Fermions in Loop Quantum Gravity and Resolution of Doubling Problem

Cong Zhang,[1,*] Hongguang Liu,[1,†] and Muxin Han[2,1,‡]

[1] *Institut für Quantengravitation, Friedrich-Alexander Universität Erlangen-Nürnberg, Staudtstr. 7/B2, 91058 Erlangen, Germany*

[2] *Department of Physics, Florida Atlantic University, 777 Glades Road, Boca Raton, FL 33431-0991, USA*

The fermion propagator is derived in detail from the model of fermion coupled to loop quantum gravity. As an ingredient of the propagator, the vacuum state is defined as the ground state of some effective fermion Hamiltonian under the background geometry given by a coherent state resembling the classical Minkowski spacetime. Moreover, as a critical feature of loop quantum gravity, the superposition over graphs is employed to define the vacuum state. It turns out that the graph superposition leads to the propagator being the average of the propagators of the lattice field theory over various graphs so that all fermion doubler modes are suppressed in the propagator. This resolves the doubling problem in loop quantum gravity. Our result suggests that the superposition nature of quantum geometry should, on the one hand, resolve the tension between fermion and the fundamental discreteness and, on the other hand, relate to the continuum limit of quantum gravity.

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I. INTRODUCTION

The physical world comprises gravity and matter. Even though the quantum theory of matters has been well-developed in standard model, the gravity therein is still classical. However, it has been suggested that a consistent theory of gravity coupled with quantum matters should contain gravity quantized [1]. On the other hand, Quantum gravity (QG) also needs matter couplings because they provide probes for empirically testing QG effects [2–4]. Particularly matter couplings with gravity provide a toolbox for early studies of the QG's ultraviolet (UV) behavior, and play essential roles in cosmology, black holes, asymptotic safety, QG phenomenology, etc [5–8].

Loop Quantum Gravity (LQG), as a promising candidate of background-independent and non-perturbative QG theory, sets the stage for matter coupling with QG [9–13]. In LQG, quantum states are given by spin networks, i.e., graphs that carry quantum spins. The graphs with the quantum spins thereon give rise to quantum geometries: the quantum numbers carried by a vertex provide the quantum volume of a chunk dual to the vertex, and the quantum numbers on the edges of the vertex give the surface area of the chunk. Hence, the graphs in LQG are interpreted as spacetime quanta and are where matters can be. Indeed, matter couplings in LQG have been extensively explored (see e.g. [14–26]). Notably, a framework of standard model coupled to LQG has been developed with the exciting feature of ultraviolet regularity [15]. In these models, the quantization procedure leads the matter fields to be located at the vertices or edges so that they are coupled to the quantum geometry given by the quantum spins. The picture constructed by the model shares several common features with the Lattice Field Theory (LFT), especially when the graphs are chosen to be cubic. However, in LQG, the graphs are dynamics, so it makes sense to consider superposition over graphs. It will be shown in our work that this difference could make LQG essentially differ from LFT.

Relating the theory of LQG coupled to matter fields to the QFT in flat/curved spacetime is an exciting and challenging topic. There have been many attempts towards this issue [27–29]. Our work addresses this issue by considering the propagator of LQG coupled to the chiral fermions [15, 25, 26]. In a propagator, an essential ingredient is the vacuum state. In the standard QFT, the vacuum state is the ground state of the Hamiltonian. To analog this definition in LQG, we meet the first obstacle that the Hamiltonian of QFT is background-dependent, where the background is classical. Hence, we need to propose a quantum spacetime state that resembles the classical background in QFT. A natural choice for such states is Thiemann's coherent states in LQG, which turn out to have desirable semiclassical features [30–32]. Thiemann's coherent states are given based on graphs like a generic state in LQG. Then an issue arises on which graphs should be selected for the coherent states. The preferred is the cubic graphs because the Hamiltonian operator has a physically satisfactory expectation value in the coherent states on cubic graphs. Given a cubic graph γ, applying the coherent state thereon peaked at the Minkowski geometry to sandwich the Hamiltonian operator, one can introduce an effective fermion Hamiltonian $\hat{H}^{\rm eff}_{F,\gamma}$ comprising only the matter field operators [16]. Indeed, since the Hamiltonian contains operators concerning matter field and background, sandwiching

* cong.zhang(AT)fuw.edu.pl

† hongguang.liu(AT)gravity.fau.de

‡ hanm(AT)fau.edu

it in the coherent state could change those background terms into their classical expectation values plus some $\hbar$-order corrections. Since the ground state $|\omega,\gamma\rangle$ of $\hat{H}^{\rm eff}_{F,\gamma}$ can be interpreted as the LQG analogy of QFT vacuum, the LQG vacuum on γ will be defined as the tensor product of the coherent state and $|\omega,\gamma\rangle$. Moreover, as an essential difference between LQG and LFT, the superposition of various cubic graphs should be employed in defining the vacuum. The superposition feature of the vacuum is expected to resolve the fermion doubling.

The fermion doubling is a problem suffered by chiral fermions in LFT. In LFT of the chiral fermion, each fermion results in 2^m fermion species on m-dimensional lattice [33, 34]. Since the fundamental discreteness of LQG makes the fermion coupling resemble the LFT to some extent, it is suspicious of the doubling problem in LQG [35]. The doubling problem is a crucial issue because it reveals the tension between fermion and the fundamental discreteness of quantum spacetime, given that the fundamental discreteness is believed to be a key feature of QG [36–38]. The confusion on fermion doubling has been long-standing in the LQG community since the first paper on LQG-fermion in 1997. The similar issue should exist in all QG approaches with discrete spacetimes. This problem is expected to be resolved by the superposition of quantum geometries [20, 39, 40]. Actually, it will be shown by our work that the doubling problem can really be resolved by the the graph-superposition vacuum state.

In the Hamiltonian formalism of general relativity, the Hamiltonian is a generator of a gauge transformation, so the dynamics depict a picture of frozen evolution. A system with this feature is called the Hamiltonian-constraint system. To get the physical evolution of this system, one introduces the notion of relational evolution, where some other fields need to be employed as a physical reference frame. In the relational evolution formulation, the Dirac observables are those functions depending on the physical reference frame fields. Thus, these Dirac observables are explained as the relational evolution of some physically interesting fields with respect to the reference frame fields. The current work will introduce the Gaussian dust fields as reference frame fields so that the physical Hamiltonian governing the relational evolution equals the Hamiltonian constraint with N=1 as the lapse function. The Hamiltonian may usually be quantized in two ways: graph-changing and graph-preserving. Generally speaking, a continuum field theory approach leads directly to the graph-changing action where the graph acted by the Hamiltonian will be changed by adding loops or vertices. Several proposals for graph-changing quantum Hamiltonian operators were considered in the literature. The graph-preserving approach is natural from the lattice discretization point of view. It makes the action of operators reducible to subspaces corresponding to the graphs. For every graph, the analytic properties of operators are much easier to study. Our work adopts the graph-preserving version but with a small amount of difference which guarantees that the resulting operator has a well-defined adjoint.

The Hamiltonian in the current work is complicated and contains not only the fermion operators but also the operators referring to the background metric. Indeed, the complicacy of the Hamiltonian is a primary obstacle to blocking LQG's progress. Due to the complicacy of the Hamiltonian, we apply the path integral formulation to deal with the evolution operator in the propagator. The path integral formula contains the integral of $e^{i/\hbar S}$ with S denoting the effective action. As shown in the body part of the paper, S is a quadratic form of the fermion field. Thus the integral concerning the fermion field takes the form of a standard fermionic Gaussian integral and is calculatable explicitly. Thus, we should pay more attention to the integral of the holonomies and fluxes, i.e., the variable referring to the background metric. This integral can be evaluated using the stationary phase approximation. To this end, one needs to vary the action to get the equation of motion $\delta S = 0$. Fortunately, this equation of motion has been well-studied in [41] for the vacuum LQG. Regardless of the slight difference between the Hamiltonian in [41] and the current work, the results in [41] can be applied to get the solution of $\delta S = 0$ adapting the boundary condition given the vacuum state. According to the stationary phase approximation algorithm, the solution endows the graphs of the vacuum state with a semiclassical dynamical geometry.

This paper is arranged as follows. In Sec. II, the classical and the quantum theories of gravity coupling fermion and Gaussian dust is reviewed, including the quantum kinematics, the coherent states, the gauge transformation and the Hamiltonian operator. In Sec. III, the path integral formulation on a single graph is introduced. Sec. IV defines the Minkowski vacuum. Then, in Sec. V we show the calculation of the propagator in detail. Finally, Sec. VI shows how the graph superposition resolves the doubling problem. The work is summarized in Sec. VII.

II. PRELIMINARY

A. Gravity coupling fermion and Gaussian dust

Let $\mathcal{M}$ be the 4-manifold homeomorphism to $\mathbb{R}\times\Sigma$ with Σ being the spatial 3-manifold. Further, we shall assume that Σ homeomorphism to $\mathbb{T}^3$ with coordinate x^a adapted to this topology. Namely, we choose the periodic boundary condition in our calculation. On $\mathcal{M}$, considering the model of gravity coupling Weyl fermion and the Gaussian dust,

the Lagrangian takes the form

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_D \tag{2.1}$$

where $\mathcal{L}_G$ is the Einstein-Hilbert Lagrangian, $\mathcal{L}_F$ describes the Wely fermion field in gravity, and $\mathcal{L}_D$ is the Lagrangian of the Gaussian dust. Here the Einstein-Hilbert Lagrangian $\mathcal{L}_G$ is written with the tetrad field e^I_μ, which endows the $\mathcal{M}$ with the metric $g_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu$. More precisely, $\mathcal{L}_G = \frac{1}{2\kappa}R[e]$, where $\kappa = 8\pi G$, and $R[e]$ denotes the curvature of $g_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu$. The fermion Lagrangian in the present work reads

$$\mathcal{L}_F = \frac{i}{2}e\left(\psi^\dagger\sigma^I e^\mu_I \nabla_\mu\psi - c.c.\right) \tag{2.2}$$

where e is the determinant of e^I_μ, σ^I denotes the covariant Pauli matrices $(\mathbb{1}_2, \sigma^i)$, and $\nabla_\mu\Psi$ is

$$\nabla_\mu\psi = \partial_\mu\psi + \frac{1}{4}\Gamma^I_{\mu J}\bar{\sigma}^I\sigma^J\psi. \tag{2.3}$$

with $\bar{\sigma}^I = (1, -\sigma^i)$ and $\Gamma^I_{\mu J}$ being the spin connection satisfying $\mathrm{d}e^I + \Gamma^I{}_J \wedge e^j = 0$. Indeed, the Lagrangian (2.2) implies that the left-handed Weyl fermion ψ is chosen for our study. The Gaussian dust comprises four scalar fields, denoted by T and S^a with $a = 1, 2, 3$. Its Lagrangian $\mathcal{L}_D$ is

$$\mathcal{L}_D = -\sqrt{|\det(g)|}\left\{\frac{1}{2}\rho[g^{\mu\nu}(\partial_\mu T)(\partial_\nu T) + 1] + (\partial_\mu T)(W_j\partial_\nu S^j)\right\}, \tag{2.4}$$

where the additional four scalar fields ρ and W_j $(j = 1, 2, 3)$ will play the role of Lagrangian multipliers and give rise to the primary constraints in the Hamiltonian theory.

Doing the Hamiltonian analysis of the Einstein-Hilbert Lagrangian, one gets a state of gravity described by the canonical pair (K^i_a, E^a_i), where E^a_i is the densitized triad filed on the initial slice Σ and K^i_a is the extrinsic curvature of Σ. To introduce the Ashtekar new variables for the loop quantization, the canonical transformation $A^i_a = \Gamma^i_a + \beta K^i_a$ should be done, where β denotes the Barbero-Immirzi parameter, and Γ^i_a is the SU(2) spin connection compatible with E^a_i. The constraint analysis of the model gives rise to the complete set of constraints containing those of the first and second classes [42]. Solving the second-class constraints by introducing the Dirac brackets, one gets a total constraint system with the first-class constraints, including the Gauss constraint, the diffeomorphism constraint and the Hamiltonian constraint. In the phase space Γ of the resulting first-class system, besides the new variables (A^i_a, E^b_j), there contain the canonical pairs (T, P), (S^j, P_j) and $(\xi, \xi^\dagger)$, where T and S^j for $j = 1, 2, 3$ are scalar fields on Σ, their conjugate momentums P and P_j are density field with weight one and the half-density field ξ relates to the fermion field by

$$\xi = \sqrt[4]{|\det(E)|}\psi\Big|_\Sigma. \tag{2.5}$$

The non-vanishing (anti-)Poisson brackets between these variables are

$$\begin{aligned}
\{A^i_a(x), E^b_j(y)\} =& \kappa\beta\delta^b_a\delta^i_j\delta(x, y),\\
\{\xi_A(x), \xi^\dagger_B(y)\}_+ =& -i\delta_{AB}\delta(x, y),\\
\{T(x), P(y)\} =& \delta(x, y),\\
\{S_i(x), P^j(y)\} =& \delta^j_i\delta(x, y).
\end{aligned} \tag{2.6}$$

The Gauss constraint G_m is independent of the dust variables and reads

$$G_m = \frac{1}{\kappa\beta}D_a E^a_m + \frac{1}{2}\xi^\dagger\sigma_m\xi. \tag{2.7}$$

It generates the SU(2) gauge transformation of the gravity and fermion fields. The diffeomorphism and Hamiltonian constraints are

$$\begin{aligned}
C^{\mathrm{tot}}_a =& H_a + P\partial_a T + P_j\partial_a S^j,\\
C^{\mathrm{tot}} =& H + \frac{P - q^{ab}\partial_a T C_b}{\sqrt{1 + q^{ab}\partial_a T\partial_b T}},
\end{aligned} \tag{2.8}$$

where H_a $(a = 1, 2, 3)$ and H denote the diffeomorphism and Hamiltonian constraints of gravity coupling fermion field, and q_{ab} is the spatial metric on the spatial manifold Σ. According to the results in [26], the constraints H_a and H are

$$\begin{aligned} H_a =& \frac{1}{\kappa\beta} E_i^b F_{ab}^i + \frac{i}{2}\Big(\xi^\dagger D_a\xi - (D_a\xi)^\dagger\xi\Big) + \beta K_a^m G_m, \\ H =& H_G + \frac{1}{2\sqrt{q}}\Big[i(\xi^\dagger E_i^a\sigma^i D_a\xi - (D_a\xi)^\dagger E_i^a\sigma^i\xi) - \beta E_i^a K_a^i\xi^\dagger\xi - \frac{1}{\beta}(1+\beta^2)D_a E_i^a\xi^\dagger\sigma^i\xi - \beta E_i^a D_a\Big(\xi^\dagger\sigma^i\xi\Big)\Big], \end{aligned} \tag{2.9}$$

with the scalar constraint of pure gravity H_G:

$$H_G = \frac{1}{2\kappa\sqrt{q}} E_i^a E_j^b \left(F_{ab}^m \epsilon_m{}^{ij} - 2(1+\beta^2)K_{[a}^i K_{b]}^j\right). \tag{2.10}$$

With the constraints, the action is expressed as

$$S = \int \mathrm{d}^4x \left(\frac{1}{\kappa\beta} E_i^a \partial_t A_a^i + \frac{i}{2}\left(\xi^\dagger\partial_t\xi - (\partial_t\xi^\dagger)\xi\right) + P_\mu\partial_t S^\mu - \lambda^m G_m - N^a C_a^{\mathrm{tot}} - N C^{\mathrm{tot}}\right), \tag{2.11}$$

with P_μ and S^μ as the abbreviation of (P, P_j) and (T, S_j) respectively, where λ^m, N^a and N are all Lagrangian multipliers.

As is well-known, a totally constrained system, like the model in the present work, leads to a scenario in which the dynamical evolution is frozen. This problematic issue can be fixed by introducing the relational evolution. To elaborate this concept, let us forget about the Gauss constraint for a moment and define Γ_C as the constraint surface given by $C_a^{\mathrm{tot}} = 0 = C^{\mathrm{tot}}$ in the phase space. Then, the physical phase space $\bar{\Gamma}$ is Γ_C modulo the gauge orbits of the constraints. Since the constraints C_a^{tot} and C^{tot} generate the spacetime diffeomorphism transformations, we can introduce a gauge fixing conditions $0 = T - t$ and $0 = \delta_j^a S^j - x^a$ for each parameter t, where x^a is a fiducial coordinate fixed once and for all on Σ. Let Γ_t denote the gauge fixed surface for t. Then, $\sigma_t : \bar{\Gamma} \to \Gamma_t$ gives an embedding of $\bar{\Gamma}$ into Γ_C. Given a function F on Γ_C that depends only on the gravity and fermion fields; we construct a family of Dirac observables by

$$t \mapsto \sigma_t^* F. \tag{2.12}$$

This equation is naturally interpreted as the relational evolution of the gravity and the fermion field with respect to the Gaussian dust [42, 43]. The physical Hamiltonian governing this evolution is

$$\mathbf{H} = \int \mathrm{d}^3x \sqrt{1 + q^{ab}\partial_a T\partial_b T}\, H + q^{ab}\partial_a T H_b = \int \mathrm{d}^3x H. \tag{2.13}$$

It is worth noting that the derivation of $\mathbf{H}$ needs the fact that F depends only on the gravity and the fermion field and that $\partial_a T = 0$.

B. quantum kinematics on cubic lattices

The current work aims at restoring the results of the quantum fermion field on the Minkowski background from fermion coupling quantum gravity. To this end, cubic lattices will be chosen for our calculation because cubic lattices can reflect the continuum properties of the Hamiltonian better than the others [44–47]. Due to this, we do not lose any essentials by concerning ourselves with only the cubic graphs. Thus, unless otherwise stated, all graphs mentioned below are cubic.

For a graph $\gamma \subset \Sigma$, the collections of edges and vertices are denoted by $E(\gamma)$ comprising $|E(\gamma)|$ elements and $V(\gamma)$ containing $|V(\gamma)|$ element, respectively. Given γ, a function ψ on $\mathrm{SU}(2)^{|E(\gamma)|}$ gives rise to a cylindrical function ψ_γ of the connection A_a^i, that is

$$\psi_\gamma(A) = \psi(\{h_e(A)\}_{e\in E(\gamma)}), \tag{2.14}$$

where $h_e(A)$ is the holonomy of A along e, i.e.,

$$h_e(A) = \mathcal{P}\exp\int_e A = 1 + \sum_{n=1}^{\infty}\int_0^1 \mathrm{d}t_n \int_0^{t_n} \mathrm{d}t_{n-1}\cdots\int_0^{t_2} \mathrm{d}t_1 A(t_1)\cdots A(t_n). \tag{2.15}$$

The space of the cylindrical functions on γ is denoted by $\mathbf{Cyl}^{(\gamma)}_G$. Two cylindrical functions $\psi^{(1)}_\gamma, \psi^{(2)}_\gamma \in \mathbf{Cyl}^{(\gamma)}_G$ have the inner product

$$\langle \psi^{(1)}_\gamma | \psi^{(2)}_\gamma \rangle = \int_{\mathrm{SU}(2)^{|E(\gamma)|}} \mathrm{d}\mu_H(\vec{h}) \psi^{(1)}(\vec{h})^* \, \psi^{(2)}(\vec{h}) \tag{2.16}$$

where $\mathrm{d}\mu_H$ is the Haar measure on $\mathrm{SU}(2)^{|E(\gamma)|}$. The Cauchy completion of the inner product space $\mathbf{Cyl}^{(\gamma)}_G$ gives rise to the Hilbert space $\mathcal{H}^G_\gamma$. $\mathcal{H}^G_\gamma$ by definition is isometric to $L^2(\mathrm{SU}(2)^{|E(\gamma)|}, \mathrm{d}\mu_H)$.

The fermion Hilbert space $\mathcal{H}^F_\gamma$ associated with γ is the tensor product of Hilbert spaces $\mathcal{H}^F_v$ located at vertices $v \in V(\gamma)$, i.e.,

$$\mathcal{H}^F_\gamma = \bigotimes_{v \in V(\gamma)} \mathcal{H}^F_v,$$

where each $\mathcal{H}^F_v$ is the Hilbert space of a fermionic oscillator of two degrees of freedom, and spanned by the orthonomal basis $\{|0,0\rangle_v, |0,1\rangle_v, |1,0\rangle_v, |1,1\rangle_v\}$; that is, the inner product of $\mathcal{H}^F_v$ reads

$${}_v\langle i_1, i_2 | j_1, j_2 \rangle_v = \delta_{i_1, j_1} \delta_{j_1, j_2}. \tag{2.17}$$

The Hilbert space $\mathcal{H}^F_v$ is graded where each $|i_1, i_2\rangle_v$ carries the degree $\mathfrak{d}(i_1, i_2) = (-1)^{i_1 + i_2}$. For the graded objects, the rule to deal with them, roughly speaking, is that an extra sign factor is added whenever the order of a product of two objects are exchanged. This rule is also applied when the tensor product of $\mathcal{H}^F_v$ and $\mathcal{H}^F_{v'}$ for different $v, v' \in V(\gamma)$ is considered (refer to [26] for more details on the tensor product of graded vector spaces).

With the Hilbert spaces $\mathcal{H}^G_\gamma$ and $\mathcal{H}^F_\gamma$, the Hilbert space $\mathcal{H}_\gamma$ of the entire system on γ is

$$\mathcal{H}_\gamma = \mathcal{H}^G_\gamma \otimes \mathcal{H}^F_\gamma. \tag{2.18}$$

A vector $\Psi \in \mathcal{H}_\gamma$ takes the general form

$$\Psi = \bigotimes_{v \in V(\gamma)} \left(\sum_{i,j} \psi_{v;ij} |i, j\rangle_v \right), \text{ with } \psi_{v;ij} \in \mathcal{H}^G_\gamma. \tag{2.19}$$

The inner product between $\Psi^{(a)} := \bigotimes_{v \in V(\gamma)} \left(\sum_{i,j} \psi^a_{v;ij} |i, j\rangle_v \right)$ (for $a = 1, 2$) is

$$\langle \Psi^{(1)} | \Psi^{(2)} \rangle = \prod_{v \in V(\gamma)} \left(\sum_{i,j} \int \mathrm{d}\mu_h \psi^{(1)}_{v;ij}(\{h_e\}_{e \in E(\gamma)})^* \, \psi^{(2)}_{v;ij}(\{h_e\}_{e \in E(\gamma)}) \right). \tag{2.20}$$

As shown in [10], $\mathcal{H}_\gamma$ admits the spin network decomposition

$$\mathcal{H}_\gamma = \bigoplus_{\vec{j}, \vec{l}} \left(\mathcal{H}^{G,(\vec{j},\vec{l})}_\gamma \otimes \mathcal{H}^F_\gamma \right) \tag{2.21}$$

where $\mathcal{H}^{G,(\vec{j},\vec{l})}_\gamma$ is the spin network subspace of $\mathcal{H}^G_\gamma$ defined by the assignments $\vec{j} = \{j_1, j_2, \cdots, j_{|E(\gamma)|}\}$ and $\vec{l} = \{l_1, \cdots, l_{|V(\gamma)|}\}$ which, respectively, assign to each edge and vertex of γ an irreducible representation of SU(2). Taking advantage of this decomposition, we define the subspace $\widetilde{\mathcal{H}}_\gamma$ of $\mathcal{H}_\gamma$ as

$$\widetilde{\mathcal{H}}_\gamma = \bigoplus_{\vec{j}', \vec{l}} \left(\mathcal{H}^{G,(\vec{j}',\vec{l})}_\gamma \otimes \mathcal{H}^F_\gamma \right). \tag{2.22}$$

where $\vec{j}'$ denotes such assignments that the spin associated to every edge is nonvanishing. Indeed, by introducing $\widetilde{\mathcal{H}}_\gamma$, we remove the states in $\mathcal{H}_\gamma$ which can be represented on a smaller graph. The introduction of $\widetilde{\mathcal{H}}_\gamma$ follows the standard procedure in LQG, where one removes the spin network states carrying vanishing spins for the direct sum decomposition of the total Hilbert space. A difference here is that we concern ourselves with only the cubic lattices so that there is no spurious vertex in our graphs. As a consequence, our case allows the assignments $\vec{l}$ of trivial

irreducible representations to vertices of γ. Taking advantage of the Hilbert spaces $\widetilde{\mathcal{H}}_\gamma$ on all (cubic) graphs γ, we get the total Hilbert space of the entire model as

$$\mathcal{H} = \bigoplus_\gamma \widetilde{\mathcal{H}}_\gamma. \tag{2.23}$$

The flux operators $\hat{p}_k^{v,e}$ on $\mathcal{H}_\gamma^G$ is defined by

$$(\hat{p}_k^{v,e}\psi)(h_{e'}(A),\cdots,h_e(A),\cdots,h_{e''}(A)) = \begin{cases} it\left.\dfrac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \psi(h_{e'}(A),\cdots,e^{-\epsilon\tau^k}h_e(A),\cdots,h_{e''}(A)), & v = s_e, \\ it\left.\dfrac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \psi(h_{e'}(A),\cdots,h_e(A)e^{\epsilon\tau^k},\cdots,h_{e''}(A)), & v = t_e, \end{cases} \tag{2.24}$$

where $\tau^k = (-i/2)(\text{Pauli matrix})^k$, and s_e and t_e denote the source and target points of e respectively, and the dimensionless parameter t is $t = \kappa\hbar/a^2$ with a being some unit of length. The multiplication operators $D^\iota_{ab}(h_e)$ for all $\iota \in \frac{1}{2}\mathbb{Z}_{>0}$ and $-\iota \leq a, b \leq \iota$ acts on $\mathcal{H}_\gamma^G$ as

$$(D^\iota_{ab}(h_e)\psi)(A) = D^\iota_{ab}(h_e(A))\psi(A)$$

where $D^\iota_{ab}(h_e(A))$ are the entries of the Wigner-D matrix $D^\iota(h_e(A))$ of $h_e(A) \in \mathrm{SU}(2)$. The commutators between theses operators are

$$\begin{aligned} [D^\iota(h_e), D^\iota(h_{e'})] &= 0 = [\hat{p}_i^{s_e,e}, p_j^{t_{e'},e'}] \\ [\hat{p}_k^{s_e,e}, \hat{p}_j^{s_{e'},e'}] &= it\delta_{ee'}\epsilon_{kj}{}^l\hat{p}_l^{s_e,e}, \\ [\hat{p}_k^{t_e,e}, \hat{p}_j^{t_{e'},e'}] &= it\delta_{ee'}\epsilon_{kjl}\hat{p}_l^{t_e,e}, \\ [D^\iota(h_{e'}), \hat{p}_j^{s_e,e}] &= it\delta_{ee'}D'^\iota(\tau^j)D^\iota(h_e), \\ [D^\iota(h_{e'}), \hat{p}_j^{t_e,e}] &= -it\delta_{ee'}D^\iota(h_e)D'^\iota(\tau^j). \end{aligned} \tag{2.25}$$

where $D'^\iota(\tau^j)$ is the corresponding representation matrix of τ^j.

On the Hilbert space $\mathcal{H}_\gamma^F$, there are the operators $\hat{\zeta}_{v,A}$ and $\hat{\zeta}^\dagger_{v,A}$ for $A = \pm$ and $v \in V(\gamma)$. For each $v \in V(\gamma)$, the space $\mathcal{H}_v^F$ is where the operators $\hat{\zeta}_{v,A}$ and $\hat{\zeta}^\dagger_{v,A}$ act. The operators $\hat{\zeta}_{v,A}$ and $\hat{\zeta}^\dagger_{v,A}$ are also graded with the degrees $\mathfrak{d}(\hat{\zeta}_{v,A}) = \mathfrak{d}(\hat{\zeta}^\dagger_{v,A}) = 1$. Their action reads

$$\begin{array}{lll} \hat{\zeta}^\dagger_{v,+}|0,i_2\rangle_v = |1,i_2\rangle, & \hat{\zeta}^\dagger_{v,+}|1,i_2\rangle_v = 0, & \forall i_2 = 0,1, \\ \hat{\zeta}_{v,+}|0,i_2\rangle_v = 0, & \hat{\zeta}_{v,+}|1,i_2\rangle_v = |0,i_2\rangle_v, & \forall i_2 = 0,1, \\ \hat{\zeta}^\dagger_{v,-}|i_1,0\rangle_v = (-1)^{i_1}|i_1,1\rangle_v, & \hat{\zeta}^\dagger_{v,-}|i_1,1\rangle_v = 0, & \forall i_1 = 0,1, \\ \hat{\zeta}_{v,-}|i_1,0\rangle_v = 0, & \hat{\zeta}_{v,-}|i_1,1\rangle_v = (-1)^{i_1}|i_1,0\rangle_v, & \forall i_1 = 0,1, \end{array} \tag{2.26}$$

where the extra sign factors are caused by the operators and the states being graded objects. Therefore, the rule for dealing with the graded objects is also applied to the graded operators. For instance, the rule leads to

$$\hat{\zeta}^\dagger_{v,A}|j_1,j_2\rangle_{v'} \otimes |i_1,i_2\rangle_v \otimes |k_1,k_2\rangle_{v''} = (-1)^{\mathfrak{d}(\hat{\zeta}^\dagger_{v,A})\mathfrak{d}(j_1,j_2)}|j_1,j_2\rangle_{v'} \otimes (\hat{\zeta}^\dagger_{v,A}|i_1,i_2\rangle_v) \otimes (|k_1,k_2\rangle_{v''} \tag{2.27}$$

where we exchange the order between $|j_1,j_2\rangle_{v'}$ and $\hat{\zeta}^\dagger_{v,A}$ on the right-hand side.

With $\hat{\zeta}_{v,A}$ and $\hat{\zeta}^\dagger_{v,A}$, we introduce the operator

$$\hat{\theta}_A(v) = \sqrt{\hbar}\,\hat{\zeta}_{v,A}. \tag{2.28}$$

Clearly, they satisfy the anti-commutation relation

$$[\hat{\theta}_A(v), \hat{\theta}^\dagger_B(v')]_+ = \hbar\delta_{AB}\delta_{v,v'} \tag{2.29}$$

where $\delta_{v,v'}$ is the Kronecker delta. In comparison with the classical Poisson brackets (2.6), equation (2.29) changes the Dirac delta $\delta(x,y)$ to Kronecker delta $\delta_{v,v'}$. Indeed, this change results from the canonical transformation [15],

$$\theta(x) = \int_{\mathbb{T}^3} \mathrm{d}^3y\sqrt{\delta(x,y)}\xi(y). \tag{2.30}$$

C. coherent states

1. LQG coherent state

Given $g \in \mathrm{SL}(2,\mathbb{C})$, it labels an SU(2) heat kernel coherent state ψ^t_g defined by

$$\psi^t_g(h) = \frac{t^{3/4} e^{-\frac{p^2}{2t}}}{\sqrt{2}\pi^{\frac{1}{4}} e^{t/8}} \sqrt{\frac{\sinh(p)}{p}} \sum_j d_j e^{-\frac{t}{2} j(j+1)} \chi_j(g h^{-1}), \tag{2.31}$$

where $\chi_j(gh^{-1})$ is the trace of gh^{-1} in the j-representation and $p = \sqrt{\vec{p}\cdot\vec{p}}$ with $g = e^{-i\vec{p}\cdot\vec{\tau}} e^{\vec{\theta}\cdot\vec{\tau}}$. The inner product between two coherent states $\psi^t_{g_1}$ and $\psi^t_{g_2}$ is

$$\langle \psi^t_{g_1} | \psi^t_{g_2} \rangle = \frac{\eta \sqrt{\sinh(p_1)\sinh(p_2)}}{\sqrt{p_1 p_2}\sinh(\eta)} e^{-\frac{p_1^2 + p_2^2 - 2\eta^2}{2t}} (1 + O(t^\infty)) \tag{2.32}$$

where η is defined by $\mathrm{tr}\left(g_1^\dagger g_2\right) = 2\cosh(\eta)$ and requiring the imaginary part of η, i.e. $\mathrm{Im}(\eta)$, is in $[0,\pi]$ [32]. These coherent states form an overcomplete basis of $L^2(\mathrm{SU}(2), \mathrm{d}\mu_H)$ with the measure

$$\mathrm{d}\nu_t(e^{i\vec{p}\cdot\vec{\tau}} e^{\vec{\theta}\cdot\vec{\tau}}) = \frac{2}{\pi t^3} \mathrm{d}\mu_H(e^{\vec{\theta}\cdot\vec{\tau}}) \mathrm{d}^3\vec{p}. \tag{2.33}$$

where $\mathrm{d}\mu_H$ denotes the SU(2) Haar measure, namely,

$$\int \mathrm{d}\nu_t(g) |\psi^t_g\rangle\langle\psi^t_g| = \mathbb{1}_{L^2(\mathrm{SU}(2), \mathrm{d}\mu_H)}, \tag{2.34}$$

Putting on each edge an SU(2) coherent state, one constructs a type of LQG coherent state by their tensor product [30]. Precisely, given $\mathbf{g} = \{g_e\}_{e\in E(\gamma)}$, the LQG coherent state $|\Psi^t_{\mathbf{g}}, \gamma\rangle \in \mathcal{H}^G_\gamma$ is

$$|\Psi^t_{\mathbf{g}}, \gamma\rangle = \bigotimes_{e\in E(\gamma)} \psi^t_{g_e}. \tag{2.35}$$

According to (2.34), the coherent states defined by (2.35) form an overcomplete basis of $\mathcal{H}^G_\gamma$, i.e.,

$$\int \mathcal{D}[\mathbf{g},\gamma] |\Psi^t_{\mathbf{g}}, \gamma\rangle\langle\Psi^t_{\mathbf{g}}, \gamma| = \mathbb{1}_{\mathcal{H}^G_\gamma}, \tag{2.36}$$

with the measure $\mathcal{D}[\mathbf{g},\gamma] = \prod_{e\in E(\gamma)} \mathrm{d}\nu_t(g_e)$. Let us decompose $g_e \in \mathrm{SL}(2,\mathbb{C})$ as $g_e = e^{ip_j(e)\tau^j} u(e)$ with $u \in \mathrm{SU}(2)$. Then, the expectation values of the holonomy and flux operators in the coherent states are

$$\begin{aligned} \langle \Psi^t_{\mathbf{g}}, \gamma | \hat{p}^{s_e,e}_j | \Psi^t_{\mathbf{g}}, \gamma\rangle =& p_j(e) + O(t), \\ \langle \Psi^t_{\mathbf{g}}, \gamma | \hat{p}^{t_e,e} | \Psi^t_{\mathbf{g}}, \gamma\rangle =& - p^{\mathrm{t}}_j(e) + O(t), \\ \langle \Psi^t_{\mathbf{g}}, \gamma | D^{\frac{1}{2}}_{ab}(h_e) | \Psi^t_{\mathbf{g}}, \gamma\rangle =& D^{\frac{1}{2}}_{ab}(u(e)) + O(t), \end{aligned} \tag{2.37}$$

where $p^{\mathrm{t}}_j(e)$, given by $p^{\mathrm{t}}_j(e)\tau^j = u(e)^{-1} p_k(e) \tau^k u(e)$, is the jth component of $u(e)^{-1} p_k(e) \tau^k u(e)$.

Given a graph γ, we need to work in $\widetilde{\mathcal{H}}^G_\gamma$, while the coherent states (2.35) are not elements in $\widetilde{\mathcal{H}}^G_\gamma$. To construct the coherent states in $\widetilde{\mathcal{H}}^G_\gamma$, let us employ the projection $\mathbf{P}_\gamma : \mathcal{H}^G_\gamma \to \widetilde{\mathcal{H}}^G_\gamma$ and define $|\widetilde{\Psi^t_{\mathbf{g}}, \gamma}\rangle \in \widetilde{\mathcal{H}}^G_\gamma$ as

$$|\widetilde{\Psi^t_{\mathbf{g}}, \gamma}\rangle := \mathbf{P}_\gamma |\Psi^t_{\mathbf{g}}, \gamma\rangle = \bigotimes_{e\in E(\gamma)} \widetilde{\psi^t_{g_e}}. \tag{2.38}$$

where $\widetilde{\psi^t_{g_e}}$ is given by

$$\widetilde{\psi^t_g} = \frac{t^{3/4} e^{-\frac{p^2}{2t}}}{\sqrt{2}\pi^{\frac{1}{4}} e^{t/8}} \sqrt{\frac{\sinh(p)}{p}} \sum_{j\neq 0} d_j e^{-\frac{t}{2} j(j+1)} \chi_j(g h^{-1}). \tag{2.39}$$

By definition, $\widetilde{\psi^t_g}$ is just ψ^t_g with removing the component of $j=0$. For $p^2 \gg t$, the norm of this component is much smaller than 1, i.e., the norm of ψ^t_g. As a consequence, $\widetilde{\psi^t_g}$ and, thus, $|\widetilde{\Psi^t_{\mathbf{g}}}, \gamma\rangle$ inherit most properties of ψ^t_g and $|\Psi^t_{\mathbf{g}}, \gamma\rangle$. In particular, $|\widetilde{\Psi^t_{\mathbf{g}}}, \gamma\rangle$ are normalized up to some $O(t^\infty)$ term, i.e.,

$$\langle\widetilde{\Psi^t_{\mathbf{g}}}, \gamma|\widetilde{\Psi^t_{\mathbf{g}}}, \gamma\rangle = 1 + O(t^\infty). \tag{2.40}$$

Moreover, the expectation values of monomials of holonomies, fluxes and volume operators in the states $\widetilde{\psi^t_g}$ and $|\widetilde{\Psi^t_{\mathbf{g}}}, \gamma\rangle$ coincide with the classical values. That is to say, the states $\widetilde{\psi^t_g}$ and $|\widetilde{\Psi^t_{\mathbf{g}}}, \gamma\rangle$ have the desirable semiclassical limit. This is implied by the

$$\langle\Psi^t_{\mathbf{g}}, \gamma|\hat{M}|\Psi^t_{\mathbf{g}}, \gamma\rangle = \langle\widetilde{\Psi^t_{\mathbf{g}}}, \gamma|\hat{M}|\widetilde{\Psi^t_{\mathbf{g}}}, \gamma\rangle + O(t^\infty). \tag{2.41}$$

where $\hat{M}$ a general monomial of fluxes, holonomies and volume operator. The derivations for (2.40) and (2.41) are quite technical and put in Appendix A, in order to keep the flow of our argument..

2. *the fermion coherent state and the coherent state of the entire system*

Give a vertex $v \in V(\gamma)$, the fermion coherent state $|\phi_{\nu_o}\rangle_v$ at v is

$$|\phi_{\nu_o}\rangle_v = \exp\left(-\frac{1}{2}\nu_o^\dagger\nu_o + \sum_{A=\pm}\hat{\zeta}^\dagger_{v,A}\nu_{o,A}\right)|0,0\rangle_v. \tag{2.42}$$

where $\nu_o = (\nu_{o,+}, \nu_{o,-})^T$ is a Grassmann-valued 2-vector, and $\nu_o^\dagger = (\nu^*_{o,+}, \nu^*_{o,-})$ with $\nu^*_{o,A}$ being the complex conjugate of $\nu_{o,A}$. It is easy to verify that $|\phi_{\nu_o}\rangle$ is an eigenstate of $\hat{\zeta}_v$, i.e.,

$$\hat{\zeta}_{v,A}|\psi_{\nu_o}\rangle_v = \nu_{o,A}|\psi_{\nu_o}\rangle_v. \tag{2.43}$$

In addition, the inner product between $|\phi_{\nu_o}\rangle_v$ and $|\phi_{\nu'_o}\rangle_v$ reads

$${}_v\langle\phi_{\nu_o}|\phi_{\nu'_o}\rangle_v = e^{-\frac{1}{2}\nu_o^\dagger\nu_o - \frac{1}{2}\nu_o'^\dagger\nu'_o + \nu_o^\dagger\nu'_o}. \tag{2.44}$$

As usual coherent states, $|\phi_{\nu_o}\rangle_v$ forms a overcomplete basis of $\mathcal{H}^F_v$,

$$\int \mathrm{d}\mu_H(\nu_o)|\phi_{\nu_o}\rangle_v\,{}_v\langle\phi_{\nu_o}| = \mathbb{1}_{\mathcal{H}^F_v}, \tag{2.45}$$

where the measure $\mathrm{d}\mu_H(\nu(v))$ is

$$\mathrm{d}\mu_H(\nu_o) = \prod_{A=\pm}\mathrm{d}\nu^*_{o,A}\mathrm{d}\nu_{o,A}. \tag{2.46}$$

Let $\nu : v \mapsto \nu(v) = (\nu_+(v), \nu_-(v))^T$ be a 2-component Grassmann-vector-valued field on $V(\gamma)$. Taking advantage of the fermion coherent state at each single vertex, we achieve the total fermion coherent state

$$|\Phi_\nu, \gamma\rangle = \bigotimes_{v\in V(\gamma)}|\phi_{\nu(v)}\rangle_v. \tag{2.47}$$

According to (2.45), the overcomplete condition is

$$\int \mathcal{D}[\nu,\gamma]|\Phi_\nu,\gamma\rangle\langle\Phi_\nu,\gamma| = \mathbb{1}_{\mathcal{H}^F_\gamma} \tag{2.48}$$

with

$$\mathcal{D}[\nu,\gamma] = \prod_{v\in V(\gamma)}\prod_{A=\pm}\mathrm{d}\nu^*_A(v)\mathrm{d}\nu_A(v). \tag{2.49}$$

A coherent state of the entire system is labelled by a pair $Z = (\mathbf{g}, \nu)$. The coherent state $|Z, \gamma\rangle \in \mathcal{H}_\gamma$ is just the tensor product

$$|Z, \gamma\rangle = |\Psi_{\mathbf{g}}, \gamma\rangle \otimes |\Phi_\nu, \gamma\rangle. \tag{2.50}$$

According to (2.36) and (2.48), the coherent state $|Z, \gamma\rangle$ admits the overcomplete condition

$$\mathbb{1}_{\mathcal{H}_\gamma} = \int \mathcal{D}[Z, \gamma] |Z, \gamma\rangle\langle Z, \gamma| \tag{2.51}$$

where $\mathcal{D}[Z, \gamma]$ is

$$\mathcal{D}[Z, \gamma] = \mathcal{D}[\nu, \gamma]\mathcal{D}[\mathbf{g}, \gamma]. \tag{2.52}$$

Moreover, the inner product between $|Z, \gamma\rangle$ and $|Z', \gamma\rangle$, by (2.32) and (2.44), is

$$\begin{aligned}\langle Z, \gamma | Z', \gamma\rangle = &\prod_{e\in E(\gamma)} \frac{\eta_e \sqrt{\sinh(p_e)\sinh(p'_e)}}{\sqrt{p_e p'_e}\sinh(\eta_e)} \times \\ &\exp\left(- \sum_{e\in E(\gamma)} \frac{p_e^2 + (p'_e)^2 - 2\eta_e^2}{2t} - \sum_{v\in V(\gamma)} \frac{\nu^\dagger(v)\nu(v) + \nu'^\dagger(v)\nu'(v) - 2\nu^\dagger(v)\nu'(v)}{2} \right)\end{aligned} \tag{2.53}$$

D. Gauge transformation and Hamiltonian operator

The Gauss constraint in terms of $\hat{\zeta}_v$ reads

$$\hat{G}_{v,m} = \hbar \left(\sum_{e \text{ at } v} \frac{\hat{p}^{v,e}_m}{t} + \hat{\zeta}_v^\dagger \frac{\sigma_m}{2} \hat{\zeta}_v \right). \tag{2.54}$$

The gauge transformation generating the Gauss constraint reads [26],

$$u \cdot \Psi = \bigotimes_{v\in V(\gamma)} \left((|1,0\rangle_v, |0,1\rangle_v)\, D^{\frac{1}{2}}(u_v) \begin{pmatrix} \psi^{(u)}_{v;10} \\ \psi^{(u)}_{v;01} \end{pmatrix} + \psi^{(u)}_{v;00}|0,0\rangle + \psi^{(u)}_{v;11}|1,1\rangle \right) \tag{2.55}$$

where $u : v \mapsto u_v \in \mathrm{SU}(2)$ is an $\mathrm{SU}(2)$ valued function on $V(\gamma)$ and $\psi^{(u)}_{v;ij} \in \mathcal{H}^G_\gamma$ is

$$\psi^{(u)}_{v;ij}(\{h_e\}_{e\in E(\gamma)}) = \psi_{v;ij}(\{u(s_e)^{-1} h_e u(t_e)\}). \tag{2.56}$$

Applying the gauge transformation (2.55) to the coherent state $|Z, \gamma\rangle$, one has

$$u|Z, \gamma\rangle = \left(\bigotimes_{e\in E(\gamma)} \psi_{u(s_e) g_e u(t_e)^{-1}} \right) \otimes \left(\bigotimes_{v\in V(\gamma)} |\phi_{u_v \cdot \nu_v}\rangle_v \right) \tag{2.57}$$

Given a graph γ, the Hamiltonian $\hat{\mathbf{H}}_\gamma$ associated with it is

$$\hat{\mathbf{H}}_\gamma = \mathbf{P}_\gamma (\hat{H}^G_\gamma + \hat{H}^F_\gamma) \mathbf{P}_\gamma, \tag{2.58}$$

where $\hat{H}^G_\gamma$ is the graph preserving Hamiltonian constraint operator of vacuum LQG, and $\hat{H}^F_\gamma$ is the graph preserving fermionic Hamiltonian containing the interaction term of gravity and fermion. One can refer to [10, 12] for more details on $\hat{H}^G_\gamma$ whose expression is given by

$$\hat{H}^G_\gamma = \frac{1}{2} \sum_{v\in V(\gamma)} \left(\hat{H}^E_\gamma(v) + (\hat{H}^E_\gamma(v))^\dagger - (1+\beta^2)(\hat{H}^L_\gamma(v) + (\hat{H}^L_\gamma(v))^\dagger) \right) \tag{2.59}$$

with

$$\begin{aligned}\hat{H}^E_\gamma(v) &= \frac{-1}{4i\kappa\beta\ell_P^2}\sum_{e(I),e(J),e(K)\text{ at }v}\epsilon^{IJK}\,\mathrm{tr}\Big(h_{\alpha_{IJ}}h^{-1}_{e(K)}[h_{e(K)},\hat{V}_v]\Big).\\ \hat{H}^L_\gamma(v) &= -\frac{4\kappa}{i\beta^7\ell_P^{10}}\sum_{e(I),e(J),e(K)\text{ at }v}\varepsilon^{IJK}\,\mathrm{tr}\Big([h_{e(I)},[\hat{V}_\gamma,\hat{H}^E_\gamma]]h^{-1}_{e(I)}[h_{e(J)},[\hat{V}_\gamma,\hat{H}^E_\gamma]]h^{-1}_{e(J)}[h_{e(K)},\hat{V}_v]h^{-1}_{e(K)}\Big)\end{aligned} \tag{2.60}$$

where α_{IJ} is the minimal loop in γ containing edges $e(I)$ and $e(J)$, $\hat{V}_v$ is the volume operator at the vertex v so that the total volume is $\hat{V}_\gamma=\sum_{v\in V(\gamma)}\hat{V}_v$ and $\hat{H}^E_\gamma=\sum_{v\in V(\gamma)}\hat{H}^E_\gamma(v)$. For $\hat{H}^F_\gamma$, we employ the one introduced in [26] but need to adapt it to make it graph-preserving. One has

$$\hat{H}^F_\gamma=\sum_{v\in V(\gamma)}\hat{H}_F(v) \tag{2.61}$$

where $\hat{H}_F(v)$ is

$$\hat{H}_F(v)=\frac{1}{2}\widehat{\sqrt{V_v^{-1}}}\left(i\left(\hat{H}^{(1)}_F(v)-\hat{H}^{(1)}_F(v)^\dagger\right)-\beta\hat{H}^{(2)}_F(v)-\frac{1+\beta^2}{\beta}\hat{H}^{(3)}_F(v)-\beta(\hat{H}^{(1)}_F(v)+\hat{H}^{(1)}_F(v)^\dagger)\right)\widehat{\sqrt{V_v^{-1}}} \tag{2.62}$$

with the explicit expression of $\hat{H}^{(i)}_F(v)$ for $i=1,2,3$ as

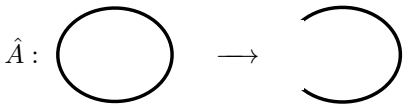
$$\begin{aligned}\hat{H}^{(1)}_F(v) =&\frac{\kappa\hbar\beta}{2t}\sum_{e\text{ at }v}\hat{\theta}^\dagger(v)\sigma^i\left(h_e\hat{\theta}(t_e)-\hat{\theta}(v)\right)\hat{p}^{v,e}_i,\\ \hat{H}^{(2)}_F(v) =&\frac{1}{i\hbar\beta^2}\left[\hat{H}^E_\gamma(v),\hat{V}_v\right]\hat{\theta}^\dagger(v)\hat{\theta}(v),\\ \hat{H}^{(3)}_F(v) =&\frac{\kappa\hbar\beta}{2t}\left(\sum_{e\text{ at }v}\hat{p}^{v,e}_i\right)\hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v).\end{aligned} \tag{2.63}$$

It is worth noting that the extra factors $1/2$ in $\hat{H}^{(1)}_F(v)$ and $\hat{H}^{(3)}_F(v)$ compared to the results in [26] come from the fact that due to each edge containing two vertices, $\hat{H}^{(1)}_F(v)$ and $\hat{H}^{(3)}_F(v)$ are counted twice in the summation over $v\in V(\gamma)$ in $\hat{H}^F_\gamma$. Moreover, Eq. (2.62) uses the inverse volume operator defined by [48]

$$\hat{V}_v^{-\frac{1}{2}}=-\frac{16}{3}\left(\frac{1}{i\ell_P^2\beta}\right)^3\sum_{e(I),e(J),e(K)\text{ at }v}\epsilon^{IJK}\,\mathrm{tr}\left([\hat{h}_{e(I)},\hat{V}_v^{1/2}]\hat{h}^{-1}_{e(I)}[\hat{h}_{e(J)},\hat{V}_v^{1/2}]\hat{h}^{-1}_{e(J)}[\hat{h}_{e(K)},\hat{V}_v^{1/2}]h^{-1}_{e(K)}\right). \tag{2.64}$$

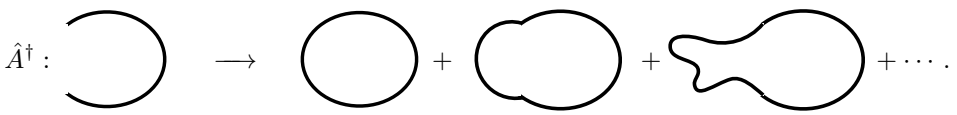
Indeed, $\hat{V}_v^{-\frac{1}{2}}$ is not always positive semiclassically, where the sign relates to the orientation $\mathrm{sgn}(\det(e))$. However, the sign is canceled in the semiclassical limit of $\hat{H}_F(v)$, since it always contains a pair of $\hat{V}_v^{-\frac{1}{2}}$.

The Hamiltonian $\hat{\mathbf{H}}_\gamma$ given by (2.58) is the same as the regular graph-preserving Hamiltonian up to the projection $\mathbf{P}_\gamma$ which is introduced based on the following considerations. At first, one needs to consider that a general operator $\hat{A}$ cannot annihilate edges of graphs if we require a densely defined adjoint operator $\hat{A}^\dagger$ in the regular LQG kinematic Hilbert space, which contains not only cubic but also all other graphs. To explain this point, let us assume an operator $\hat{A}$ which changes each graph γ to a new graph $\hat{A}[\gamma]$ by erasing an edge e_γ of γ, i.e., $\hat{A}[\gamma]=\gamma-\{e_\gamma\}$. Fix a graph $\mathring{\gamma}$ and consider the action of the adjoint $\hat{A}^\dagger$ on $A[\mathring{\gamma}]$. By definition, the result of $A[\mathring{\gamma}]$ acted by $\hat{A}^\dagger$, denoted by $\hat{A}^\dagger[A[\mathring{\gamma}]]$, is a linear combination of such graphs γ that $A[\gamma]=A[\mathring{\gamma}]$. Consequently, $\hat{A}^\dagger[A[\mathring{\gamma}]]$ cannot be defined because there are uncountable infinitely many such graphs γ. As an example, let us consider an operator $\hat{A}$ which always erases a segment of a loop, like



(2.65)

Then, one has



(2.66)

Indeed, in $\hat{H}^E_\gamma(v)$, the loop holonomy $h_{\alpha_{IJ}}$ (see (2.60)) to regularize the curvature plays the same role as $\hat{A}$ in the above example for the loop which has a segment e carrying a small spin $j_e = 1/2$. Second, the operators $\hat{H}^G_\gamma$ and $\hat{H}^F_\gamma$ can change graphs just by the holonomy operators therein, because these holonomies shift spins and, thus, can annihilate edges carrying spin 1/2. Thus, the projection in $\hat{\mathbf{H}}_\gamma$ makes $\hat{\mathbf{H}}_\gamma$ different from $\hat{H}^G_\gamma + \hat{H}^F_\gamma$ by their actions on spin network states with small spins, while these states are purely quantum. Thus, the difference between $\hat{\mathbf{H}}_\gamma$ and $\hat{H}^G_\gamma + \hat{H}^F_\gamma$ can only be presented at the quantum geometry level.

Taking advantage of $\hat{\mathbf{H}}_\gamma$, the Hamiltonian operator associated with a single graph, we define the total physical Hamiltonian which could act on the total Hilbert space $\mathcal{H}$ as

$$\hat{\mathbf{H}} = \sum_\gamma \hat{\mathbf{H}}_\gamma. \tag{2.67}$$

Due to projection operator, $\hat{\mathbf{H}}_\gamma$ kills the Hilbert spaces $\widetilde{\mathcal{H}}_{\gamma'}$ for all $\gamma' \neq \gamma$. Thus, the action of $\hat{\mathbf{H}}$ on $\widetilde{\mathcal{H}}_\gamma$ is the same as that of $\hat{\mathbf{H}}_\gamma$. This fact will bring us great convenience for our future calculations.

III. PATH INTEGRAL FORMULATION ON A SINGLE GRAPH

For getting the transition amplitude between arbitrary states in $\widetilde{\mathcal{H}}_\gamma$ for a given γ, it is sufficient to calculate

$$A^\gamma_{Z',Z} = \langle Z', \gamma | \exp\left[-\frac{i}{\hbar} T \hat{\mathbf{H}}_\gamma\right] | Z, \gamma\rangle, \tag{3.1}$$

due to $\widetilde{\mathcal{H}}_\gamma$ being a subspace of $\mathcal{H}_\gamma$ in which $|Z,\gamma\rangle$ forms an overcomplete basis. Note that despite $|Z,\gamma\rangle$ being not in $\widetilde{\mathcal{H}}_\gamma$, $\hat{\mathbf{H}}_\gamma$ still has a well-defined action on it. Dividing the time interval $[0,T]$ into N slices with length $\delta\tau$ such that $T = N\delta\tau$, inserting the resolution of identity (2.51), and employing the approximation $\exp\left(-\frac{i}{\hbar}\delta\tau\hat{\mathbf{H}}_\gamma\right) \cong 1 - \frac{i}{\hbar}\delta\tau\hat{\mathbf{H}}_\gamma$, we finally get

$$A^{\gamma,N}_{Z',Z} = \int \left(\prod_{i=1}^{N-1} \mathcal{D}[Z_i,\gamma]\right) \prod_{i=0}^{N-1} \langle Z_{i+1},\gamma|Z_i,\gamma\rangle \exp\left[-\frac{i}{\hbar}\sum_{i=0}^{N-1}\delta\tau \frac{\langle Z_{i+1},\gamma|\hat{\mathbf{H}}_\gamma|Z_i,\gamma\rangle}{\langle Z_{i+1},\gamma|Z_i,\gamma\rangle}\right] \tag{3.2}$$

with $Z_0 = Z$ and $Z_N = Z'$. By employing (2.53), we get

$$\begin{aligned}\langle Z_{i+1},\gamma|Z_i,\gamma\rangle =& \xi(\mathbf{g}_{i+1},\mathbf{g}_i)\exp\left(-\sum_{e\in E(\gamma)}\frac{p_{i+1}(e)^2 + p_i(e)^2 - 2\eta_{i+1,i}(e)}{2t}\right)\\ &\exp\left(-\sum_{v\in V(\gamma)}\left(\frac{1}{2}\nu^\dagger_{i+1}(v)\nu_{i+1}(v) + \frac{1}{2}\nu^\dagger_i(v)\nu_i(v) - \nu^\dagger_{i+1}(v)\nu_i(v)\right)\right)\end{aligned} \tag{3.3}$$

where $p_i(e)$ parametrize $\mathbf{g}_i(e)$ as above, $\eta_{i+1,i}(e)$ is given correspondingly by $\mathrm{tr}\big(\mathbf{g}_{i+1}(e)^\dagger \mathbf{g}_i(e)\big) = 2\cosh(\eta_{i+1,i}(e))$, and $\xi(\mathbf{g}_{i+1},\mathbf{g}_i)$ is

$$\xi(\mathbf{g}_{i+1},\mathbf{g}_i) = \prod_{e\in E(\gamma)} \frac{\eta_{i+1,i}(e)\sqrt{\sinh(p_{i+1}(e))\sinh(p_i(e))}}{\sqrt{p_{i+1}(e)p_i(e)}\sinh(\eta_{i+1,i}(e))}. \tag{3.4}$$

Substituting (3.3) into the expression of $A^{\gamma,N}_{Z',Z}$, we have

$$A^{\gamma,N}_{Z',Z} = \int_{\mathcal{R}(Z',Z)} \prod_{i=1}^{N-1} \mathcal{D}[Z_i,\gamma]\Xi(\mathbf{g},\gamma) e^{S_N[\mathbf{g},\nu]/t}, \tag{3.5}$$

where $\mathcal{R}_\gamma(Z',Z)$ is the set of all polygon paths satisfying $Z[0] = Z$ and $Z[T] = Z'$, $\Xi[\mathbf{g}]$ is given by $\Xi[\mathbf{g}] = \prod_{i=0}^{N-1}\xi(\mathbf{g}_{i+1},\mathbf{g}_i)$, and S_N is given by

$$S_N[\mathbf{g},\nu] = \sum_{i=0}^{N-1} K_\gamma(Z_{i+1},Z_i) - \frac{i\kappa}{a^2}\delta\tau\sum_{i=0}^{N-1}\frac{\langle Z_{i+1},\gamma|\hat{\mathbf{H}}_\gamma|Z_i,\gamma\rangle}{\langle Z_{i+1},\gamma|Z_i,\gamma\rangle}, \tag{3.6}$$

with

$$K_\gamma(Z_{i+1},Z_i)=\sum_{e\in E(\gamma)}\left(\eta_{i+1,i}(e)^2-\frac{1}{2}\left(p_{i+1}(e)^2+p_i(e)^2\right)\right)\\ +t\sum_{v\in V(\gamma)}\left(\nu^\dagger_{i+1}(v)\nu_i(v)-\frac{1}{2}\left(\nu^\dagger_{i+1}(v)\nu_{i+1}(v)+\nu^\dagger_i(v)\nu_i(v)\right)\right). \tag{3.7}$$

Taking advantage of $A^{\gamma,N}_{Z',Z}$, we have

$$A^\gamma_{Z',Z}=\lim_{N\to\infty}A^{\gamma,N}_{Z',Z}. \tag{3.8}$$

With $A^\gamma_{Z',Z}$, It can be obtained for the transition amplitude between any initial state $|\Psi_{\rm in},\gamma\rangle$ and final state $|\Psi_{\rm out},\gamma\rangle$. Here, $|\Psi_{\rm in},\gamma\rangle$ and $|\Psi_{\rm out},\gamma\rangle$ can be in either $\mathcal{H}_\gamma$ or $\widetilde{\mathcal{H}}_\gamma$. More precisely, the transition amplitude from $|[\Psi_{\rm in}],\gamma\rangle$ to $|[\Psi_{\rm out}],\gamma\rangle$ is

$$\begin{aligned}&\langle[\Psi_{\rm out}],\gamma|\exp\left[\frac{i}{\hbar}T\hat{\mathbf{H}}_\gamma\right]|[\Psi_{\rm in}],\gamma\rangle\\ =&\mathcal{N}^{(\rm in)}_\gamma\mathcal{N}^{(\rm out)}_\gamma\int{\rm d}\mu_H(u)\mathcal{D}[Z',\gamma]\mathcal{D}[Z,\gamma]\langle\Psi_{\rm out},\gamma|Z',\gamma\rangle A^\gamma_{Z',Z}\langle Z,\gamma|u|\Psi_{\rm in},\gamma\rangle,\end{aligned} \tag{3.9}$$

where $|[\Psi_{\rm in}],\gamma\rangle$ and $|[\Psi_{\rm out}],\gamma\rangle$ denote the gauge invariant correspondence of $|\Psi_{\rm in},\gamma\rangle$ and $|\Psi_{\rm out},\gamma\rangle$, i.e.,

$$|[\Psi_s],\gamma\rangle=\mathcal{N}^{(s)}_\gamma\int{\rm d}\mu_H(u)\,u|\Psi_s,\gamma\rangle,\ \text{for } s={\rm in},{\rm out}, \tag{3.10}$$

with $\mathcal{N}^{(s)}_\gamma$ being the normalization factor

$$\mathcal{N}^{(s)}_\gamma=\left|\int{\rm d}\mu_H(u)\langle\Psi_s,\gamma|u|\Psi_s,\gamma\rangle\right|^{-\frac{1}{2}}. \tag{3.11}$$

The current work will concern ourselves with the n-point correlation functions which are objects taking the form

$$\begin{aligned}B=&\langle Z',\gamma|\prod_{j=1}^n\exp\left[\frac{i}{\hbar}\hat{\mathbf{H}}_\gamma T_j\right]\hat{O}_j\exp\left[-\frac{i}{\hbar}\hat{\mathbf{H}}_\gamma T_j\right]|Z,\gamma\rangle\\ =&\langle Z',\gamma|\exp\left[\frac{i}{\hbar}\hat{\mathbf{H}}_\gamma T_n\right]\prod_{j=1}^n\hat{O}_j\exp\left[-\frac{i}{\hbar}\hat{\mathbf{H}}_\gamma(T_j-T_{j-1})\right]|Z,\gamma\rangle\end{aligned} \tag{3.12}$$

where the sequence of moments T_k for $0\le k\le n$ satisfy $T_n\ge T_{n-1}\ge\cdots\ge T_0=0$ and $\hat{O}_k$ is an operator polynomial of holonomies, volume operators and fermion field operators. The flow of evolution contained in B is forward between T_0 and T_n but backward between T_n and T_{n+1}. Since a backward evolution is equivalent a forward one with a minus Hamiltonian, we can rewrite B as

$$B=\langle Z',\gamma|\exp\left[-\frac{i}{\hbar}\hat{\mathbf{H}}^{(T)}_\gamma(T_{n+1}-T_n)\right]\prod_{j=1}^n\hat{O}_j\exp\left[-\frac{i}{\hbar}\hat{\mathbf{H}}^{(T)}_\gamma(T_j-T_{j-1})\right]|Z,\gamma\rangle \tag{3.13}$$

with $T_{n+1}=2T_n\ge T_n$, where the time-dependent Hamiltonian $\hat{\mathbf{H}}^{(T)}_\gamma$ is

$$\hat{\mathbf{H}}^{(T)}_\gamma=\begin{cases}\hat{\mathbf{H}}_\gamma, & T<T_n\\ -\hat{\mathbf{H}}_\gamma, & T>T_n.\end{cases} \tag{3.14}$$

In (3.13), the evolution flow becomes forward all the time, while the price is the time-dependent Hamiltonian $\hat{\mathbf{H}}^{(T)}_\gamma$. Now B can be calculated by applying the standard path integral formulation. We only need to proceed with steps analogous to calculating $A^\gamma_{Z,Z'}$; we divide the interval $[T_{k-1},T_k]$ for all $1\le k\le n+1$ into N_k slices such that $T_k-T_{k-1}=N_k\delta\tau$ and then insert resolution of identity. Since the Hamiltonian in the present case becomes time

dependent, the final path integral formula will involve an action slightly different from the previous one in $A^{\gamma}_{Z,Z'}$. With straightforward derivation, one obtains the final result taking the form.

$$B = \lim_{N_i\to\infty}\int_{\mathcal{R}(Z',Z)}\prod_{i=1}^{N+n-1}\mathcal{D}[Z_i,\gamma]\left(\prod_{k=1}^{n}\frac{\langle Z_{\mathfrak{f}(k)+1},\gamma|\hat{O}_k|Z_{\mathfrak{f}(k)},\gamma\rangle}{\langle Z_{\mathfrak{f}(k)+1},\gamma|Z_{\mathfrak{f}(k)},\gamma\rangle}\right)\Xi[\mathbf{g}]e^{\frac{1}{t}\widetilde{S}_{\vec{N}}[\mathbf{g},\nu]} \tag{3.15}$$

where $N=\sum_{j=1}^{n+1}N_j$, an extra resolution of identity is inserted at each moment T_k to sandwich $\hat{O}_k$, $\mathfrak{f}(k)=k-1+\sum_{i=1}^{k}N_i$ is employed to count the resolution of identity before $\hat{O}_k$ and the action $\widetilde{S}_{\vec{N}}[\mathbf{g},\nu]$ for $\vec{N}=(N_1,N_2,\cdots,N_{n+1})$ is

$$\widetilde{S}_{\vec{N}}[\mathbf{g},\nu]=\sum_{j=0}^{N+n-1}K_{\gamma}(Z_{j+1},Z_j)-\frac{i\kappa}{a^2}\sum_{j=0}^{N+n-1}\delta\tau_j\frac{\langle Z_{j+1},\gamma|\hat{\mathbf{H}}_{\gamma}|Z_j,\gamma\rangle}{\langle Z_{j+1},\gamma|Z_j,\gamma\rangle} \tag{3.16}$$

with $Z_0=Z$, $Z_{N+n}=Z'$ and $\delta\tau_j$ given by

$$\delta\tau_j=\begin{cases}0 & j=\mathfrak{f}(k),\ \forall 1\leq k\leq n,\\ \delta\tau & j\leq\mathfrak{f}(n)-1 \text{ and } j\neq\mathfrak{f}(k),\ \forall 1\leq k\leq n,\\ -\delta\tau & \mathfrak{f}(n)+1\leq j\leq N+n-1.\end{cases} \tag{3.17}$$

In (3.17), $\delta\tau_j$ takes $-\delta\tau$ due to the Hamiltonian taking $-\hat{\mathbf{H}}_{\gamma}$ after T_n; and $\delta\tau_{\mathfrak{f}(k)}$ takes 0 because the matrix element of $\hat{O}_k$ contributes only the kinetic term to the action.

Finally, with (3.15), one can formulate $\exp\left[-\frac{i}{\hbar}\hat{\mathbf{H}}^{(T)}_{\gamma}(T_{n+1}-T_n)\right]\prod_{j=1}^{n}\hat{O}_j\exp\left[-\frac{i}{\hbar}\hat{\mathbf{H}}^{(T)}_{\gamma}(T_j-T_{j-1})\right]$ sandwiched by any initial and final states, analogous to (3.9).

IV. MINKOWSKI VACUUM ON CUBIC LATTICES

We seek some vacuum state in which the quantum gravity state resembles the classical Minkowski geometry. A natural candidate is a state comprising the coherent states (see Sec. II C) peaked at the Minkowski geometry. To employ these coherent states, an issue that the coherent states are graph-dependent arises so that one seems have to choose some preferred graphs. Then physical results might depend on the choice of graphs. To minimize the dependence on the choice of graphs, we propose considering the superposition of graphs. Furthermore, since the concerning background is semiclassical, it is reasonable to prioritize those graphs that reflect the semiclassical properties better than others. Taking into account of the topology of Σ and the desirable form of the expectation value of the Hamiltonian operator on cubic lattice [46], we would consider superposition states in which cubic lattices carry significant weight. As a typical representative of such states, the one comprising only cubic graphs is thus employed.

As discussed above, given a cubic graph γ, the vacuum state $|\Omega,\gamma\rangle\in\widetilde{\mathcal{H}}_{\gamma}$ on γ should take the form

$$|\Omega,\gamma\rangle=\mathbf{P}_{\gamma}|\Psi^t_{\mathbf{g}^{\gamma}_M},\gamma\rangle\otimes|\omega,\gamma\rangle=\widetilde{|\Psi^t_{\mathbf{g}^{\gamma}_M},\gamma\rangle}\otimes|\omega,\gamma\rangle \tag{4.1}$$

where $|\Psi^t_{\mathbf{g}^{\gamma}_M},\gamma\rangle\in\mathcal{H}^G_{\gamma}$ is the coherent state peaking at the Minkowski geometry on γ and $|\omega,\gamma\rangle$ is some state in $\mathcal{H}^F_{\gamma}$. To define $\mathbf{g}^{\gamma}_M$ precisely, we need to introduce a Euclidean metric δ_{ij} on Σ such that the perimeter of Σ along each axis is ℓ_o. Then, for γ being the cubic graph with edges parallel to the axes and the fiducial lattice spacing μ in the fiducial metric, $\mathbf{g}^{\gamma}_M$ is given by

$$\mathbf{g}^{\gamma}_M(e)=e^{-i\mathring{p}_{\gamma}\tau_e} \tag{4.2}$$

where $\tau_e=\tau_j$ for those edges e along the j-axis direction and $\mathring{p}_{\gamma}=P\mu^2/(a^2\beta)$ with $P\delta_{ij}$ being the dynamical interior geometry on γ endowed by $|\widetilde{\Psi^t_{\mathbf{g}^{\gamma}_M}},\gamma\rangle$. In (4.1), the state is defined in $\widetilde{\mathcal{H}}_{\gamma}$, setting the stage for superposition over graphs. Moreover, as discussed at the end of Sec. II C 1, the projection $\mathbf{P}_{\gamma}$ will not destroy the semiclassical features of $|\Psi^t_{\mathbf{g}^{\gamma}_M},\gamma\rangle$. That is to say, $|\widetilde{\Psi^t_{\mathbf{g}^{\gamma}_M},\gamma}\rangle$ can play the role of $|\Psi^t_{\mathbf{g}^{\gamma}_M},\gamma\rangle$ to endow the graph γ with a semiclassical Minkowski geometry. For $|\omega,\gamma\rangle$, its role indicate that $|\Omega,\gamma\rangle$ is a vacuum. Thus $|\omega,\gamma\rangle$ is defined as the ground state of the effective fermion Hamiltonian $\hat{H}^{\text{eff}}_{F,\gamma}$ in the semiclassical Minkowski background, where $\hat{H}^{\text{eff}}_{F,\gamma}$ is given by

$$\langle\Phi|\hat{H}^{\text{eff}}_{F,\gamma}|\Phi'\rangle=-\langle\widetilde{\Psi^t_{\mathbf{g}^{\gamma}_M}}\otimes\Phi,\gamma|\mathbf{P}_{\gamma}\hat{H}^F_{\gamma}\mathbf{P}_{\gamma}|\widetilde{\Psi^t_{\mathbf{g}^{\gamma}_M}}\otimes\Phi',\gamma\rangle,\ \forall\Phi,\Phi'\in\mathcal{H}^F_{\gamma}. \tag{4.3}$$

This definition together with (2.41) leads $\hat{H}^{\rm eff}_{F,\gamma}$ to be

$$\hat{H}^{\rm eff}_{F,\gamma} = \frac{i\hbar}{2}\frac{1}{a\sqrt{\mathring{p}_\gamma\beta}}\sum_{v\in V(\gamma)}\sum_{e\in E_v(\gamma)}\left(\hat{\zeta}^\dagger_v\sigma^e\hat{\zeta}_{v+\delta_e} - \hat{\zeta}^\dagger_{v+\delta_e}\sigma^e\hat{\zeta}_v\right) \tag{4.4}$$

where $E_v(\gamma) = \{e^+_x(v), e^+_y(v), e^+_z(v)\}$ is the set of the three edges starting from v and directing to the positive directions, and δ_e and σ^e for the edge e along the kth direction are defined by $(\delta_e)_i := \delta_{k,i}$ and $\sigma^e := \sigma^k$.

To diagonalize $\hat{H}^{\rm eff}_{F,\gamma}$, we introduce the operators

$$\hat{\xi}_{\vec{k}} = \Theta(\vec{k})\widehat{\widehat{\zeta}}_{\vec{k}} \tag{4.5}$$

where $\widehat{\widehat{\zeta}}_{\vec{k}}$ is the Fourier transformation of $\hat{\zeta}_v$, i.e.,

$$\widehat{\widehat{\zeta}}_{\vec{k}} = \sqrt{\frac{\mu^3}{\ell_o^3}}\sum_{v\in V(\gamma)}\hat{\zeta}_v e^{-i\frac{2\pi}{\ell_o}\vec{k}\cdot\vec{x}_v}$$

with $\vec{x}_v \in (\mu\mathbb{Z})^3$ denoting the coordinate of v, and $\Theta(\vec{k})$ is

$$\Theta(\vec{k}) = \begin{pmatrix} \frac{\sin\left(\frac{2\pi\mu}{\ell_o}k^3\right)-\mathfrak{s}(\vec{k})}{\sin\left(\frac{2\pi\mu}{\ell_o}k^1\right)-i\sin\left(\frac{2\pi\mu}{\ell_o}k^2\right)}\sqrt{\frac{\mathfrak{s}(\vec{k})+\sin\left(\frac{2\pi\mu}{\ell_o}k^3\right)}{2\mathfrak{s}(\vec{k})}}, & \sqrt{\frac{\mathfrak{s}(\vec{k})+\sin\left(\frac{2\pi\mu}{\ell_o}k^3\right)}{2\mathfrak{s}(\vec{k})}} \\ \frac{\sin\left(\frac{2\pi\mu}{\ell_o}k^3\right)+\mathfrak{s}(\vec{k})}{\sin\left(\frac{2\pi\mu}{\ell_o}k^1\right)-i\sin\left(\frac{2\pi\mu}{\ell_o}k^2\right)}\sqrt{\frac{\mathfrak{s}(\vec{k})-\sin\left(\frac{2\pi\mu}{\ell_o}k^3\right)}{2\mathfrak{s}(\vec{k})}}, & \sqrt{\frac{\mathfrak{s}(\vec{k})-\sin\left(\frac{2\pi\mu}{\ell_o}k^3\right)}{2\mathfrak{s}(\vec{k})}} \end{pmatrix} =: \begin{pmatrix} \Theta(\vec{k})_{+,+} & \Theta(\vec{k})_{+,-} \\ \Theta(\vec{k})_{-,+} & \Theta(\vec{k})_{-,-} \end{pmatrix}, \tag{4.6}$$

with

$$\mathfrak{s}(\vec{k}) = \sqrt{\sum_{m=1}^3 \sin^2\left(\frac{2\pi\mu}{\ell_o}k^m\right)}. \tag{4.7}$$

Here, the total number $\ell_o/\mu = N$ of vertices along each direction is assumed to be even[1]. The range of k^a for $a = 1, 2, 3$ is chosen as $k^a \in K[\gamma] = [-\frac{\ell_o}{2\mu}, \frac{\ell_o}{2\mu} - 1] \cap \mathbb{Z}$, i.e. $\vec{k} \in K[\gamma]^3$. Moreover, the matrix $\Theta(\vec{k})$ is actually introduced to diagonalize the matrix $\sum_{j=1}^3 \sin\left(\frac{2\pi\mu}{\ell_o}k^j\right)\sigma^j$, i.e.,

$$\sum_{j=1}^3 \sin\left(\frac{2\pi\mu}{\ell_o}k^j\right)\sigma^j = \Theta(\vec{k})^\dagger \begin{pmatrix} -\mathfrak{s}(\vec{k}) & 0 \\ 0 & \mathfrak{s}(\vec{k}) \end{pmatrix}\Theta(\vec{k}). \tag{4.8}$$

A straightforward calculation gives the properties of the operators $\hat{\xi}_{\vec{k},A}$

$$\begin{aligned} [\hat{\xi}_{\vec{k},A}, \hat{\xi}_{\vec{k}',B}]_+ &= 0 = [\hat{\xi}^\dagger_{\vec{k},A}, \hat{\xi}^\dagger_{\vec{k}',B}]_+, \\ [\hat{\xi}^\dagger_{\vec{k},A}, \hat{\xi}_{\vec{k}',B}]_+ &= \delta_{AB}\delta_{\vec{k},\vec{k}'}, \\ \hat{\xi}_{\vec{k},A}|O,\gamma\rangle &= 0, \end{aligned} \tag{4.9}$$

where $|O,\gamma\rangle$ denotes the unphysical vacuum state, i.e.,

$$|O,\gamma\rangle = \bigotimes_{v\in V(\gamma)}|0,0\rangle_v. \tag{4.10}$$

The effective Hamiltonian operator $\hat{H}^{\rm eff}_{F,\gamma}$ in terms of $\hat{\xi}_{\vec{k}}$ is

$$\hat{H}^{\rm eff}_{F,\gamma} = \frac{\hbar}{a\sqrt{\mathring{p}_\gamma\beta}}\sum_{\vec{k}\in K[\gamma]^3}\left(\mathfrak{s}(\vec{k})\hat{\xi}^\dagger_{\vec{k},+}\hat{\xi}_{\vec{k},+} - \mathfrak{s}(\vec{k})\hat{\xi}^\dagger_{\vec{k},-}\hat{\xi}_{\vec{k},-}\right). \tag{4.11}$$

---

[1] As shown below, the lattice we consider will be the refinement of an initial cubic lattice γ_0. Given the initial cubic lattice γ_0 with N_0^3 vertices, after the lattice refinements of n-steps, the refined cubic lattice γ_n has N_n^3 vertices, where $N_n = 2^n N_0$ is always even for $n \geq 1$. Therefore we are mostly interested in the case with an even number of vertices in each direction on the lattice. Moreover, the assumption also guarantees that the vacuum state is Bosonic.

Due to the algebra (4.9), the operator $\hat{\xi}^\dagger_{\vec{k},A}\hat{\xi}_{\vec{k},A}$ for each $A=\pm$ has the eigenvalues 0 and 1. Thus the ground state $|\omega,\gamma\rangle$ of $\hat{H}^{\rm eff}_{F,\gamma}$ satisfies

$$\hat{\xi}^\dagger_{\vec{k},-}|\omega,\gamma\rangle = 0 = \hat{\xi}_{\vec{k},+}|\omega,\gamma\rangle, \tag{4.12}$$

and takes eigenvalue

$$\omega(\gamma) = -\frac{\hbar}{a\sqrt{\mathring{p}_\gamma\beta}}\sum_{\vec{k}\in K[\gamma]^3}\mathfrak{s}(\vec{k}). \tag{4.13}$$

According to (4.12), $|\omega,\gamma\rangle$ can be expressed explicitly as

$$|\omega,\gamma\rangle = \mathrm{sgn}(\gamma)\prod_{\vec{k}\in K[\gamma]^3}\hat{\xi}^\dagger_{\vec{k},-}|O,\gamma\rangle, \tag{4.14}$$

with $\mathrm{sgn}(\gamma)$ being either 1 or -1 such that

$$\langle\omega,\gamma|\Phi_\nu,\gamma\rangle = \exp\left(-\sum_{v\in V(\gamma)}\frac{1}{2}\nu^\dagger(v)\nu(v)\right)\prod_{\vec{k}\in K[\gamma]^3}\sum_{B=\pm}\Theta(\vec{k})_{-,B}\widetilde{\nu}_B(\vec{k}). \tag{4.15}$$

where $\tilde{\nu}$ is the Fourier transformation of ν,

$$\widetilde{\nu}(\vec{k}) = \sqrt{\frac{\mu^3}{\ell_o^3}}\sum_{v\in V(\gamma)}\nu(v)e^{-i\frac{2\pi}{\ell_o}\vec{k}\cdot\vec{x}_v}. \tag{4.16}$$

Indeed, one can verify easily that

$$\langle\omega,\gamma|\Phi_\nu,\gamma\rangle = (\text{a sign factor})\times\mathrm{sgn}(\gamma)\exp\left(-\sum_{v\in V(\gamma)}\frac{1}{2}\nu^\dagger(v)\nu(v)\right)\prod_{\vec{k}\in K[\gamma]^3}\sum_{B=\pm}\Theta(\vec{k})_{-,B}\widetilde{\nu}_B(\vec{k}) \tag{4.17}$$

which allows us to choose a convention of $\mathrm{sgn}(\gamma)$ such that (4.15) holds. The assumption of the even number of vertices in each lattice ensures that there are an even number of modes $\vec{k}$ in $K[\gamma]^3$. Therefore, the ground state given by (4.14) is Bosonic.

The final vacuum state $|\Omega\rangle$ will be the superposition of $|\Omega,\gamma\rangle$ on various cubic graphs. To define $|\Omega\rangle$ precisely, let us fix an initial lattice $\gamma_1=\gamma$ with fiducial lattice spacing $\mu_1\equiv\mu$ and define γ_n as a lattice refinement of γ_1 such that its the fiducial lattice spacing is μ_1/n (see Fig. 1). Let $L\gg 1$ be the total number of the refinements. Then, the final vacuum state is defined as

$$|\Omega\rangle = \sum_{n=1}^{n} w_n|\Omega,\gamma_n\rangle \tag{4.18}$$

where w_n is the satisfying $\sum_{n=1}^{L}|w_n|^2=1$. Here the vacuum state is defined for the future calculation of the propagator . The requirement that γ_n is a lattice refinement of γ_1 guarantees that all graphs in the superposition contribute to the final result. Note that $|\Omega\rangle$ is not gauge invariant. The gauge invariant vacuum $|[\Omega]\rangle$ is given by the group averaging of $|\Omega\rangle$, i.e.,

$$|[\Omega]\rangle = \sum_{n=1}^{L} w_n|[\Omega],\gamma_n\rangle \tag{4.19}$$

where $|[\Omega],\gamma_n\rangle$ is the normalized gauge invariant projection of $|\Omega,\gamma\rangle$, i.e,

$$|[\Omega],\gamma_n\rangle = \mathcal{N}_n\int \mathrm{d}\mu_H(u)u|\Omega,\gamma\rangle \tag{4.20}$$

with $\mathcal{N}_n$ denoting the normalization factor.

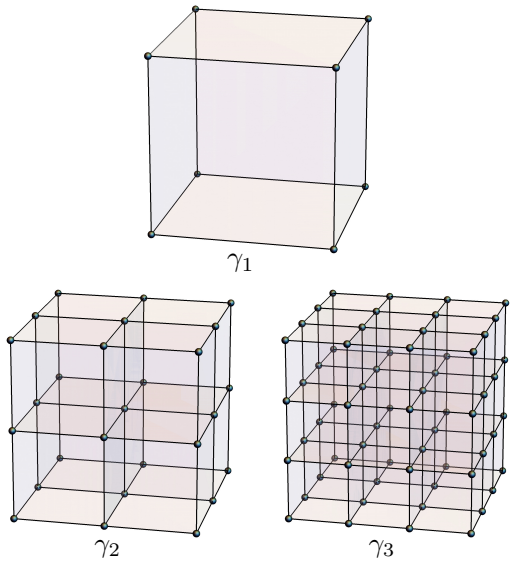


FIG. 1. Lattice refinement of a cell of γ_1 to γ_2 and γ_3. The fiducial lattice spacing of γ_n is μ/n such that the number of vertices N_n along each direction on γ_n satisfies $N_n = nN_1$, with N_1 be that of γ_1.

V. PROPAGATOR ON MINKOWSKI BACKGROUND

Another ingredient for propagator is the field operator. Due to (2.5) and (2.28), it is natural to define the fermion field operator as

$$\hat{\psi}_A(v) = \sqrt{\hbar}\hat{V}_v^{-\frac{1}{2}}\hat{\zeta}_{v,A}, \forall A = \pm, \tag{5.1}$$

where $\hat{V}_v^{-\frac{1}{2}}$ is given in (2.64). As the statement below (2.64), there is a sign ambiguity in $\hat{V}_v^{-\frac{1}{2}}$. However, as far as the propagator is concerned, $\hat{\psi}(v)$ and, thus, $\sqrt{\hbar}\hat{V}_v^{-\frac{1}{2}}$ alway appear in pair. Thus, the sign ambiguity does not make any essential. For convenience, we introduce the notation

$$\begin{aligned}\hat{\psi}_A(\tau, v) =& \exp\left[\frac{i}{\hbar}\tau\hat{\mathbf{H}}\right]\hat{\psi}_A(v)\exp\left[-\frac{i}{\hbar}\tau\hat{\mathbf{H}}\right],\\ \hat{\psi}^\dagger_A(\tau, v) =& \exp\left[\frac{i}{\hbar}\tau\hat{\mathbf{H}}\right]\hat{\psi}^\dagger_A(v)\exp\left[-\frac{i}{\hbar}\tau\hat{\mathbf{H}}\right].\end{aligned} \tag{5.2}$$

Applying $\hat{\psi}_A(\tau, v)$ and $\hat{\psi}^\dagger_A(\tau, v)$, the operator $\hat{G}_{\vec{A}}(\vec{\tau}, \vec{v})$ for the propagator is given by

$$\hat{G}_{\vec{A}}(\vec{\tau}, \vec{v}) := \mathcal{T}\left(\hat{\psi}_{A_1}(\tau_1, v_1)\hat{\psi}^\dagger_{A_2}(\tau_2, v_2)\right) \tag{5.3}$$

where the vectors $\vec{A}$, $\vec{\tau}$ and $\vec{v}$ denote (A_1, A_2), (τ_1, τ_2) and (v_1, v_2) respectively and $\mathcal{T}$ is the time ordering operator to put the operators at latter time to the left. It should be spelled out that we concern ourselves with only the case where v_1 and v_2 are both in $V(\gamma_1)$ so that all graphs have contribution to the final results. Note that the operator $\hat{G}_{\vec{A}}(\vec{\tau}, \vec{v})$ is not gauge invariant. To get a gauge invariant operator, we need to choose a smeared function $\hat{F}^{\vec{A}}$ to do contraction with $\hat{G}_{\vec{A}}(\vec{\tau}, \vec{v})$. A concrete example of $\hat{F}^{\vec{A}} \equiv \hat{F}^{(A_1,A_2)}$ is

$$\hat{F}^{\vec{A}} = \sum_\gamma \mathbf{P}_\gamma D^{\frac{1}{2}}_{A_2A_1}(h_{v_2\to v_1})\mathbf{P}_\gamma \tag{5.4}$$

where $h_{v_2\to v_1}$ denotes a holonomy along a path in γ_1 from v_2 to v_1. Here that the path of the holonomy $h_{v_2\to v_1}$ is required to be in γ_1 ensures that $\hat{F}^{\vec{A}}$ is a well-defined operator in all $\widetilde{\mathcal{H}}_{\gamma_n}$. It is worthing noting that our calculation shown below does not depend on the explicit expression of $\hat{F}^{\vec{A}}$. It will be obtained that, as far as the leading order

is concerned, the final result is just the classical value of $\hat{F}^{\vec{A}}$ times some kernel $G_{\vec{A}}(\vec{\tau},\vec{v})$, where the kernel $G_{\vec{A}}(\vec{\tau},\vec{v})$ restores the propagator function of the regular QFT. Using $\widehat{G[F]}(\vec{\tau},\vec{v})$ to denote $\hat{F}^{\vec{A}}\hat{G}_{\vec{A}}(\vec{\tau},\vec{v})$, we define the propagator as

$$G[F](\vec{\tau},\vec{v}) = \langle[\Omega]|\widehat{G[F]}(\vec{\tau},\vec{v})|[\Omega]\rangle. \tag{5.5}$$

Substituting (4.19) into (5.5), one gets

$$G[F](\vec{\tau},\vec{v}) = \sum_{m,m'=1}^{L} w^*_{m'} w_m \langle[\Omega],\gamma_{m'}|\widehat{G[F]}(\vec{\tau},\vec{v})|[\Omega],\gamma_m\rangle. \tag{5.6}$$

In the operator $\widehat{G[F]}(\vec{\tau},\vec{v})$ the evolution operator $\exp\left[-\frac{i}{\hbar}\tau\hat{\mathbf{H}}\right]$ is involved. According to (2.67) and (2.58), each Hilbert space $\widetilde{\mathcal{H}}_\gamma$ is preserved by $\hat{\mathbf{H}}$. As a consequence, the evolution operator $\exp\left[-\frac{i}{\hbar}\tau\hat{\mathbf{H}}\right]$ preserves each single $\widetilde{\mathcal{H}}_\gamma$. This fact, together with $v_1, v_2 \in V(\gamma_1)$ and the definition of $\hat{F}^{\vec{A}}$, leads to that $\widehat{G[F]}(\vec{\tau},\vec{v})|[\Omega],\gamma_m\rangle$ is in $\widetilde{\mathcal{H}}_{\gamma_m}$. Since $\widetilde{\mathcal{H}}_{\gamma_m}$ is orthogonal to $\widetilde{\mathcal{H}}_{\gamma_{m'}}$ for $m \neq m'$, (5.6) can be simplified as

$$G[F](\vec{\tau},\vec{v}) = \sum_{m=1}^{L} |w_m|^2 \langle[\Omega],\gamma_{m'}|\widehat{G[F]}(\vec{\tau},\vec{v},\gamma_m)|[\Omega],\gamma_m\rangle. \tag{5.7}$$

where $\widehat{G[F]}(\vec{\tau},\vec{v},\gamma) := \mathbf{P}_\gamma D^{\frac{1}{2}}_{A_2A_1}(h_{v_2\to v_1})\hat{G}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)\mathbf{P}_\gamma$ is given by

$$\hat{G}_{\vec{A}}(\vec{\tau},\vec{v},\gamma) := \mathcal{T}\left(\hat{\psi}_{A_1}(\tau_1,v_1,\gamma)\hat{\psi}^\dagger_{A_2}(\tau_2,v_2,\gamma)\right) \tag{5.8}$$

$$\hat{\psi}_A(\tau,v,\gamma) = \exp\left[\tfrac{i}{\hbar}\tau\hat{\mathbf{H}}_\gamma\right]\hat{\psi}_A(v)\exp\left[-\tfrac{i}{\hbar}\tau\hat{\mathbf{H}}_\gamma\right], \tag{5.9}$$

$$\hat{\psi}_A(\tau,v,\gamma)^\dagger = \exp\left[\tfrac{i}{\hbar}\tau\hat{\mathbf{H}}_\gamma\right]\hat{\psi}^\dagger_A(v)\exp\left[-\tfrac{i}{\hbar}\tau\hat{\mathbf{H}}_\gamma\right]. \tag{5.10}$$

According to (5.7), the propagator $G[F](\vec{\tau},\vec{v})$ is completely encoded in

$$\begin{aligned} G[F](\vec{\tau},\vec{v},\gamma) :=& \langle[\Omega],\gamma|\widehat{G[F]}(\vec{\tau},\vec{v},\gamma)|[\Omega],\gamma\rangle \\ =& \mathcal{N}_\gamma \int \mathrm{d}\mu_H(u)\langle\Omega,\gamma|\widehat{G[F]}(\vec{\tau},\vec{v},\gamma)u|\Omega,\gamma\rangle \end{aligned} \tag{5.11}$$

where $u : V(\gamma) \ni v \mapsto u(v) \in \mathrm{SU}(2)$ denotes the gauge transformation field and the normalization factor is given by

$$\mathcal{N}_\gamma^{-1} = \int \mathrm{d}\mu_H(u)\langle\Omega,\gamma|u|\Omega,\gamma\rangle. \tag{5.12}$$

We need to apply the path integral formulation to calculation $G[F](\vec{\tau},\vec{v},\gamma)$. As in regular QFT, the path integral will be calculated by stationary phase approximation which needs to solve the equation of motion obtained by vanishing the variation of the effective action. In LQG, without considering the fermion field, the equation of motion and its solution have been well studied in [41], despite the different boundary states used therein. Actually, without fermion, the results in [41] can be applied directly to calculate the transition amplitude between the coherent state $|\Psi^t_{\mathbf{g}^\gamma_M},\gamma\rangle$. However, in the current work, our boundary state contains $\mathbf{P}_\gamma|\Psi^t_{\mathbf{g}^\gamma_M},\gamma\rangle$ rather than $|\Psi^t_{\mathbf{g}^\gamma_M},\gamma\rangle$. Thus, to relate our work with the results in [41], we introduce $|\Omega_0,\gamma\rangle$ as

$$|\Omega_0,\gamma\rangle = |\Psi^t_{\mathbf{g}^\gamma_M},\gamma\rangle \otimes |\omega,\gamma\rangle \tag{5.13}$$

so that our vacuum state satisfy

$$|\Omega,\gamma\rangle = \mathbf{P}_\gamma|\Omega_0,\gamma\rangle. \tag{5.14}$$

Then, we define $G^0[F](\vec{\tau},\vec{v},\gamma)$ as

$$G^0[F](\vec{\tau},\vec{v},\gamma) := \mathcal{N}_\gamma \int \mathrm{d}\mu_H(u)\langle\Omega_0,\gamma|\widehat{G[F]}(\vec{\tau},\vec{v},\gamma)u|\Omega_0,\gamma\rangle. \tag{5.15}$$

By (5.4), the projection contained in $\hat{F}^{\vec{A}}$ leads directly to

$$G^0[F](\vec{\tau},\vec{v},\gamma) = G[F](\vec{\tau},\vec{v},\gamma). \tag{5.16}$$

Nonetheless, we would do the following discussion to see that $G^0[F](\vec{\tau},\vec{v},\gamma)$ and $G[F](\vec{\tau},\vec{v},\gamma)$ are the same up to a $O(t^\infty)$ term for a general $\hat{F}^{\vec{A}}$ without $\mathbf{P}_\gamma$ contained. We have

$$\begin{aligned}
&G^0[F](\vec{\tau},\vec{v},\gamma) - G[F](\vec{\tau},\vec{v},\gamma)\\
=&\mathcal{N}_\gamma \int \mathrm{d}\mu_H(u)\langle\Omega_0,\gamma|\widehat{G[F]}(\vec{\tau},\vec{v},\gamma)u(1-\mathbf{P}_\gamma)|\Omega_0,\gamma\rangle\\
&+\mathcal{N}_\gamma \int \mathrm{d}\mu_H(u)\langle\Omega_0,\gamma|(1-\mathbf{P}_\gamma)\widehat{G[F]}(\vec{\tau},\vec{v},\gamma)u|\Omega_0,\gamma\rangle\\
&+\mathcal{N}_\gamma \int \mathrm{d}\mu_H(u)\langle\Omega_0,\gamma|(1-\mathbf{P}_\gamma)\widehat{G[F]}(\vec{\tau},\vec{v},\gamma)u(1-\mathbf{P}_\gamma)|\Omega_0,\gamma\rangle
\end{aligned} \tag{5.17}$$

Note that our fermion field operators $\hat{\psi}_A(v)$ and $\hat{\psi}^\dagger_A(v)$ are constructed on discrete graphs. This fact leads to that $\hat{\psi}_A(v)$ and $\hat{\psi}^\dagger_A(v)$ are bounded. Indeed, they are bounded by 1, i.e., $\|\hat{\psi}_A(v)\| = \|\hat{\psi}^\dagger_A(v)\| < 1$. As a consequence, $\hat{G}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)$ is bounded by 1. We thus have, for instance,

$$\begin{aligned}
&\left|\langle\Omega_0,\gamma|\widehat{G[F]}(\vec{\tau},\vec{v},\gamma)u(1-\mathbf{P}_\gamma)|\Omega_0,\gamma\rangle\right|^2\\
\leq&\|(1-\mathbf{P}_\gamma|\Omega_0,\gamma\rangle\|^2\left\|\hat{F}^{\vec{A}}|\Omega_0,\gamma\rangle\right\|^2.
\end{aligned} \tag{5.18}$$

The term $\left\|\hat{F}^{\vec{A}}|\Omega_0,\gamma\rangle\right\|^2$ is just the expectation value of $\hat{F}^{\vec{A}}(\hat{F}^{\vec{A}})^\dagger$ in the state $|\Omega_0,\gamma\rangle$ which is finite. For the term $\|(1-\mathbf{P}_\gamma|\Omega_0,\gamma\rangle\|^2$, according to (2.40), it is a $O(t^\infty)$ term. Thus, we get

$$\left|\langle\Omega_0,\gamma|\widehat{G[F]}(\vec{\tau},\vec{v},\gamma)u(1-\mathbf{P}_\gamma)|\Omega_0,\gamma\rangle\right|^2 = O(t^\infty). \tag{5.19}$$

The same arguments are applied to the other terms in (5.17). It thus concludes that

$$G[F](\vec{\tau},\vec{v},\gamma) = G^0[F](\vec{\tau},\vec{v},\gamma) + O(t^\infty). \tag{5.20}$$

Moreover, proceeding the same step analogous to (5.17), $\mathcal{N}_\gamma^{-1}$ can be simplified as

$$\begin{aligned}
\mathcal{N}_\gamma^{-1} &= \int \mathrm{d}\mu_H(u)\langle\Omega_0,\gamma|\mathbf{P}_\gamma u\mathbf{P}_\gamma|\Omega_0,\gamma\rangle\\
&= \int \mathrm{d}\mu_H(u)\langle\Omega_0,\gamma|u|\Omega_0,\gamma\rangle + O(t^\infty).
\end{aligned} \tag{5.21}$$

We finally have

$$G[F](\vec{\tau},\vec{v},\gamma) = \langle[\Omega_0],\gamma|\widehat{G[F]}(\vec{\tau},\vec{v},\gamma)|[\Omega_0],\gamma\rangle + O(t^\infty) \tag{5.22}$$

where $|[\Omega_0],\gamma\rangle$ denotes the normalized gauge invariant projection of $|\Omega_0,\gamma\rangle$. The right hand side of (5.22), i.e., $\langle[\Omega_0],\gamma|\widehat{G[F]}(\vec{\tau},\vec{v},\gamma)|[\Omega_0],\gamma\rangle$ is calculable by employing the results in [41], as shown in the following.

A. Calculation of $G^0[F](\vec{\tau},\vec{v},\gamma)$

To perform the time ordering operation in (5.8), we introduce the map $\mathfrak{i}:\{1,2\}\to\{1,2\}$ to reordering τ_1 and τ_2 such that $\tau_{\mathfrak{i}(1)} > \tau_{\mathfrak{i}(2)}$. Then, we have

$$\hat{G}_{\vec{A}}(\vec{\tau},\vec{v},\gamma) = \mathrm{sgn}(\vec{\tau})\exp\left[\frac{i}{\hbar}\tau_{\mathfrak{i}(2)}\hat{\mathbf{H}}_\gamma\right]\hat{Y}^{(v_{\mathfrak{i}(2)})}_{A_{\mathfrak{i}(2)}}\exp\left[-\frac{i}{\hbar}(\tau_{\mathfrak{i}(2)}-\tau_{\mathfrak{i}(1)})\hat{\mathbf{H}}_\gamma\right]\hat{Y}^{(v_{\mathfrak{i}(1)})}_{A_{\mathfrak{i}(1)}}\exp\left[-\frac{i}{\hbar}\tau_{\mathfrak{i}(1)}\hat{\mathbf{H}}_\gamma\right] \tag{5.23}$$

where $\mathrm{sgn}(\vec{\tau})$ is the sign factor generated by time ordering, i.e., $\mathrm{sgn}(\vec{\tau})=1$ for $\tau_1>\tau_2$, and -1 otherwise, and $\hat{Y}^{(v)}_A$ denotes either $\hat{\psi}_{A_1}(v_1)$ or $\hat{\psi}^\dagger_{A_2}$, explicitly given by

$$\hat{Y}^{(v_i)}_{A_i}=\begin{cases}\hat{\psi}_{A_1}(v_1), & i=1,\\ \hat{\psi}^\dagger_{A_2}(v_2), & i=2.\end{cases} \tag{5.24}$$

Then, $G^0[F](\vec{\tau},\vec{v},\gamma)$ can be calculated directly by applying (3.15). One have

$$G^0[F](\vec{\tau},\vec{v},\gamma)=\lim_{N_i\to 0}G^0_{\vec{N}}[F](\vec{\tau},\vec{v},\gamma) \tag{5.25}$$

with

$$\begin{aligned}&G^0_{\vec{N}}[F](\vec{\tau},\vec{v},\gamma)\\ =&\mathcal{N}_\gamma\int \mathrm{d}\mu_H(u)\int\prod_{i=0}^{N+3}\mathcal{D}[Z_i,\gamma]\frac{\langle Z_{N+3},\gamma|\hat{F}^{\vec{A}}|Z_{N+2},\gamma\rangle}{\langle Z_{N+3},\gamma|Z_{N+2},\gamma\rangle}\left(\mathrm{sgn}(\vec{\tau})\prod_{j=1}^{2}\frac{\langle Z_{\mathfrak{f}(j)+1},\gamma|\hat{Y}^{(v_{\mathfrak{i}(j)})}_{A_{\mathfrak{i}(j)}}|Z_{\mathfrak{f}(j)},\gamma\rangle}{\langle Z_{\mathfrak{f}(j)+1},\gamma|Z_{\mathfrak{f}(j)},\gamma\rangle}\right)\times\\ &\Xi[\mathbf{g}]e^{\frac{1}{t}\widetilde{S}_{\vec{N}}[\mathbf{g},\nu]}\langle\Omega_0,\gamma|Z_{N+3},\gamma\rangle\langle Z_0,\gamma|u|\Omega_0,\gamma\rangle.\end{aligned} \tag{5.26}$$

Eq. (5.26) can be obtained with the same steps for (3.15) except for some subtleties. At first, due to the operator $\hat{F}^{\vec{A}}$, we need to insert one more resolution of identity than in (3.15). This fact gives rise to an additional kinetic term $K_\gamma(Z_{N+3},Z_{N+2})$ in the action. In other words, the action $\widetilde{S}_{\vec{N}}[\mathbf{g},\nu]$ in (5.26) is not the same as the one obtained by directly applying (3.15) for our concrete case. Instead, they differ due to the additional kinetic term $K_\gamma(Z_{N+3},Z_{N+2})$ in (5.26). However, this slight difference will not cause any essential in our following calculation. Second, we use the convention that N_k denotes the number of slices dividing the interval $[\tau_{\mathfrak{i}(k-1)},\tau_{\mathfrak{i}(k)}]$ for $1\le k\le 3$, where $\tau_{\mathfrak{i}(0)}=0$ and $\tau_{\mathfrak{i}(3)}=2\tau_{\mathfrak{i}(2)}$ as did for (3.15). Under this convention, for the case $\tau_2<\tau_1$ so that $\tau_{\mathfrak{i}(1)}=\tau_2$ and $\tau_{\mathfrak{i}(2)}=\tau_1$, the number of slices dividing $[0,\tau_2]$ is denoted by N_1 and the number of slices dividing $[\tau_2,\tau_1]$ is denoted by N_2.

To evaluate $G^0_{\vec{N}}[F]$, let us consider $\langle\Omega_0,\gamma|Z_{N+3},\gamma\rangle$ and $\langle Z_0,\gamma|u|\Omega_0,\gamma\rangle$ at first. By definition, $\langle\Omega_0,\gamma|Z_{N+3},\gamma\rangle$ can be simplified as

$$\begin{aligned}\langle\Omega_0,\gamma|Z_{N+3},\gamma\rangle=&\left(\prod_{e\in E(\gamma)}\frac{\mathring{\eta}_{N+3}(e)\sqrt{\sinh(p_{N+3}(e))\sinh(\mathring{p}_\gamma)}}{\sqrt{p_{N+3}(e)\mathring{p}_\gamma}\sinh(\mathring{\eta}_{N+3}(e))}\right)\left(\prod_{\vec{k}\in K[\gamma]^3}\sum_{A=\pm}\Theta(\vec{k})_{-,A}\widetilde{\nu}_{N+3,A}(\vec{k})\right)\times\\ &\exp\left(-\sum_{e\in E(\gamma)}\frac{p_{N+3}(e)^2+(\mathring{p}_\gamma)^2-2\mathring{\eta}_{N+3}(e)^2}{2t}-\frac{1}{2}\sum_{v\in V(\gamma)}\nu^\dagger_{N+3}(v)\nu_{N+3}(v)\right)\end{aligned} \tag{5.27}$$

where $\mathring{\eta}_{N+3}(e)$ is given by $2\cosh(\mathring{\eta}_{N+3}(e))=\mathrm{tr}\big(\mathbf{g}_{N+3}(e)e^{-i\mathring{p}_\gamma\tau_e}\big)$, and $\widetilde{\nu}_{N+3}$ is the Fourier transformation of ν_{N+3}. Similarly, we have

$$\begin{aligned}\langle Z_0,\gamma|u|\Omega_0,\gamma\rangle=&\left(\prod_{e\in E(\gamma)}\frac{\mathring{\eta}^u_0(e)\sqrt{\sinh(p_0(e))\sinh(\mathring{p}_\gamma)}}{\sqrt{p_0(e)\mathring{p}_\gamma}\sinh(\mathring{\eta}^u_0(e))}\right)\left(\prod_{\vec{k}\in K[\gamma]^3}\sum_{A=\pm}\Theta(\vec{k})_{-,A}\widetilde{\nu}^u_{0,A}(\vec{k})\right)^*\times\\ &\exp\left(-\sum_{e\in E(\gamma)}\frac{p_0(e)^2+(\mathring{p}_\gamma)^2-2(\mathring{\eta}^u_0(e))^2}{2t}-\frac{1}{2}\sum_{v\in V(\gamma)}\nu^\dagger_0(v)\nu_0(v)\right)\end{aligned} \tag{5.28}$$

where $\mathring{\eta}^u_0(e)$ is given by $2\cosh(\mathring{\eta}^u_0(e))=\mathrm{tr}\big(g_0(e)^\dagger u e^{-i\mathring{p}_\gamma\tau_e}\big)$ and $\widetilde{\nu}^u_0$ denote the Fourier transformation of ν^u_0, with ν^u_0 given by $(\nu^u_0)_A(v)=(u(v)^{-1})_A{}^B\nu_B(v)$ being the gauge transform of ν. Substituting (5.27), (5.28) into (5.26), we have

$$\begin{aligned}G^0_{\vec{N}}[F](\vec{\tau},\vec{v},\gamma)=&\mathcal{N}_\gamma\int\mathrm{d}\mu_H(u)\int\prod_{i=0}^{N+3}\mathcal{D}[Z_i,\gamma]\frac{\langle Z_{N+3},\gamma|\hat{F}^{\vec{A}}|Z_{N+2},\gamma\rangle}{\langle Z_{N+3},\gamma|Z_{N+2},\gamma\rangle}\left(\mathrm{sgn}(\vec{\tau})\prod_{j=1}^{2}\frac{\langle Z_{\mathfrak{f}(j)+1},\gamma|\hat{Y}^{(v_{\mathfrak{i}(j)})}_{A_{\mathfrak{i}(j)}}|Z_{\mathfrak{f}(j)},\gamma\rangle}{\langle Z_{\mathfrak{f}(j)+1},\gamma|Z_{\mathfrak{f}(j)},\gamma\rangle}\right)\times\\ &\Xi'[\mathbf{g},u]e^{\frac{1}{t}\widetilde{S}'_{\vec{N}}[\mathbf{g},\nu,u]}\prod_{\vec{k}\in K[\gamma]^3}\sum_{A=\pm}\Theta(\vec{k})_{-,A}\widetilde{\nu}_{N+3,A}(\vec{k})\left(\prod_{\vec{k}\in K[\gamma]^3}\sum_{A=\pm}\Theta(\vec{k})_{-,A}\widetilde{\nu}^u_{0,A}(\vec{k})\right)^*\end{aligned} \tag{5.29}$$

with

$$\begin{aligned}\widetilde{S}'_{\vec{N}}[\mathbf{g},\nu,u] =&\widetilde{S}_{\vec{N}}[\mathbf{g},\nu] - \sum_{e\in E(\gamma)} \frac{p_{N+3}(e)^2+(\mathring{p}_\gamma)^2-2\mathring{\eta}_{N+3}(e)^2}{2} - \frac{t}{2}\sum_{v\in V(\gamma)} \nu^\dagger_{N+3}(v)\nu_{N+3}(v)\\ &- \sum_{e\in E(\gamma)} \frac{p_0(e)^2+(\mathring{p}_\gamma)^2-2(\mathring{\eta}^u_0(e))^2}{2} - \frac{t}{2}\sum_{v\in V(\gamma)} \nu^\dagger_0(v)\nu_0(v)\end{aligned} \tag{5.30}$$

and

$$\Xi'[\mathbf{g},u] = \Xi[\mathbf{g}] \prod_{e\in E(\gamma)} \frac{\mathring{\eta}_{N+3}(e)\sqrt{\sinh(p_{N+3}(e))\sinh(\mathring{p}_\gamma)}}{\sqrt{p_{N+3}(e)\mathring{p}_\gamma}\sinh(\mathring{\eta}_{N+3}(e))} \frac{\mathring{\eta}^u_0(e)\sqrt{\sinh(p_0(e))\sinh(\mathring{p}_\gamma)}}{\sqrt{p_0(e)\mathring{p}_\gamma}\sinh(\mathring{\eta}^u_0(e))}. \tag{5.31}$$

To further simplify the expression (5.29), we turn to the term

$$\mathrm{sgn}(\vec{\tau})\prod_{j=1}^2 \frac{\langle Z_{\mathfrak{f}(j)+1},\gamma|\hat{Y}^{(v_{\mathfrak{i}(j)})}_{A_{\mathfrak{i}(j)}}|Z_{\mathfrak{f}(j)},\gamma\rangle}{\langle Z_{\mathfrak{f}(j)+1},\gamma|Z_{\mathfrak{f}(j)},\gamma\rangle} = \mathrm{sgn}(\vec{\tau})\frac{\langle Z_{\mathfrak{f}(2)+1},\gamma|\hat{Y}^{(v_{\mathfrak{i}(2)})}_{A_{\mathfrak{i}(2)}}|Z_{\mathfrak{f}(2)},\gamma\rangle}{\langle Z_{\mathfrak{f}(2)+1},\gamma|Z_{\mathfrak{f}(2)},\gamma\rangle}\frac{\langle Z_{\mathfrak{f}(1)+1},\gamma|\hat{Y}^{(v_{\mathfrak{i}(1)})}_{A_{\mathfrak{i}(1)}}|Z_{\mathfrak{f}(1)},\gamma\rangle}{\langle Z_{\mathfrak{f}(1)+1},\gamma|Z_{\mathfrak{f}(1)},\gamma\rangle}.$$

By definition of $\mathrm{sgn}(\vec{\tau})$, it can be canceled if we can reorder the right hand sight such that the term with $\hat{Y}^{(v_1)}_{A_1}$ is always put on the right. To write the result in a compact form, let us introduce a function $\mathfrak{n}(i)$ defined as follows: if $\tau_1<\tau_2$, $\mathfrak{n}(i)=\mathfrak{f}(i)$ for all $i=1,2$; otherwise, $\mathfrak{n}(1)=\mathfrak{f}(2)$ and $\mathfrak{n}(2)=\mathfrak{f}(1)$. Then, a straightforward calculation gives us

$$\mathrm{sgn}(\vec{\tau})\prod_{j=1}^2 \frac{\langle Z_{\mathfrak{f}(j)+1},\gamma|\hat{Y}^{(v_{\mathfrak{i}(j)})}_{A_{\mathfrak{i}(j)}}|Z_{\mathfrak{f}(j)},\gamma\rangle}{\langle Z_{\mathfrak{f}(j)+1},\gamma|Z_{\mathfrak{f}(j)},\gamma\rangle} = \left(\prod_{j=1}^{2n}\frac{\langle Z_{\mathfrak{n}(j)+1},\gamma|\hat{V}^{-\frac{1}{2}}_{v_j}|Z_{\mathfrak{n}(j)},\gamma\rangle}{\langle Z_{\mathfrak{n}(j)+1},\gamma|Z_{\mathfrak{n}(j)},\gamma\rangle}\right)\mathcal{F}_{\vec{A}} \tag{5.32}$$

where $\mathcal{F}_{\vec{A}} = \hbar\nu_{\mathfrak{n}(1),A_1}(v_1)\nu^*_{\mathfrak{n}(2)+1,A_2}(v_2)$ and we used (5.24), (5.1) and (2.43). Hence $G^0_{\vec{N}}[F]$ can be expressed as

$$\begin{aligned}G^0_{\vec{N}}[F](\vec{\tau},\vec{v},\gamma) =&\mathcal{N}_\gamma\int \mathrm{d}\mu_H(u)\int\prod_{i=0}^{N+3}\mathcal{D}[Z_i,\gamma]\frac{\langle Z_{N+3},\gamma|\hat{F}^{\vec{A}}|Z_{N+2},\gamma\rangle}{\langle Z_{N+3},\gamma|Z_{N+2},\gamma\rangle}\left(\prod_{j=1}^2\frac{\langle Z_{\mathfrak{n}(j)+1},\gamma|\hat{V}^{-\frac{1}{2}}_{v_j}|Z_{\mathfrak{n}(j)},\gamma\rangle}{\langle Z_{\mathfrak{n}(j)+1},\gamma|Z_{\mathfrak{n}(j)},\gamma\rangle}\right)\times\\ &\Xi'[\mathbf{g},u]e^{\frac{1}{t}\widetilde{S}'_{\vec{N}}[\mathbf{g},\nu,u]}\mathcal{F}_{\vec{A}}\prod_{\vec{k}\in K[\gamma]^3}\sum_{A=\pm}\Theta(\vec{k})_{-,A}\widetilde{\nu}_{N+3,A}(\vec{k})\left(\prod_{\vec{k}\in K[\gamma]^3}\sum_{A=\pm}\Theta(\vec{k})_{-,A}\widetilde{\nu}^u_{0,A}(\vec{k})\right)^*\end{aligned} \tag{5.33}$$

To calculate (5.33), let us divide $\widetilde{S}'_{\vec{N}}[\mathbf{g},\nu,u]$ into two parts as

$$\widetilde{S}'_{\vec{N}}[\mathbf{g},\nu,u] = S^G_{\vec{N}}[\mathbf{g},u] + S^F_{\vec{N}}[\mathbf{g},\nu] \tag{5.34}$$

where $S^G_{\vec{N}}[\mathbf{g},u]$ is the effective action of the vacuum LQG and $S^F_{\vec{N}}[\mathbf{g},\nu]$ is the effective action of fermion coupled to gravity. The explicit expression of $S^G_{\vec{N}}[\mathbf{g},u]$ and $S^F_{\vec{N}}[\mathbf{g},\nu]$ can be obtained by referring to (5.30), (3.16), and (3.7). We have

$$\begin{aligned}S^G_{\vec{N}}[\mathbf{g},u] =& \sum_{j=-1}^{N+3}\sum_{e\in E(\gamma)}\left(\eta_{i+1,i}(e)^2-\frac{1}{2}\left(p_{i+1}(e)^2+p_i(e)^2\right)\right)\\ &-\frac{i\kappa}{a^2}\sum_{j=0}^{N+2}\delta\tau_j\frac{\langle\mathbf{g}_{j+1},\gamma|\mathbf{P}_\gamma\hat{H}^G_\gamma\mathbf{P}_\gamma|\mathbf{g}_j,\gamma\rangle}{\langle\mathbf{g}_{j+1},\gamma|\mathbf{g}_j,\gamma\rangle}\end{aligned} \tag{5.35}$$

with $\eta_{N+4,N+3}(e)=\mathring{\eta}_{2N+2n+1}(e)$ and $\eta_{0,-1}(e)=\mathring{\eta}^u_0(e)$ and

$$\begin{aligned}S^F_{\vec{N}}[\mathbf{g},\nu] =& t\left(\sum_{j=0}^{N+2}\sum_{v\in V(\gamma)}\nu^\dagger_{j+1}(v)\nu_j(v)-\sum_{j=0}^{N+3}\sum_{v\in V(\gamma)}\nu^\dagger_j(v)\nu_j(v)\right)\\ &-\frac{i\kappa}{a^2}\sum_{j=0}^{N+2}\delta\tau_j\frac{\langle Z_{j+1},\gamma|\mathbf{P}_\gamma\hat{H}^F_\gamma\mathbf{P}_\gamma|Z_j,\gamma\rangle}{\langle Z_{j+1},\gamma|Z_j,\gamma\rangle}.\end{aligned} \tag{5.36}$$

Note that $S^F_{\vec{N}}$ is a quadratic form of the fermion field $\nu_j(v)$ due to $\hat{H}^F_\gamma$ is quadratic in terms of the operator $\hat{\theta}(v)=\sqrt{\hbar}\hat{\zeta}_v$ by (2.62). Thus, the integral of fermion fields in (5.33) is the standard Gaussian integral, which can be performed in principle. Let us use $I[\mathbf{g},u]$ to denote the result of the fermion integral. It depends on $\mathbf{g}$ and u because, for instance, the holonomies and fluxes are contained in the coefficient matrix of $S^F_{\vec{N}}$. Using $I[\mathbf{g},u]$, we can write $G^0_{\vec{N}}[F]$ in the following form

$$G^0_{\vec{N}}[F]=\int \mathrm{d}\mu_H(u)\int\prod_{i=0}^{N+3}\mathcal{D}[\mathbf{g}_i,\gamma]W[\mathbf{g},u]I[\mathbf{g},u]e^{\frac{1}{t}S^G_{\vec{N}}[\mathbf{g},u]} \tag{5.37}$$

where $W[\mathbf{g},u]$ represents some function of $\mathbf{g}$ and u, and its explicit expression does not affect the present discussion. Since we concern ourselves with the leading order of $G^0_{\vec{N}}[F]$ in t, the integral of (5.37) will be calculated by the stationary phase approximation method. To this end, we need to expand $S^G_{\vec{N}}[\mathbf{g},u]$ to power series of t, i.e., $S^G_{\vec{N}}[\mathbf{g},u]=S_0+tS_1+\cdots$. Then the stationary phase approximation method needs us to study the equation of motion $\delta S_0=0$. The equation of motion has been well-studied in [41]. According to the result of [41], in the continuous limit, i.e., $\delta\tau\ll 1$, $\delta S_0=0$ gives rise to the Hamilton's equation with respect to the Hamiltonian given by the leading order, i.e., $O(t^0)$-term, of the Hamiltonian operator's expectation value in the coherent states. In our work, the Hamiltonian is $\mathbf{P}_\gamma\hat{H}^G_\gamma\mathbf{P}_\gamma$ whose expectation value is the same as that of $\hat{H}^G_\gamma$ in the leading order. According to [41], the leading order of the expectation value of $\hat{H}^G_\gamma$ in the coherent state is the same as its classical value H^G_γ. The classical value H^G_γ is H_G (see (2.10)) regularized with the holonomies and fluxes on γ. According to these facts and the boundary condition resulting from the gravity part $|\Psi^t_{\mathbf{g}^\gamma_M},\gamma\rangle$ of the boundary states $|\Omega_0,\gamma\rangle$, the solution to the equation of motion $\delta S_0=0$ is $\mathbf{g}_j=\mathbf{g}^\gamma_M$ for all $0\leq j\leq N+3$ and $u=\mathbb{1}$. Hence, the solution endows γ with a Minkowski geometry at any moment $\tau_j\equiv j\delta\tau$. Therefore, as far as the leading order of $G^0_{\vec{N}}[F]$ in t is concerned, (5.37) is

$$G^0_{\vec{N}}[F]=\left(\int \mathrm{d}\mu_H(u)\int\prod_{0}^{N+3}\mathcal{D}[\mathbf{g}_i,\gamma]W[\mathbf{g},u]e^{\frac{1}{t}S^G_{\vec{N}}[\mathbf{g},u]}\right)I(\mathbf{g}^\gamma_M,\mathbb{1})(1+O(t)). \tag{5.38}$$

Returning to our specific calculation, (5.38) implies

$$\begin{aligned}G^0_{\vec{N}}[F](\vec{\tau},\vec{v},\gamma)=&\mathcal{N}_\gamma\langle\Psi^t_{\mathbf{g}^\gamma_M},\gamma|\hat{F}^{\vec{A}}|\Psi^t_{\mathbf{g}^\gamma_M},\gamma\rangle\left(\prod_{j=1}^{2}\langle\Psi^t_{\mathbf{g}^\gamma_M},\gamma|\hat{V}^{-\frac{1}{2}}_{v_j}|\Psi^t_{\mathbf{g}^\gamma_M},\gamma\rangle\right)\\&A^{G,\vec{N}}_\gamma\mathfrak{F}^{\vec{N}}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)(1+O(t))\end{aligned} \tag{5.39}$$

where we assumed reasonably that $\hat{F}^{\vec{A}}$ does not contain any fermion operators, $A^{G,\vec{N}}_\gamma$ reads

$$A^{G,\vec{N}}_\gamma=\int \mathrm{d}\mu_H(u)\prod_{i=0}^{N+3}\mathcal{D}[\mathbf{g}_i,\gamma]\Xi'[\mathbf{g},u]e^{S^G_{\vec{N}}[\mathbf{g},u]/t} \tag{5.40}$$

and $\mathfrak{F}^{\vec{N}}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)$ is

$$\begin{aligned}\mathfrak{F}^{\vec{N}}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)=&\int\prod_{i=0}^{N+3}\mathcal{D}[\nu_i,\gamma]e^{\frac{1}{t}S^F_{\vec{N}}[\mathbf{g}^\gamma_M,\nu]}\mathcal{F}_{\vec{A}}\prod_{\vec{k}\in K[\gamma]^3}\sum_{A=\pm}\Theta(\vec{k})_{-,A}\widetilde{\nu}_{2N+2n+1,A}(\vec{k})\times\\&\left(\prod_{\vec{k}\in K[\gamma]^3}\sum_{A=\pm}\Theta(\vec{k})_{-,A}\widetilde{\nu}_{0,A}(\vec{k})\right)^*.\end{aligned} \tag{5.41}$$

According to the discussion below (3.13), the limit of $A^{G,\vec{N}}_\gamma$ as $\delta\tau\to 0$ has the physical interpretation that the initial $\int\mathrm{d}\mu_H(u)u|\Psi^t_{\mathbf{g}^\gamma_M},\gamma\rangle$ evolves by the Hamiltonian $\mathbf{P}_\gamma\hat{H}^G_\gamma\mathbf{P}_\gamma$ from $\tau=0$ to $\max(\tau_1,\tau_2)$, then continues evolving, however, by the minus Hamiltonian $-\mathbf{P}_\gamma\hat{H}^G_\gamma\mathbf{P}_\gamma$ to $2\max(\tau_1,\tau_2)$, and meets the final state $\langle\Psi^t_{\mathbf{g}^\gamma_M},\gamma|$. Thus, one has

$$\lim_{N_i\to\infty}A^{G,\vec{N}}_\gamma=\int\mathrm{d}\mu_H(u)\langle\Psi_{\mathbf{g}^\gamma_M},\gamma|u|\Psi_{\mathbf{g}^\gamma_M},\gamma\rangle\equiv\mathcal{N}'_\gamma. \tag{5.42}$$

According to (5.41), $\mathfrak{F}^{\vec{N}}(\vec{\tau},\vec{v},\gamma)$ is a standard Fermionic Gaussian integral, and thus is calculable. We put the step-by-step derivation and the results in Appendix B. With that result, we can introduce $\mathfrak{F}(\vec{\tau},\vec{v},\gamma)$ as

$$\mathfrak{F}(\vec{\tau},\vec{v},\gamma) := \lim_{N_i\to\infty} \mathfrak{F}^{\vec{N}}(\vec{\tau},\vec{v},\gamma) \tag{5.43}$$

For the factor $\mathcal{N}_\gamma$ in (5.39), inserting a resolution of identity in (5.21) gives rise to

$$\mathcal{N}_\gamma^{-1} = \int \mathrm{d}\mu_H(u) \int \mathcal{D}[Z,\gamma]\langle\Omega_0,\gamma|Z,\gamma\rangle\langle Z,\gamma|u|\Omega_0,\gamma\rangle + O(t^\infty). \tag{5.44}$$

Then, applying (5.27) and (5.28) with letting $Z_{N+3} = Z = Z_0$, one may simplify the integral in (5.44) to a Gaussian like one. The stationary phase approximation analysis as did for the integral (5.37) gives that the integrand is peaked at $u = \mathbb{1}$. Following the arguments analogous to (5.38), one gets

$$\mathcal{N}_\gamma^{-1} = \langle\Psi_{\mathbf{g}_\gamma^M},\gamma|u|\Psi_{\mathbf{g}_\gamma^M},\gamma\rangle(1+O(t)) = \mathcal{N}'_\gamma(1+O(t))$$

which leads to

$$\mathcal{N}_\gamma\mathcal{N}'_\gamma = 1 + O(t). \tag{5.45}$$

Combining (5.45), (5.43), (5.42), and the result of the expectation value of $\hat{V}_{v_j}^{-\frac{1}{2}}$, and taking the limit $N_i\to\infty$, we get

$$G^0[F](\vec{\tau},\vec{v},\gamma) = \frac{1}{(a^2\mathring{p}_\gamma\beta)^{3/2}}\langle\hat{F}^{\vec{A}}\rangle\mathfrak{F}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)(1+O(t)). \tag{5.46}$$

Recalling the relation (5.20) and substituting the explicit expression of $\mathfrak{F}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)$ given by (B22), we get the propagator on γ

$$\begin{aligned} G(\vec{\tau},\vec{v},\gamma) =& \frac{\hbar}{V_{\mathrm{tot}}}\sum_{\vec{k}\in K[\gamma]^3}\exp\left(i\frac{2\pi}{\ell_o}\vec{k}\cdot(\vec{x}_{v_1}-\vec{x}_{v_2})\right)\times \\ &\left(\theta(\tau_1-\tau_2)\frac{\mathfrak{s}(\vec{k})\sigma^0 - \sum_{m=1}^3\sin\left(\frac{2\pi\mu}{\ell_o}k^m\right)\sigma^m}{2\mathfrak{s}(\vec{k})}e^{-i\omega(\vec{k})(\tau_1-\tau_2)}\right. \\ &\left. -\,\theta(\tau_2-\tau_1)\frac{\mathfrak{s}(\vec{k})\sigma^0 + \sum_{m=1}^3\sin\left(\frac{2\pi\mu}{\ell_o}k^m\right)\sigma^m}{2\mathfrak{s}(\vec{k})}e^{-i\omega(\vec{k})(\tau_2-\tau_1)}\right) \end{aligned} \tag{5.47}$$

where $V_{\mathrm{tot}} = (P\ell_o)^{3/2}$ is the total volume of γ under the geometry endowed by the coherent state $|\Psi_{\mathbf{g}_M^\gamma},\gamma\rangle$ and $\omega(\vec{k})$, by (B23), is

$$\omega(\vec{k}) = \frac{\mathfrak{s}(\vec{k})}{\sqrt{P}\mu} \tag{5.48}$$

with μ being the lattice spacing under the fiducial metric. Note that $G(\vec{\tau},\vec{v},\gamma)$ is the matrix $\{G_{\vec{A}}(\vec{\tau},\vec{v},\gamma)\}_{A_1,A_2=\pm}$ with $\{G_{\vec{A}}(\vec{\tau},\vec{v},\gamma)$ given by $G[F](\vec{\tau},\vec{v},\gamma) = G_{\vec{A}}(\vec{\tau},\vec{v},\gamma)\langle\hat{F}^{\vec{A}}\rangle$. Moreover, V_{tot} is independent of graphs and actually can be interpreted as the volume of Σ under the dynamics metric $P\delta_{ij}$. Furthermore, $\sqrt{P}\mu$ in (5.48) is the lattice spacing of γ under the dynamical metric $P\delta_{ij}$. With $G(\vec{\tau},\vec{v},\gamma)$, the final propagator after superposition over graphs is

$$G(\vec{\tau},\vec{v}) = \sum_{m=1}^L |w_m|^2 G(\vec{\tau},\vec{v},\gamma_m). \tag{5.49}$$

VI. FERMION DOUBLING AND ITS RESOLUTION

For the convenience of our further discussion, let us rescale the coordinate on Σ from x^i to $x'^i = \sqrt{P}x^i$ so that the dynamical metric becomes δ_{ij} in the prime coordinate. Besides, we rescale k^j by $k'^j = 2\pi k^j/(\sqrt{P}\ell_o)$ so that $\vec{k}'$

takes values in the fundamental Brillouin zone (FBZ) of γ which is denoted by FBZ(γ). Here, $\vec{k}' \in$ FBZ(γ) means $-\pi/a \leq k'^j \leq \pi/a$ and $k'^j = -\pi/a + 2n\pi/\sqrt{P}\ell_o$ for all $j = 1, 2, 3$, where $n \in \mathbb{Z}$ and $a = \sqrt{P}\mu$ being the lattice spacing under the dynamical metric. In what follows, for convenience we will still use the unprimed letters to denote the rescaled variables. With the rescaled variables, $G(\vec{\tau}, \vec{v}, \gamma)$ in (5.47) can be written as

$$G(\vec{\tau}, \vec{v}, \gamma) = \frac{\hbar}{V_{\rm tot}} \sum_{\vec{k} \in {\rm FBZ}(\gamma)} \int \frac{{\rm d}\omega}{2\pi i} G_\gamma(\omega, \vec{k}) e^{-i\omega(\tau_1 - \tau_2) + i\vec{k}\cdot(\vec{x}_{v_1} - \vec{x}_{v_2})} \tag{6.1}$$

with $G_\gamma(\omega, \vec{k})$ given by

$$G_\gamma(\omega, \vec{k}) = \frac{-\omega\sigma^0 + \frac{1}{a}\sum_{m=1}^3 \sin(ak^m)\sigma^m}{\omega^2 - \frac{1}{a^2}\sum_{m=1}^3 \sin^2(ak^m) + i\epsilon} \tag{6.2}$$

As usual in QFT, $G_\gamma(\omega, \vec{k})$ has the doubling problem. A physical mode at, saying, $(\omega, \vec{k}) = (\omega, k, 0, 0)$ satisfying $\omega = \frac{1}{a}\sin(ak) > 0$, implies another spurious doubler mode at $(\omega, \frac{\pi}{a} - k, 0, 0)$, so the fermion species is doubled in each direction on the lattice. The fermion doubling problem causes the trouble of the continuum limit of fermions on lattices, and is intractably linked to chirality by the Nielsen-Ninomiya no-go theorem [34]. However, in LQG, the lattices are dynamical and define the quantum states, so that the superposition over lattices makes sense. As shown in the previous sections, once we consider the vacuum given by the superposition of graphs, the resulting propagator is the average of those on the lattices. More precisely, according to (5.49), the propagator after the superposition of graph is

$$G(\vec{\tau}, \vec{v}) = \frac{\hbar}{V_{\rm tot}} \sum_{\vec{k} \in {\rm FBZ}(\gamma_L)} \int \frac{{\rm d}\omega}{2\pi i} \sum_{m=1}^L |w_m|^2 \chi_m(\vec{k}) G_{\gamma_m}(\omega, \vec{k}) e^{-i\omega(\tau_1 - \tau_2) + i\vec{k}\cdot(\vec{x}_{v_1} - \vec{x}_{v_2})}. \tag{6.3}$$

where $\chi_m(\vec{k})$ is the characteristic function given by

$$\chi_m(\vec{k}) = \begin{cases} 1, & \vec{k} \in {\rm FBZ}(\gamma_m), \\ 0, & \text{othereise.} \end{cases} \tag{6.4}$$

Eq. (6.3) implies the propagator in the Fourier space is

$$G(\omega, \vec{k}) = \sum_{m=1}^L |w_m|^2 \chi_m(\vec{k}) G_{\gamma_m}(\omega, \vec{k}). \tag{6.5}$$

The assumptions of Nielsen-Ninomiya theorem is clearly violated by summing over lattices. Thus, the superposition over graphs may help to resolve the doubling problem. This can be seen from the following heuristic discussion. For each graph γ_m, the physical mode, saying $(\omega_o, \vec{k}) = (\omega_o, k, 0, 0)$, occurs around $k = \frac{1}{a_m}\arcsin(a_m\omega_o) \approx \omega_o$. Since the physical mode is independent of the graphs, the average in (6.5) will keep the value of $G(\omega_o, \vec{k})$ around the physical mode unchanged. However, for the doubler modes, the one associated with γ_m occurs at $k = \pi/a_m - \frac{1}{a_m}\arcsin(a_m\omega_o) \approx \pi/a_m - \omega_o$ which depends on the graph. Therefore, the average in (6.5) will suppress $G(\omega_o, \vec{k})$ around the doubler modes. This discussion explains the mechanism that the superposition over graphs resolves the doubling problem. The issues can also be checked by a numerical investigation. In the numerical experiment, we choose $w_m = 1/\sqrt{L}$ for all m and set the parameters as $\omega_o = 50, a_1 = 2^{-50}, L = 1.2 \times 10^5, \epsilon = 10^{-4}$. Then, the results show that $|G_{++}(\omega_o, \vec{k})| < 4.2$ at the doubler modes on all $\{\gamma_n\}_{n=1}^L$, while $|G_{++}(\omega_o, \vec{k}_o)| \simeq 5 \times 10^5$ (equals ω_o/ϵ) at the physical mode. Moreover, as shown in Fig.2, L goes large, $G(\omega_o, \vec{k})$ remains large and constant at the physical mode k_o, while it is suppressed at the doubler mode. The coincidence between $G(\omega_o, \vec{k}_n)$ for the doubler modes $\vec{k}_n$ and $\omega_o/(\epsilon L)$ shown in FIG. 2 indicates that $G(\omega_o, \vec{k})$ at the doubler mode is indeed only dominated by one term in the sum. In absence of the doubling mode, $G(\omega_o, \vec{k})$ is peaked at the physical mode $\vec{k} = \vec{k}_o$. In any neighborhood of $\vec{k}_o$ and with sufficiently large L, $G(\omega_o, \vec{k})$ approximates the continuum fermion propagator arbitrarily well, due to the well-known result $\lim_{L\to\infty} \frac{1}{L}\sum_{n=1}^L f_n = \lim_{n\to\infty} f_n$ for any sequence $\{f_n\}$.

VII. OUTLOOK AND DISCUSSION

In this paper, we investigate the propagator of the LQG coupling fermion field. Since the LQG graphs own the physical meanings, the vacuum state in LQG can thus be defined as a superposition of the states on various graphs.

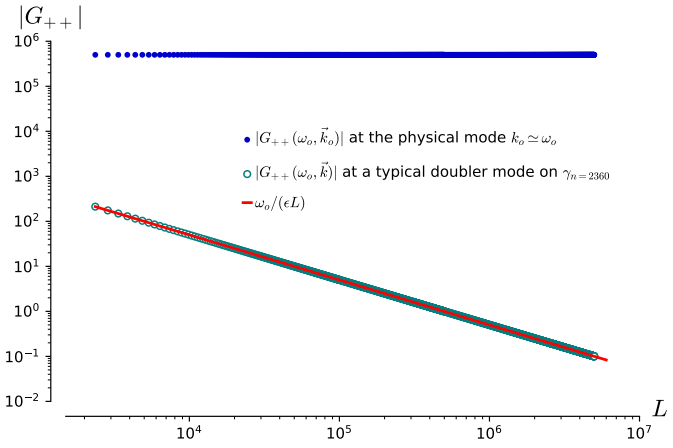


FIG. 2. Log-log plots of $|G_{++}(\omega_o, \vec{k}_o)|$ at the physical mode $k_o \simeq \omega_o$ (blue dots), and $|G_{++}(\omega_o, \vec{k})|$ at a typical doubler mode on γ_n with $n = 2360$ (green circles). The parameters are $\omega_o = 50, a_1 = 2^{-50}$ $\epsilon = 10^{-4}$. The red line draws $\omega_o/(\epsilon L)$ as a function of L. In this calculation, we choose $\hbar = 1$.

Applying the path integral formulation, we calculate the propagator step by step. Finally, it is shown that the graph-superposition feature results in the LQG fermion propagator being an average of the LFT fermion propagators over lattices. In the resulting fermion propagator, the doubler modes are suppressed by the average, but the physical mode remains unchanged. This result supports the fermion in LQG being free of the doubling problem. Moreover, according to our results, the superposition nature of quantum geometry should be a key to resolving the tension between fermion and the fundamental discreteness of QG. In addition, the superposition of lattices bringing the fermion propagator close to its continuous limit suggests that the LQG's continuous limit should also consider the superposition of lattices. This picture is similar to that suggested by group field theory [49].

In LQG, a state on γ can be rewritten on a larger graph γ' by letting the extra edges in γ' but not in γ take vanishing spins. Thus, the states $|\Omega, \gamma_n\rangle$ for all $1 \leq n \leq L$ can be thought of as a state on γ_L. Consequently, the superposition vacuum state $|\Omega\rangle$ is a state on γ_L. The action of the Hamiltonian on $|\Omega, \gamma_n\rangle$ is defined by considering the state as a one on γ_n. Thus the Hamiltonian acts on $|\Omega, \gamma_n\rangle$ in such a way that the objects, like loops, in the Hamiltonian operator, are chosen as the minimal ones in γ_n. However, a minimal loop in γ_n is not the minimal one in γ_L. That is to say, if one repeats the above calculation but regards the state as a one on γ_L, the Hamiltonian becomes nonlocal. This argument relates the doubling problem resolution proposed by the current work to the locality violation. It will be left as our future work to derive the nonlocal Hamiltonian.

Interestingly, the propagator (6.5) suggests that the quantum geometry provides a soft UV cut-off to fermions. Let us scale $\vec{k}$ large so that $\vec{k}$ is outside FBZ(γ_m) for certain m. Then the factor $\chi_n(\vec{k})$ makes all terms with $n \leq m$ in (6.5) vanish. Consequently, as we scale $\vec{k}$ larger and larger, fewer and fewer terms in the summation of the propagator survive and, thus, the propagator approaches 0. This result is consistent with the expectation that QG should regularize the UV behavior of matter fields.

Finally, let us discuss the path integral formulation introduced in Sec. III. The path integral formula is derived following the standard procedure. Then, an issue of whether the path integral formula is well-defined arises. This is a generally existing issue in the path integral formulation and can be explored from the following two perspectives. The first is to consider the path integral as the limit of a N-dimensional integral as N approaches ∞ (see (3.8)). From this point of view, one needs to pay attention to the existence of the limit. It should be noted that this perspective does not view the path integral formula as an integral over the space of paths. Due to this, the other attempt is to construct a measure on the space of paths to rewrite the formula as an integral over paths. This attempt has been achieved in quantum mechanics for the coherent state path integral formulation [50]. The process is quite similar to defining the integral $\int_{\mathbb{R}} e^{ix} \mathrm{d}x$ as the limit of $\int_{\mathbb{R}} e^{-x^2/\lambda^2 + ix} \mathrm{d}x/(\lambda\pi)$ as $\lambda \to 0^+$. The difference is that, in the path integral formula, the term analogous to $e^{-x^2/\lambda^2} \mathrm{d}x$ in the heuristic example finally gives rise to the Wiener measure in the space of paths. Then, it should be asked if this process can be applied to LQG to achieve a mathematically rigorous path integral formula.

VIII. ACKNOWLEDGEMENT

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Appendix A: semiclassical properties of $\widetilde{\psi_g^t}$ and $|\widetilde{\Psi_{\mathbf{g}}^t},\gamma\rangle$

To begin with, we show that $|\widetilde{\Psi_{\mathbf{g}}^t},\gamma\rangle$ is normalized up to some negligible term. One has

$$\langle\widetilde{\psi_g^t}|\widetilde{\psi_g^t}\rangle = 1 - \frac{t^{3/2}}{2\pi^{\frac{1}{2}}e^{t/4}}\frac{\sinh(p)}{p}e^{-\frac{p^2}{t}} = 1 + O(t^\infty). \tag{A1}$$

Thus, $1 - \||\widetilde{\Psi_{\mathbf{g}}^t},\gamma\rangle\|^2$ can be estimated by

$$\begin{aligned} 1 - \langle\widetilde{\Psi_{\mathbf{g}}^t},\gamma|\widetilde{\Psi_{\mathbf{g}}^t},\gamma\rangle &= 1 - \prod_{e\in E(\gamma)}\left(1 - \frac{t^{3/2}}{2\pi^{\frac{1}{2}}e^{t/4}}\frac{\sinh(p_e)}{p_e}e^{-\frac{p_e^2}{t}}\right) \\ &\leq \prod_{e\in E(\gamma)}\left(1 + \frac{t^{3/2}}{2\pi^{\frac{1}{2}}e^{t/4}}\frac{\sinh(p_e)}{p_e}e^{-\frac{p_e^2}{t}}\right) - 1. \end{aligned} \tag{A2}$$

Leting p_o be the minimum in $\{p_e\}_{e\in E(\gamma)}$, we can simplify (A2) to be

$$1 - \langle\widetilde{\Psi_{\mathbf{g}}^t},\gamma|\widetilde{\Psi_{\mathbf{g}}^t},\gamma\rangle \leq \left(1 + \frac{t^{3/2}}{2\pi^{\frac{1}{2}}e^{t/4}}\frac{\sinh(p_o)}{p_o}e^{-\frac{p_o^2}{t}}\right)^{|E(\gamma)|} - 1. \tag{A3}$$

To estimate the right hand side of this equation further, we may relate $|E(\gamma)|$ and the parameter t as follows. At first, since we concern ourselves with only the cubic graphs, $|E(\gamma)|$, i.e., the number of edges of a graph γ, is proportional to $1/\mu^3$ where μ is the lattice spacing of γ. Then, we note $p_e \sim \mu^2$ in that p_e is the integration of the classical flux over the surface dual to e. Thus, $\mu^4 \gg t$ for $p_e^2 \gg t$. As a consequence, $|E(\gamma)| \sim 1/\mu^3 \ll 1/t^{3/4}$. Combining this result with (A3), we finally obtain

$$1 - \langle\widetilde{\Psi_{\mathbf{g}}^t},\gamma|\widetilde{\Psi_{\mathbf{g}}^t},\gamma\rangle \leq \left(1 + \frac{t^{3/2}}{2\pi^{\frac{1}{2}}e^{t/4}}\frac{\sinh(p_o)}{p_o}e^{-\frac{p_o^2}{t}}\right)^{t^{3/4}} - 1 \leq c\,\frac{\sinh(p_o)}{p_o}t^{3/4}e^{-\frac{p_o^2}{t}}, \tag{A4}$$

for some constant c. This result implies

$$\langle\widetilde{\Psi_{\mathbf{g}}^t},\gamma|\widetilde{\Psi_{\mathbf{g}}^t},\gamma\rangle = 1 + O(t^\infty). \tag{A5}$$

Next, we show that the expectation values of monomials of holonomies, fluxes and volume operators in the states $\widetilde{\psi_g^t}$ and $|\widetilde{\Psi_{\mathbf{g}}^t},\gamma\rangle$ coincide with the classical values. That is to say, the states $\widetilde{\psi_g^t}$ and $|\widetilde{\Psi_{\mathbf{g}}^t},\gamma\rangle$ have the desirable semiclassical limit. To show this, it is sufficient to do the calculations on a graph γ comprising n edges which intersects at a vertex v as their common end point. The orientation of γ will be chosen such that all edges are outgoing. Indeed, for $n = 6$, this graph is just a single vertex of a cubic graph carrying the edges ending at it. Define an operator $\hat{\mathfrak{J}}_e$ on an edge e as

$$\hat{\mathfrak{J}}_e = \sqrt{\sum_{i=1}^{3}(\hat{p}_i^{v,e})^2}, \tag{A6}$$

and consider the operators taking the form $\prod_{l=1}^m \hat{p}_{i_l}^{v_l,e_l}$. Given a state $\psi \in \mathcal{H}_\gamma^G$, it can be expanded as a linear combination of the basis $\prod_{e\in E(\gamma)} D^{j_e}_{m_e n_e}(h_e)$,

$$\psi = \sum_{\vec{j},\vec{m},\vec{n}} \psi_{\vec{j},\vec{m},\vec{n}} \prod_{e\in E(\gamma)} D^{j_e}_{m_e n_e}(h_e). \tag{A7}$$

where $\vec{j}$, for instance, is the abbreviation of $\{j_e\}_{e\in E(\gamma)}$. Then, for ψ in the common domain of $\hat{p}_3^{v',e}$ and $\hat{\mathfrak{J}}_e$, one has

$$\|\hat{p}_3^{v',e}\psi\|^2=\sum_{\vec{j},\vec{m},\vec{n}}\frac{|\psi_{\vec{j},\vec{m},\vec{n}}|^2}{\prod_{\vec{j}}d_j}|tm_e|^2\leq\sum_{\vec{j},\vec{m},\vec{n}}\frac{|\psi_{\vec{j},\vec{m},\vec{n}}|^2}{\prod_{\vec{j}}d_{j_e}}t^2j_e(j_e+1)=\|\hat{\mathfrak{J}}_e\psi\|^2, \tag{A8}$$

where the calculation is done by assuming $v'=s_e=v$ but the inequality still holds for $v'=t_e$. For the operators $\hat{p}_i^{v',e}$ with $i=1,2$, we may take their eigenstates as the basis for the expansion so that the same calculation is applied to result in the same conclusion. Hence we conclude

$$\|\hat{p}_i^{v',e}\psi\|\leq\|\hat{\mathfrak{J}}_e\psi\|,\ \forall i=1,2,3,v'=v,t_e \text{ and } e\in E(\gamma). \tag{A9}$$

Moreover, due to

$$[\hat{\mathfrak{J}}_e,\hat{p}_i^{v',e'}]=0,\forall i=1,2,3,v'=v,t_{e'} \text{ and } e,e'\in E(\gamma), \tag{A10}$$

one gets

$$\left\|\prod_{l=1}^m\hat{p}_{i_l}^{v_l,e_l}\psi\right\|\leq\left\|\hat{\mathfrak{J}}_{e_1}\prod_{l=2}^m\hat{p}_{i_l}^{v_l,e_l}\psi\right\|=\left\|\prod_{l=2}^m\hat{p}_{i_l}^{v_l,e_l}\hat{\mathfrak{J}}_{e_1}\psi\right\|\leq\cdots\leq\left\|\prod_{l=1}^m\hat{\mathfrak{J}}_{e_l}\psi\right\| \tag{A11}$$

for all ψ in the common domain.

The volume operator $\hat{V}_v$ at v is given by

$$\begin{aligned}\hat{V}_v=&\kappa_0\sqrt{|\hat{Q}_v|},\\ \hat{Q}_v=&\frac{1}{6}\sum_{e,e',e'' \text{ at } v}\epsilon_{ijk}\epsilon(e,e',e'')\hat{p}_i^{v,e}\hat{p}_j^{v,e'}\hat{p}_k^{v,e''},\end{aligned} \tag{A12}$$

where κ_0 is a constant and $\epsilon(e,e',e'')=0,\pm1$ depending on the orientation of $e\wedge e'\wedge e''$ [51]. Given a state $\psi\in\mathcal{H}_\gamma^G$ lying in the domain of $\hat{V}_v$, one has

$$\|\hat{V}_v\psi\|^2=\langle\psi,\hat{V}_v^2\psi\rangle\leq\|\psi\|\|\hat{V}_v^2\psi\|=\|\psi\|\sqrt{\langle\psi,\hat{V}_v^4\psi\rangle}. \tag{A13}$$

Substituting the definition of $\hat{V}_v$ to get $\langle\psi,\hat{V}_v^4\psi\rangle=\kappa_0^4\langle\psi,\hat{Q}_v^2\psi\rangle=\kappa_0^4\|\hat{Q}_v\psi\|^2$, one obtains

$$\|\hat{V}_v\psi\|^2\leq\kappa_0^2\|\psi\|\|\hat{Q}_v\psi\|. \tag{A14}$$

Then, employing the triangle inequality and (A10) result in

$$\|\hat{Q}_v\psi\|\leq\sum_{e,e',e'' \text{ at } v}|\epsilon(e,e',e'')|\,\|\hat{\mathfrak{J}}_e\hat{\mathfrak{J}}_{e'}\hat{\mathfrak{J}}_{e''}\psi\|. \tag{A15}$$

For convenience, we introduce an operator $\hat{\Omega}_{e,e',e''}$ as

$$\hat{\Omega}_{e,e',e''}=|\epsilon(e,e',e'')|(\hat{\mathfrak{J}}_e+1)(\hat{\mathfrak{J}}_{e'}+1)(\hat{\mathfrak{J}}_{e''}+1). \tag{A16}$$

Because of $\hat{\mathfrak{J}}_e>0$, one has $(|\epsilon(e,e',e'')|\hat{\mathfrak{J}}_e\hat{\mathfrak{J}}_{e'}\hat{\mathfrak{J}}_{e''})^2\leq(\mathfrak{Q}_{e,e',e''})^2$ and $(\mathfrak{Q}_{e,e',e''})^2>1$, which implies

$$|\epsilon(e,e',e'')|\|\hat{\mathfrak{J}}_e\hat{\mathfrak{J}}_{e'}\hat{\mathfrak{J}}_{e''}\psi\|\leq\|\hat{\Omega}_{e,e',e''}\psi\| \tag{A17}$$

and

$$\|\psi\|\leq\|\hat{\Omega}_{e,e',e''}\psi\|. \tag{A18}$$

Combing (A15) and(A17), we have

$$\|\hat{Q}_v\psi\|\leq\sum_{e,e',e''}\|\hat{\Omega}_{e,e',e''}\psi\| \tag{A19}$$

This equation leads to

$$\|\hat{V}_v\psi\|^2 \leq \kappa_0^2 \sum_{e,e',e'' \text{ at } v} \|\psi\|\|\hat{\mathfrak{Q}}_{e,e',e''}\psi\| \leq \kappa_0^2 \sum_{e,e',e''} \|\hat{\mathfrak{Q}}_{e,e',e''}\psi\|^2 \leq \kappa_0^2 \left(\sum_{e,e',e''} \|\hat{\mathfrak{Q}}_{e,e',e''}\psi\| \right)^2 \tag{A20}$$

where the last step uses $\|\hat{\mathfrak{Q}}_{e,e',e''}\psi\| > 0$.

Thanks to (A20), we now can employ the relative bound of monomials of fluxes and volume operators. Consider the operators $\hat{K}_1^{(m)} = \prod_{l=1}^m \hat{p}_{i_l}^{v_l,e_l}\hat{V}_v$ and $\hat{K}_2^{(m)} = \hat{V}_v \prod_{l=1}^m \hat{p}_{i_l}^{v_l,e_l}$. They satisfy

$$\begin{aligned}
\|\hat{K}_1^{(m)}\psi\| &\leq \left\| \prod_{l=1}^m \hat{\mathfrak{J}}_{e_l}\hat{V}_v\psi \right\| = \left\| \hat{V}_v \prod_{l=1}^m \hat{\mathfrak{J}}_{e_l}\psi \right\| \leq \kappa_0 \sum_{e,e',e'' \text{ at } v} \left\| \mathfrak{Q}_{e,e',e''} \prod_{l=1}^m \hat{\mathfrak{J}}_{e_l}\psi \right\|, \\
\|\hat{K}_2^{(m)}\psi\| &\leq \kappa_0 \sum_{e,e',e'' \text{ at } v} \left\| \mathfrak{Q}_{e,e',e''} \prod_{l=1}^m \hat{p}_{i_l}^{v_l,e_l}\psi \right\| \leq \kappa_0 \sum_{e,e',e'' \text{ at } v} \left\| \mathfrak{Q}_{e,e',e''} \prod_{l=1}^m \hat{\mathfrak{J}}_{e_l}\psi \right\|.
\end{aligned} \tag{A21}$$

Since a general monomial of fluxes and volume operators takes the form

$$\hat{K} = \prod_{i=1}^n \hat{K}_{j_i}^{(m_i)}, \tag{A22}$$

we can obtain

$$\begin{aligned}
\|\hat{K}\psi\| &\leq \kappa_0 \sum_{e_1,e_1',e_1''} \left\| \hat{\mathfrak{Q}}_{e_1,e_1',e_1''} \prod_{k=1}^{m_1} \hat{\mathfrak{J}}_{e_k} \prod_{i=2}^n \hat{K}_{j_i}^{(m_i)}\psi \right\| \\
&= \kappa_0 \sum_{e_1,e_1',e_1''} \left\| \prod_{i=2}^n \hat{K}_{j_i}^{(m_i)} \hat{\mathfrak{Q}}_{e_1,e_1',e_1''} \prod_{k=1}^{m_1} \hat{\mathfrak{J}}_{e_k}\psi \right\| \\
&\leq \cdots \\
&\leq \kappa_0^n \sum_{\substack{e_1,e_1',e_1'' \\ e_2,e_2',e_2'' \\ \cdots \\ e_n,e_n',e_n''}} \left\| \prod_{i=1}^n \left(\hat{\mathfrak{Q}}_{e_i,e_i',e_i''} \prod_{k_i=1}^{m_i} \hat{\mathfrak{J}}_{e_{k_i}} \right) \psi \right\|.
\end{aligned} \tag{A23}$$

Now let us investigate the relative bound of operators $\hat{K}D_{ab}^{\frac{1}{2}}(h_e)$ and $D_{ab}^{\frac{1}{2}}(h_e)\hat{K}$ by applying (A23). Because of $\|D_{ab}^{\frac{1}{2}}(h_e)\| \leq 1$, we get

$$\|D_{ab}^{\frac{1}{2}}(h_e)\hat{K}\psi\| \leq \|\hat{K}\psi\| \leq \kappa_0^n \sum_{\substack{e_1,e_1',e_1'' \\ e_2,e_2',e_2'' \\ \cdots}} \left\| \prod_{i=1}^n \left(\hat{\mathfrak{Q}}_{e_i,e_i',e_i''} \prod_{k_i=1}^{m_i} \hat{\mathfrak{J}}_{e_{k_i}} \right) \psi \right\|. \tag{A24}$$

Moreover, for the operator $\hat{K}D_{ab}^{\frac{1}{2}}(h_e)$, we have

$$\|\hat{K}D_{ab}^{\frac{1}{2}}(h_e)\psi\| \leq \kappa_0^n \sum_{\substack{e_1,e_1',e_1'' \\ e_2,e_2',e_2'' \\ \cdots}} \left\| \prod_{i=1}^n \left(\hat{\mathfrak{Q}}_{e_i,e_i',e_i''} \prod_{k_i=1}^{m_i} \hat{\mathfrak{J}}_{e_{k_i}} \right) D_{ab}^{\frac{1}{2}}(h_e)\psi \right\|. \tag{A25}$$

For the term whose norm is involved in the right hand side, we get

$$\begin{aligned}
&\prod_{i=1}^n \left(\hat{\mathfrak{Q}}_{e_i,e_i',e_i''} \prod_{k_i=1}^{m_i} \hat{\mathfrak{J}}_{e_{k_i}} \right) D_{ab}^{\frac{1}{2}}(h_e)\psi \\
&= \sum_{\vec{j},\vec{m},\vec{n}} \psi_{\vec{j},\vec{m},\vec{n}} \sum_{J_e = j_e \pm \frac{1}{2}} d_{J_e} \mathfrak{F}(\{j_{\tilde{e}}\}_{\tilde{e}\neq e}, J_e)(-1)^{a+m_e-b-n_e} \begin{pmatrix} \frac{1}{2} & j_e & J_e \\ a & m_e & -(a+m_e) \end{pmatrix} \begin{pmatrix} \frac{1}{2} & j_e & J_e \\ b & n_e & -(b+n_e) \end{pmatrix} \times \\
&D_{(a+m_e)(b+n_e)}^{J_e}(h_e) \otimes \bigotimes_{\tilde{e}\neq e} D_{m_{\tilde{e}},n_{\tilde{e}}}^{j_{\tilde{e}}}(h_e),
\end{aligned} \tag{A26}$$

where $\mathfrak{F}(\{j_{\tilde{e}}\}_{\tilde{e}\neq e}, J_e)$ denotes the eigenvalue of the operator $\prod_{i=1}^{n}\left(\hat{\Omega}_{e_i,e_i',e_i''}\prod_{k_i=1}^{m_i}\hat{\mathfrak{J}}_{e_{k_i}}\right)$ acting on $D^{J_e}_{(a+m_e)(b+n_e)}(h_e)\otimes\bigotimes_{\tilde{e}\neq e}D^{j_{\tilde{e}}}_{m_{\tilde{e}},n_{\tilde{e}}}(h_e)$. Then a straightforward calculation gives

$$\left\|\prod_{i=1}^{n}\left(\hat{\Omega}_{e_i,e_i',e_i''}\prod_{k_i=1}^{m_i}\hat{\mathfrak{J}}_{e_{k_i}}\right)D^{\frac{1}{2}}_{ab}(h_e)\psi\right\|^2 \leq 4\sum_{\vec{j},\vec{m},\vec{n}}\frac{|\psi_{\vec{j},\vec{m},\vec{n}}|}{\prod_{\tilde{e}}d_{j_{\tilde{e}}}}\left(\mathfrak{F}(\{j_{\tilde{e}}\}_{\tilde{e}\neq e}, j_e+\frac{1}{2})^2+\mathfrak{F}(\{j_{\tilde{e}}\}_{\tilde{e}\neq e}, j_e-\frac{1}{2})^2\right) \tag{A27}$$

where it is necessary to employ the inequality

$$\left|d_j\begin{pmatrix}\frac{1}{2} & j\pm\frac{1}{2} & j\\ a & m & -(a+m)\end{pmatrix}\begin{pmatrix}\frac{1}{2} & j\pm\frac{1}{2} & j\\ b & n & -(b+n)\end{pmatrix}\right|^2\leq 2 \tag{A28}$$

Since $\mathfrak{F}(\{j_{\tilde{e}}\}_{\tilde{e}\neq e}, j_e\pm\frac{1}{2})$ is a polynomial of $\sqrt{j_{\tilde{e}}(j_{\tilde{e}}+1)}$ (for all $\tilde{e}\neq e$) and $\sqrt{(j_e\pm\frac{1}{2})(j_e\pm\frac{1}{2}+1)}$, there exists a positive constant β such that

$$\mathfrak{F}(\{j_{\tilde{e}}\}_{\tilde{e}\neq e}, j_e+\frac{1}{2})^2+\mathfrak{F}(\{j_{\tilde{e}}\}_{\tilde{e}\neq e}, j_e-\frac{1}{2})^2\leq 2\,\mathfrak{F}(\{j_{\tilde{e}}\}_{\tilde{e}\neq e}, j_e)^2+\beta^2, \forall\vec{j}. \tag{A29}$$

Therefore, we have

$$\left\|\prod_{i=1}^{n}\left(\hat{\Omega}_{e_i,e_i',e_i''}\prod_{k_i=1}^{m_i}\hat{\mathfrak{J}}_{e_{k_i}}\right)D^{\frac{1}{2}}_{ab}(h_e)\psi\right\|^2\leq 2\left\|\prod_{i=1}^{n}\left(\hat{\Omega}_{e_i,e_i',e_i''}\prod_{k_i=1}^{m_i}\hat{\mathfrak{J}}_{e_{k_i}}\right)\psi\right\|^2+\beta^2\|\psi\|^2 \tag{A30}$$

This implies

$$\left\|\prod_{i=1}^{n}\left(\hat{\Omega}_{e_i,e_i',e_i''}\prod_{k_i=1}^{m_i}\hat{\mathfrak{J}}_{e_{k_i}}\right)D^{\frac{1}{2}}_{ab}(h_e)\psi\right\|\leq\sqrt{2}\left\|\prod_{i=1}^{n}\left(\hat{\Omega}_{e_i,e_i',e_i''}\prod_{k_i=1}^{m_i}\hat{\mathfrak{J}}_{e_{k_i}}\right)\psi\right\|+\beta\|\psi\|. \tag{A31}$$

Substituting this inequality into (A25), one finally obtains

$$\|\hat{K}D^{\frac{1}{2}}_{ab}(h_e)\psi\|\leq\sqrt{2}\kappa_0^n\sum_{\substack{e_1,e_1',e_1''\\ e_2,e_2',e_2''\\ \cdots}}\left\|\prod_{i=1}^{n}\left(\hat{\Omega}_{e_i,e_i',e_i''}\prod_{k_i=1}^{m_i}\hat{\mathfrak{J}}_{e_{k_i}}\right)\psi\right\|+\beta'\|\psi\|. \tag{A32}$$

for some constant β'. For a general monomial $\hat{M}$ of fluxes, holonomies and volume operator, one only needs to do the derivations (A24) and (A32) successively to get

$$\|\hat{M}\psi\|\leq\alpha\kappa_0^N\sum_{\substack{e_1,e_1',e_1''\\ e_2,e_2',e_2''\\ \cdots\\ e_N,e_N',e_N''}}\left\|\prod_{i=1}^{N}\hat{\Omega}_{e_i,e_i',e_i''}\widehat{\mathscr{J}}\psi\right\|+\beta''\|\psi\|, \tag{A33}$$

where α and β'' are some constants depending on the explicit expression of $\hat{M}$, N is the number of volume operators in $\hat{M}$, and $\widehat{\mathscr{J}}$ is the operator

$$\widehat{\mathscr{J}}:=\hat{M}\Big|_{D^{\frac{1}{2}}_{ab}(h_e)\to 1,\hat{V}_v\to 1,\hat{p}_i^{v',e}\to\hat{\mathfrak{J}}_e}. \tag{A34}$$

Now we can study the expectation value of $\hat{M}$ in the coherent state $|\widetilde{\Psi^t_{\mathbf{g}}},\gamma\rangle$. A straightforward calculation gives

$$\begin{aligned}&\left|\langle\Psi^t_{\mathbf{g}},\gamma|\hat{M}|\Psi^t_{\mathbf{g}},\gamma\rangle-\langle\widetilde{\Psi^t_{\mathbf{g}}},\gamma|\hat{M}|\widetilde{\Psi^t_{\mathbf{g}}},\gamma\rangle\right|\\ \leq&2\left\|(1-\mathbf{P}_\gamma)|\Psi^t_{\mathbf{g}},\gamma\rangle\right\|\left\|\hat{M}|\Psi^t_{\mathbf{g}},\gamma\rangle\right\|+\left\|(1-\mathbf{P}_\gamma)|\Psi^t_{\mathbf{g}},\gamma\rangle\right\|\left\|\hat{M}(1-\mathbf{P}_\gamma)|\Psi^t_{\mathbf{g}},\gamma\rangle\right\|.\end{aligned} \tag{A35}$$

Here $\left\|\hat{M}|\Psi^t_{\mathbf{g}},\gamma\rangle\right\|^2$ is just the expectation value of $\hat{M}^\dagger\hat{M}$ in the regular complexifier coherent state. Then employing the results in [30] results in

$$\left\|\hat{M}|\Psi^t_{\mathbf{g}},\gamma\rangle\right\|^2 = M(\mathbf{g})^* M(\mathbf{g}) + O(t) \tag{A36}$$

where $M(\mathbf{g})$ denotes the classical value of the observable $\hat{M}$ at $\mathbf{g}$. For $\left\|\hat{M}(1-\mathbf{P}_\gamma)|\Psi^t_{\mathbf{g}},\gamma\rangle\right\|$ in (A35), (A33) leads to

$$\left\|\hat{M}(1-\mathbf{P}_\gamma)|\Psi^t_{\mathbf{g}},\gamma\rangle\right\| \leq \alpha\kappa_0^N \sum_{\substack{e_1,e_1',e_1''\\ e_2,e_2',e_2''\\ \cdots\\ e_N,e_N',e_N''}} \left\|\prod_{i=1}^N \hat{\mathfrak{Q}}_{e_i,e_i',e_i''}\widehat{\mathscr{J}}(1-\mathbf{P}_\gamma)|\Psi^t_{\mathbf{g}},\gamma\rangle\right\| + \beta''\|(1-\mathbf{P}_\gamma)|\Psi^t_{\mathbf{g}},\gamma\rangle\| \tag{A37}$$

Due to $[\hat{\mathfrak{J}}_e, P_\gamma]=0$ for all $e\in E(\gamma)$, one has

$$\left\|\prod_{i=1}^N \hat{\mathfrak{Q}}_{e_i,e_i',e_i''}\widehat{\mathscr{J}}(1-\mathbf{P}_\gamma)|\Psi^t_{\mathbf{g}},\gamma\rangle\right\| = \left\|\prod_{i=1}^N \hat{\mathfrak{Q}}_{e_i,e_i',e_i''}\widehat{\mathscr{J}}|\Psi^t_{\mathbf{g}},\gamma\rangle\right\| - \left\|\prod_{i=1}^N \hat{\mathfrak{Q}}_{e_i,e_i',e_i''}\widehat{\mathscr{J}}\mathbf{P}_\gamma|\Psi^t_{\mathbf{g}},\gamma\rangle\right\|. \tag{A38}$$

By definition, $\prod_{i=1}^N \hat{\mathfrak{Q}}_{e_i,e_i',e_i''}\widehat{\mathscr{J}}$ takes the form

$$\prod_{i=1}^N \hat{\mathfrak{Q}}_{e_i,e_i',e_i''}\widehat{\mathscr{J}} = \prod_{e\in E(\gamma)} P_{k_e}(\hat{\mathfrak{J}}_e) \tag{A39}$$

where $P_{k_e}(\hat{\mathfrak{J}}_e)$ is a polynomial of $\hat{\mathfrak{J}}_e$ with degree k_e. Let us define $\mathbf{P}_e$ as the projection

$$\mathbf{P}_e : \bigoplus_{j_e\geq 0} \mathcal{H}_{j_e}\otimes\mathcal{H}_{j_e}^* \to \bigoplus_{j_e\neq 0} \mathcal{H}_{j_e}\otimes\mathcal{H}_{j_e}^* \tag{A40}$$

so that $\mathbf{P}_\gamma = \prod_{e\in E(\gamma)}\mathbf{P}_e$. We have

$$\left\|\prod_{i=1}^N \hat{\mathfrak{Q}}_{e_i,e_i',e_i''}\widehat{\mathscr{J}}\mathbf{P}_\gamma|\right\| = \prod_{e\in E(\gamma)} \|P_{k_e}(\hat{\mathfrak{J}}_e)\mathbf{P}_e\psi^t_{g_e}\| \tag{A41}$$

Because of $[\mathbf{P}_e,\hat{\mathfrak{J}}_e]=0$, one has

$$\begin{aligned}\|P_{k_e}(\hat{\mathfrak{J}}_e)\mathbf{P}_e\psi^t_{g_e}\| =& \|P_{k_e}(\hat{\mathfrak{J}}_e)\psi^t_{g_e}\| - \|P_{k_e}(\hat{\mathfrak{J}}_e)(1-\mathbf{P}_e)\psi^t_{g_e}\| \\ =& |P_{k_e}(p_e)| + O(t) - \frac{t^{3/4}e^{-\frac{p_e^2}{2t}}}{\sqrt{2}\pi^{\frac{1}{4}}e^{t/8}}\sqrt{\frac{\sinh(p_e)}{p_e}}|P_{k_e}(0)| \\ =& |P_{k_e}(p_e)| - O(t^\infty),\end{aligned} \tag{A42}$$

where we used $\|P(\hat{\mathfrak{J}}_e)\psi^t_{g_e}\|^2 = \langle\psi_{g_e^t}P_{k_e}(\hat{\mathfrak{J}}^e)P_{k_e}(\hat{\mathfrak{J}}^e)^\dagger\psi^t_{g_e}\rangle = |P_{k_e}(p_e)|^2 + O(t)$. Then combining (A38) (A39), (A41), and (A42), we get

$$\left\|\prod_{i=1}^N \hat{\mathfrak{Q}}_{e_i,e_i',e_i''}\widehat{\mathscr{J}}(1-\mathbf{P}_\gamma)|\Psi^t_{\mathbf{g}},\gamma\rangle\right\| = \prod_{e\in E(\gamma)}|P_{k_e}(p_e)| - \prod_{e\in E(\gamma)}(|P_{k_e}(p_e)| - O(t^\infty)) = O(t^\infty). \tag{A43}$$

Finally, substituting (A36), (A43) an (A37) into (A35), we get

$$\left|\langle\Psi^t_{\mathbf{g}},\gamma|\hat{M}|\Psi^t_{\mathbf{g}},\gamma\rangle - \langle\widetilde{\Psi^t_{\mathbf{g}}},\gamma|\hat{M}|\widetilde{\Psi^t_{\mathbf{g}}},\gamma\rangle\right| = O(t^\infty) \tag{A44}$$

where we used

$$\left\|(1-\mathbf{P}_\gamma)|\Psi^t_{\mathbf{g}},\gamma\rangle\right\| = 1 - \langle\widetilde{\Psi}^t_{\mathbf{g}},\gamma|\widetilde{\Psi}^t_{\mathbf{g}},\gamma\rangle = O(t^\infty). \tag{A45}$$

Therefore, the expectation value of $\hat{M}$ in $|\widetilde{\Psi^t_{\mathbf{g}}},\gamma\rangle$ is the same as that in the regular compexifier coherent state up to some $O(t^\infty)$ term, i.e.,

$$\langle\Psi^t_{\mathbf{g}},\gamma|\hat{M}|\Psi^t_{\mathbf{g}},\gamma\rangle = \langle\widetilde{\Psi^t_{\mathbf{g}}},\gamma|\hat{M}|\widetilde{\Psi^t_{\mathbf{g}}},\gamma\rangle + O(t^\infty). \tag{A46}$$

Appendix B: Calculation of $\mathfrak{F}^{\vec{N}}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)$

1. The Hession matrix of $S^F_{\vec{N}}[\mathbf{g}^\gamma_M,\nu]$

For the integral in (5.41), we need the Hession matrix $\mathscr{H}^{\vec{N}}_F$ given by

$$S^F_{\vec{N}}[\mathbf{g}^\gamma_M,\nu]=\sum_{j,j'=0}^{N+3}\sum_{v,v'\in V(\gamma)}\nu^\dagger_{j'}(v')(\mathscr{H}^{\vec{N}}_F)_{(v',j'),(v,j)}\nu_j(v). \tag{B1}$$

As shown in (5.36), $S^F_{\vec{N}}[\mathbf{g}^\gamma_M,\nu]$ contains the matrix element of the Hamiltonian operator $\mathbf{P}_\gamma\hat{H}^F_\gamma\mathbf{P}_\gamma$ between $|Z_j,\gamma\rangle$ and $\langle Z_{j+1},\gamma|$. Since the stationary phase of the gravity path integral has fixed the background geometry to the Minkowski geometry for all moments, $|Z_j,\gamma\rangle$ for each j thus takes the form $|Z_j,\gamma\rangle=|\Psi^t_{\mathbf{g}^M_\gamma},\gamma\rangle\otimes|\Phi_{\nu_j},\gamma\rangle$. As a consequence, sandwiched by $|Z_j,\gamma\rangle$ and $\langle Z_{j+1},\gamma|$, the holonomy and flux operators in $\mathbf{P}_\gamma\hat{H}^F_\gamma\mathbf{P}_\gamma$ becomes their expectation value in the states $\mathbf{P}_\gamma|\Psi^t_{\mathbf{g}^M_\gamma},\gamma\rangle$. Since we are concerned with only the leading order of the integral (5.41), we thus only need to calculate the leading order of the holonomy and flux operators' expectation values. According to (2.41), the leading order of these expectation values are the same as that in the states $|\Psi^t_{\mathbf{g}^M_\gamma},\gamma\rangle$ and thus compatible with their classical value. We thus have

$$\begin{aligned}&\frac{\langle\Psi^t_{\mathbf{g}^\gamma_M},\gamma|\otimes\langle\nu_{k+1}|\mathbf{P}_\gamma\hat{H}_F\mathbf{P}_\gamma|\nu_k\rangle\otimes|\Psi^t_{\mathbf{g}^\gamma_M},\gamma\rangle}{\langle\nu_{k+1}|\nu_k\rangle}\\ =&\frac{i\hbar}{2}\frac{1}{a\sqrt{\mathring{p}_\gamma\beta}}\sum_{v\in V(\gamma)}\sum_{e\in E_v(\gamma)}\Big(\nu^\dagger_{k+1}(v)\sigma^e\nu_k(v+\delta_e)-\nu^\dagger_{k+1}(v+\delta_e)\sigma^e\nu_k(v)\Big)\end{aligned} \tag{B2}$$

where we use the same notations as in (4.4). Substituting (B2) into (5.36) and comparing the result with the definition (B1), we get

$$\frac{(\mathscr{H}^{\vec{N}}_F)_{(v',j'),(v,j)}}{t}=\delta_{v',v}\delta_{j',j+1}-\delta_{j',j}\delta_{v',v}+\frac{\delta\tau_j}{2a\sqrt{\mathring{p}_\gamma\beta}}\sum_{e\in E_v(\gamma)}(\delta_{v',v-\delta_e}-\delta_{v',v+\delta_e})\delta_{j',j+1}\sigma^e. \tag{B3}$$

The Hessian matrix $\mathscr{H}^{\vec{N}}_F$ can be block diagonalized under the Bogoliubov transformation,

$$\widehat{\nu}_j(\vec{k})=\sqrt{\frac{\mu^3}{\ell_o^3}}\sum_{v\in V(\gamma)}\Theta(\vec{k})\nu_j(v)e^{-i\frac{2\pi}{\ell_o}\vec{k}\cdot\vec{x}_v}, \tag{B4}$$

which gives rise to the transform of $\mathscr{H}^{\vec{N}}_F$

$$\frac{\widehat{\mathscr{H}^{\vec{N}}_F}(\vec{k},A)_{j',j}}{t}=-\delta_{j',j}+\delta_{j',j+1}\left(1-A\frac{i\delta\tau_j}{a\sqrt{\mathring{p}_\gamma\beta}}\mathfrak{s}(\vec{k})\right),\ \forall A=\pm, \tag{B5}$$

so that

$$S^F_N=\sum_{\substack{\vec{k}\in K[\gamma]^3\\S=\pm}}\sum_{j,j'=0}^{N+3}\widehat{\nu}_{j',S}(\vec{k})^*\widehat{\mathscr{H}^{\vec{N}}_F}(\vec{k},S)_{j',j}\widehat{\nu}_{j,S}(\vec{k}). \tag{B6}$$

Substituting (B4) and (B6) into (5.41) leads to

$$\begin{aligned}\mathfrak{F}^{\vec{N}}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)=\int\mathcal{D}[\hat{\nu},\gamma]\exp\left(\frac{1}{t}\sum_{\substack{\vec{k}\in K[\gamma]^3\\S=\pm}}\sum_{j,j'=0}^{N+3}\widehat{\nu}_{j',S}(\vec{k})^*\widehat{\mathscr{H}^{\vec{N}}_F}(\vec{k},S)_{j',j}\widehat{\nu}_{j,S}(\vec{k})\right)\times\\ \mathcal{F}_{\vec{A}}\prod_{\vec{k}\in K[\gamma]^3}\widehat{\nu}_{N+3,-}(\vec{k})\left(\prod_{\vec{k}\in K[\gamma]^3}\widehat{\nu}_{0,-}(\vec{k})\right)^*,\end{aligned} \tag{B7}$$

where $\mathcal{D}[\widehat{\nu},\gamma]$ is

$$\mathcal{D}[\widehat{\nu},\gamma]=\prod_{i=0}^{N+3}\prod_{\vec{k}\in K[\gamma]^3}\prod_{A=\pm}\mathrm{d}\widehat{\nu}_{i,A}(\vec{k})\mathrm{d}\widehat{\nu}_{i,A}(\vec{k})^*. \tag{B8}$$

Let $\widehat{\mathcal{F}}_{\vec{B}}$ be the transform of $\mathcal{F}_{\vec{A}}$ under the transformation (B4), i.e.,

$$\widehat{\mathcal{F}}_{\vec{B}}(\vec{k}_1,\vec{k}_2)=\hbar\,\widehat{\nu}_{\mathfrak{n}(1),B_1}(\vec{k}_1)\widehat{\nu}^*_{\mathfrak{n}(2)+1,B_2}(\vec{k}_2),\ \text{with}\ \vec{k}=(k_1,k_2). \tag{B9}$$

With $\widehat{\mathcal{F}}_{\vec{B}}$, we defined $\widehat{\mathfrak{F}}^{\vec{N}}_{\vec{B}}(\vec{\tau},\vec{k}_1,\vec{k}_2,\gamma)$ as

$$\begin{aligned}\widehat{\mathfrak{F}}^{\vec{N}}_{\vec{B}}(\vec{\tau},\vec{k}_1,\vec{k}_2,\gamma)=\int\mathcal{D}[\hat{\nu},\gamma]\exp\left(\frac{1}{t}\sum_{\substack{\vec{k}'\in K[\gamma]^3\\S=\pm}}\sum_{j,j'=0}^{N+3}\widehat{\nu}_{j',S}(\vec{k}')^*\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k}',S)_{j',j}\widehat{\nu}_{j,S}(\vec{k})\right)\times\\ \widehat{\mathcal{F}}_{\vec{B}}(\vec{k}_1,\vec{k}_2)\prod_{\vec{k}''\in K[\gamma]^3}\widehat{\nu}_{N+3,-}(\vec{k}'')\left(\prod_{\vec{k}'''\in K[\gamma]^3}\widehat{\nu}_{0,-}(\vec{k}''')\right)^*\end{aligned} \tag{B10}$$

such that

$$\begin{aligned}\mathfrak{F}^{\vec{N}}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)=&\frac{\mu^3}{\ell_o^3}\sum_{(\vec{k}_1,\vec{k}_2)\in K_2[\gamma]}\left(\Theta(\vec{k}_1)^\dagger_{A_1,B_1}\Theta(\vec{k}_2)_{B_2A_{l'}}\right)\times\\ &\exp\left(i\frac{2\pi}{\ell_o}\left(\vec{k}_1\cdot\vec{x}_{v_1}-\vec{k}_2\cdot\vec{x}_{v_2}\right)\right)\widehat{\mathfrak{F}}^{\vec{N}}_{\vec{B}}(\vec{\tau},\vec{k}_1,\vec{k}_2,\gamma).\end{aligned} \tag{B11}$$

where $K_2[\gamma]$ denotes $K[\gamma]^3\times K[\gamma]^3$.

To perform the integral in (B10), we employ the the result for Gaussian integral of complex fermions [52] as follows

$$\int D[\psi,\psi^\dagger]\left(\prod_{\alpha=1}^{r}\psi^*_{i_\alpha}\psi_{j_\alpha}\right)\exp\left[\psi^\dagger A\psi\right]=\det(A)\det\left((A^{-T})_{IJ}\right), \tag{B12}$$

with $\mathcal{D}[\psi,\psi^\dagger]:=\prod_j\mathrm{d}\psi_j\mathrm{d}\psi^*_j$, $I:=(i_1,i_2,\cdots,i_r)$ and $J:=(j_1,j_2,\cdots,j_r)$. According to this formula, we study the determinant and the inverse of the Hessian matrix $\widehat{\mathscr{H}_F}^{\vec{N}}$. Writing the matrix in the matrix form, we get

$$\frac{\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},S)}{t}=\begin{pmatrix}-1&0&0&0&\cdots&0\\ \mathfrak{a}_1(\vec{k},S)&-1&0&0&\cdots&0\\ 0&\mathfrak{a}_2(\vec{k},S)&-1&0&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&\cdots&\mathfrak{a}_{N+2}(\vec{k},S)&-1&0\\ 0&0&\cdots&0&\mathfrak{a}_{N+3}(\vec{k},S)&-1\end{pmatrix}, \tag{B13}$$

with

$$\mathfrak{a}_j(\vec{k},S)=1-S\frac{i\delta\tau_j}{a\sqrt{\mathring{p}_\gamma\beta}}\mathfrak{s}(\vec{k}). \tag{B14}$$

The fact that this matrix is lower triangular results in

$$\det\left(\frac{\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},S)}{t}\right)=(-1)^{N+4}=\pm1. \tag{B15}$$

Moreover, one can check that the inverse of $\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},S)$ is given by

$$t(\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},S)^{-1})_{mn}=\begin{cases}0, & m<n,\\ 1, & m=n,\\ -\prod_{l=n+1}^{m}\mathfrak{a}_l(\vec{k},S), & m>n,\end{cases} \tag{B16}$$

2. result of $\widehat{\mathfrak{F}}^{\vec{N}}_{\vec{B}}(\vec{\tau},\vec{k}_1,\vec{k}_2,\gamma)$

Let us now turn to the integral in (B10). Since the Hession matrix has been block diagonalized with respect to modes, the integral in (B10) thus can be calculated mode by mode by applying (B12). At first, for $\vec{k}_1\neq\vec{k}_2$ the integral of $\prod_{i=0}^{N+3}\prod_{A=\pm}\mathrm{d}\widehat{\nu}_{i,A}(\vec{k}_1)\mathrm{d}\widehat{\nu}_{i,A}(\vec{k}_1)^*$ vanishes $\widehat{\mathfrak{F}}^{\vec{N}}_{\vec{B}}(\vec{\tau},\vec{k}_1,\vec{k}_2,\gamma)$ clearly. We thus only need to consider the case with $\vec{k}_1=\vec{k}_2$. Then, for the mode $\vec{k}$ satisfying $\vec{k}\neq\vec{k}_1$, the integral of $\prod_{i=0}^{N+3}\prod_{A=\pm}\mathrm{d}\widehat{\nu}_{i,A}(\vec{k})\mathrm{d}\widehat{\nu}_{i,A}(\vec{k})^*$, denoted by $I_{\vec{k}}$, is

$$I_{\vec{k}}=-\prod_{S=\pm}\det\left(\frac{\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},S)}{t}\right)t(\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},-)^{-1})_{N+3,0}=\prod_{l=1}^{N+3}\mathfrak{a}_l(\vec{k},-).$$

According to (B14), for $\delta\tau\ll1$ and, equivalently, $N\gg1$, we have

$$I_{\vec{k}}=\exp\left(i\frac{\mathfrak{s}(\vec{k})}{a\sqrt{\mathring{p}_\gamma\beta}}\sum_{j=1}^{N+3}\delta\tau_j\right)+O(\delta\tau)=1+O(\delta\tau)\tag{B17}$$

where in the last step we applied $\sum_{j=1}^{N+3}\delta\tau_j=0$ resulting from (3.17). For the mode $\vec{k}_1$, using $I_{\vec{k}_1,\vec{B}}$ to denote the result, we have

$$\begin{aligned}\hbar^{-1}I_{\vec{k}_1,\vec{B}}=&\det\begin{pmatrix}\delta_{B_1,B_2}t(\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},B_1)^{-1})_{\mathfrak{n}(1),\mathfrak{n}(2)+1} & \delta_{B_1,-}t(\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},-)^{-1})_{\mathfrak{n}(1),0}\\ \delta_{B_2,-}t(\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},-)^{-1})_{N+3,\mathfrak{n}(2)+1} & t(\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},-)^{-1})_{N+3,0}\end{pmatrix}\\ =&\det\begin{pmatrix}\delta_{B_1,B_2}\theta(\tau_1-\tau_2)e^{-B_1i\frac{\mathfrak{s}(\vec{k}_1)}{a\sqrt{\mathring{p}_\gamma\beta}}(\tau_1-\tau_2)} & \delta_{B_1,-}e^{i\frac{\mathfrak{s}(\vec{k}_1)}{a\sqrt{\mathring{p}_\gamma\beta}}\tau_1}\\ \delta_{B_2,-}e^{-i\frac{\mathfrak{s}(\vec{k}_1)}{a\sqrt{\mathring{p}_\gamma\beta}}\tau_2} & 1\end{pmatrix}+O(\delta\tau)\end{aligned}\tag{B18}$$

where we used the same arguments as in (B17) and the step function $\theta(\tau_1-\tau_2)$ results from that $t(\widehat{\mathscr{H}_F}^{\vec{N}}(\vec{k},B_1)^{-1})_{\mathfrak{n}(1),\mathfrak{n}(2)+1}$ is proportional to $\theta(\mathfrak{n}(1)-\mathfrak{n}(2))$. Performing the determinant and doing some straightforwards calculations, we finally get

$$\hbar^{-1}I_{\vec{k}_1,\vec{B}}=\delta_{B_1,+}\delta_{B_2,+}\theta(\tau_1-\tau_2)e^{-i\frac{\mathfrak{s}(\vec{k}_1)}{a\sqrt{\mathring{p}_\gamma\beta}}(\tau_1-\tau_2)}-\delta_{B_1,-}\delta_{B_2,-}\theta(\tau_2-\tau_1)e^{-i\frac{\mathfrak{s}(\vec{k}_1)}{a\sqrt{\mathring{p}_\gamma\beta}}(\tau_2-\tau_1)}\tag{B19}$$

Combining the results (B17) and (B19), one obtains

$$\widehat{\mathfrak{F}}^{\vec{N}}_{\vec{B}}(\vec{\tau},\vec{k}_1,\vec{k}_2,\gamma)=\delta_{k_1,k_2}\hbar\left(\delta_{B_1,+}\delta_{B_2,+}\theta(\tau_1-\tau_2)e^{-i\frac{\mathfrak{s}(\vec{k}_1)}{a\sqrt{\mathring{p}_\gamma\beta}}(\tau_1-\tau_2)}-\delta_{B_1,-}\delta_{B_2,-}\theta(\tau_2-\tau_1)e^{-i\frac{\mathfrak{s}(\vec{k}_1)}{a\sqrt{\mathring{p}_\gamma\beta}}(\tau_2-\tau_1)}\right)+O(\delta\tau).\tag{B20}$$

Substituting this result into (B11) and noting that

$$\pm\Theta(\vec{k})^\dagger_{A_1,\pm}\Theta(\vec{k})_{\pm,A_2}=\frac{\pm\mathfrak{s}(\vec{k})(\sigma^0)_{A_1,A_2}-\sum_{m=1}^3\sin\left(\frac{2\pi\mu}{\ell_o}k^m\right)(\sigma^m)_{A_1,A_2}}{2\mathfrak{s}(\vec{k})},\tag{B21}$$

we get the result of $\mathfrak{F}^{\vec{N}}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)$,

$$\begin{aligned}\mathfrak{F}^{\vec{N}}(\vec{\tau},\vec{v},\gamma)=&\frac{\mu^3}{\ell_o^3}\hbar\sum_{\vec{k}\in K[\gamma]^3}\exp\left(i\frac{2\pi}{\ell_o}\vec{k}\cdot(\vec{x}_{v_1}-\vec{x}_{v_2})\right)\times\\ &\left(\theta(\tau_1-\tau_2)\frac{\mathfrak{s}(\vec{k})\sigma^0-\sum_{m=1}^3\sin\left(\frac{2\pi\mu}{\ell_o}k^m\right)\sigma^m}{2\mathfrak{s}(\vec{k})}e^{-i\omega(\vec{k})(\tau_1-\tau_2)}\right.\\ &\left.-\theta(\tau_2-\tau_1)\frac{\mathfrak{s}(\vec{k})\sigma^0+\sum_{m=1}^3\sin\left(\frac{2\pi\mu}{\ell_o}k^m\right)\sigma^m}{2\mathfrak{s}(\vec{k})}e^{-i\omega(\vec{k})(\tau_2-\tau_1)}\right)+O(\delta\tau).\end{aligned}\tag{B22}$$

where $\mathfrak{F}^{\vec{N}}(\vec{\tau},\vec{v},\gamma) := \lim_{N_i\to\infty}\mathfrak{F}^{\vec{N}}(\vec{\tau},\vec{v},\gamma)$ with $\mathfrak{F}^{\vec{N}}(\vec{\tau},\vec{v},\gamma)$ denoting the matrix taking $\mathfrak{F}^{\vec{N}}_{\vec{A}}(\vec{\tau},\vec{v},\gamma)$ for $A_1, A_2 = \pm$ as its entries and we defined $\omega(\vec{k})$ as

$$\omega(\vec{k}) = \frac{\mathfrak{s}(\vec{k})}{a\sqrt{\mathring{p}_\gamma\beta}}. \tag{B23}$$

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