Parameterized Barrier Functions to Guarantee Safety Under Uncertainty

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***Abstract*—Deploying safety-critical controllers in practice necessitates the ability to modulate uncertainties in control systems. In this context, robust control barrier functions—in a variety of forms—have been used to obtain safety guarantees for uncertain systems. Yet the differing types of uncertainty experienced in practice have resulted in a fractured landscape of robustification—with a variety of instantiations depending on the structure of the uncertainty. This letter proposes a framework for generalizing these variations into a single form: *parameterized barrier functions (PBFs)*, which yield safety guarantees for a wide spectrum of uncertainty types. This leads to controllers that enforce robust safety guarantees while their conservativeness scales by the parameterization. To illustrate the generality of this approach, we show that input-to-state safety (ISSf) is a special case of the PBF framework, whereby improved safety guarantees can be given relative to ISSf.**

***Index Terms*—Safety guarantee, uncertainty, robust control barrier functions.**

I. Introduction

CONTROL barrier functions (CBFs) [1] have become a popular tool for synthesizing safety-critical controllers due to their generality and relative ease of synthesis and implementation. Safety is encoded by a single scalar inequality constraint, by which CBFs provide an easy-to-compute condition that implies the safety of the system when satisfied. The efficacy of this approach has been demonstrated in a variety of applications such as multi-agent systems [2], robotic manipulators [3], autonomous vessels [4] and autonomous trucks [5]. One of the main challenges in obtaining formal safety guarantees with CBFs in practice is uncertainty: both of the internal model used to synthesize the CBF controller, and the external environment with which the system interacts. Since CBFs use models to calculate safe actions, a mismatch between a system and its model can lead degradations in safety [6].

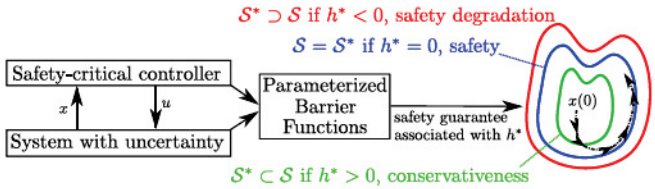

Fig. 1. Illustration of the parameterized barrier function (PBF) framework. PBF establishes safety guarantees for a wide range of robust safety-critical controllers designed for systems with uncertainties.

Robustness against uncertainties is typically achieved by introducing a robustifying term in the CBF condition. In one of the first works on robust CBFs [7], a robustifying term was added based upon a bound on the uncertainties with the result being robust safety. Later, different observer and identification techniques have been proposed to alleviate the conservativeness of robust controllers by estimating the uncertainty, or considering specific classes of uncertainties [8], [9], [10], [11], [12], [13]. Data-driven methods account for uncertainties in a similar fashion where a sufficient condition for the safety is found using properties of uncertainties [14], [15], [16]. Learning can also be utilized to estimate a robustifying term in an episodic fashion [17]—this has been deployed successfully on robotic systems [18]. In the case of stochastic estimation techniques, probabilistic safety guarantees are obtained using chance constraints with the standard deviation of the process used as an upper confidence bound [19], [20].

To quantify the effect of uncertainties on safety, it is important to characterize how adding a robustness term impacts the ability to satisfy a safety constraint. On one hand, various studies focused on ensuring safety w.r.t. a *more conservative* constraint for robustness [9], [10], [11]. Similar concepts were investigated to improve the feasibility of a stability problem with control Lyapunov functions before in [21], and of a safety problem with CBFs more recently in [22]. On the other hand, input-to-state safety (ISSf) [23], inspired from input-to-state stability [24], enforces safety w.r.t. a *relaxed* safety constraint that is parameterized by the robustifying term. Arbitrarily

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small relaxation in the safety constraint can be achieved, and the robustness-safety trade-off can be improved with less conservative "tunable" ISSf conditions [25]. ISSf has proven to be useful when implementing CBFs in practice; for example, in safety aware control of quadrupeds [26], and control of full-scale trucks [5]. However, the fact that ISSf relaxes the safety constraint prevents the analysis of conservative controller performance that may occur when robustness terms overcome the uncertainty.

The first goal of this letter is to generalize the concept of robust CBFs across various forms of robustification methods in the literature. Second, we formulate *parameterized barrier functions (PBFs)*, where safety guarantees are parameterized relative to a given safety constraint with a generalized robust CBF. The parameterization allows for the relaxation of the strict safety condition expressed in the robust CBF formulation. This gives us flexibility to establish safety guarantees for other levels sets of the CBF in the case that the nominal robust CBF conditions are not met. Third, we connect the ISSf framework to PBFs and show that it is possible to obtain improved safety guarantees for ISSf-CBF-based controllers. An inverted pendulum example is used throughout to illustrate the key concepts.

II. Background

Consider a nonlinear control system of the form:

$$\dot{x} = f(t,x) + g(t,x)u, \tag{1}$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$. The functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous in x and piece-wise continuous in t. A feedback controller $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$, $u = k(t,x)$, that is locally Lipschitz continuous in x and piece-wise continuous in t implies there exists a time interval $I(t_0, x_0) \subseteq \mathbb{R}$ for each initial condition $x(t_0) = x_0$ such that the closed loop system has a unique solution $x(t)$ for all $t \in I(t_0, x_0)$ [27]. For convenience we take $t_0 = 0$ and assume that the solution exists for all time, that is, $I(0, x_0) = [0, \infty)$ for all $x_0 \in \mathbb{R}^n$.

Safety is formally defined as the forward invariance of a set in the state space. We define the 0-superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathcal{H}$, $\mathcal{H} \subseteq \mathbb{R}$:

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid h(x) \geq 0\}, \tag{2}$$

such that $\mathcal{S}$ is nonempty and has no isolated points, and we say that the system (1) with a controller $u = k(t,x)$ is safe w.r.t. the set $\mathcal{S}$ if $x_0 \in \mathcal{S} \implies x(t) \in \mathcal{S}$ for all $t \geq 0$ and $x_0 \in \mathcal{S}$. Control barrier functions [1] give us tools to synthesize controllers with safety guarantees.

Definition 1 [1]: A continuously differentiable function h is a *control barrier function (CBF)* for (1) on $\mathcal{S}$ if 0 is a regular value[1] and there exists a function $\alpha \in \mathcal{K}^{\mathrm{e}}_{\infty}$[2] such that the following holds for all $t \geq 0$ and $x \in \mathcal{S}$:

$$\sup_{u\in\mathbb{R}^m} \left[\nabla h(x) f(t,x) + \nabla h(x) g(t,x) u\right] > -\alpha(h(x)). \tag{3}$$

[1]If $h(x) = q \implies \nabla h(x) \neq 0$, then q is a regular value of h.

[2]Function $\alpha : \mathbb{R} \to \mathbb{R}$ belongs to *extended class-*$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}^{\mathrm{e}}_{\infty}$) if it is continuous, strictly increasing, $\alpha(0) = 0$, and $\lim_{r\to\pm\infty} \alpha(r) = \pm\infty$.

The existence of a CBF implies that the set of controllers:

$$K_{\mathrm{CBF}}(t,x) = \{u \in \mathbb{R}^m \mid \nabla h(x) f(t,x) + \nabla h(x) g(t,x) u \geq -\alpha(h(x))\} \tag{4}$$

is not empty, and the main result in [1] states that controllers taking values in K_{CBF} ensure safety:

Theorem 1 [1]: Let h be a CBF for (1) on $\mathcal{S}$. Then, any controller $u = k(t,x) \in K_{\mathrm{CBF}}(t,x)$ renders (1) safe w.r.t. $\mathcal{S}$.

III. Robust Control Barrier Functions

Safety guarantees established by CBFs may deteriorate in the presence of an uncertainty in the model. Consider:

$$\dot{x} = f(t,x) + g(t,x)u + \tilde{f}(t,x) + \tilde{g}(t,x)u, \tag{5}$$

where the unknown functions $\tilde{f} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{g} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are assumed to be locally Lipschitz in x and piece-wise continuous in t. Uncertainties $\tilde{f}$ and $\tilde{g}$ are often called as *additive* and *multiplicative* uncertainties, respectively, emphasizing their relationship with the input u in the dynamics. Their effect on safety is seen in $\dot{h}$:

$$\dot{h}(t,x,u) = \overbrace{\nabla h(x) f(t,x) + \nabla h(x) g(t,x) u}^{\dot{h}_{\mathrm{n}}(t,x,u)} + \underbrace{\nabla h(x)\tilde{f}(t,x)}_{L_{\tilde{f}}h(t,x)} + \underbrace{\nabla h(x)\tilde{g}(t,x)}_{L_{\tilde{g}}h(t,x)} u, \tag{6}$$

where $\dot{h}_{\mathrm{n}}$ denotes the *known* portion of $\dot{h}$ while the Lie derivatives $L_{\tilde{f}}h(t,x)$ and $L_{\tilde{g}}h(t,x)$ are unknown. A controller $u = k(t,x) \in K_{\mathrm{CBF}}(t,x)$ yields:

$$\dot{h}(t,x,k(t,x)) \geq -\alpha(h(x)) + L_{\tilde{f}}h(t,x) + L_{\tilde{g}}h(t,x)k(t,x), \tag{7}$$

and no longer satisfies the condition $\dot{h} \geq -\alpha(h)$.

In the literature this problem is often addressed by adding a compensation term to the safety constraint (3) for robustness against the uncertainty. To capture this term for a variety of approaches, we generalize the notion of robust CBF, which was first proposed in [7] using a specific compensation term for a specific type of uncertainty.

Definition 2: A continuously differentiable function h is a *robust control barrier function (RCBF)* for (5) on $\mathcal{S}$ if 0 is a regular value of h and there exist functions $\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\alpha \in \mathcal{K}^{\mathrm{e}}_{\infty}$ such that the following holds for all $t \geq 0$ and $x \in \mathcal{S}$:

$$\sup_{u\in\mathbb{R}^m} \left[\dot{h}_{\mathrm{n}}(t,x,u) - \sigma(t,x,u)\right] > -\alpha(h(x)). \tag{8}$$

The compensation term σ allows one to cancel the undesired effects of uncertainties on safety. Similar to CBFs, the existence of a RCBF yields a set of robustly safe controllers:

$$K_{\mathrm{RCBF}}(t,x) = \{u \in \mathbb{R}^m \mid \dot{h}_{\mathrm{n}}(t,x,u) - \sigma(t,x,u) \geq -\alpha(h(x))\}, \tag{9}$$

and the following theorem, generalized from [7], gives a sufficient condition to obtain robust safety results:

Theorem 2: Let h be a RCBF for (5) on $\mathcal{S}$ with σ satisfying:

$$L_{\tilde{f}}h(t,x) + L_{\tilde{g}}h(t,x)u + \sigma(t,x,u) \geq 0, \tag{10}$$

for all $t \geq 0$, $x \in \partial\mathcal{S}$ and $u \in \mathbb{R}^m$. Then, any controller $u = k(t,x) \in K_{\mathrm{RCBF}}(t,x)$ renders (5) safe w.r.t. $\mathcal{S}$.

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TABLE I
A BRIEF SUMMARY OF ROBUST CONTROL BARRIER FUNCTION (RCBF) AND INPUT-TO-STATE SAFETY (ISSf) BASED METHODS FOR ROBUST SAFETY-CRITICAL CONTROL, WITH THE CORRESPONDING σ TERM USED IN (9), TO PROVIDE SAFETY WITH ROBUSTNESS AGAINST THE UNCERTAINTIES IN (5)

		Method Summary	$\sigma(t,x,u)$
RCBF	[7]	Bounded uncertainty: $\|\tilde{f}(t,x)\| \leq p$.	$\|\nabla h(x)\|p$
	[28]	Bounded uncertainty: $\|\tilde{f}(t,x)\| \leq p$, continuous non-increasing κ with $\kappa(0)=1$	$\kappa(h(x))\|\nabla h(x)\|p$
	[15]	$\tilde{f}$ is a convex hull of functions $\psi_i(x), i=1,\cdots,q$, $\tilde{g}$ is a convex hull of functions $\rho_i(x), i=1,\cdots,q$,	$-\min_{i\in\{1,..,q\}} \nabla h(x)\phi_i(x)$ $-\min_{i\in\{1,..,q\}} \nabla h(x)\rho_i(x)u$
	[12]	$[\tilde{f}, (\tilde{g}\ \mathrm{diag}(u))^\top]^\top = \psi(x,u)\theta$, and $\exists A, b$ s.t. $A\theta \leq b$	$\inf_{A\theta\leq b} \nabla h(x)\psi(x,u)\theta$
	[13]	Sector bounded nonlinear perturbation at input, i.e., $\exists\alpha,\beta$ defining $g_s=(\alpha+\beta)g/2$ and $\theta=(\beta-\alpha)/(\beta+\alpha)$.	$(L_g h(x) - L_{g_s}h(x))\,u + \theta\|u\|\|L_{g_s}h(x)\|$
	[8]	$\hat{f}$ estimates $\tilde{f}$ and b_d defines an error band.	$-\nabla h(x)\hat{f}(x) + b_d(t)$
	[9]	$b(t,x) = L_{\tilde{f}}h(t,x)$ is Lipschitz in t (constant L_b), $\hat{b}$ estimates b and k_b is estimation gain.	$-\hat{b}(t,x) + L_b/k_b$
	[11]	$L_{\tilde{f}}h(t,x) = \rho(t,x)\theta$, $\|\theta\| \leq \overline{\theta}$ $\hat{\theta}$ estimates θ and $\tilde{\theta}_U$ is upper error bound.	$\min\left\{\|\rho(t,x)\|\overline{\theta}, -\rho(t,x)\hat{\theta}(t) + \|\rho(t,x)\|\tilde{\theta}_U(t)\right\}$
	[14]	$\tilde{f}$ and $\tilde{g}$ are Lipschitz in x (with $L_{\tilde{f}}, L_{\tilde{g}}$), $\exists N$ data x_i, u_i and $\dot{x}_i$ so $\tilde{F}_i = \dot{x}_i - f(x_i) + g(x_i)u_i$	$\sum_{i=1}^{N}\left(\lambda_i^T\tilde{F}_i - \|\lambda_i\|(L_{\tilde{f}} + L_{\tilde{g}}\|u_i\|)\|x-x_i\|\right)$, λ_i are Lagrange multipliers.
	[16]	$\dot{h}_n(t,x,u) = 0$, $\dot{h}$ is Lipschitz in x and u (with L_x, L_u), $\exists N$ data x_i, u_i and $\dot{h}_i$	$\min_{i\in[1\cdots N]}\left[-\dot{h}_i + L_x\|x-x_i\| + L_u\|u-u_i\|\right]$
ISSf	[25]	Bounded uncertainty, continuously differentiable $\epsilon(r) > 0$ with $\frac{\mathrm{d}\epsilon}{\mathrm{d}r}(r) \geq 0$	$\frac{\|\nabla h(x)\|^2}{\epsilon(h(x))}$

Remark 1: Robust safety-critical controller design is often formulated as the optimization problem:

$$k(t,x) = \underset{u\in\mathbb{R}^m}{\operatorname{argmin}} \|u - k_d(t,x)\|^2 \quad \text{s.t.} \quad \dot{h}_n(t,x,u) - \sigma(t,x,u) \geq -\alpha(h(x)), \tag{11}$$

where $k_d : \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}^m$ is a desired controller.

A plethora of methods has been proposed in the literature to design σ; see a list in Table I for RCBF-based methods as well as an input-to-state safety-based method that will be described in Section V. To illustrate robust safety we now consider a certain class of uncertainty with bounded additive term and no multiplicative term:

$$\|\tilde{f}(t,x)\| \leq p, \quad \tilde{g}(t,x) = 0, \tag{12}$$

for all $t \geq 0$, $x \in \mathbb{R}^n$ with some bound $p \geq 0$. Inspired by [7], we will use a compensation term of the form:

$$\sigma(t,x,u) = \|\nabla h(x)\|p, \tag{13}$$

which, along with (12), implies that (10) is satisfied, and thus the controller (11) with (13) keeps the set $\mathcal{S}$ safe. Results obtained through Theorem 2 are illustrated using an inverted pendulum example with a time varying uncertainty.

Example 1: Consider the inverted pendulum in Fig. 2(a) that consists of a massless rod of length l and a concentrated mass m. The pendulum is actuated with a torque u, while an unknown external force $F(t)$ is acting horizontally on the mass. With the angle θ, angular velocity $\dot{\theta}$, and state $x = \begin{bmatrix}\theta & \dot{\theta}\end{bmatrix}^\top$, the equation of motion of the pendulum reads:

TABLE II
PARAMETERS USED FOR EXAMPLE 1

$g = 10$ m/s^2	$m = 2$ kg	$l = 1$ m
$\overline{F} = 2$ N	$q_1 = 4$ 1/rad	$q_2 = 2$ s/rad
$\alpha_c = 8$ 1/s	$K_p = 0.6$ 1/s^2	$K_d = 0.6$ 1/s

$$\dot{x} = \underbrace{\begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 \end{bmatrix}}_{f(t,x)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}}_{g(t,x)} u + \underbrace{\begin{bmatrix} 0 \\ \frac{F(t)}{ml}\cos x_1 \end{bmatrix}}_{\tilde{f}(t,x)}, \tag{14}$$

where the parameter g is the gravitational acceleration. All the parameters used in this example are given in Table II.

The external force $F(t)$ yields an additive uncertainty $\tilde{f}$ ($\tilde{g}(t,x) \equiv 0$). We assume that there exists an upper bound $\overline{F}$ such that $|F(t)| \leq \overline{F}$, $\forall t \geq 0$, which yields $p = \frac{\overline{F}}{ml}$. A piece-wise continuous force is considered for simulations:

$$F(t) = \overline{F}\big(1 - 2s(t-5) + s(t-10) + s(t-15)\big), \tag{15}$$

where s is the Heaviside function.

We seek to design a control torque u such that we keep the pendulum upright within a given safe region of angles, even with the disturbance $F(t)$. The set $\mathcal{S}$ is defined using:

$$h(x) = 1 - \frac{1}{2}x^\top A x, \quad A = \begin{bmatrix} 2q_1^2 & q_1q_2 \\ q_1q_2 & 2q_2^2 \end{bmatrix}, \tag{16}$$

with parameters $q_1, q_2 > 0$ given in Table II. The resulting set $\mathcal{S}$ is the black ellipse in Fig. 2(b). Note that $\nabla h(x) = 0$ only if $x = 0$, while $h(0) = 1$, thus 0 is a regular value of h.

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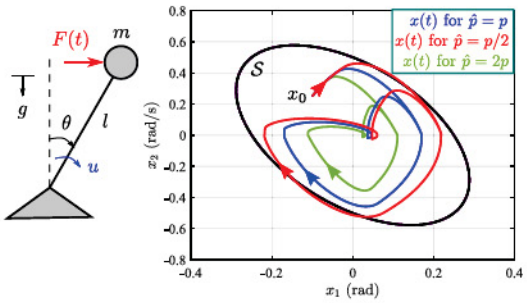

Fig. 2. (a) Inverted pendulum model. (b) The safe set $\mathcal{S}$ (black ellipse) and simulated trajectories (colored curves) with the controller (11) for different values of the estimated uncertainty bounds; cf. (13) and (18).

A desired controller is selected as:

$$k_{\mathrm{d}}(x) = ml^2\big(-g/l \sin x_1 - K_{\mathrm{p}} x_1 - K_{\mathrm{d}} x_2\big) \tag{17}$$

with parameters $K_{\mathrm{p}}, K_{\mathrm{d}} > 0$. We use (11) as robust safety-critical controller with σ in (13) and $\alpha(r) = \alpha_c r$, $\alpha_c > 0$. Simulation results are depicted in Fig. 2(b) as a blue curve. The controller successfully keeps the system safe w.r.t. $\mathcal{S}$.

To achieve robust safety, the compensation term σ is typically designed based on certain properties of $\tilde{f}$ and $\tilde{g}$ such as the upper bound p in (12); see Table I. In practice, these properties may be hard to estimate, thus the compensation (10) required for robust safety may not be realized. For example, if p in (12) is not known precisely, one may rely on an estimation $\hat{p}$ of p instead, with the compensation term:

$$\sigma(t, x, u) = \|\nabla h(x)\| \hat{p}. \tag{18}$$

Then, under-approximating the size of the uncertainty may yield safety degradation, while over-approximation may induce conservative behavior that is not captured by Theorem 2. We illustrate these for the inverted pendulum problem.

Example 2: Consider the system in Example 1 and (18). If the uncertainty is under-approximated ($\hat{p} < p$), (10) is not satisfied and Theorem 2 cannot establish safety guarantees. Indeed, simulations capture safety degradation where $x(t)$ leaves $\mathcal{S}$; see the red curve in Fig. 2(b) for $\hat{p} = p/2$. If the uncertainty is over-approximated ($\hat{p} > p$), (10) holds and Theorem 2 implies that the set $\mathcal{S}$ is safe. The corresponding simulation results, depicted in Fig. 2(b) as a green curve for $\hat{p} = 2p$, comply with this. However, we observe conservative behavior where $x(t)$ evolves inside a smaller subset of $\mathcal{S}$.

IV. Parameterized Barrier Functions

To quantify safety degradation and conservativeness emerging from non-ideal compensation of uncertainties, we extend the RCBF-based safety guarantees by introducing the concept of *parameterized barrier function*.

Our key idea is to establish safety guarantees for other superlevel sets of h than $\mathcal{S}$. Thus, we introduce the set:

$$\mathcal{S}^* \triangleq \big\{x \in \mathbb{R}^n \;\big|\; H(x, h^*) \geq 0\big\}, \tag{19}$$

$$\partial\mathcal{S}^* \triangleq \big\{x \in \mathbb{R}^n \;\big|\; H(x, h^*) = 0\big\}, \tag{20}$$

where the function $H : \mathbb{R}^n \times \mathcal{H} \to \mathbb{R}$ is given as:

$$H(x, h^*) \triangleq h(x) - h^*; \tag{21}$$

with h defining $\mathcal{S}$ in (2) and a parameter $h^* \in \mathcal{H} \subseteq \mathbb{R}$ to be determined. We have $\mathcal{S}^* \supset \mathcal{S}$ if $h^* < 0$, $\mathcal{S}^* = \mathcal{S}$ if $h^* = 0$, and $\mathcal{S}^* \subset \mathcal{S}$ if $h^* > 0$; see Fig. 1. We assume that the set $\mathcal{S}^*$ is nonempty and has no isolated points for any $h^* \in \mathcal{H}$.

Definition 3: Function H is a *parameterized barrier function (PBF)* for (5) on $\mathcal{S}^*$ if h is a RCBF for (5) on $\mathcal{S}^*$ and h^* is a regular value of h.

Next, we state conditions for the safety of (5) w.r.t. $\mathcal{S}^*$, to ultimately capture safety degradation and conservativeness.

Theorem 3: Let H be a PBF for (5) on $\mathcal{S}^*$ with $h^* \in \mathcal{H}$ and σ satisfying:

$$L_{\tilde{f}} h(t, x) + L_{\tilde{g}} h(t, x) u + \sigma(t, x, u) \geq \alpha(h^*), \tag{22}$$

for all $t \geq 0$, $x \in \partial\mathcal{S}^*$ and $u \in \mathbb{R}^m$. Then, any controller $u = k(t, x) \in K_{\mathrm{RCBF}}(t, x)$ renders (5) safe w.r.t. $\mathcal{S}^*$.

Proof: H is continuously differentiable since h is a RCBF and h^* is a constant, and we have:

$$\dot{H}(t, x, u) = \dot{h}_{\mathrm{n}}(t, x, u) + L_{\tilde{f}} h(t, x) + L_{\tilde{g}} h(t, x) u. \tag{23}$$

For any controller $u = k(t, x) \in K_{\mathrm{RCBF}}(t, x)$ this yields:

$$\begin{aligned}\dot{H}(t, x, k(t, x)) \geq &-\alpha(h(x)) + \sigma(t, x, k(t, x)) \\ &+ L_{\tilde{f}} h(t, x) + L_{\tilde{g}} h(t, x) k(t, x).\end{aligned} \tag{24}$$

Considering $x \in \partial\mathcal{S}^*$, i.e., $h(x) = h^*$, (22) implies that

$$\dot{H}(t, x, k(t, x)) \geq 0. \tag{25}$$

Since h^* is a regular value of h we have that 0 is a regular value of H. Thus, the rest of the proof follows from [29]. ■

Remark 2: Theorem 3 *relaxes* the sufficient condition (10), which is based on certain known properties of the unknown functions $\tilde{f}$ and $\tilde{g}$, and consequently establishes more accurate safety guarantees. Indeed, Theorem 2 is a special case of Theorem 3 with $h^* = 0$. If (22) holds with $h^* < 0$, condition (10) may not hold and Theorem 2 cannot establish safety. Still, Theorem 3 provides safety guarantees w.r.t. $\mathcal{S}^* \supset \mathcal{S}$, hence it quantifies safety degradation. If (22) holds with $h^* > 0$, condition (10) also holds, and Theorem 2 establishes safety w.r.t. $\mathcal{S}$. However, Theorem 3 provides safety w.r.t. $\mathcal{S}^* \subset \mathcal{S}$, hence it quantifies conservativeness.

More accurate safety guarantees can be established for the compensation term in (18) through Theorem 3 as follows.

Corollary 1: Consider (5) with (12) and (18). Assume that there exist $\underline{\delta}, \overline{\delta} : \mathcal{H} \to \mathbb{R}_{\geq 0}$ such that for any $h^* \in \mathcal{H}$:

$$\underline{\delta}(h^*) \leq \|\nabla h(x)\| \leq \overline{\delta}(h^*), \tag{26}$$

$\forall x \in \partial\mathcal{S}^*$. If H is a PBF for (5) on $\mathcal{S}^*$ with h^* defined by:

$$\alpha(h^*) = \begin{cases} \overline{\delta}(h^*)(\hat{p} - p) & \text{if } \hat{p} < p, \\ \underline{\delta}(h^*)(\hat{p} - p) & \text{if } \hat{p} > p, \end{cases} \tag{27}$$

then $u = k(t, x) \in K_{\mathrm{RCBF}}(t, x)$ renders (5) safe w.r.t. $\mathcal{S}^*$.

Proof: The choice (18) of the robustifying term σ implies:

$$L_{\tilde{f}} h(t, x) + L_{\tilde{g}} h(t, x) u + \sigma(t, x, u) \geq \|\nabla h(x)\|(\hat{p} - p). \tag{28}$$

This leads to (22) by using (26) and (27), and the rest of the proof follows from Theorem 3. ■

Remark 3: The value of h^* given by (27) quantifies safety degradation and conservativeness. If the uncertainty is under-approximated ($\hat{p} < p$), (27) yields $h^* < 0$ and $\mathcal{S}^* \supset \mathcal{S}$, while over-approximation ($\hat{p} > p$) leads to $h^* > 0$ and $\mathcal{S}^* \subset \mathcal{S}$.

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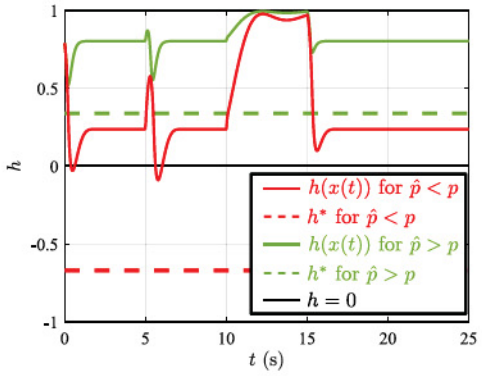

Fig. 3. The time evolution of the RCBF, $h(x(t))$, for the simulations of Example 2, and the safety guarantees established in Example 3 using the PBF framework in the form of a lower bound h^* for $h(x(t))$.

Example 3: Consider the setup of Example 2. Based on (16), we get $\nabla h(x) = -Ax$, and it can be shown that (26) holds for any $h^* \in \mathcal{H} = (-\infty, 1]$ and for all $x \in \partial\mathcal{S}^*$ with $\underline{\delta}(h^*) = \sqrt{2\underline{\lambda}(1-h^*)}$ and $\overline{\delta}(h^*) = \sqrt{2\overline{\lambda}(1-h^*)}$, where $0 < \underline{\lambda} \leq \overline{\lambda}$ are the eigenvalues of A. Since any $h^* < 1$ is a regular value of h, Corollary 1 establishes safety w.r.t. the set $\mathcal{S}^*$ with h^* given by (27).

The value of h^* is depicted in Fig. 3 with dashed line along with $h(x(t))$ corresponding to the simulated trajectories in Fig. 2(b). Observe that for the case of under-approximation ($\hat{p} < p$, red) the PBF framework successfully quantifies safety degradation by a lower bound $h^* < 0$ for $h(x(t))$, which complies with the simulation results. For the case of over-approximation ($\hat{p} > p$, green) the bound $h^* > 0$ captures the safe but conservative system behavior.

V. Input-to-State Safety via PBFs

A well-known existing concept proposed to characterize safety degradation is input-to-state safety (ISSf) [23].[3] In this section we show that ISSf is a special case of the PBF framework (restricted to $h^* < 0$). Then, we propose a method that endows ISSf-CBF-based controllers with more accurate safety guarantees (including $h^* = 0$ and $h^* > 0$).

In essence, ISSf gives ways to quantify safety degradation in the presence of a bounded disturbance such as (12). This inspired PBFs, as ISSf considers safety degradation in the context of safety guarantees for another superlevel set of h:

$$\mathcal{S}_{\text{ISSf}} = \left\{ x \in \mathbb{R}^n \middle| h(x) - \alpha^{-1}\left(-\epsilon(h(x))p^2/4\right) \geq 0 \right\}, \quad (29)$$

with a continuously differentiable function $\epsilon : \mathcal{H} \to \mathbb{R}_{>0}$ that satisfies $\frac{\mathrm{d}\epsilon}{\mathrm{d}r}(r) \geq 0$, $\forall r \in \mathcal{H}$ and $\alpha^{-1} \in \mathcal{K}_\infty^{\mathrm{e}}$ [25]. ISSf-CBFs provide controllers with safety guarantees w.r.t. $\mathcal{S}_{\text{ISSf}}$.

Definition 4: A continuously differentiable function h is an *input-to-state safe control barrier function (ISSf-CBF)* for (5) if there exist a function $\alpha \in \mathcal{K}_\infty^{\mathrm{e}}$ such that the following holds for all $t \geq 0$ and $x \in \mathbb{R}^n$:

$$\sup_{u \in \mathbb{R}^m} \left[\dot{h}_{\mathrm{n}}(t, x, u)\right] > -\alpha(h(x)) + \frac{\|\nabla h(x)\|^2}{\epsilon(h(x))}. \quad (30)$$

[3]Although ISSf was originally proposed for matched input disturbances, in this letter we extend it for additive type of uncertainties f.

Remark 4: While the original ISSf formulation in [23] considers $\frac{\mathrm{d}\epsilon}{\mathrm{d}r}(r) = 0$, our work in [5] shows that controller performance can be improved by choosing $\frac{\mathrm{d}\epsilon}{\mathrm{d}r}(r) > 0$ through experiments with a full-scale automated truck.

Reference [25, Th. 3] establishes safety for (5) w.r.t. $\mathcal{S}_{\text{ISSf}}$, if the controller takes values in the non-empty set:

$$K_{\text{ISSf}}(t, x) = \left\{ u \in \mathbb{R}^m \mid \dot{h}_{\mathrm{n}}(t, x, u) \geq -\alpha(h(x)) + \frac{\|\nabla h(x)\|^2}{\epsilon(h(x))} \right\}. \quad (31)$$

In the next theorem, we link ISSf-CBFs to the PBF framework and establish the same result via PBFs.

Theorem 4: If h is an ISSf-CBF for (5) with (12), then H is a PBF for this system on $\mathcal{S}^*$ with:

$$\sigma(t, x, u) = \frac{\|\nabla h(x)\|^2}{\epsilon(h(x))}, \quad (32)$$

and $h^* < 0$ being the unique solution of:

$$h^* - \alpha^{-1}\left(-\epsilon(h^*)p^2/4\right) = 0. \quad (33)$$

Furthermore, any controller $u = k(t, x) \in K_{\text{ISSf}}(t, x)$ renders (5) safe w.r.t. $\mathcal{S}^* = \mathcal{S}_{\text{ISSf}} \supset \mathcal{S}$.

Proof: First, we observe that (33) has a unique solution h^* based on the monotonicity properties of α^{-1} and ϵ. Furthermore, $\mathcal{S}^* = \mathcal{S}_{\text{ISSf}}$ based on (29) and (33), while the property $\epsilon(r) > 0$ for all $r \in \mathcal{H}$ yields $h^* < 0$ and $\mathcal{S}^* \supset \mathcal{S}$. Moreover, h^* is a regular value of h thanks to the strict inequality in (30); please refer to the proof of in [25, Th. 3] for details. Hence, comparing (30) with (8) and (32) establishes that H is a PBF. Finally, by noticing that $\frac{\|\nabla h(x)\|^2}{\epsilon(h(x))} - \|\nabla h(x)\|p \geq -\frac{\epsilon(h(x))p^2}{4}$, (12) and (32) yield:

$$L_{\tilde{g}}h(t, x)u + L_{\tilde{f}}h(t, x) + \sigma(t, x, u) \geq -\epsilon(h(x))p^2/4. \quad (34)$$

This inequality and (33) imply that condition (22) in Theorem 3 holds, therefore (5) is safe w.r.t. $\mathcal{S}^*$. ■

Next, we derive more accurate safety guarantees for ISSf-CBF-based controllers via the PBF framework.

Corollary 2: Consider (5) with (12) and (32). Assume that there exists $\underline{\delta} : \mathcal{H} \to \mathbb{R}_{\geq 0}$ such that for any $h^* \in \mathcal{H}$:

$$\underline{\delta}(h^*) \leq \|\nabla h(x)\|, \quad (35)$$

$\forall x \in \partial\mathcal{S}^*$. If H is a PBF for (5) on $\mathcal{S}^*$ with h^* satisfying:

$$h^* - \alpha^{-1}\left(\underline{\delta}(h^*)^2/\epsilon(h^*) - \underline{\delta}(h^*)p\right) = 0, \quad (36)$$

and $\epsilon(h^*) \leq 2\underline{\delta}(h^*)/p$ holds, then $u = k(t, x) \in K_{\text{RCBF}}(t, x)$ renders (5) safe w.r.t. $\mathcal{S}^* \subseteq \mathcal{S}_{\text{ISSf}}$.

Proof: Based on (12), (32) and $\epsilon(h^*) \leq 2\underline{\delta}(h^*)/p$, it can be shown that the following holds for all $x \in \partial\mathcal{S}^*$:

$$L_{\tilde{g}}h(t, x)u + L_{\tilde{f}}h(t, x) + \sigma(t, x, u) \geq \frac{\underline{\delta}(h^*)^2}{\epsilon(h^*)} - \underline{\delta}(h^*)p. \quad (37)$$

Using (36) gives (22), and Theorem 3 yields safety w.r.t. $\mathcal{S}^*$. Moreover, $h^* \geq \alpha^{-1}(-\frac{\epsilon(h^*)p^2}{4})$ holds, thus $\mathcal{S}^* \subseteq \mathcal{S}_{\text{ISSf}}$. ■

Remark 5: Since $\mathcal{S}^* \subseteq \mathcal{S}_{\text{ISSf}}$, the PBF framework provides a tighter safety guarantee than ISSf theory. Indeed, all cases of $h^* < 0$, $h^* = 0$ and $h^* > 0$ can occur in (36), corresponding to safety degradation, safety and conservativeness.

Example 4: Consider the inverted pendulum problem in Example 1. We utilize the controller (11) with σ in (32),

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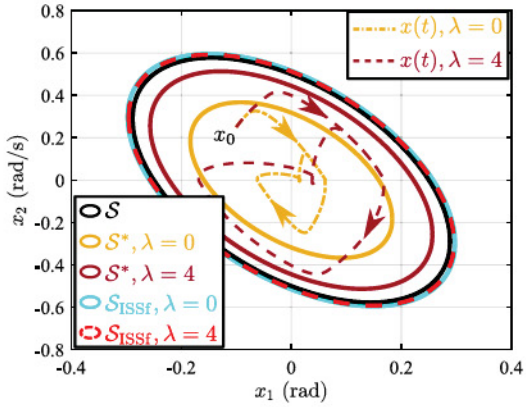

Fig. 4. Safety guarantees established in Example 4 for $\epsilon_0 = 1$ with $\lambda = 0$ and $\lambda = 4$ using the ISSf approach ($\boldsymbol{\mathcal{S}}_{\text{ISSf}}$) and the PBF framework ($\boldsymbol{\mathcal{S}}^*$). While the ISSf case (cyan and red ellipses) does not capture the conservative behavior, the PBF framework yields more accurate safety guarantees (orange and brown ellipses) for simulated trajectories (orange and brown curves).

$\epsilon(r) = \epsilon_0 e^{\lambda r}$, $\epsilon_0 > 0$ and $\lambda \geq 0$. Two simulation results are given in Fig. 4, with $\epsilon_0 = 1$, $\lambda = 0$ (orange dashed-dotted curve), and $\epsilon_0 = 1$, $\lambda = 4$ (brown dashed curve). Both simulated trajectories stay within $\mathcal{S}$. Indeed, while the former parameter pair yields a more conservative result, introducing λ alleviates the conservativeness as discussed in Remark 4.

Boundaries of the corresponding $\mathcal{S}_{\text{ISSf}}$ sets, calculated by solving (33), are also plotted by cyan solid and red dashed ellipses. As expected, these sets obtained from the ISSf theory fail to evaluate the conservativeness. The boundaries of the sets $\mathcal{S}^*$, after solving (36), are plotted by orange and brown solid lines in Fig. 4. Indeed, they are more accurate bounds on the trajectories of the system. This shows that the PBF framework provides flexibility to quantify conservativeness.

VI. Conclusion

This letter focused on establishing safety guarantees for control systems with uncertainties. We proposed parameterized barrier functions (PBFs) that generalize existing robust control barrier function (RCBF) formulations. We highlighted that the PBF framework offers flexibility to evaluate not only safety, but safety degradation and conservativeness of RCBF-based controllers. Moreover, we showed that input-to-state safety (ISSf) can be viewed as a special case of the PBF framework, and we derived improved safety guarantees for ISSf-CBF-based controllers. Future research may extend the PBF framework to sampled-data systems to analyze robustness against inter-sampling effects in continuous-time.

References

[1] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 3861–3876, Aug. 2017.

[2] L. Lindemann and D. V. Dimarogonas, "Control barrier functions for multi-agent systems under conflicting local signal temporal logic tasks," *IEEE Contr. Syst. Lett.*, vol. 3, pp. 757–762, 2019.

[3] W. S. Cortez, D. Oetomo, C. Manzie, and P. Choong, "Control barrier functions for mechanical systems: Theory and application to robotic grasping," *IEEE Trans. Control Syst. Technol.*, vol. 29, no. 2, pp. 530–545, Mar. 2021.

[4] E. H. Thyri, E. A. Basso, M. Breivik, K. Y. Pettersen, R. Skjetne, and A. M. Lekkas, "Reactive collision avoidance for ASVs based on control barrier functions," in *Proc. Conf. Control Technol. Appl. (CCTA)*, 2020, pp. 380–387.

[5] A. Alan, A. J. Taylor, C. R. He, A. D. Ames, and G. Orosz, "Control barrier functions and input-to-state safety with application to automated vehicles," 2022, *arXiv:2206.03568*.

[6] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames, "Robustness of control barrier functions for safety critical control," *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 54–61, 2015.

[7] M. Jankovic, "Robust control barrier functions for constrained stabilization of nonlinear systems," *Automatica*, vol. 96, pp. 359–367, Oct. 2018.

[8] M. Black and D. Panagou, "Safe control design for unknown nonlinear systems with Koopman-based fixed-time identification," 2022, *arXiv:2212.00624*.

[9] A. Alan, T. G. Molnar, E. Daş, A. D. Ames, and G. Orosz, "Disturbance observers for robust safety-critical control with control barrier functions," *IEEE Contr. Syst. Lett.*, vol. 7, pp. 1123–1128, 2023.

[10] B. T. Lopez, J.-J. E. Slotine, and J. P. How, "Robust adaptive control barrier functions: An adaptive and data-driven approach to safety," *IEEE Contr. Syst. Lett.*, vol. 5, pp. 1031–1036, 2021.

[11] A. Isaly, O. S. Patil, R. G. Sanfelice, and W. E. Dixon, "Adaptive safety with multiple barrier functions using integral concurrent learning," in *Proc. Amer. Control Conf. (ACC)*, 2021, pp. 3719–3724.

[12] M. H. Cohen, C. Belta, and R. Tron, "Robust control barrier functions for nonlinear control systems with uncertainty: A duality-based approach," in *Proc. 61st IEEE Conf. Decis. Control (CDC)*, 2022, pp. 174–179.

[13] J. Buch, S.-C. Liao, and P. Seiler, "Robust control barrier functions with sector-bounded uncertainties," *IEEE Contr. Syst. Lett.*, vol. 6, pp. 1994–1999, 2022.

[14] A. J. Taylor, V. D. Dorobantu, S. Dean, B. Recht, Y. Yue, and A. D. Ames, "Towards robust data-driven control synthesis for nonlinear systems with actuation uncertainty," in *Proc. 60th IEEE Conf. Decis. Control (CDC)*, 2021, pp. 6469–6476.

[15] Y. Emam, P. Glotfelter, S. Wilson, G. Notomista, and M. Egerstedt, "Data-driven robust barrier functions for safe, long-term operation," *IEEE Trans. Robot.*, vol. 38, no. 3, pp. 1671–1685, Jun. 2022.

[16] Z. Jin, M. Khajenejad, and S. Z. Yong, "Robust data-driven control barrier functions for unknown continuous control affine systems," *IEEE Contr. Syst. Lett.*, vol. 7, pp. 1309–1314, 2023.

[17] A. J. Taylor, A. Singletary, Y. Yue, and A. D. Ames, "Learning for safety-critical control with control barrier functions," in *Proc. Mach. Learn. Res. (PMLR)*, vol. 120, 2020, pp. 708–717.

[18] N. Csomay-Shanklin, R. K. Cosner, M. Dai, A. J. Taylor, and A. D. Ames, "Episodic learning for safe bipedal locomotion with control barrier functions and projection-to-state safety," in *Proc. Mach. Learn. Res. (PMLR)*, vol. 144, 2021, pp. 1041–1053.

[19] F. Castañeda, J. J. Choi, B. Zhang, C. J. Tomlin, and K. Sreenath, "Pointwise feasibility of Gaussian process-based safety-critical control under model uncertainty," in *Proc. 60th IEEE Conf. Decis. Control (CDC)*, 2021, pp. 6762–6769.

[20] P. Akella, S. X. Wei, J. W. Burdick, and A. D. Ames, "Learning disturbances online for risk-aware control: Risk-aware flight with less than one minute of data," in *Proc. Mach. Learn. Res. (PMLR)*, vol. 211, 2023, pp. 665–678.

[21] M. Lazar, "Flexible control Lyapunov functions," in *Proc. Amer. Control Conf. (ACC)*, 2009, pp. 102–107.

[22] W. Xiao, C. Belta, and C. G. Cassandras, "Adaptive control barrier functions," *IEEE Trans. Autom. Control*, vol. 67, no. 5, pp. 2267–2281, May 2022.

[23] S. Kolathaya and A. D. Ames, "Input-to-state safety with control barrier functions," *IEEE Contr. Syst. Lett.*, vol. 3, pp. 108–113, 2019.

[24] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Syst. Control Lett.*, vol. 24, no. 5, pp. 351–359, 1995.

[25] A. Alan, A. J. Taylor, C. R. He, G. Orosz, and A. D. Ames, "Safe controller synthesis with tunable input-to-state safe control barrier functions," *IEEE Contr. Syst. Lett.*, vol. 6, pp. 908–913, 2022.

[26] R. Cosner et al., "Safety-aware preference-based learning for safety-critical control," in *Proc. Mach. Learn. Res. (PMLR)*, vol. 168, 2022, pp. 1020–1033.

[27] L. Perko, *Differential Equations and Dynamical Systems*, vol. 7. New York, NY, USA: Springer, 2013.

[28] D. R. Agrawal and D. Panagou, "Safe and robust observer-controller synthesis using control barrier functions," *IEEE Contr. Syst. Lett.*, vol. 7, pp. 127–132, 2023.

[29] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*. Cham, Switzerland: Birkhäuser, 2008.

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