Assume that u is a critical point of Q which is integrable as above. The function $N(\sigma)$ is monotone increasing in σ , and for almost all σ we have*

$$N'(\sigma) = \frac{2\sigma}{I_\sigma(u)} (I_\sigma(u_r)I_\sigma(u) - \langle u_r, u \rangle_\sigma^2)$$

where u_r denotes the radial derivative of u and $\langle \cdot, \cdot \rangle_\sigma$ denotes the ρ -weighted L^2 inner product taken on $C \cap \partial B_\sigma(0)$. The limit of $N(\sigma)$ as σ goes to 0 exists and is finite. The function $N(\sigma)$ is equal to a constant $N(0)$ if and only if u is homogeneous of degree $N(0)$.

PROOF. The argument can be done variationally and combines two distinct deformations of the function u . The first involves a radial deformation of C ; precisely, let $\zeta(r)$ be a function which is nonnegative, decreasing, and has support in $B_\sigma(0)$. Let X denote the vector field on \mathbb{R}^n given by $X = \zeta(r)x$ where x denotes the position vector. The flow F_t of X then preserves C , and we may write

$$Q_\sigma(u \circ F_t) = \int_{C \cap B_\sigma(0)} (|\nabla_t u|^2 - (q \circ F_t)u^2) \rho \circ F_t \, d\mu_t$$

where we have used a change of variable and ∇_t and μ_t denotes the gradient operator and volume measure with respect to $F_t^*(g)$ where g is the induced metric on C from \mathbb{R}^n . Differentiating with respect to t and setting $t = 0$ we obtain

$$0 = \int_C \{ \langle -\mathcal{L}_X g, du \otimes du \rangle - X(q)u^2 \} \rho + (|\nabla u|^2 - qu^2) (X(\rho) + \rho \operatorname{div}(X)) \} \, d\mu$$

where \mathcal{L} denotes the Lie derivative. By direct calculation we have $X(q) = -2\zeta q$, $X(\rho) = p\zeta\rho$, $\operatorname{div}(X) = r\zeta'(r) + k\zeta$, and $\mathcal{L}_X g = 2r\zeta'(r)(dr \otimes dr) + 2\zeta g$. Substituting in this information and collecting terms we have

$$0 = \int_C \{ (p+k-2)\zeta(|\nabla u|^2 - qu^2) + r\zeta'(|\nabla u|^2 - 2u_r^2 - qu^2) \} \rho \, d\mu.$$

Letting ζ approach the characteristic function of $B_\sigma(0)$ this implies

$$\begin{aligned} (p+k-2)Q_\sigma(u) &= \sigma \int_{C \cap \partial B_\sigma(0)} (|\nabla u|^2 - 2u_r^2 - qu^2) \rho \, d\mu_{k-1} \\ &= \sigma \frac{dQ_\sigma(u)}{d\sigma} - 2\sigma \int_{C \cap \partial B_\sigma(0)} u_r^2 \rho \, d\mu_{k-1}. \end{aligned}$$

The second ingredient we need comes from the deformation $u_t = (1 + t\zeta(r))u$ where ζ is as above. Since $\dot{u} = \zeta u$ this deformation implies

$$0 = \int_C (\langle \nabla u, \nabla(\zeta u) \rangle - q\zeta u^2) \rho \, d\mu.$$

Expanding this and letting ζ approach the characteristic function of $B_\sigma(0)$ we have

$$Q_\sigma(u) = \int_{C \cap \partial B_\sigma(0)} uu_r \rho \, d\mu_{k-1}.$$

The proof will now follow by combining these. First we have

$$N'(\sigma) = I_\sigma(u)^{-2} \{ (Q_\sigma + \sigma Q'_\sigma) I_\sigma - \sigma Q_\sigma I'_\sigma \}.$$

Substituting in for the terms involving derivatives this implies

$$\begin{aligned} N'(\sigma) &= I_\sigma^{-2} \{ (Q_\sigma + (p+k-2)Q_\sigma) I_\sigma - Q_\sigma(p+k-1)I_\sigma \} \\ &\quad + 2\sigma I_\sigma^{-2} \left\{ \int_{C \cap \partial B_\sigma(0)} u_r^2 \rho \, d\mu_{k-1} - Q_\sigma^2 I_\sigma \right\}. \end{aligned}$$

Since the first term on the right is 0, we may write this as

$$N'(\sigma) = 2I_\sigma(u)^{-1} (I_\sigma(u)I_\sigma(u_r) - \langle u_r, u \rangle_\sigma^2)$$

which is the desired formula.

To see that $N(\sigma)$ is bounded from below as σ goes to 0 we can observe that

$$N(\sigma) = \frac{1}{2}\sigma \frac{d}{d\sigma} \log(\bar{I}_\sigma(u)), \quad \bar{I}_\sigma(u) = \frac{\int_{C \cap \partial B_\sigma(0)} u^2 \rho \, d\mu_{k-1}}{\int_{C \cap \partial B_\sigma(0)} \rho \, d\mu_{k-1}},$$

and the monotonicity expresses the condition that the function $\log \bar{I}_\sigma(u)$ is a convex function of $t = \log \sigma$. Since this function is defined for all $t \leq 0$, and by the coarea formula for any $\sigma_1 > 0$, there is a $\sigma \in [\sigma_1, 2\sigma_1]$ so that $I_\sigma(u) \leq c\sigma^{-1}$ it follows that there is a sequence $t_i = \log \sigma_i$ tending to $-\infty$ such that $\bar{I}_{\sigma_i}(u) \leq c\sigma_i^{-K}$ for some $K > 0$. Thus we have the function $\log \bar{I}_{\sigma_i}(u) \leq -ct_i$. It follows that the slope (that is $N(\sigma)$) of the convex function $\log \bar{I}_\sigma(u)$ is bounded from below as t tends to $-\infty$.

Now if $N(\sigma) = N(0)$ is constant, we must have equality in the Schwartz inequality for each σ , and hence we would have $u_r = f(r)u$ for some function $f(r)$. Now this implies that $Q_\sigma = f(\sigma)I_\sigma$ and hence we have $rf(r) = N(0)$. Therefore it follows that $f(r) = r^{-1}N(0)$, and $ru_r = N(0)u$ so u is homogeneous of degree $N(0)$ by Euler's formula. \square

We will need to extend the usual monotonicity formula for the volume of minimal submanifolds to the setting in which the submanifold under consideration minimizes a weighted volume with a homogeneous weight function within a partially regular cone. Precisely, let C be a $k+1$ dimensional cone in \mathbb{R}^n with a singular set \mathcal{S} of Hausdorff dimension at most $k-2$. Let ρ be a positive weight function which is homogeneous of degree p ; i.e. we have $\rho(\lambda x) = \lambda^p \rho(x)$ for $x \in C$ and $\lambda > 0$. Assume that ρ is smooth and positive on the regular set of C , and that ρ is locally integrable with respect to Hausdorff measure on C .

THEOREM 3.7. *Let Σ be a hypersurface in a $k+1$ dimensional cone C which minimizes the weighted volume V_ρ for a homogeneous weight function ρ . We then have the monotonicity formula*

$$\frac{d}{d\sigma}(\sigma^{-k-p} \text{Vol}_\rho(\Sigma \cap B_\sigma(0))) = \int_{\Sigma \cap \partial B_\sigma(0)} r^{-p-k-2} |x^\perp|^2 \rho \, d\mu_{k-1}$$

where x^\perp denotes the component of the position vector x perpendicular to Σ .

PROOF. We take a function $\zeta(r)$ which is decreasing, nonnegative, and equal to 0 for $r > \sigma$, and we consider the vector field $X = \zeta x$ where x denotes the position vector. The first variation formula for the ρ -weighted volume then implies

$$0 = \int_{\Sigma} (X(\rho) + \text{div}_\Sigma(X)\rho) \, d\mu_k.$$

Since ρ is homogeneous we have $X(\rho) = p\zeta\rho$, and by direct calculation $\text{div}_\Sigma(X) = k\zeta + r^{-1}\zeta'|x^T|^2$ where x^T denotes the component of x tangential to Σ . Thus we have

$$0 = \int_{\Sigma} \{(p+k)\zeta + r^{-1}\zeta'|x^T|^2\} \rho \, d\mu_k$$

Taking ζ to approximate the characteristic function of $B_\sigma(0)$ we may write this

$$(p+k) \text{Vol}_\rho(\Sigma \cap B_\sigma(0)) = \sigma \frac{d}{d\sigma} \text{Vol}_\rho(\Sigma \cap B_\sigma(0)) - \int_{\Sigma \cap \partial B_\sigma(0)} r^{-1} |x^\perp|^2 \rho \, d\mu_{k-1}$$

where x^\perp is the component of x normal to Σ in C . Note that $r^2 = |x^T|^2 + |x^\perp|^2$ because C is a cone and so x is tangential to C . This may be rewritten as the desired monotonicity formula and completes the proof. \square

We now show that there can be no tangent minimal 2-slicing with C_2 having an isolated singularity at $\{0\}$.

THEOREM 3.8. *If C_2 is a cone lying in a tangent minimal 2-slicing such that $C_2 \sim \{0\} \subseteq \mathcal{R}_2$, then C_2 is a plane and $\mathcal{R}_2 = C_2$.*

PROOF. From the eigenvalue estimate of Theorem 2.6 we have

$$\int_{C_2} \left(\frac{3}{4} \sum_{j=3}^n |\nabla_2 \log u_j|^2 - R_2 \right) \varphi^2 d\mu_2 \leq 4 \int_{C_2} |\nabla_2 \varphi|^2 d\mu_2$$

for test functions φ with compact support in $C_2 \sim \{0\}$. Since C_2 is a two dimensional cone we have $R_2 = 0$ away from the origin, and hence we have

$$\int_{C_2} \sum_{j=3}^n |\nabla_2 \log u_j|^2 \varphi^2 d\mu_2 \leq c \int_{C_2} |\nabla_2 \varphi|^2 d\mu_2.$$

Letting r denote the distance to the origin, we take ε and R so that $0 < \varepsilon \ll R$ and choose φ to be a function of r which is equal to 0 for $r \leq \varepsilon^2$, equal to 1 for $\varepsilon \leq r \leq R$, and equal to 0 for $r \geq R^2$. In the range $\varepsilon^2 \leq r \leq \varepsilon$ we choose

$$\varphi(r) = \frac{\log(\varepsilon^{-2}r)}{\log(\varepsilon^{-1})}$$

and for $R \leq r \leq R^2$

$$\varphi(r) = \frac{\log(R^2r^{-1})}{\log R}.$$

Thus for $\varepsilon^2 \leq r \leq \varepsilon$ we have $|\nabla_2 \varphi|^2 = (r|\log \varepsilon|)^{-2}$ and for $R \leq r \leq R^2$ we have $|\nabla_2 \varphi|^2 = (r \log R)^{-2}$. It thus follows that

$$\int_{C_2} |\nabla_2 \varphi|^2 d\mu_2 \leq c(|\log \varepsilon|^{-1} + (\log R)^{-1}).$$

Thus we may let ε tend to 0 and R tend to ∞ to conclude that the functions u_3, \dots, u_n are constant on C_2 . This implies that C_2 has zero mean curvature and hence is a plane. If all of the cones C_3, \dots, C_{n-1} are regular near the origin, then it follows that $0 \in \mathcal{R}_2$, and we have completed the proof. Otherwise there is a C_m for $m \geq 3$ which denotes the largest dimensional cone in the minimal 2-slicing for which the origin is a singular point. It follows that C_m is a volume minimizing cone in $\mathbb{R}^{m+1} = C_{m+1}$, and hence u_m must be homogeneous of a negative degree (see Lemma 3.10 below) contradicting the fact that u_m is constant along C_2 . This completes the proof. \square

Completion of proof of Theorem 3.5: We first prove the compactness of the Σ_k in the sense of (3.3) under the assumption that we have the partial regularity of bounded minimal $(k+1)$ -slicings and the compactness (both (3.3) and (3.4)) for $j \geq k+1$. We need the following lemma.

LEMMA 3.9. *Assume that both the compactness and partial regularity hold for $(k+1)$ -slicings. Given any $x \in \mathcal{S}_{k+1}$, there are constants c and r_0 (depending on x and Σ_{k+1}) so that for $r \in (0, r_0]$ we have*

$$\int_{\Sigma_{k+1} \cap B_{2r}(x)} u_{k+1}^2 \rho_{k+2} d\mu_{k+1} \leq cr^2 \int_{\Sigma_{k+1} \cap B_r(x)} P_{k+1} u_{k+1}^2 \rho_{k+2} d\mu_{k+1},$$

and

$$Vol_{\rho_{k+2}}(\Sigma_{k+1} \cap B_{2r}(x)) \leq c Vol_{\rho_{k+2}}(\Sigma_{k+1} \cap B_r(x)).$$

PROOF. Since the left hand side of the inequality is continuous under convergence and the right hand side is lower semicontinuous (Fatou's theorem) it is enough to establish the inequality for $r = 1$ on a cone C_{k+1} . This we can do by a compactness argument since we can normalize

$$\int_{C_{k+1} \cap B_1(0)} u_{k+1}^2 \rho_{k+2} d\mu_{k+1} = 1$$

and if we had a sequence of singular cones for which the right hand side tends to zero we would have a limiting cone C_{k+1} on which $P_{k+1} = 0$. It follows that u_{k+2}, \dots, u_{n-1} are constant on C_{k+1} . Note that the highest dimensional *singular* cone in the slicing C_{n_0} is minimal and hence u_{n_0} is homogeneous of a negative degree (see Lemma 3.10 below). Therefore if $n_0 > k + 1$ we have a contradiction. Therefore we conclude that C_{k+1} is minimal and C_{k+2}, \dots, C_{n-1} are planes. Thus it follows that $\tilde{A}_{k+1} = A_{k+1} = 0$ and hence C_{k+1} is also a plane. Thus the cones are regular sufficiently far out in the sequence; a contradiction. The second inequality follows easily by reduction to cones. This proves the bounds. \square

Given a sequence $(\Sigma_j^{(i)}, u_j^{(i)})$ of Λ -bounded minimal k - slicings, we may apply the inductive assumption to obtain a subsequence (with the same notation) for which the corresponding sequence of $(k + 1)$ -slicings converges in the sense of (3.3) and (3.4). By standard compactness theorems we may assume that $\Sigma_k^{(i)}$ converges on compact subsets of $\Omega \sim \mathcal{S}_{k+1}$ to a limiting submanifold Σ_k which minimizes Vol_{ρ_k} (and is therefore regular outside a closed set of dimension at most $k - 7$). To establish (3.3) we choose a neighborhood U of \mathcal{S}_{k+1} such that

$$Vol_{\rho_{k+2}}(\Sigma_{k+1} \cap \bar{U}) < \varepsilon.$$

We apply Lemma 3.9 and compactness to find a finite collection of points $x_\alpha \in \mathcal{S}_{k+1}$ and balls $B_{r_\alpha}(x_\alpha) \subset U$ so that

$$\int_{\Sigma_{k+1} \cap B_{2r_\alpha}(x_\alpha)} u_{k+1}^2 \rho_{k+2} d\mu_{k+1} < cr_\alpha^2 \int_{\Sigma_{k+1} \cap B_{r_\alpha}(x_\alpha)} P_{k+1} u_{k+1}^2 \rho_{k+2} d\mu_{k+1}$$

and

$$Vol_{\rho_{k+2}}(\Sigma_{k+1} \cap B_{2r_\alpha}(x_\alpha)) < c Vol_{\rho_{k+2}}(\Sigma_{k+1} \cap B_{r_\alpha}(x_\alpha)).$$

Now apply the Besicovitch covering lemma to extract a finite number of disjoint collections \mathcal{B}_α , $\alpha = 1, \dots, K$ of such balls whose union covers \mathcal{S}_{k+1} . If V denotes the union of these balls, then V is a neighborhood of \mathcal{S}_{k+1} , and hence for i sufficiently large we have $\mathcal{S}_{k+1}^{(i)} \subset V$. Because of convergence of the left sides and lower semicontinuity of the right side, we have for i

sufficiently large

$$\begin{aligned} & \int_{\Sigma_{k+1}^{(i)} \cap B_{2r_\alpha}(x_\alpha)} (u_{k+1}^{(i)})^2 \rho_{k+2}^{(i)} d\mu_{k+1} \\ & < cr_\alpha^2 \int_{\Sigma_{k+1}^{(i)} \cap B_{r_\alpha}(x_\alpha)} P_{k+1}^{(i)} (u_{k+1}^{(i)})^2 \rho_{k+2}^{(i)} d\mu_{k+1} \end{aligned}$$

and

$$Vol_{\rho_{k+2}}^{(i)}(\Sigma_{k+1}^{(i)} \cap B_{2r_\alpha}(x_\alpha)) < c Vol_{\rho_{k+2}}^{(i)}(\Sigma_{k+1}^{(i)} \cap B_{r_\alpha}(x_\alpha)).$$

By the coarea formula, for each such ball $B_{r_0}(x)$ we may find $s \in [r_0, 2r_0]$ (s depending on i) so that

$$Vol_{\rho_{k+1}}^{(i)}(\Sigma_{k+1}^{(i)} \cap \partial B_s(x)) \leq 2r_0^{-1} \int_{\Sigma_{k+1}^{(i)} \cap B_{2r_0}(x)} u_{k+1}^{(i)} \rho_{k+2}^{(i)} d\mu_{k+1}.$$

Using the minimizing property of $\Sigma_k^{(i)}$ and simple inequalities we find

$$\begin{aligned} Vol_{\rho_{k+1}}^{(i)}(\Sigma_k^{(i)} \cap B_{r_0}) & \leq \varepsilon_1^{-1} \int_{\Sigma_{k+1}^{(i)} \cap B_{2r_0}(x)} \rho_{k+2}^{(i)} d\mu_{k+1} \\ & + \varepsilon_1 r_0^{-2} \int_{\Sigma_{k+1}^{(i)} \cap B_{2r_0}(x)} (u_{k+1}^{(i)})^2 \rho_{k+2}^{(i)} d\mu_{k+1} \end{aligned}$$

for any $\varepsilon_1 > 0$. Applying the inequalities above and summing over the balls (using disjointness and a bound on K) we find

$$\begin{aligned} & Vol_{\rho_{k+1}}^{(i)}(\Sigma_k^{(i)} \cap V) \\ & \leq c\varepsilon_1^{-1} Vol_{\rho_{k+2}}^{(i)}(\Sigma_{k+1}^{(i)} \cap \bar{U}) + c\varepsilon_1 \int_{\Sigma_{k+1}^{(i)}} P_{k+1}^{(i)} (u_{k+1}^{(i)})^2 \rho_{k+2}^{(i)} d\mu_{k+1}. \end{aligned}$$

For i sufficiently large this implies

$$Vol_{\rho_{k+1}}^{(i)}(\Sigma_k^{(i)} \cap V) \leq c\varepsilon_1^{-1}\varepsilon + c\varepsilon_1,$$

so that we may fix ε_1 sufficiently small and then choose ε as small as we wish to make the right hand side smaller than any preassigned amount. Since we have

$$\lim_{i \rightarrow \infty} Vol_{\rho_{k+1}}^{(i)}(\Sigma_k^{(i)} \sim V) = Vol_{\rho_{k+1}}(\Sigma_k \sim V),$$

we can conclude that $\lim_{i \rightarrow \infty} Vol_{\rho_{k+1}}^{(i)}(\Sigma_k^{(i)}) = Vol_{\rho_{k+1}}(\Sigma_k)$ establishing (3.3).

Now assume that we have established the partial regularity of all bounded minimal $(k+1)$ -slicings and that we have proven the compactness for the Σ_k in the sense of (3.3). We can then use the results we have obtained above together with dimension reduction to prove partial regularity for Σ_k . Precisely, we have $\dim(\mathcal{S}_k) \leq k-2$, and if $\dim(\mathcal{S}_k) > k-3$, then we can choose a number d with

$$k-3 < d < \dim(\mathcal{S}_k),$$

and go to a point $x \in \mathcal{S}_k$ of density for the measure \mathcal{H}_∞^d (since $\mathcal{H}_\infty^d(\mathcal{S}_k) > 0$). Taking successive tangent cones in the standard way and using the upper-semicontinuity of $\mathcal{H}_\infty^d(\mathcal{S}_k)$ we would eventually produce a minimal 2-slicing by cones such that $C_2 \times \mathbb{R}^{k-2}$ has singular set with Hausdorff dimension at most $k-2$ (by partial regularity of $(k+1)$ -slicings) and greater than $k-3$. Therefore the cone C_2 must have an isolated singularity at the origin. This in turn contradicts Theorem 3.8. Therefore it follows that $\dim(\mathcal{S}_k) \leq k-3$ and Σ_k is partially regular.

The final step of the proof is to show that the compactness statement holds for the u_k under the assumption that it holds for (Σ_j, u_j) for $j \geq k+1$ and also for Σ_k (as established above). Assume that we have a sequence of minimal k -slicings such that the associated $(k+1)$ -slicings and $\Sigma_k^{(i)}$ converge on compact subsets in the sense of (3.3) and (3.4). We choose a compact domain U which is admissible for (Σ_j, U_j) and a nested sequence of domains U_i admissible for $(\Sigma_j^{(i)}, u_j^{(i)})$. We work with a connected component of $\Sigma_k \cap U$ which by abuse of notation we call by the same name Σ_k .

We may assume that the $u_k^{(i)}$ converge uniformly to u_k on compact subsets of $\Omega \sim \mathcal{S}_k$ (where we can write $\Sigma_k^{(i)}$ locally as a normal graph over Σ_k and compare corresponding values of $u_k^{(i)}$ to u_k). In particular, if W is a compact subdomain of $\Omega \cap \mathcal{R}_k$ we have convergence of weighted L^2 norms of $u_k^{(i)}$ to the corresponding L^2 norm of u_k on W . If U is any compact subdomain of Ω and $\eta > 0$, then by Proposition 3.1 applied with $\mathcal{S} = \mathcal{S}_k$ we can find an open neighborhood V of $\mathcal{S} \cap \bar{U}$ so that for i sufficiently large $\mathcal{S}_k^{(i)} \cap \bar{U} \subset V$, and

$$\int_{\Sigma_k^{(i)} \cap V} (u_k^{(i)})^2 \rho_{k+1}^{(i)} d\mu_k \leq \eta \int_{\Sigma_k^{(i)} \cap \Omega} [|\nabla_k u_k^{(i)}|^2 + (1 + P_k^{(i)})(u_k^{(i)})^2] \rho_{k+1}^{(i)} d\mu_k.$$

The same inequality holds for the limit, and by the boundedness of the sequence the integral on the right is uniformly bounded. Thus by choosing η small enough we can make the right hand side less than any prescribed $\varepsilon > 0$. On the other hand if we take $W = U \setminus \bar{V}$ we then have convergence of the weighted L^2 norms on W , so we can make the difference as small as we wish on W . It follows that the difference of L^2 norms can be made arbitrarily small on U . This completes the proof that the weighted L^2 integrals converge.

Completing the proof will require the construction of a proper locally Lipschitz function Ψ_k on \mathcal{R}_k such that $u_k |\nabla_k \Psi_k|$ is bounded in $L^2(\Sigma_k)$. We give the construction of such a function in Proposition 3.11 below. It also follows that we may construct a subsequence so that $\Psi_k^{(i)}$ are uniformly close to Ψ_k on compact subsets of $\mathbb{R}^N \sim \mathcal{S}_k$ for i large. We can now prove the second part of the convergence (3.4). Assume that $U \subset U_1 \subset \Omega$ are compact domains. We let $\varepsilon > 0$ we may choose a neighborhood V of \mathcal{S}_k so small that $\int_{V \cap \bar{U}_1} u_k^2 \rho_{k+1} d\mu_k < \varepsilon$. Because Ψ_k is proper on \mathcal{R}_k , we may choose Λ sufficiently large that $E_k(\Lambda) \subset V$ where $E_k(\Lambda)$ is the subset of Σ_k on which

$\Psi_k > \Lambda$. We now let $\gamma(t)$ be a nondecreasing Lipschitz function such that $\gamma(t) = 0$ for $t < \Lambda$, $\gamma(t) = 1$ for $t > \Lambda$, and $\gamma'(t) \leq \Lambda^{-1}$. We let φ be a spatial cutoff function which is 1 on U , 0 outside U_1 , and has bounded gradient. We then have the inequality by Proposition 3.2

$$\int_{\Sigma_k^{(i)}} (|\nabla_k \psi_k^{(i)}|^2 + P_k^{(i)}(\psi_k^{(i)})^2) \rho_k^{(i)} d\mu_j \leq cQ_k(\psi_k^{(i)}, \psi_k^{(i)})$$

where $\psi_k^{(i)} = \varphi(\gamma \circ \Psi_k^{(i)})u_k^{(i)}$. Since the support of $\psi_k^{(i)}$ is contained in V for i sufficiently large we then have

$$\begin{aligned} & \int_{\Sigma_k^{(i)}} (|\nabla_k \psi_k^{(i)}|^2 + P_k^{(i)}(\psi_k^{(i)})^2) \rho_{k+1}^{(i)} d\mu_j \\ & \leq c \int_{\Sigma_k^{(i)} \cap V} (1 + \Lambda^{-2} |\nabla_k \Psi_k^{(i)}|^2) (u_k^{(i)})^2 \rho_{k+1}^{(i)} d\mu_k. \end{aligned}$$

Since we have convergence of the L^2 norms of $u_k^{(i)}$ and boundedness of the L^2 norms of $u_k^{(i)} |\nabla_k \Psi_k^{(i)}|$, we then conclude that

$$\int_{\Sigma_k^{(i)}} (|\nabla_k \psi_k^{(i)}|^2 + P_k^{(i)}(\psi_k^{(i)})^2) \rho_{k+1}^{(i)} d\mu_j \leq c\varepsilon + c\Lambda^{-2}.$$

If we let V_1 be a neighborhood of \mathcal{S}_k such that $\Sigma_k \cap V_1 \subset E_k(3\Lambda)$, then for i sufficiently large we will have $\Sigma_k^{(i)} \cap V_1 \subset E_k^{(i)}(2\Lambda)$ and hence

$$\int_{\Sigma_k^{(i)} \cap V_1} (|\nabla_k u_k^{(i)}|^2 + P_k^{(i)}(u_k^{(i)})^2) \rho_{k+1}^{(i)} d\mu_j \leq c\varepsilon + c\Lambda^{-2}.$$

Since this can be made arbitrarily small, we have shown (3.4) and completed the proof of Theorem 3.5. \square

We will need the following lemma concerning minimal cones $C_m \subset \mathbb{R}^{m+1}$.

LEMMA 3.10. *Assume that C_m is a volume minimizing cone in \mathbb{R}^{m+1} and that u_m is a positive minimizer for Q_m which is homogeneous of degree d on C . There is a positive constant c depending only on m so that $d \leq -c$.*

PROOF. We first observe that

$$\begin{aligned} Q_m(\varphi, \varphi) &= S_m(\varphi, \varphi) + \frac{3}{8} \int_{C_m} |A_m|^2 \varphi^2 d\mu_m \\ &= \int_{C_m} \left(|\nabla_m \varphi|^2 - \frac{5}{8} |A_m|^2 \varphi^2 \right) d\mu_m. \end{aligned}$$

We write $u_m = r^d v(\xi)$ where $\xi \in S^m$, and we observe that the equation for u_m evaluated at $r = 1$ becomes

$$0 = \Delta_m u_m + \frac{5}{8} u_m = \Delta v + \frac{5}{8} |A_m|^2 v + d(d + m - 2)v$$

where we let $\Sigma = C \cap S^m$ and Δ the Laplace operator on Σ . Thus v satisfies the eigenvalue equation $\Delta v + 5/8 |A_m|^2 v = -\mu v$ where $d(d + m - 1) = \mu$.

This implies that $d = 1/2(1 - m + \sqrt{(m-1)^2 + 4\mu})$ or $d = 1/2(1 - m - \sqrt{(m-1)^2 + 4\mu})$. Since v and $|\nabla v|$ are in $L^2(\Sigma)$ we must have $\mu < 0$ and this implies that $d < 0$. To prove the negative upper bound on d recall that the set of volume minimizing cones is a compact set, and we have proven the compactness theorem above for the L^2 norms, so if we had a sequence $(C_m^{(i)}, u_m^{(i)})$ such that $d^{(i)}$ tends to 0 we could extract a convergent subsequence of the $(\Sigma^{(i)}, v^{(i)})$ which converges to (Σ, v) where we could normalize $\int_{\Sigma^{(i)}} (v^{(i)})^2 d\mu_{m-1} = 1$ (hence $\int_{\Sigma} v^2 d\mu_{m-1} = 1$). Since we have smooth convergence on compact subsets of the complement of the singular set of Σ we would then have $\Delta v + 5/8|A_m|^2 v = 0$ and therefore we would have $\mu = 0$ for the limiting cone, a contradiction. \square

As the final topic of this section we construct the proper functions which were used in the proof of Theorem 3.5. This result will also be used in the next section.

PROPOSITION 3.11. *Suppose we have a Λ -bounded minimal k -slicing in Ω . There exists a positive function Ψ_k which is locally Lipschitz on \mathcal{R}_k and such that for any domain U compactly contained in Ω , the function Ψ_k is proper on $\mathcal{R}_k \cap \bar{U}$. Moreover, the function $u_k |\nabla_k \Psi_k|$ is bounded in $L^2(\Sigma_k \cap U)$ for any domain U compactly contained in Ω .*

PROOF. We define $\Psi_k = \max\{1, \log u_k, \log u_{k+1}, \dots, \log u_{n-1}\}$ and we show that it has the properties claimed. First note that Ψ_k is locally Lipschitz on \mathcal{R}_k since it is the maximum of a finite number of smooth functions on \mathcal{R}_k . The bound

$$\int_{\Sigma_k \cap U} (u_k |\nabla_k \Psi_k|)^2 \rho_{k+1} d\mu_k \leq \max_{k \leq j \leq n-1} \int_{\Sigma_k \cap U} (u_k |\nabla_k \log u_j|)^2 \rho_{k+1} d\mu_k$$

together with Proposition 3.2 implies the $L^2(\Sigma_k)$ bound claimed on Ψ_k . (Note that we may replace φ by φu_k in the first inequality of Proposition 3.2 where φ is a cutoff function which is equal to 1 on U .)

It remains to prove that Ψ_k is proper on $\mathcal{R}_k \cap \bar{U}$. Since \bar{U} is compact it suffices to show that for any $x_0 \in \mathcal{S}_k \cap \bar{U}$ we have

$$\lim_{x \rightarrow x_0} \Psi_k(x) = \infty.$$

If we let $m \geq k$ be the largest integer such that Σ_m is singular at x_0 , then there is an open neighborhood V of x_0 in which Σ_m is a volume minimizing hypersurface in a smooth Riemannian manifold. We will show that u_m tends to infinity at x_0 by first showing that this is true for any homogeneous approximation of u_m at x_0 . In order to construct homogeneous approximations we need to have the compactness theorem for this top dimensional case, but our proof of compactness used the result we are trying to prove, so we must find another argument for establishing (3.4) since (3.3) is a standard result for volume minimizing hypersurfaces in smooth manifolds. Our proof of the first part of (3.4) did not require the function Ψ_k , so we need only deal with

the second part. First recall that $\dim(\mathcal{S}_m) \leq m - 7$, so it follows from a standard result that given any $\varepsilon, \delta > 0$ and $a \in (0, 7)$ we can find a Lipschitz function ψ so that $\psi = 1$ in a neighborhood of \mathcal{S}_m , $\psi(x) = 0$ for points x with $\text{dist}(x, \mathcal{S}_m) \geq \delta$, and

$$\int_{\Sigma_m \cap V} |\nabla_m \psi|^a d\mu_m < \varepsilon^a.$$

We show that

$$\int_{\Sigma_m \cap V} |\nabla_m \psi|^2 u_m^2 d\mu_m \leq c\varepsilon^2.$$

If we can establish this inequality, then we can complete the proof of compactness for $k = m$ in the set V as in the proof of Theorem 3.5. To establish the inequality, we observe that the equation satisfied by u_m is of the form

$$\Delta_m u_m + 5/8 |A_m|^2 u_m + q u_m = 0$$

where q is a bounded function (since Σ_m is volume minimizing in a smooth manifold). On the other hand the stability implies that

$$\int_{\Sigma_m} |A_m|^2 \varphi^2 d\mu_m \leq \int_{\Sigma_m} (|\nabla \varphi|^2 + c\varphi^2) d\mu_m.$$

We may then replace φ by $u_m^{8/5} \varphi$ and use the equation for u_m to obtain

$$\int_{\Sigma_m} |\nabla_m(u_m)^{8/5}|^2 \varphi^2 d\mu_m \leq c \int_{\Sigma_m} u_m^{16/5} (|\nabla_m \varphi|^2 + \varphi^2) d\mu_m.$$

We may then apply the Sobolev inequality for minimal submanifolds to conclude that u_m satisfies

$$\int_{\Sigma_m \cap V} u_m^{\frac{16m}{5(m-2)}} d\mu_m \leq c.$$

We then apply the Hölder inequality to obtain

$$\int_{\Sigma_m \cap V} |\nabla_m \psi|^2 u_m^2 d\mu_m \leq \|\nabla_m \psi\|_{\frac{16m}{3m+10}}^2 \|u_m\|_{\frac{16m}{5(m-2)}}^2.$$

Setting $a = \frac{16m}{3m+10} < 7$ we have from above

$$\int_{\Sigma_m \cap V} |\nabla_m \psi|^2 u_m^2 d\mu_m \leq c\varepsilon^2$$

as desired.

Thus we have the compactness theorem for (Σ_m, u_m) in V and we can construct tangent cones to Σ_m at x_0 and homogeneous approximations to u_m at x_0 . By Lemma 3.10 any such homogeneous approximation v_m has strictly negative degree $d \leq -c$ on its cone C_m of definition. If we let $\mathcal{R}_m(C)$ denote the regular set of C , then it follows that for any $\mu > 1$, we have

$$\inf_{\mathcal{R}_m(C) \cap B_{\alpha\sigma}(0)} v_m \geq \mu \inf_{\mathcal{R}_m(C) \cap B_\sigma(0)} v_m$$

for a fixed constant $\alpha \in (0, 1)$ depending on μ , but independent of which cone and which homogeneous approximation we choose. Note that $\Delta_m u_m \leq c u_m$

and $\Delta_m v_m \leq 0$, so by the mean value inequality on volume minimizing hypersurfaces (see [BG]) we have

$$u_m(x) \geq cr^{-m} \int_{\Sigma_m \cap B_r(x)} u_m d\mu_m, \quad v_m(x) \geq cr^{-m} \int_{C_m \cap B_r(x)} v_m d\mu_m$$

for any r so that $B_r(x_0)$ is compactly contained in V . It follows that the essential infima of both u_m and v_m are positive on any compact subset. We now show that there exists $\alpha \in (0, 1)$ such that

$$\inf_{\mathcal{R}_m \cap B_{\alpha\sigma}(x_0)} u_m \geq 2 \inf_{\mathcal{R}_m \cap B_\sigma(x_0)} u_m$$

for σ sufficiently small. If we establish this, we have finished the proof that u_m tends to infinity at x_0 and hence we will have the desired properness conclusion for Ψ_k . To establish this inequality we observe that if $(\Sigma_m^{(i)}, u_m^{(i)})$ is a sequence converging to (Σ_m, u_m) in the sense of (3.3) and (3.4) and K is a compact set such that $\mathcal{R}_m \cap K \neq \emptyset$ we have

$$\inf_{\mathcal{R}_m \cap K} u_m \leq \liminf_{i \rightarrow \infty} \inf_{\mathcal{R}_m^{(i)} \cap K} u_m^{(i)} \leq \limsup_{i \rightarrow \infty} \inf_{\mathcal{R}_m^{(i)} \cap K} u_m^{(i)} \leq c \inf_{\mathcal{R}_m \cap K} u_m$$

for a fixed constant c . The first and second inequalities are obvious, and to get the third we observe that for a small radius r and any $x \in \mathcal{R}_m \cap K$ we have from above

$$u_m(x) \geq cr^{-m} \int_{\Sigma_m \cap B_r(x)} u_m d\mu_m,$$

and hence for i sufficiently large

$$u_m(x) \geq cr^{-m} \int_{\Sigma_m^{(i)} \cap B_r(x)} u_m^{(i)} d\mu_m \geq \varepsilon_0 \inf_{\Sigma_m^{(i)} \cap B_r(x)} u_m^{(i)}$$

for a positive constant ε_0 . This establishes the third inequality. The proof can now be completed by using rescalings at x_0 which converge to (C_m, v_m) for some cone and homogeneous function together with the corresponding result for the homogeneous case. \square

4. Existence of minimal k -slicings

The main purpose of this section is to prove Theorem 2.4. We begin with the construction of the eigenfunction u_k assuming the Σ_k has already been constructed and is partially regular in the sense that $\dim(\mathcal{S}_k) \leq k-3$. We define the Hilbert spaces \mathcal{H}_k and $\mathcal{H}_{k,0}$ as in the last section, namely, \mathcal{H}_k (respectively $\mathcal{H}_{k,0}$) is the completion in $\|\cdot\|_{0,1}$ of the Lipschitz functions with compact support in $\mathcal{R}_k \cap \bar{\Omega}$ (respectively $\mathcal{R}_k \cap \Omega$). In order to handle boundary effects we also assume that there is a larger domain Ω_1 which contains $\bar{\Omega}$ as a compact subset and that the k -slicing is defined and boundaryless in Ω_1 . Note that this is automatic if $\partial\Sigma_j = \emptyset$. Thus $\mathcal{H}_{k,0}$ consists of those functions in \mathcal{H}_k with 0 boundary data on $\Sigma_k \cap \Omega$. The quadratic form Q_k is nonnegative

definite on the Lipschitz functions with compact support in $\mathcal{R}_k \cap \Omega$, and so the standard Schwartz inequality holds for any pair of such functions φ, ψ

$$(4.1) \quad Q_k(\varphi, \psi) \leq \sqrt{Q_k(\varphi, \varphi)} \sqrt{Q_k(\psi, \psi)}.$$

We now have the following result.

THEOREM 4.1. *The function $Q_k(\varphi, \psi)$ is continuous with respect to the norm $\|\cdot\|_{0,1}$ in both variables and therefore extends as a continuous nonnegative definite bilinear form on $\mathcal{H}_{k,0}$. The Schwartz inequality (4.1) holds for $\varphi, \psi \in \mathcal{H}_{k,0}$. The function $Q_k(\varphi, \varphi)$ is strongly continuous and weakly lower semicontinuous on $\mathcal{H}_{k,0}$.*

PROOF. From Proposition 3.2 we have for φ_1, φ_2 Lipschitz functions with compact support in $\mathcal{R}_k \cap \Omega$

$$Q_k(\varphi_1 - \varphi_2, \varphi_1 - \varphi_2) \leq c \|\varphi_1 - \varphi_2\|_{1,k}^2,$$

so it follows from (4.1) that

$$|Q_k(\varphi_1, \psi) - Q_k(\varphi_2, \psi)| \leq \sqrt{Q_k(\varphi_1 - \varphi_2, \varphi_1 - \varphi_2)} \sqrt{Q_k(\psi, \psi)}.$$

Combining these we see that Q_k is continuous in the first slot, and since it is symmetric in both slots. Therefore Q_k extends as a continuous nonnegative definite bilinear form on $\mathcal{H}_{k,0}$ and the Schwartz inequality holds on $\mathcal{H}_{k,0}$ by continuity.

To complete the proof we must prove that $Q_k(\varphi, \varphi)$ is weakly lower semicontinuous on $\mathcal{H}_{k,0}$. Note that the square norm $\|\varphi\|_{0,k}^2 + Q_k(\varphi, \varphi)$ is equivalent to $\|\varphi\|_{1,k}^2$ by Proposition 3.2. Therefore these have the same bounded linear functionals and hence determine the same weak topology on $\mathcal{H}_{k,0}$. Assume we have a sequence $\varphi \in \mathcal{H}_{k,0}$ which converges weakly to $\varphi \in \mathcal{H}_{k,0}$. We then have for any $\psi \in \mathcal{H}_{k,0}$

$$Q_k(\varphi, \psi) = \lim_{i \rightarrow \infty} Q_k(\varphi_i, \psi).$$

This implies that for i sufficiently large

$$Q_k(\varphi, \varphi) = Q_k(\varphi - \varphi_i, \varphi) + Q_k(\varphi_i, \varphi) \leq \varepsilon + \sqrt{Q_k(\varphi_i, \varphi_i)} \sqrt{Q_k(\varphi, \varphi)}$$

for any chosen $\varepsilon > 0$. It follows that

$$Q_k(\varphi, \varphi) \leq \sqrt{Q_k(\varphi, \varphi)} \liminf_{i \rightarrow \infty} \sqrt{Q_k(\varphi_i, \varphi_i)}$$

which implies the desired weak lower semicontinuity. \square

In order to construct a lowest eigenfunction u_k we will need the following Rellich-type compactness theorem.

THEOREM 4.2. *The inclusion of $\mathcal{H}_{k,0}$ into $L^2(\Sigma_k)$ is compact in the sense that any bounded sequence in $\mathcal{H}_{k,0}$ has a convergent subsequence in $L^2(\Sigma_k)$.*

PROOF. This statement follows from Proposition 3.1 and the standard Rellich theorem. Assume that we have a bounded sequence $\varphi_i \in \mathcal{H}_{k,0}$; that is, $\|\varphi_i\|_{1,k}^2 \leq c$. We may extend the φ_i to Ω_1 by taking $\varphi_i = 0$ in $\Omega_1 \sim \Omega$, and by the standard Rellich compactness theorem we may assume by extracting a subsequence that the φ_i converge in L^2 norm on compact subsets of $\bar{\Omega} \sim \mathcal{S}_k$ and weakly in $\mathcal{H}_{k,0}$ to a limit $\varphi \in \mathcal{H}_{k,0}$. We show that φ_i converges to φ in $L^2(\Sigma_k)$. Given any $\varepsilon_1 > 0$, we can choose $\varepsilon > 0$, $\delta > 0$ in Proposition 3.1 so that for each i we have

$$\left(\int_{\Sigma_k \cap V} \varphi_i^2 \rho_{k+1} d\mu_k \right)^{1/2} \leq \varepsilon_1/3$$

where V is an open neighborhood of $\mathcal{S}_k \cap \bar{\Omega}$. The Fatou theorem then implies

$$\left(\int_{\Sigma_k \cap V} \varphi^2 \rho_{k+1} d\mu_k \right)^{1/2} \leq \varepsilon_1/3$$

Since $K = (\Sigma_k \sim V) \cap \bar{\Omega}$ is a compact subset of $\bar{\Omega} \sim \mathcal{S}_k$, we have for i sufficiently large

$$\left(\int_K (\varphi_i - \varphi)^2 \rho_{k+1} d\mu_k \right)^{1/2} \leq \varepsilon_1/3.$$

Combining these bounds we find

$$\|\varphi_i - \varphi\|_0 \leq \left(\int_K (\varphi_i - \varphi)^2 \rho_{k+1} d\mu_k \right)^{1/2} + \left(\int_{\Sigma_k \cap V} (\varphi_i - \varphi)^2 \rho_{k+1} d\mu_k \right)^{1/2} \leq \varepsilon_1$$

for i sufficiently large. This completes the proof. \square

We are now ready to prove the existence, positivity, and uniqueness of u_k on $\Sigma_k \cap \Omega$.

THEOREM 4.3. *The quadratic form Q_k on $\mathcal{H}_{k,0}$ has discrete spectrum with respect to the $L^2(\Sigma_k)$ inner product and may be diagonalized in an orthonormal basis for $L^2(\Sigma_k)$. The eigenfunctions are smooth on $\mathcal{R}_k \cap \Omega$, and if we choose a first eigenfunction u_k , then u_k is nonzero on $\mathcal{R}_k \cap \Omega$ and is therefore either strictly positive or strictly negative since $\mathcal{R}_k \cap \Omega$ is connected. Furthermore any first eigenfunction is a multiple of u_k which we may take to be positive.*

PROOF. This follows from the standard minmax variational procedure for defining eigenvalues and constructing eigenfunctions. For example, to construct the lowest eigenvalue and eigenfunction we let

$$\lambda_k = \inf \{ Q_k(\varphi, \varphi) : \varphi \in \mathcal{H}_{k,0}, \|\varphi\|_{0,k} = 1 \}.$$

By Theorem 4.2 and Theorem 4.1 we may achieve this infimum with a function $u_k \in \mathcal{H}_{k,0}$ with $\|u_k\|_{0,k} = 1$. The Euler-Lagrange equation for u_k is then the eigenfunction equation with eigenvalue λ_k . The higher eigenvalues and eigenfunctions can be constructed by imposing orthogonality constraints

with respect the $L^2(\Sigma_k)$ inner product. We omit the standard details. The smoothness on $\mathcal{R}_k \cap \Omega$ follows from elliptic regularity theory.

The fact that a lowest eigenfunction u is nonzero follows from the fact that if $u \in \mathcal{H}_{k,0}$ then $|u| \in \mathcal{H}_{k,0}$ and $Q_k(u, u) = Q_k(|u|, |u|)$ a property which can be easily checked on the dense subspace of Lipschitz functions with compact support in $\mathcal{R}_k \cap \Omega$ and then follows by continuity. The multiplicity one property of the lowest eigenspace follows from this property in the usual way. We omit the details. \square

We now come to the existence results. We first discuss Theorem 2.4 and we then generalize the existence proof to a more precise form. Suppose X is a closed k -dimensional oriented manifold with $k < n$. We assume that Σ_n is a closed oriented n -manifold and that there is a smooth map $F : \Sigma_n \rightarrow X \times T^{n-k}$ of degree $s \neq 0$. We let Ω denote a (unit volume) volume form of X and let $\Theta = F^*\Omega$ so that Θ is a closed k -form on Σ_n . We let t^p for $p = k+1, \dots, n$ denote the coordinates on the circles and we assume they are periodic with period 1. For $p = k+1, \dots, n$ we let ω^p be the closed 1-form $\omega^p = F^*(dt^p)$. The assumption on the degree of F implies that $\int_{\Sigma_n} \Theta \wedge \omega^{k+1} \wedge \dots \wedge \omega^n = s$.

We will need the following elementary lemma.

LEMMA 4.4. *Suppose N^m is a closed oriented Riemannian manifold and let Ω be its volume form. Given any open set U of N which is not dense in N , the form Ω is exact on U . Moreover, given an open set V compactly contained in U , we can find a closed m -form Ω_1 which agrees with Ω on $M \setminus U$ and such that $\Omega_1 = 0$ in V .*

PROOF. Let f be a smooth function which is equal to 1 in U and such that $\int_N f d\Omega = 0$. Let u be a solution of $\Delta u = f$ and let θ be the $(m-1)$ -form $\theta = *du$. We then have $d\theta = d * du = (\Delta u)\Omega$, so we have $d\theta = \Omega$ on U .

To prove the last statement, we let ζ be a smooth cutoff function which is equal to 1 in V and has compact support in U . We then define $\Omega_1 = \Omega - d(\zeta * du)$. We then have $\Omega_1 = 0$ in V and Ω_1 differs from Ω by an exact form. \square

We now restate the existence theorem.

THEOREM 4.5. *For a manifold $M = \Sigma_n$ as described above, there is a Λ -bounded, partially regular, minimal k -slicing. Moreover, if $k \leq j \leq n-1$ and Σ_j is regular, then $\int_{\Sigma_j} \Theta \wedge \omega^{k+1} \wedge \dots \wedge \omega^j = s$.*

PROOF. We begin with the 1-form ω^n and we integrate to get a map $u_n : \Sigma_n \rightarrow S^1$ so that $\omega^n = du_n$. Let t be a regular value of u_n and consider the hypersurface $S_n = u_n^{-1}(t)$. Because the map F has degree s and we have normalized our forms in $X \times T^{n-k}$ to have integral 1, we see that $\int_{S_n} \Theta \wedge \omega^{k+1} \wedge \dots \wedge \omega^{n-1} = s$. Let Σ_{n-1} be a least volume cycle in Σ_n with

the property that $\int_{\Sigma_n} \Theta \wedge \omega^{k+1} \wedge \dots \wedge \omega^{n-1} = s$. The existence follows from standard results of geometric measure theory.

Now suppose for $j \geq k$ we have constructed a partially regular minimal $j+1$ slicing with the property that there is a form Θ_{j+1} of compact support which is cohomologous to $\Theta \wedge \omega^{k+1} \wedge \dots \wedge \omega^{j+1}$ such that $\int_{\Sigma_{j+1}} \Theta_{j+1} = s$. Since the slicing is partially regular, we have that the Hausdorff dimension of \mathcal{S}_{j+1} is at most $j-2$, so it follows that the image $F_j(\mathcal{S}_{j+1})$ under the projection map $F_j : \Sigma_n \rightarrow X \times T^{j-k}$ is a compact set of Hausdorff dimension at most $j-2$. It follows from Lemma 4.4 that the form $\Omega \wedge dt^{k+1} \wedge \dots \wedge dt^j$ is exact in a neighborhood U of $F_j(\mathcal{S}_{j+1})$, given a neighborhood V of $F_j(\mathcal{S}_{j+1})$ which is compact in U we can find a form Ω_j which is cohomologous to $\Omega \wedge dt^{k+1} \wedge \dots \wedge dt^j$ and vanishes in V . Pulling back we see that $\Theta_j = F^* \Omega_j$ vanishes in a neighborhood of \mathcal{S}_{j+1} and is cohomologous to $\Theta \wedge \omega^{k+1} \wedge \dots \wedge \omega^j$. We let u_{j+1} be the map gotten by integrating ω^{j+1} and consider its restriction to Σ_{j+1} . Since u_{j+1} is in L^2 with respect to the weight ρ_{j+2} , we see that $\rho_{j+1} = u_{j+1} \rho_{j+2}$ is integrable on Σ_{j+1} . It then follows from the coarea formula that we can find a regular value t of u_{j+1} in \mathcal{R}_{j+1} so that the hypersurface $S_j \subset \Sigma_{j+1}$ given by $S_j = u_{j+1}^{-1}(t)$ has finite ρ_{j+1} -weighted volume and satisfies $\int_{S_j} \Theta_j = s$. We can then solve the minimization problem for the ρ_{j+1} -weighted volume among integer multiplicity rectifiable currents T with support in Σ_{j+1} , with no boundary in \mathcal{R}_{j+1} , and with $T(\Theta_j) = s$. A minimizer for this problem gives us Σ_j and completes the inductive step for the existence. \square

REMARK 4.1. The existence proof above does not specify the homology class of the minimizers even if the minimizers are smooth since we are minimizing among cycles for which the integral of Θ_j is fixed. In general there may be homology classes for which the integral of Θ_j vanishes. We have chosen the class to do the minimization in order to avoid a precise discussion of the homology of the singular spaces in which we are working. In the following we give a more precise existence theorem which specifies the homology classes and allows them to be general integral homology classes, possibly torsion classes.

We now formulate and prove a more general existence theorem for minimal k slicings. In the theorem we let $[\Sigma_n]$ denote the fundamental homology class in $H_n(\Sigma_n, \mathbb{Z})$ and, for a cohomology class $\alpha \in H^p(\Sigma_n, \mathbb{Z})$, we let $\alpha \cap [\Sigma_n]$ denote its Poincaré dual in $H_{n-p}(M, \mathbb{Z})$.

THEOREM 4.6. *Let Σ_n be a smooth oriented manifold of dimension n and let k be an integer with $1 \leq k \leq n-1$. Let $\alpha^1, \dots, \alpha^{n-k}$ be cohomology classes in $H^1(\Sigma_n, \mathbb{Z})$, and suppose that $\alpha^{n-k} \cap \alpha^{n-k-1} \cap \dots \cap \alpha^1 \cap [\Sigma_n] \neq 0$ in $H_n(\Sigma_n, \mathbb{Z})$. There exists a partially regular minimal k slicing with Σ_j representing the homology class $\alpha^{n-j} \cap \dots \cap \alpha^1 \cap [\Sigma_n]$.*

PROOF. Assume that we are given a partially regular Λ -bounded minimal $(k+1)$ -slicing which represents $\alpha_1, \dots, \alpha_{n-k-1}$. We thus have the weight

function ρ_{k+1} defined on Σ_{k+1} which we use to produce Σ_k . From the partial regularity the singular set \mathcal{S}_{k+1} of Σ_{k+1} has Hausdorff dimension at most $k-2$.

We consider the class of integer multiplicity rectifiable currents which are relative cycles in $H_k(\Sigma_n, \mathcal{S}_{k+1}, \mathbb{Z})$; that is, for any $k-1$ form θ of compact support in $\Sigma_{k+1} \setminus \mathcal{S}_{k+1}$ we have $T(d\theta) = 0$. Because the set \mathcal{S}_{k+1} has zero $k-1$ dimensional Hausdorff measure we have $H_k(\Sigma_n, \mathbb{Z}) = H_k(\Sigma_n, \mathcal{S}_{k+1}, \mathbb{Z})$. This follows because a current which is a relative cycle T in $\Sigma_n \setminus \mathcal{S}_{k+1}$ is also a cycle in Σ_n since ∂T is zero since it is unchanged by adding a set of $k-1$ measure zero.

We use ρ_{k+1} weighted volume to set up a minimization problem. We consider the class of relative cycles T with support contained in Σ_{k+1} which have finite weighted mass; that is, $T = (S_k, \Theta, \xi)$ where S_k is a countably k -rectifiable set, Θ a μ_k -measurable integer valued function on S_k , and ξ a μ_k -measurable map from S_k to $\wedge^k \mathbb{R}^N$ such that $\xi(x)$ is a unit simple vector for μ_k a.e. $x \in S_k$. Such a k -current T_k is ρ_{k+1} -finite if

$$\text{Vol}_{\rho_{k+1}}(T_k) \equiv \int_{S_k} \rho_{k+1} |\Theta| d\mu_k < \infty.$$

Since we have already constructed Σ_{k+1} so that it is Λ -bounded we have

$$\int_{\Sigma_{k+1}} \rho_{k+1} d\mu_{k+1} \leq \Lambda.$$

Now we can find a smooth closed hypersurface H_k which is Poincaré dual to α_k , and we may perturb it and use the coarea formula in a standard way to arrange that $\bar{\Sigma}_k \equiv \Sigma_{k+1} \cap H_k$ is a smooth embedded submanifold away from \mathcal{S}_{k+1} and

$$\int_{\bar{\Sigma}_k} \rho_{k+1} d\mu_k \leq c.$$

In particular the associated current $\bar{T}_k \equiv (\bar{\Sigma}_k, 1, \bar{\xi})$ (where $\bar{\xi}$ is the oriented unit tangent plane of $\bar{\Sigma}_k$) is ρ_{k+1} -finite and is a competitor in our variational problem.

The standard theory of integral currents now allows us to construct a minimizer for our variational problem which gives us the next slice Σ_k which could be disconnected and with integer multiplicity. Thus Σ_k represents the homology class $\alpha^{n-k} \cap \dots \cap \alpha^1 \cap [\Sigma_n]$. This completes the proof of Theorem 4.6. \square

5. Application to scalar curvature problems

In this section we prove two theorems for manifolds with positive scalar curvature. The first of these is for compact manifolds and the second is the Positive Mass Theorem for asymptotically flat manifolds. Our first theorem which we will need to prove the Positive Mass Theorem is the following.

THEOREM 5.1. *Let M_1 be any closed oriented n -manifold. The manifold $M = M_1 \# T^n$ does not have a metric of positive scalar curvature.*

PROOF. Such a manifold M has admits a map $F : M \rightarrow T^n$ of degree 1, and so by Theorem 2.4 there exists a closed minimal 1-slicing of M in contradiction to Theorem 2.7. \square

We also prove the following more general theorem.

THEOREM 5.2. *Assume that M is a compact oriented n -manifold with a metric of positive scalar curvature. If $\alpha_1, \dots, \alpha_{n-2}$ are classes in $H^1(M, \mathbb{Z})$ with the property that the class σ_2 given by $\sigma_2 = \alpha_{n-2} \cap \alpha_{n-3} \cap \dots \cap \alpha_1 \cap [M] \in H_2(M, \mathbb{Z})$ is nonzero, then the class σ_2 can be represented by a sum of smooth two spheres. If α_{n-1} is any class in $H^1(M, \mathbb{Z})$, then we must have $\alpha_{n-1} \cap \sigma_2 = 0$. In particular, if M has classes $\alpha_1, \dots, \alpha_{n-1}$ with $\alpha_{n-1} \cap \dots \cap \alpha_1 \cap [M] \neq 0$, then M cannot carry a metric of positive scalar curvature.*

PROOF. By the existence and regularity results of Sections 3 and 4, there is a minimal 2-slicing so that $\Sigma_2 \in \sigma_2$ is regular and satisfies the eigenvalue bound of Theorem 2.6. Choosing $\varphi = 1$ on any given component of Σ_2 and applying the Gauss-Bonnet theorem we see that each component must be topologically S^2 .

In particular it follows that for any other $\alpha_{n-1} \in H^1(M, \mathbb{Z})$ we have that $\alpha_{n-1} \cap \sigma_2$ is a class in $H_1(\Sigma_2, \mathbb{Z})$, and therefore is zero. \square

We now prove a Riemannian version of the positive mass theorem. Assume that M is a complete manifold with the property that there is a compact subset $K \subset M$ such that $M \setminus K$ is a union of a finite number of connected components each of which is an asymptotically flat end. This means that each of the components is diffeomorphic to the exterior of a compact set in \mathbb{R}^n and admits asymptotically flat coordinates x^1, \dots, x^n in which the metric g_{ij} satisfies

$$(5.1) \quad g_{ij} = \delta_{ij} + O(|x|^{-p}), \quad |x| |\partial g_{ij}| + |x|^2 |\partial^2 g_{ij}| = O(|x|^{-p}), \quad |R| = O(|x|^{-q})$$

where $p > (n-2)/2$ and $q > n$. Under these assumptions the ADM mass is well defined by the formula (see [Sc] for the n dimensional case)

$$m = \frac{1}{4(n-1)\omega_{n-1}} \lim_{\sigma \rightarrow \infty} \int_{S_\sigma} \sum_{i,j} (g_{ij,i} - g_{ii,j}) \nu_j \, d\xi(\sigma)$$

where S_σ is the euclidean sphere in the x coordinates, $\omega_{n-1} = \text{Vol}(S^{n-1}(1))$, and the unit normal and volume integral are with respect to the euclidean metric. We may now state the Positive Mass Theorem.

THEOREM 5.3. *Assume that M is an asymptotically flat manifold with $R \geq 0$. For each end it is true that the ADM mass is nonnegative. Furthermore, if any of the masses is zero, then M is isometric to \mathbb{R}^n .*

PROOF. The theorem can be reduced to the case when there is a single end by capping off the other ends keeping the scalar curvature nonnegative. We will show only that $m \geq 0$, and the equality statement can be derived from this (see [SY2]). We will reduce the proof to the compact case using results of [SY3] and an observation of J. Lohkamp.

PROPOSITION 5.4. *If the mass of M is negative, there is a metric of nonnegative scalar curvature on M which is euclidean outside a compact set. This produces a metric of positive scalar curvature on a manifold \hat{M} which is gotten by replacing a ball in T^n by the interior of a large ball in M .*

PROOF. Results of [SY3] and [Sc] imply that if $m < 0$ we can construct a new metric on M with nonnegative scalar curvature, negative mass, and which is conformally flat and scalar flat near infinity. In particular, we have $g = u^{4/(n-2)}\delta$ near infinity where u is a euclidean harmonic function which is asymptotic to 1. Thus u has the expansion

$$u(x) = 1 + \frac{m}{|x|^{n-2}} + O(|x|^{1-n})$$

where m is the mass. Now we use an observation of Lohkamp. Since $m < 0$, we can choose $0 < \varepsilon_2 < \varepsilon_1$ and σ sufficiently large so that we have $u(x) < 1 - \varepsilon_1$ for $|x| = \sigma$ and $u(x) > 1 - \varepsilon_2$ for $|x| \geq 2\sigma$. If we define $v(x) = u(x)$ for $|x| \leq \sigma$ and $v(x) = \min\{1 - \varepsilon_2, u(x)\}$ for $|x| > \sigma$, then we see that $v(x)$ is weakly superharmonic for $|x| \geq \sigma$, so may be approximated by a smooth superharmonic function with $v(x) = u(x)$ for $|x| \leq \sigma$ and $v(x) = 1 - \varepsilon_2$ for $|x|$ sufficiently large. The metric which agrees with the original inside S_σ and is given by $v^{4/(n-2)}\delta$ outside then has nonnegative scalar curvature and is euclidean near infinity.

By extending this metric periodically we then produce a metric on \hat{M} with nonnegative scalar curvature which is not Ricci flat. Therefore the metric can be perturbed to have positive scalar curvature. \square

Using this result the theorem follows from Theorem 5.2 since the standard 1-forms on T^n can be pulled back to \hat{M} to produce the $\alpha_1, \dots, \alpha_{n-1}$ of that theorem. This completes the proof of Theorem 5.3. \square

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