Fast Methods for Ranking Synthetic BECs

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***Abstract*—We gather existing methods that are used to compare and rank the BECs synthesized by a polar code constructor, compare them, and propose new methods that compare synthetic BECs faster.**

I. Introduction

The discovery of polar coding enabled 5G and other coding tasks. The formalism introduced by Arıkan [1] came with a large panel of theoretical and practical challenges. A fundamental one is about the reliability of the synthetic channels W^α, where W is the underlying physical channel and α is a binary string, and how to order the set $\{W^\alpha : \alpha \in \{0,1\}^m\}$. Further, we shall write $\alpha \succcurlyeq \gamma$ when W^α is favored over W^γ for all W.

From general channel models (W being a BMS channel) to particular ones (W being a BEC), several rules were proposed. Rule Set A (RS-A) was initially used by Mori and Tanaka [2] to construct polar codes over BMS channels and is generated by $1 \succcurlyeq 0$. Later, Schurch [3] and Bardet et al. [4] added $10 \succcurlyeq 01$ over BMS channels, which we call Rule Set B (RS-B).

Other works focus on BECs. Dragoi and Cristescu [5] introduced a family of infinitely many new rules: $100^k01 \succcurlyeq 010^k10$ for any integer $k \geq 0$; they are dubbed Rule Set C (RS-C). A recent work [6] generalizes RS-C to even more rules that are not cataloged here. Wu and Siegel [7] conjectured that $0^k1^{2^k} \succcurlyeq 1^k0^{2^k}$ is true for any integer $k \geq 0$, which is claimed to be proved in [8]; we call them Rule Set D (RS-D). Kahraman [9] proposed that the first 2^k terms of the Thue–Morse sequence should be inferior to their bitwise complement for all integer $k \geq 0$. This family begins with (a) $0 \preccurlyeq 1$; (b) $01 \preccurlyeq 10$; (c) $0110 \preccurlyeq 1001$; (d) $01101001 \preccurlyeq 10010110$. Observe that (a), (b), and (c) can be explained by RS-A, RS-B, and RS-C, respectively. We call these Rule Set E (RS-E).

Order relations for polar codes have several applications. Bardet et al. [4] conducted the first investigation of automorphism groups of polar codes that respect RS-A and RS-B. The automorphism group then plays a role in parallelized decoding of polar codes [10]–[16]. Another application involves construction of polar codes. Mondelli, Hassani, and Urbanke showed in [17] that the reliability of all but o(block length) synthetic channels can be inferred using RS-A and RS-B, hence the sub-linearity of the complexity of code construction. (For other approaches of code construction, see [18] for discussions on Gaussian approximation.)

As for a better understanding of W^α itself, the sharp transition of $I(W^\alpha)$ with respect to $I(W)$ was analyzed in [19]. The position of the threshold was investigated in [20]. Alongside, the link between network reliability and synthetic channels over BECs was established. The behaviors for $I(W)$ very close to 0 and 1 were studied in [21], while the average behavior of $I(W^\alpha)$ we studied in [5].

On a parallel track, He et al. proposed *beta expansion* [22] that bypasses partial order relations but aims directly at construction of codes. The idea is to "evaluate" a binary string 1001 to a real number $1001_\beta := 1\cdot\beta^3+0\cdot\beta^2+0\cdot\beta^1+1\cdot\beta^0$ before comparing the evaluations. Simulations over AWGN channels determined that a good value of β is $2^{1/4}$.

Eventually, we could retrieve the maximum amount of information about the poset using all the techniques and all the total/partial/pre-order relations. Our problem can be restated as follows: how to decide whether α and γ are comparable or not with respect to $\preccurlyeq$ for all pairs (α,γ) of binary strings with length at most m. We attack this problem from three sides: (a) generate as many comparable pairs as possible, (b) generate as many non-comparable pairs as possible, and (c) generate total orders that (almost) extend the targeted partial order.

One unusual thing we do is not to restrict the scope to comparing bit strings of the same length, as most of the existing works did. Instead, we study *intergenerational* comparisons. Firstly, we observe that some intrageneration inequalities are consequences of intergenerational ones; an example is that $011 \succcurlyeq 10$ leads to $0011 \succcurlyeq 010 \succcurlyeq 1000$. Secondly, intergenerational inequalities compares synthetic channels for relaxed polar codes [23], pruned polar codes [24], and parallelized low-latency polar codes [25].

While this paper is more of a survey than a research paper, we still make several contributions along the way:

- We show that RS-E is true over BECs (Proposition 11).
- We show that RS-C and RS-E are compatible with beta expansion. But RS-D becomes not compatible with beta expansion as the string length approaches infinity (Propositions 34 to 36).
- We propose a total preorder, $\succcurlyeq_{\mathrm{fst}}$, that is easy to compute, compatible with all rule sets, and only slightly different from $\succcurlyeq$ (Section VI-C).
- We present evidence that the most appropriate β value is related to the scaling exponent of polar codes (Section V-B).

This paper is organized as follows. Section II defines the reliability poset. Section III goes over the rule sets. Section IV discusses Bernstein basis. Section V discusses beta expansion. Section VI studies threshold behavior.

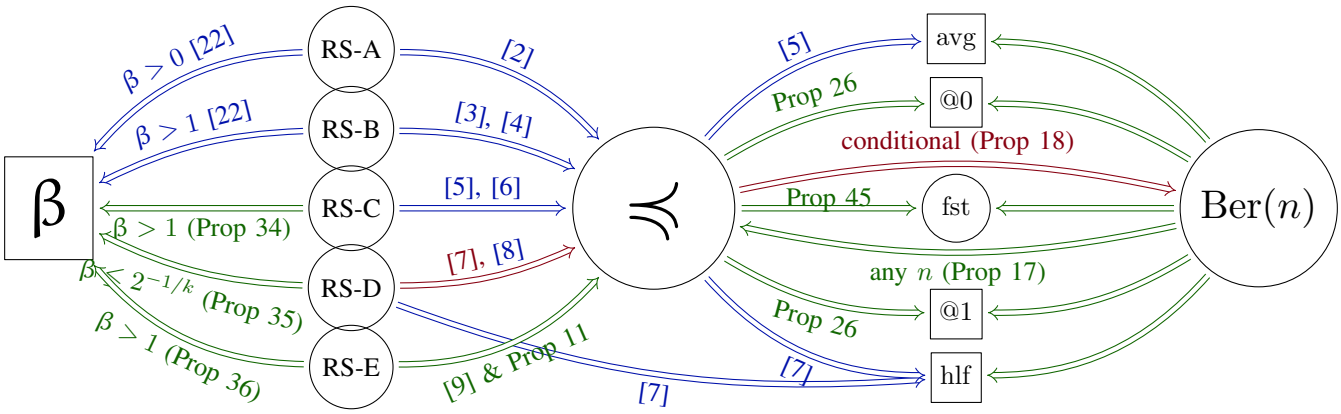


Fig. 1: Ordering relations for polar codes. Inspired by [5, Figure 5]. Rectangles are total preorders. Circles are partial preorders. Arrows are implications, e.g., $\alpha \preccurlyeq \gamma$ implies $\alpha \le_{\text{avg}} \gamma$. Blue for existing works; green for ours; red for conjectures. The arrow from RS-D to $\preccurlyeq$ is claimed to be proved in [8].

II. The Reliability Poset of Synthetic BECs

Given a BEC W with capacity $I(W) = x \in [0,1]$, the polar transformation synthesizes two channels, W^0 and W^1; they are BECs with capacities

$$I(W^0) = I_0(x) \coloneqq x^2 \quad \text{and} \quad I(W^1) = I_1(x) \coloneqq 1-(1-x)^2,$$

respectively. To study the recursion of polar transformations, we define $I(x) \coloneqq x$ and

$$I_{a_1 a_2 \cdots a_\ell}(x) \coloneqq I_{a_2 \cdots a_\ell}(I_{a_1}(x))$$

for any bit string $a_1 a_2 \cdots a_\ell \in \{0,1\}^\ell$. This way, $I_\alpha(x)$ is the capacity of W^α for any $\alpha \in \{0,1\}^\ell$. We call it the *reliability polynomial* of α.

We want to study if W^α is no worse than W^γ for any BEC W, and this for any pair of bit strings α and γ. We define a binary relation $\succcurlyeq$.

Definition 1: For any $\alpha \in \{0,1\}^\ell$ and $\gamma \in \{0,1\}^m$, where ℓ and m are nonnegative integers, we write $\alpha \succcurlyeq \gamma$ and say α *outperforms* γ if

$$I_\alpha(x) \ge I_\gamma(x) \quad \text{for all } x \in [0,1].$$

Many works have studied $\succcurlyeq$. So one might be surprised when learning that we are the first to prove the following.

Theorem 2: $\succcurlyeq$ is a partial order on $\bigcup_{m=0}^{\infty} \{0,1\}^m$.

Proof: Reflexivity: since $I_\alpha(x) \ge I_\alpha(x)$, $\succcurlyeq$ is reflexive.

Transitivity: since $I_\alpha(x) \ge I_\gamma(x)$ and $I_\gamma(x) \ge I_\delta(x)$ implies $I_\alpha(x) \ge I_\delta(x)$, $\succcurlyeq$ is transitive.

Antisymmetry: this is the nontrivial part. Suppose $\alpha \succcurlyeq \gamma \succcurlyeq \alpha$, then $I_\alpha(x) \ge I_\gamma(x) \ge I_\alpha(x)$, which implies $I_\alpha(x) - I_\gamma(x) = 0$ for all $x \in [0,1]$. By the fundamental theorem of algebra, any nonzero polynomial has finitely many roots; but $I_\alpha - I_\gamma$ has infinitely many so it must be the zero polynomial. Now proving antisymmetry boils down to proving $\alpha = \gamma$ given $I_\alpha = I_\gamma$. We prove it as a standalone lemma, Lemma 3. ∎

Lemma 3: If $I_\alpha = I_\gamma$ as polynomials, $\alpha = \gamma$ as bit strings.

Proof: We will prove the contrapositive. Suppose $\alpha = \kappa 0 \lambda$ and $\gamma = \kappa 1 \mu$. That is, they both begin with κ but disagree afterward. By the fundamental theorem of algebra, there exists a complex number $z \in \mathbf{C}$ such that $I_\kappa(z) = -1$. Now $I_\alpha(z) = I_{0\lambda}(I_\kappa(z)) = I_{0\lambda}(-1) = I_\lambda(I_0(-1)) = I_\lambda(1) = 1$. On the other hand, $I_\gamma(z) = I_{1\mu}(I_\kappa(z)) = I_{1\mu}(-1) = I_\mu(I_1(-1)) = I_\mu(-3)$. Note that $|I_0(x)|$ and $|I_1(x)|$ will be ≥ 3 if $|x| \ge 3$. So it is impossible that I_μ will send -3 to 1. This implies $I_\gamma(z) \ne 1 = I_\alpha(z)$; they are two different polynomials. ∎

See the center of Fig. 2 for the partial order $\succcurlyeq$ on $\{0,1\}^8$.

III. Predefined Rule Sets

By *rules* we mean simple inequalities that can be used to generate more inequalities. One conditional and two unconditional rules were known very early and they hold for a more general context: BMS channels and channel degradation.

Proposition 4 (Concatenation [26, Lemma 4.7]): If $\alpha \succcurlyeq \gamma$ and $\kappa \succcurlyeq \lambda$, then $\alpha\kappa \succcurlyeq \gamma\lambda$.

Proposition 5 (Rule Set A [2]): $1 \succcurlyeq 0$.

Proposition 6 (Rule Set B [3], [4]): $10 \succcurlyeq 01$.

And then there are relations proposed with BECs in mind.

Proposition 7 (Duality [7, Corollary 4]): If $\alpha \succcurlyeq \gamma$, then $\bar{\alpha} \preccurlyeq \bar{\gamma}$. Bar denotes bitwise complement, e.g., $\overline{1000} = 0111$.

Proposition 8 (Rule Set C [5, Definition 8], [6, Section VII.B]): $100^k01 \succcurlyeq 010^k10$ for any integer $k \ge 0$.

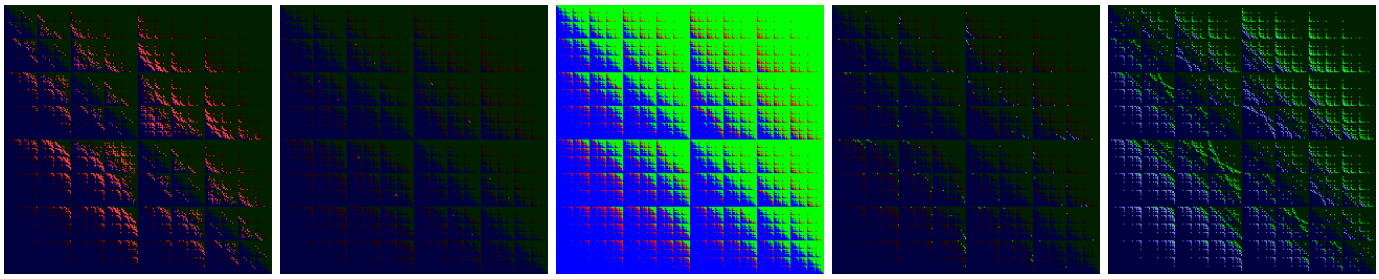
Conjecture 9 (Rule Set D [7, (45)]): $0^k1^{2^k} \succcurlyeq 1^k0^{2^k}$ for any integer $k \ge 0$. (This is claimed to be proved in [8].)

Proposal 10 (Rule Set E [9]): $\tau_1 \cdots \tau_{2^k} \preccurlyeq \overline{\tau_1 \cdots \tau_{2^k}}$ for any integer $k \ge 0$. Here, $\tau = 01101001\ldots$ is the Thue–Morse sequence [27, OEIS: A010060]: τ_n is the parity of the number of 1's in the binary representation of n.

Remark: the significance of Proposal 10 is that it generalizes RS-A, RS-B, and RS-C in an interesting way but, concerning the chaotic nature of τ_n, it is not obvious how to prove it. We succeed in finding a proof.

Proposition 11: $\tau_1 \cdots \tau_{4k} \preccurlyeq \overline{\tau_1 \cdots \tau_{4k}}$ for any integer $k \ge 1$.

Proof: Let $o \coloneqq 0110$ and $\iota \coloneqq 1001$. Use brute force to verify (a) $o \preccurlyeq \iota$, (b) $o\iota \preccurlyeq \iota o$, and (c) $o\iota\iota \preccurlyeq \iota o o$. Next use the



(a) All rule sets (b) Bernstein ($\mathrm{Ber}(256)$) (c) Standard comparison (d) Fast preorder (e) Beta expansion

Fig. 2: Binary relations visualized via incidence matrices. Each picture has 256 rows and 256 columns indexed by $\{0,1\}^8$ lexicographically. Blue means row index $\succcurlyeq$ column index. Green means row index $\preccurlyeq$ column index. Red means incomparable. Dimmed pixels are those that agree with (c). Note that (b) contains sixteen red pixels not dimmed. But as the n in $\mathrm{Ber}(n)$ increases, all pixels will eventually agree with (c).

fact that τ is *cube-free* (it does not contain $\iota\iota\iota$) to infer that $\tau = o\iota\iota\ o\iota\ o\ o\iota\iota\ o\ o\iota\ \cdots$ can be written as a product of o, $o\iota$, and $o\iota\iota$. Hence (a), (b), and (c) are all we need to prove $\tau_1 \cdots \tau_{4k} \preccurlyeq \overline{\tau_1 \cdots \tau_{4k}}$. ∎

Proposition 12 (Rule Set F): $011 \succcurlyeq 10$.

Proof: $I_{011}(x) - I_{10}(x) = x^3(x-1)^2(4+x-2x^2-x^3)$; which is nonnegative over $x \in [0,1]$. ∎

See Fig. 3 for how progressively adding new rules unlocks more comparable pairs. Notice how RS-F, despite of its simplicity, manages to make a difference on top of all other rules.

IV. Bernstein Bases

Bernstein bases and Bernstein coefficients for polar codes were studied in [5], [21], [28], [29]. Bernstein coefficients form the closest representation of the shape of a function as they are the "control points" of a Bézier curve [30]. They are also studied in the context of reliability polynomials [31], [32]. In this section, we discuss the relation between Bernstein coefficients and the partial order $\succcurlyeq$.

Definition 13 ([33], [34]): The Bernstein basis of degree n is the set $\{\binom{n}{i}x^i(1-x)^{n-i} : i = 0,\dots,n\}$. It is a basis for polynomials of degree at most n.

Definition 14: Let f be a polynomial with degree at most n. Let $\mathrm{Ber}(n,f)$ be the expansion of f in terms of the Bernstein basis of degree n:

$$f(x) = \sum_{i=0}^{n} B_i \binom{n}{i} x^i (1-x)^{n-i}.$$

B_i are called the *Bernstein coefficients*. The collection of rescaled coefficients $N_i := B_i\binom{n}{i}$ is called the *N-form* of f.

A. Coefficient-wise partial order

We define an approximation of $\succcurlyeq$ using Bernstein basis.

Definition 15: Let $\alpha \in \{0,1\}^\ell$ and $\gamma \in \{0,1\}^m$. Suppose $n \geq \max(2^\ell, 2^m)$. We say $\alpha \succcurlyeq_{\mathrm{Ber}(n)} \gamma$ if the coefficients in $\mathrm{Ber}(n, I_\alpha - I_\gamma)$ are all nonnegative.

Proposition 16: $\alpha \succcurlyeq_{\mathrm{Ber}(n)} \gamma$ iff $\bar{\alpha} \preccurlyeq_{\mathrm{Ber}(n)} \bar{\gamma}$.

Proposition 17: So long as $n \geq \max(2^\ell, 2^m)$, we have $\alpha \succcurlyeq_{\mathrm{Ber}(n)} \gamma$ implying $\alpha \succcurlyeq \gamma$.

Proposition 17 indicates that we can use $\succcurlyeq_{\mathrm{Ber}(n)}$ to generate some comparable pairs. Fig. 2 shows that it almost enumerates all pairs. This, as we see it, is because the converse is almost true.

Theorem 18 ([35]): Given $I_\alpha(x) > I_\gamma(x)$ for all $x \in (0,1)$, we have $\alpha \succcurlyeq_{\mathrm{Ber}(n)} \gamma$ for sufficiently large n.

Proof: This is a direct consequence of the classical results regarding whether a positive polynomial over $(0,1)$ has positive Bernstein coefficients. Bernstein proved that this is indeed true if n is made large enough. Subsequent works addressed how large n needs to be. See [35, Section 1] for discussion. ∎

There is another (in fact, more effective) approach to prove inequalities using Bernstein basis: subdivision.

Theorem 19 ([36]): If f is a degree-n polynomial positive over $[0,1]$, then there are division points $0 = d_0 < \ldots < d_\ell = 1$ such that the Bernstein coefficients of f with respect to each subinterval $[d_i, d_{i+1}]$ are all positive.

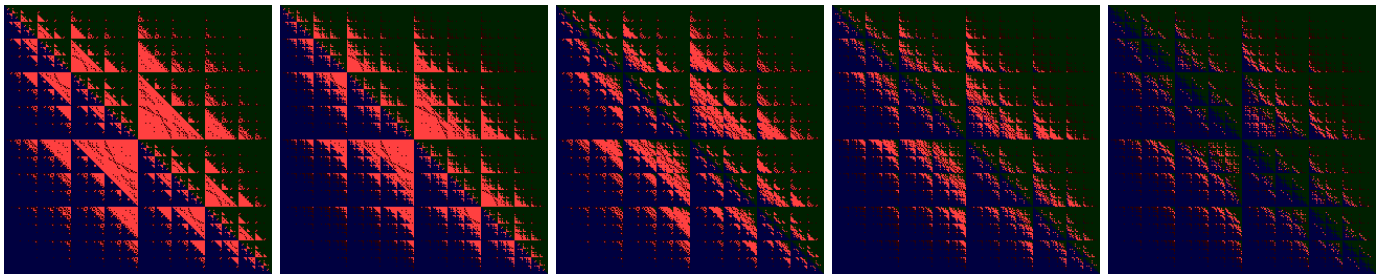
In [36], Theorem 19 is extended to an algorithm that outputs (a) the division points to certify that $f > 0$, or $f < 0$ or (b) a subinterval $[a,b]$ plus a factor $g \mid f$ such that $g(a)g(b) < 0$ to witness that f has roots in $[0,1]$. This is a very reliable method to prove or disprove any inequality of the form $\alpha \succcurlyeq \gamma$ in finite time, and is the very method used to produce Fig. 2 and the proof of Proposition 11.

There is a library worth of literature that studies Bernstein coefficients, especially when they play the role of the number of size-i subsets in an upward-closed family. The following nontrivial fact, in particular, adds one more reason to why we should study them.

Theorem 20 (Sperner and Kruskal–Katona [31, Section 5]): Bernstein coefficients B_i are non-decreasing in i.

B. The first nontrivial coefficient

Definition 21: Let f be a nonzero polynomial of degree at most n. Let N_i be the N-form of $\mathrm{Ber}(n,f)$. The *exponent* of f is the smallest index i such that N_i is nonzero. The *mantissa* of f is $N_{\text{the exponent of } f}$.



(a) RS-A and B (b) RS-A, B, and C (c) RS-A, B, and D (d) RS-A, B, C, and D (e) RS-A, B, C, D, and F

Fig. 3: Same setting as Fig. 2. As more rule sets are considered, less incomparable pairs wander.

Notice that $f(\varepsilon) = \text{mantissa}\cdot\varepsilon^{\text{exponent}}(1+O(\varepsilon))$ when $\varepsilon \to 0$.

Theorem 22 ([21, Theorem 3]): Let $\alpha \in \{0,1\}^\ell$. Let $n \geq 2^\ell$. Let N_i be the N-form of $\mathrm{Ber}(n, I_\alpha)$. Then the exponent of I_α is 2^z, where z is the number of zeros in α; moreover, the mantissa of I_α is such that

$$\log_2(N_{2^z}) = \sum_{k=1}^{\ell-z} 2^{\text{number of 0's to the right of the } k\text{th 1 in } \alpha}.$$

Sketch: Note that $I_0(x) = x^2$ and $I_1(x) = 2x + O(x^2)$. So I_0 doubles the exponent and I_1 maintains it. Moreover, I_0 squares the mantissa and I_1 doubles it. ∎

Studying exponents and mantissas gives us the leverage to determine whether W^α or W^γ is better when the underlying W is very reliable. This is the theme of [21, Section IV-C]. On the other hand, it can also be used to generate necessary conditions for the comparability of α and γ. The following proposition gives an easy class of incomparable pairs.

Proposition 23: If α has more ones than γ does and α has more zeros than γ does, then α and γ are incomparable by $\succcurlyeq$, i.e., $\alpha \not\succcurlyeq \gamma$ and $\alpha \not\preccurlyeq \gamma$.

Proposition 23 is not useful if one wants to compare α and γ having the same length. So in the next subsection we are going to craft some examples leveraging the knowledge of mantissas.

C. Use knowledge of coefficients to construct incomparables

Definition 24: We write $\alpha \geq_{@0} \gamma$ if $I_\alpha - I_\gamma = 0$ or the mantissa of $I_\alpha - I_\gamma$ is positive. We write $\alpha \sim_{@0} \gamma$ if I_α and I_γ share the same exponent and mantissa. We write $\alpha >_{@0} \gamma$ if I_α has a smaller exponent than I_γ does, or they have the same exponent but the former has a higher mantissa.

$\geq_{@0}$, $\sim_{@0}$, and $>_{@0}$ imply that $\lim_{x\to 0} I_\alpha(x)/I_\gamma(x)$ is ≥ 0, $= 1$, or > 1, respectively. There are also "@1" versions, which ask for the same criteria at the neighborhood of $x = 1$.

Definition 25: We write $\alpha \geq_{@1} \gamma$ if $\bar\alpha \leq_{@0} \bar\gamma$. We write $\alpha \sim_{@1} \gamma$ if $\bar\alpha \sim_{@0} \bar\gamma$. We write $\alpha >_{@1} \gamma$ if $\bar\alpha <_{@0} \bar\gamma$.

Proposition 26: If $\alpha \succcurlyeq \gamma$ then $\alpha \geq_{@0} \gamma$ and $\alpha \geq_{@1} \gamma$.

Example usage of Proposition 26: 100001 and 011000 are incomparable under $\succcurlyeq$ because $100001 >_{@0} 011000$ and $100001 <_{@1} 011000$.

Proposition 27: If $\alpha >_{@0} \gamma$ and $\kappa \sim_{@0} \lambda$, then $\alpha\kappa \geq_{@0} \gamma\lambda$ and $\kappa\alpha \geq_{@0} \lambda\gamma$.

Constructing more incomparable pairs becomes easy thanks to Proposition 27. For instance, $(\kappa 100001\lambda, \kappa 011000\lambda)$ is an incomparable pair for arbitrary κ and λ.

V. Beta Expansion

Beta expansion [22] was defined not to partial-order the channels, but to give, for each block length, a totally-ordered list so a user will simply take the first k strings to construct a code of dimension k. This is essentially a curve-fitting problem. In [37] the authors models this with more parameters.

In this paper, we discuss whether beta expansion is compatible with the rule sets covered in Section III. We also try to "explain" beta expansion using scaling exponent.

Definition 28: Let β be a positive real number. The beta expansion of any string $\alpha = a_1 \cdots a_\ell \in \{0,1\}^\ell$ is defined as

$$\alpha_\beta := \sum_{i=1}^{\ell} a_i \beta^{\ell-i}.$$

Definition 29: We say $\alpha \geq_\beta \gamma$ if $\alpha_\beta \geq \gamma_\beta$ for the beta value understood from the context.

A. Compatibility of beta and rule sets

Proposition 30: If $\alpha \geq_\beta \gamma$ and $\kappa \geq_\beta \lambda$ while κ and λ sharing the same length, then $\alpha\kappa \geq_\beta \gamma\lambda$.

Proposition 31 ([22, Proposition 3]): $1 \geq_\beta 0$ if $\beta \geq 0$.

Proposition 32 ([22, Proposition 3]): $10 \geq_\beta 01$ if $\beta \geq 1$.

Proposition 33 ([22, Proposition 1]): For α and γ of the same length, $\alpha \geq_\beta \gamma$ iff $\bar\alpha \leq_\beta \bar\gamma$.

Proposition 34: $100^k01 \geq_\beta 010^k10$ for all k if $\beta \geq 1$.

Proof: $(100^k01)_\beta - (010^k10)_\beta = (\beta-1)(\beta^{k+2}-1)$, which is nonnegative as $\beta \geq 1$. ∎

Proposition 35: $0^k1^{2^k} \geq_\beta 1^k0^{2^k}$ implies $\beta \leq 2^{-1/k}$. So beta expansion and RS-D are not compatible due to $\lim_{k\to\infty} 2^{-1/k} = 1$.

Proof: $1 \leq (0^k1^{2^k})_\beta/(1^k0^{2^k})_\beta \leq \beta^{-k} + \beta^{-2k} + \beta^{-3k} + \cdots = \beta^{-k}/(1-\beta^{-k})$. This implies $\beta^{-k} \geq 1/2$. ∎

Proposition 36: $\tau_1 \cdots \tau_{2^k} \leq_\beta \overline{\tau_1 \cdots \tau_{2^k}}$ if $\beta \geq 1$.

Proof: $(\overline{\tau_1 \cdots \tau_{2^k}})_\beta - (\tau_1 \cdots \tau_{2^k})_\beta = (\beta^{2^0}-1)(\beta^{2^1}-1) \times \cdots (\beta^{2^{k-1}}-1)$, which is nonnegative as $\beta \geq 1$. ∎

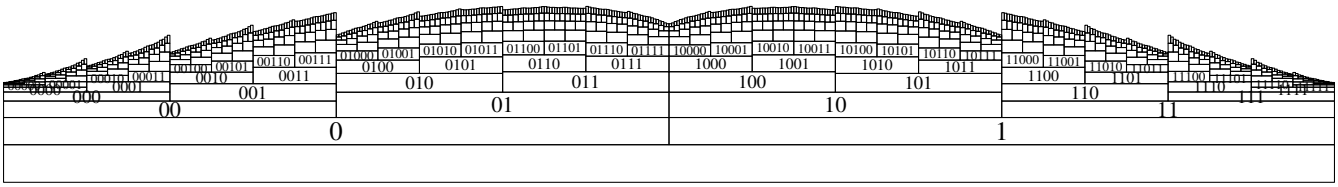


Fig. 4: How to select beta? The height of a rectangle labeled α is $I_{\alpha 0}^{-1}(1/2) - I_{\alpha 1}^{-1}(1/2)$. This number is the *influence* of the last digit on the halfway point. The choice $\beta = 2^{1/4}$ represents an estimate that the average height of the rectangles on level ℓ is about $2^{-\ell/4}$. That being said, (a) $\beta = 2^{1/3.6}$ seems to be a better choice, at least for BECs; (b) since the influence depends on α, neither $2^{1/4}$ nor $2^{1/3.6}$ provides rigorous ordering.

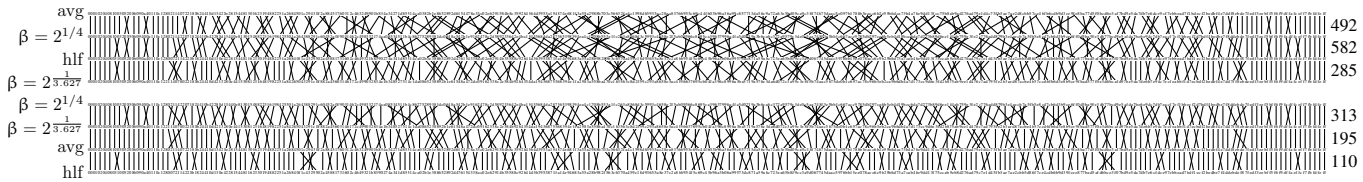


Fig. 5: Matchings among four total orders on $\{0,1\}^8$. The string 11110000 is shorthanded as f0. The Kendall tau distances (the numbers of crossings) are marked on the right.

B. A theoretical connection between beta and scaling

Scaling exponent is the answer to the following question: at block length 2^ℓ, how many synthetic channels "stay mediocre"? That is, how many $\gamma \in \{0,1\}^m$ are such that $\varepsilon < H(W^\gamma) < 1 - \varepsilon$. As it turns out, this number is about $2^{m-m/4}$ for AWGN channels and $2^{m-m/3.627}$ for BECs [38], [39].

In beta expansion, the ith bit of α is endowed with an "influence" of strength $\beta^{\ell-i}$ (see also Fig. 4). But, it is very clear that a_i is influential only if $W^{a_1 \cdots a_{i-1}}$ is mediocre. Hence, a hand-waving argument is that $\beta^{\ell-i}$ should be in proportion to the fraction of mediocre channels, which is $2^{-i/4}$ for AWGN channels. This suggests $\beta \approx 2^{1/4}$, coinciding with the value recommended in [22].

To prove our point, we turn to BECs. Since the fraction of mediocre channels is now $2^{-i/3.627}$, we hypothesize that $\beta = 2^{1/3.627}$ should be "more suitable" than $\beta = 2^{1/4}$. We measure the Kendall tau distance [40] among four total orders: $\geq_\beta$ using $\beta = 2^{1/4}$ or $\beta = 2^{1/3.627}$, and $\geq_{\mathrm{avg}}$ and $\geq_{\mathrm{hlf}}$ defined in Section VI. We see in Fig. 5 that $2^{1/3.627}$, avg, and hlf lead to similar total orders while $\beta = 2^{1/4}$ leads to an outlier. This indicates that it is appropriate to sync β with the scaling exponent.

VI. Threshold—Area or Halfway Point?

Inspired by beta expansion, we define total orders so that the first k strings will likely form a good polar code. Or, a more conservative approach is to include the first $k/2$ strings and carefully examine the next k strings (cf. [41]). For either goal, a representative total order is in need.

A. Average

Definition 37 ([5, Definition 10]): Define $\mathrm{avg}(\alpha)$ to be $\int_0^1 I_\alpha(x)\,dx$. We write $\alpha \geq_{\mathrm{avg}} \gamma$ when $\mathrm{avg}(\alpha) \geq \mathrm{avg}(\gamma)$.

Proposition 38 ([5, Proposition 7]): $\alpha \geq_{\mathrm{avg}} \gamma$ if $\alpha \succcurlyeq \gamma$.

Proposition 39 ([5, Lemma 5]): $\alpha \geq_{\mathrm{avg}} \gamma$ iff $\bar{\alpha} \leq_{\mathrm{avg}} \bar{\gamma}$.

Proposition 40: $\mathrm{avg}(\alpha 0) + \mathrm{avg}(\alpha 1) = 2\mathrm{avg}(\alpha)$.

Proposition 40 implies that, if $A_1, A_2, \ldots$ are iid random bits, then $\mathrm{avg}(A_1 \cdots A_t)$ is a martingale as t increases. This martingale is bounded and hence converging. The limit is the ϑ function in [20, Proposition 3] and the z^* function in [39, Lemma 11]. Hence $\mathrm{avg}(A_1 \cdots A_t)$ can also be seen as the Doob martingale of z^*.

B. Halfway point

Definition 41: Define $\mathrm{hlf}(\alpha) \coloneqq I_\alpha^{-1}(1/2)$. We say $\alpha \geq_{\mathrm{hlf}} \gamma$ if $\mathrm{hlf}(\alpha) \leq \mathrm{hlf}(\gamma)$.

Proposition 42: $\alpha \succcurlyeq \gamma$ implies $\alpha \geq_{\mathrm{hlf}} \gamma$.

Proposition 43: $\alpha \geq_{\mathrm{hlf}} \gamma$ iff $\bar{\alpha} \leq_{\mathrm{hlf}} \bar{\gamma}$.

The halfway point was studied in [7], [20]. It and avg are eventually the same function as the length of string goes to infinity because the reliability polynomials experience hard thresholds [19]. Halfway point outshines average because it is as easy to compute as beta expansion (cf. [29]).

C. A Fast preorder

We conclude this paper with an intersection of three preorders inspired by Section IV-C.

Definition 44: We say $\alpha \succcurlyeq_{\mathrm{fst}} \gamma$ if all three bullets are met:

- $\alpha >_{@0} \gamma$ or $\alpha \sim_{@0} \gamma$;
- $\alpha >_{@1} \gamma$ or $\alpha \sim_{@1} \gamma$;
- $\alpha \geq_{\mathrm{hlf}} \gamma$.

Proposition 45: $\alpha \succcurlyeq \gamma$ implies $\alpha \succcurlyeq_{\mathrm{fst}} \gamma$.

Proposition 46: $\alpha \succcurlyeq_{\mathrm{fst}} \gamma$ iff $\bar{\alpha} \preccurlyeq_{\mathrm{fst}} \bar{\gamma}$.

$\succcurlyeq_{\mathrm{fst}}$ is easy to compute as $>_{@0}, \sim_{@0}, >_{@1}, \sim_{@1}$ involve only counting 0 and 1 and $\geq_{\mathrm{hlf}}$ is a matter of chaining I_0^{-1} and I_1^{-1}. The fast partial order is empirically a very successful approximation of $\succcurlyeq$. See Fig. 2 for how unnoticeable the difference is.

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