

RIGIDITY OF NONPOSITIVELY CURVED MANIFOLDS WITH CONVEX BOUNDARY

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ABSTRACT. We show that a compact Riemannian 3-manifold M with strictly convex simply connected boundary and sectional curvature $K \leq a \leq 0$ is isometric to a convex domain in a complete simply connected space of constant curvature a , provided that $K \equiv a$ on planes tangent to the boundary of M . This yields a characterization of strictly convex surfaces with minimal total curvature in Cartan-Hadamard 3-manifolds, and extends some rigidity results of Greene-Wu, Gromov, and Schroeder-Strake. Our proof is based on a recent comparison formula for total curvature of Riemannian hypersurfaces, which also yields some dual results for $K \geq a \geq 0$.

1. INTRODUCTION

A *Cartan-Hadamard* manifold $\mathcal{H}$ is a complete simply connected Riemannian n -space with sectional curvature $K \leq 0$. Greene and Wu [9, 10] and Gromov [3, Sec. 5] showed that, when $n \geq 3$, these spaces exhibit remarkable rigidity properties, analogous to those observed earlier by Mok, Siu, and Yau [13, 18] in Kähler geometry. In particular, a fundamental result is that if K vanishes outside a compact set $C \subset \mathcal{H}$, then $\mathcal{H}$ is isometric to Euclidean space $\mathbf{R}^n$. More generally, if $K \leq a \leq 0$ on $\mathcal{H}$, and $K \equiv a$ on $\mathcal{H} \setminus C$, then $K \equiv a$ on $\mathcal{H}$ [9, p. 734] [17]. We extend this result when $n = 3$:

Theorem 1.1. *Let M be a compact Riemannian 3-manifold with nonempty $\mathcal{C}^2$ boundary ∂M and sectional curvature $K \leq a \leq 0$. Suppose that ∂M is strictly convex, each component of ∂M is simply connected, and $K \equiv a$ on planes tangent to ∂M . Then M is isometric to a convex domain in a Cartan-Hadamard manifold of constant curvature a . In particular, M is diffeomorphic to a ball.*

Strictly convex here means that the second fundamental form of ∂M is positive definite with respect to the outward normal. For $n = 3$, this theorem immediately implies the rigidity results mentioned above, by letting M be a geodesic ball in $\mathcal{H}$ containing C .

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Schroeder and Strake [15] had established this result for $a = 0$ (and only for $n = 3$) refining earlier work of Schroeder and Ziller [14]. The simply connected assumption on components of ∂M is necessary, as can be seen by considering a tubular neighborhood of a closed geodesic in a hyperbolic manifold.

As an application of Theorem 1.1 we obtain the following characterization for strictly convex surfaces with minimal total curvature. We say that an oriented closed (compact, connected, without boundary) hypersurface $\Gamma \subset \mathcal{H}$ is *strictly convex* if its second fundamental form $\mathbb{I}$ is positive definite. Then Γ is embedded, bounds a convex domain, and is simply connected [1]. The *total Gauss-Kronecker curvature* of Γ is given by $\mathcal{G}(\Gamma) := \int_{\Gamma} \det(\mathbb{I})$, and $|\Gamma|$ denotes the area of Γ .

Corollary 1.2. *Let $\mathcal{H}$ be a 3-dimensional Cartan-Hadamard manifold with curvature $K \leq a \leq 0$, and $\Gamma \subset \mathcal{H}$ be a $\mathcal{C}^2$ closed strictly convex surface. Then*

$$(1) \quad \mathcal{G}(\Gamma) \geq 4\pi - a|\Gamma|,$$

with equality only if $K \equiv a$ on the convex domain bounded by Γ .

Proof. By Gauss' equation $\det(\mathbb{I}_p) = K_{\Gamma}(p) - K(T_p\Gamma)$ for all $p \in \Gamma$, where K_{Γ} is the intrinsic curvature of Γ , and $T_p\Gamma$ is the tangent plane of Γ at p . Since Γ is simply connected, $\int_{\Gamma} K_{\Gamma} = 4\pi$ by Gauss-Bonnet theorem. Thus

$$(2) \quad \mathcal{G}(\Gamma) = 4\pi - \int_{p \in \Gamma} K(T_p\Gamma) \geq 4\pi - a|\Gamma|.$$

If equality holds in (1), then it also holds in (2), which forces $K \equiv a$ on tangent planes of Γ . Theorem 1.1, applied to the convex domain bounded by Γ , completes the proof. $\square$

For $a = 0$, the last result is stated in [3, p. 66] and follows from [15, Thm. 2]. Gromov's approach to the rigidity theorems mentioned above [3, Sec. 5], which are further developed in [14, 15], was based on extension of isometric embeddings in locally symmetric spaces. In most of these results the rank of the space is required to be bigger than 1, which precludes negative upper bounds for curvature. The arguments of Greene and Wu [9] on the other hand involve volume comparison theory, which applies readily to various curvature bounds [9, p. 734]; see Seshadri [17]. Here we develop a different approach via recent work on total curvature of Riemannian hypersurfaces [7, 8], which also yields some results for the dual case $K \geq a \geq 0$; see Note 3.1. Generalizing (1) to dimensions $n > 3$ is an important open problem with applications to the isoperimetric inequality; see [7] for more references and background in this area.

2. PROOF OF THEOREM 1.1

The proof consists of three parts. First we use the comparison formula developed in [7, 8] to show that $K \equiv a$ on a neighborhood of ∂M (which we do not a priori assume to be connected). Then it follows that M is isometric to a convex domain in a Cartan-Hadamard manifold $\overline{M}$, which has constant curvature a outside M (in particular ∂M is connected). Finally enclosing M in a geodesic ball $B \subset \overline{M}$ and shrinking the radius of B completes the proof via the first part of the argument. The first part also involves the Gauss-Bonnet theorem, which is why we need to assume that $n = 3$. Other aspects of the proof work in all dimensions $n \geq 3$.

(Part I). Let Γ be a component of ∂M , $d_\Gamma: M \rightarrow \mathbf{R}$ be the distance function of Γ , and $\Gamma_t := d_\Gamma^{-1}(t)$ be the parallel surface of Γ at distance t . Since Γ is $\mathcal{C}^2$, there exists $\varepsilon > 0$ such that Γ_t is $\mathcal{C}^2$ for $t \in [0, \varepsilon]$, see [5]. In particular, for $t \in [0, \varepsilon]$, the principal curvatures κ_i^t of Γ_t with respect to the outward normal ν_t are well-defined. By assumption, $\kappa_i^0 > 0$, which yields that $\kappa_i^t > 0$, assuming ε is sufficiently small. Let e_i^t be a choice of orthogonal principal directions for κ_i^t , and K_i^t be the sectional curvatures of M for planes spanned by ν_t and e_i^t . Let $\Omega_\varepsilon \subset M$ be the domain bounded in between Γ and Γ_ε , and recall that $\mathcal{G}(\Gamma_t)$ denote the total Gauss-Kronecker curvature of Γ_t , i.e., the integral of $\kappa_1^t \kappa_2^t$ over Γ_t . Since $|\nabla d_\Gamma|$ is constant on Γ_t , the comparison formula in [8, Thm. 3.1] reduces to

$$\mathcal{G}(\Gamma) - \mathcal{G}(\Gamma_\varepsilon) = - \int_{\Omega_\varepsilon} (\kappa_1^t K_2^t + \kappa_2^t K_1^t).$$

Let $H^t := \kappa_1^t + \kappa_2^t$ denote the mean curvature of Γ_t . Since $\kappa_i^t \geq 0$,

$$(3) \quad \mathcal{G}(\Gamma) - \mathcal{G}(\Gamma_\varepsilon) \geq -a \int_{\Omega_\varepsilon} H^t = -a(|\Gamma| - |\Gamma_\varepsilon|).$$

The last equality is due to Stokes theorem, since $H^t = \operatorname{div}(\nabla d_\Gamma)$ and $|\nabla d_\Gamma| = 1$ (more formally, the above inequality holds on $\Omega_\varepsilon \setminus \Omega_s$, for $0 < s < \varepsilon$, and we may take the limit as $s \rightarrow 0^+$). On the other hand, by Gauss' equation and Gauss-Bonnet theorem,

$$(4) \quad \begin{aligned} \mathcal{G}(\Gamma) &= 4\pi - \int_{p \in \Gamma} K(T_p \Gamma) = 4\pi - a|\Gamma|, \\ \mathcal{G}(\Gamma_\varepsilon) &= 4\pi - \int_{p \in \Gamma_\varepsilon} K(T_p \Gamma_\varepsilon) \geq 4\pi - a|\Gamma_\varepsilon|. \end{aligned}$$

Hence

$$\mathcal{G}(\Gamma) - \mathcal{G}(\Gamma_\varepsilon) \leq -a(|\Gamma| - |\Gamma_\varepsilon|).$$

So equality holds in (3) which forces $K_i^t \equiv a$ on Ω_ε for $i = 1, 2$. Furthermore, equality in (3) implies equality in (4), which yields that $K \equiv a$ on tangent planes of Γ_ε . So $K \equiv a$ on a triplet of mutually orthogonal planes at each point of Γ_ε . It follows that $K \equiv a$

with respect to all planes with footprint on Γ_ε , since $K \leq a$. As this argument holds for all $\varepsilon' \leq \varepsilon$, we conclude that $K \equiv a$ on Ω_ε .

(Part II). Let $\mathcal{H}$ be the 3-dimensional Cartan-Hadamard manifold of constant curvature a . Then Ω_ε is locally isometric to $\mathcal{H}$. Furthermore, since Γ is simply connected, so is Ω_ε . Thus there exists an isometric immersion $f: \Omega_\varepsilon \rightarrow \mathcal{H}$, by a standard monodromy argument. In particular, $f(\Gamma)$ forms a closed immersed surface in $\mathcal{H}$ with positive principal curvatures. Consequently, by a result of Alexander [1, Thm. 1], see also [15, Lem. 1], f embeds Γ into the boundary of a convex domain $C \subset \mathcal{H}$. Let $C' \subset \mathcal{H}$ be the closure of $\mathcal{H} \setminus C$. Using the diffeomorphism f between $f(\Gamma) = \partial C'$ and Γ , we may glue C' to M along Γ to obtain a smooth manifold with one fewer boundary component. Repeating this procedure for each component Γ of ∂M yields an extension of M to a complete manifold $\overline{M}$ of nonpositive curvature. Now pick a point $p \in \overline{M}$. By Cartan-Hadamard theorem, the exponential map $\exp_p: T_p \overline{M} \rightarrow \overline{M}$ is a covering. Let X be a component of $\overline{M} \setminus M$. Note that X is simply connected since, by Schoenflies theorem, it is homeomorphic to the complement of a ball in $\mathbf{R}^3$. Let X' be a component of $\exp_p^{-1}(X)$. Since X is simply connected, X' is homeomorphic to X . In particular $\partial X'$ is an embedded topological sphere. Thus X' is the complement of a bounded set in $T_p \overline{M}$. Since any two such sets must intersect, it follows that $\overline{M} \setminus M$ is connected, and $X' = \exp_p^{-1}(X)$. In particular $\overline{M} \setminus M = X$, which is simply connected. Consequently $\exp_p: X' \rightarrow X$ is one-to-one, which yields that it is one-to-one everywhere, since $\exp_p$ is a covering map. Hence $\overline{M}$ is simply connected, and therefore is a Cartan-Hadamard manifold. Finally, since $\overline{M} \setminus M$ is connected, ∂M is connected. So M forms a convex domain in $\overline{M}$ by Alexander's result [1, Thm. 1].

(Part III). It remains to show that $K \equiv a$ on $\overline{M}$. By construction $K \equiv a$ on $\overline{M} \setminus M$. So we just need to check that $K \equiv a$ on M . Fix a point o in $\overline{M}$ and let $B_r \subset \overline{M}$ be the geodesic ball of radius r centered at o . If r is large enough, so that $M \subset B_r$, then $K \equiv a$ outside B_r . Let r_0 be the infimum of $r > 0$ such that $K \equiv a$ on $\overline{M} \setminus B_r$. If $r_0 = 0$ we are done. Otherwise, since $a \leq 0$, ∂B_{r_0} forms a strictly convex surface by Hessian comparison (the principal curvatures of ∂B_{r_0} are bigger than those of a sphere of the same radius in $\mathbf{R}^3$ [12, 1.7.3]). Thus we may apply the result of Part I to B_{r_0} to obtain that $K \equiv a$ on a neighborhood of ∂B_{r_0} in B_{r_0} , which contradicts the definition of r_0 . So we conclude that $K \equiv a$ everywhere, which completes the proof.

3. NOTES

Note 3.1. Part I of the proof of Theorem 1.1 works just as well for nonnegative curvature, i.e., suppose that $K \geq a \geq 0$ on M and $K \equiv a$ on tangent planes of ∂M , then

virtually the same argument shows that $K \equiv a$ on an open neighborhood of ∂M . Thus if ∂M contracts to a point through strictly convex surfaces, then $K \equiv a$ on M as we showed in Part III. This may be considered a dual version of Theorem 1.1. For instance if M is a geodesic ball of radius r in a space with $K \leq b$, then it satisfies the contraction property provided that $r \leq \pi/(2\sqrt{b})$, by Hessian comparison [12, 1.7.3]. More generally, the required contraction may be achieved via a curvature flow when maximum value of K is not too large compared to principal curvatures of ∂M [2]. Furthermore note that when $a = 0$, and ∂M is simply connected, M may be extended to a nonnegatively curved manifold $\overline{M}$ which is flat outside M , as discussed in Part II of the above proof. Then $\overline{M}$ is isometric to $\mathbf{R}^3$ by [9, Thm. 1]. So M will be flat, as had been observed earlier in [15, p. 486]. See [14, 16] for more rigidity results for nonnegative curvature

Note 3.2. Once Part I of the proof of Theorem 1.1 has been established, and it is known a priori that M is simply connected with connected boundary Γ , one may complete the argument more directly by covering M with a continuous family of strictly convex surfaces Γ_t with $\Gamma_0 = \Gamma$. For instance, we may let Γ_t be level sets of a strictly convex function on M , see [4, Lem. 1]. Alternatively, one may use harmonic mean curvature flow, i.e., set $\Gamma_t := X(\Gamma, t)$ for $X: \Gamma \times [0, T) \rightarrow M$ given by

$$\frac{\partial}{\partial t} X(p, t) = \frac{-1}{1/\kappa_1^t(p) + 1/\kappa_2^t(p)} \nu_t(p), \quad X(p, 0) = p,$$

where ν_t is the outward normal of Γ_t and κ_i^t are its principal curvatures. Xu showed [19], see also Gulliver and Xu [11], that Γ_t converges to a point o as $t \rightarrow T$, and remains strictly convex throughout [19, Prop. 19]. So Γ_t always moves inward, foliating the region $M \setminus \{o\}$. The stated regularity requirement in [19] is that Γ be $\mathcal{C}^\infty$, which we may assume is the case after a perturbation of Γ [6, Lem. 3.3], since by Part I we have $K \equiv a$ near Γ .

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