Three-phase Transformer Modeling

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***Abstract*— We present a simple and transparent method to derive the admittance matrices of three-phase banks of single-phase transformers in an unbalanced setting. The basic idea is to explicitly separate a transformer model into an *internal model* that specifies the characteristics of the constituent single-phase transformers, and a *conversion rule* that maps its internal variables to its terminal variables. Its admittance matrix can then be derived by eliminating the internal variables from the set of equations describing the internal model and the conversion rule. The method is general and provides rigor and clarity, facilitating extensions. It also reveals a transformer model that is unified and modular.**

I. INTRODUCTION

Motivation. Unbalanced three-phase systems are subtle because currents and voltages in different phases are coupled; see, e.g., [1, Chapter 11] for transmission systems and [2] for distribution systems. Their modeling has become increasingly important as the need for analyzing low-voltage network rises. There are often two common sources of confusion in three-phase systems. First a three-phase device can be in Y or Δ configuration. The neutral of a Y-configured device may or may not have a neutral wire, it may or may not be grounded, and if grounded, the grounding impedance may or may not be negligible. As a consequence, for instance, a simple transformer model in a single-phase setting proliferates into 9 standard transformers in a three-phase setting (each of the primary and secondary side can be in grounded Y, Y, or Δ configuration); see e.g. [3, Table 1]. These variants often seem disparate (they are not). Second, the voltages and currents across single-phase devices internal to Δ configuration are observed externally only through a linear map that is not invertible. In many applications, we control the *internal* currents or power flows of the single-phase devices on the secondary circuit, e.g., controlling the charging currents of electric vehicle chargers in Δ configuration. Our control decisions interact with other devices over the network, however, only through *terminal* voltages and currents observable externally of the three-phase devices. Many models in the literature, however, assume we control directly the terminal variables e.g. [4]. The interplay between internal and terminal variables of a three-phase device sometimes seems confusing, and is the key to the modeling and analysis of unbalanced systems with both Y and Δ-configured devices and transformers.

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We have developed such a modeling approach in [5, 6] that addresses both issues. It makes transparent the unified structure of three-phase systems and shows that models and properties of single-phase networks have direct extensions to three-phase networks. The basic idea is to explicitly separate a device/transformer model into an *internal model* that specifies the characteristics of a single-phase device or transformer, and a *conversion rule* that maps its internal variables to its terminal variables. An external model of the device or transformer can then be derived by eliminating the internal variables from the set of equations describing the internal model and the conversion rule. The internal model depends only on the behavior of the constituent single-phase components (e.g. non-ideal voltage sources, ZIP loads, different single-phase transformer models) regardless of their configurations. The conversion rule depends only on their configurations regardless of the type of components.

This separation provides two benefits. First it facilitates the modeling of secondary distribution circuits where usually only the end devices are directly controllable, not the currents or powers at the transformers. Second it leads to an explicit and systematic way to exploit common structures across different device/transformer variants and derive their external models that are general, unifying, and simple. This approach is established in detail in [5]. In [6] we introduce the basic framework, focusing on ideal devices (voltage sources and ZIP loads) without transformers, and deriving single-phase analysis for a balanced network. In this paper we apply the method to the modeling of three-phase banks of single-phase transformers (as opposed to three-phase common-core transformers) and re-derive their admittance matrices in the literature. In a tutorial paper [7] we will present models for nonideal devices and illustrate our approach by applying it to solving power flow through backward forward sweep and formulating three-phase optimal power flow problems.

Literature. Three-phase load flow solvers have been developed since at least the 1960s, e.g., see [8] for solution in the sequence coordinate and [9, 10] the phase coordinate. A three-phase network is equivalent to a single-phase circuit where each node is indexed by a (bus, phase) pair [10]. Single-phase power flow algorithms such as Newton Raphson [11] or Fast Decoupled methods [12] can be directly applied to the equivalent circuit. The main difference with a single-phase network is the modeling of three-phase components in the equivalent circuit, such as models for three-phase lines [2, 13–15], transformers and co-generators [3, 10, 13, 16][2, Ch 8][1, Ch 7.4][17][18–21], constant-power devices [1, Chapter 11], as well as voltage regulators, and loads [2], etc.

In this paper we focus on the modeling of three-phase banks of single-phase transformers. There are two types of transformer models.

The first type models a single-phase transformer as consisting of a leakage admittance and a shunt admittance in series with an ideal transformer. The circuit model of a three-phase transformer is then assembled by connecting three copies of this single-phase model in Y or Δ configurations. This is developed in, e.g., [3, 10, 13, 16][2, Ch 8][1, Ch 7.4]. In particular [10] analyzes transformers in YY, $\Delta\Delta$, $Y\Delta$ and open $\Delta\Delta$ configurations where a Y configuration may be solidly grounded or ungrounded, and [3] analyzes transformers in Y_gY_g and in $Y_g\Delta$ configurations (but listed models for all 9 standard transformer variants). Here Y_g means solidly grounded Y configuration. Instead of deriving the transformer admittance matrices in the phase coordinate, [17] derives them in the sequence coordinate and then transforms them into the phase coordinate.

The second type models a single-phase transformer as consisting of two ideal transformers connected by a unitary voltage network. As for the first type, the circuit model of a three-phase transformer is again assembled by connecting three copies of this single-phase model in Y or Δ configurations. The idea of decomposing a nonideal transformer into two ideal transformers connected by a unitary voltage network is first mentioned, but not explored, in [10]. It is developed in detail in [18] where the unitary network is a Π circuit with a leakage (series) admittance and two shunt admittances. The 3×3 leakage and shunt admittance matrices are not necessarily diagonal, thus capturing magnetic coupling between windings of different phases. The unitary voltage network that models leakage fluxes and core losses can be quite general e.g. [19]. Instead of Π circuit, the unitary voltage network in [20] uses a T circuit model. This method is extended rigorously in [21] for modeling a large number of transformer variants.

Outline and contribution. In this paper we apply our general method to re-derive the admittance matrices in the literature for both types of three-phase transformer models. The idea of deriving a transformer admittance matrix by connecting its internal variables and terminal variables has appeared in various forms in the literature. We formalize and make explicit the derivation procedure. This provides clarity and rigor, and facilitates extensions. For example, while the derivation in [3, 10] analyzes the circuit of each three-phase transformer variant, our approach applies more uniformly to all variants and significantly simplifies the derivation. The models in [3, 10] also assume leakage admittances and turns ratios to be identical across phases a, b, c and shunt admittances to be zero. These assumptions are unnecessary in our approach.

We start in Section II by reviewing both types of transformer models in the single-phase setting, which are then extended to the three-phase setting. We apply our method to deriving the admittance matrices for the first type of three-phase transformer models in Section III, and for the second type of transformer models with unitary voltage networks in Section IV. We conclude in Section V.

II. Review: single-phase transformer

A. Circuit model

A single-phase transformer converts voltage and current from the primary side to the secondary side according to the turns ratio of the primary and secondary coils. Our starting point is the following system of equations that describes its steady-state behavior.
Nonideal elements:

$$\begin{aligned} v_j &= r_j i_j + L_{lj}\frac{di_j}{dt} + \hat{v}_j, \qquad & \hat{v}_j &= L_m \frac{d\hat{i}_m}{dt} \\ v_k &= r_k i_k + L_{lk}\frac{di_k}{dt} + \hat{v}_k \end{aligned}$$

Ideal transformer:

$$\hat{v}_k = \frac{N_k}{N_j}\hat{v}_j, \qquad -i_k = \frac{N_j}{N_k}\left(i_j - \hat{i}_m\right)$$

where v_j, v_k are the AC voltages, in the time domain, across the primary and secondary terminals respectively, and i_j, i_k are the currents entering the primary coil and the secondary coil respectively. The current $\hat{i}_m$ is called the primary magnetizing current and the voltages $\hat{v}_j, \hat{v}_k$ arise from the rate of change of mutual flux between the primary and secondary coils. The number of turns of the primary and secondary coils are N_j and N_k respectively. Their voltage gain is $n := N_j/N_k$ and turns ratio is $a := 1/n$ (even though a is used to denote both a phase and a turns ratio its meaning should be clear from the context). This set of equations in the phasor domain is the following.
Nonideal elements:

$$V_j = z^p I_j + \hat{V}_j, \qquad \hat{I}_m = y^m \hat{V}_j \tag{1a}$$
$$V_k = z^s I_k + \hat{V}_k \tag{1b}$$

Ideal transformer:

$$\hat{V}_k = \frac{N_k}{N_j}\hat{V}_j, \qquad -I_k = \frac{N_j}{N_k}\left(I_j - \hat{I}_m\right) \tag{1c}$$

where the series impedances $z^p := r_j + \omega L_{lj}$ and $z^s := r_k + \omega L_{lk}$ model the core losses and leakage fluxes in the primary and secondary circuits respectively, and the shunt admittance $y^m := 1/(\omega L_m) = R/(\omega N_j^2)$ models the finite permeability of the core. The model (1) can be interpreted as the circuit in Figure 1. Variables with hats denote internal variables.

We now derive two popular transformer models from the circuit model in Figure 1.

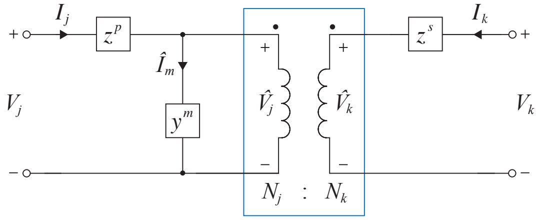
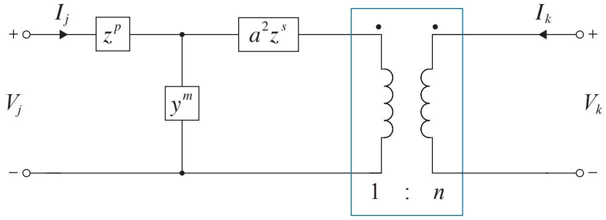
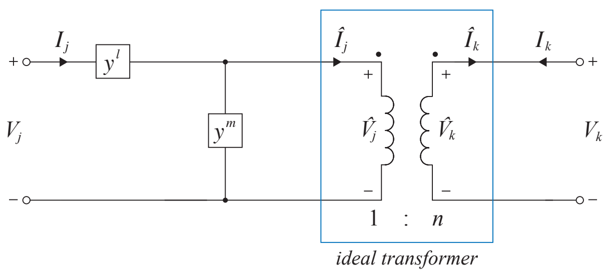


Fig. 1: Circuit model of single-phase nonideal transformer. The dashed box represents an ideal transformer with $a := N_j/N_k$.



(a) T equivalent circuit



(b) Simplified circuit

Fig. 2: The T equivalent circuit and its approximation. The internal variables $(\hat{V}_j, \hat{I}_j)$, $(\hat{V}_k, \hat{I}_k)$ and terminal variables (V_j, V_j^n, I_j), (V_k, V_k^n, I_k).

B. Simplified model

We can refer the series impedance z^s in Figure 1 on the secondary side to the primary side as $a^2 z^s$ where a is the turns ratio. The resulting model, shown in Figure 2(a), is called the T equivalent circuit of the transformer. It is equivalent in the sense that the admittance matrices that map the terminal voltages (V_j, V_k) to the terminal currents (I_j, I_k) are the same in both models (their internal variables may take different values). If the shunt admittance matrix y^m is negligible compared with the series admittance $1/a^2 z^s$ then the end-to-end behavior will be approximately the same if we switch the order of these two elements and combine $a^2 z^s$ with z^p into the series (leakage) impedance $z^l := z^p + a^2 z^s$ or the series (leakage) admittance $y^l := 1/z^l = 1/(z^p + a^2 z^s)$, as shown in Figure 2(b). We will call this the *simplified model* or the *simplified circuit*.

Remark 1 (Approximation error)**.** Let T and $\hat{T}$ denote the transmission matrices of the T equivalent circuit in Figure 2(a) and that of the simplified circuit in Figure 2(b) respectively that maps (V_k, I_k) to (V_j, I_j). If $\epsilon := |a^2 z^s y^m| \ll 1$ then it can be shown [5] that the relative approximation error is upper bounded by

$$\frac{\|T - \hat{T}\|}{\|T\|} \;<\; \epsilon \;\ll\; 1$$

where the matrix norm is $\|A\| := \sum_{i,j} |A_{ij}|$. $\square$

To derive the admittance matrix of the simplified model in Figure 2(b), let the currents entering/leaving and the voltages across the ideal transformer be denoted by variables with a hat: $\left(\hat{V}_j, \hat{I}_j\right)$, $\left(\hat{V}_k, \hat{I}_k\right)$. They are the *internal* variables. The transformer gains are

$$\hat{V}_k \;=\; n\hat{V}_j, \qquad \hat{I}_k \;=\; \frac{1}{n}\hat{I}_j \;=:\; a\hat{I}_j \tag{2a}$$

This is an *internal model* of the single-phase (ideal) transformer.

The *terminal voltages* $\left(V_j, V_j^n, V_k, V_k^n\right)$ are defined with respect to the ground. We emphasize that, while the internal voltages $(\hat{V}_j, \hat{V}_k)$ are defined to be the voltage drops across the ideal transformer windings, the terminal voltages $\left(V_j, V_j^n, V_k, V_k^n\right)$ are defined with respect to a common reference point (the ground); in particular the primary and secondary windings are not assumed to be grounded. If the neutral of terminal j is solidly grounded, then $V_j^n := 0$. The *terminal currents* (I_j, I_k) are defined to be the sending-end currents from buses j and k respectively to the other side, as shown in Figure 2(b). The terminal and internal variables are related by the *conversion rule*:

$$I_j \;=\; y^l\left(V_j - V_j^n - \hat{V}_j\right), \quad I_j \;=\; y^m\hat{V}_j + \hat{I}_j \tag{2b}$$

$$\hat{V}_k \;=\; V_k \;-\; V_k^n, \qquad \hat{I}_k \;=\; -I_k \tag{2c}$$

Eliminating the internal variables from (2) yields an external model that relates the terminal variables:

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2(y^l + y^m) \end{bmatrix}}_{Y} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} \;-\; \begin{bmatrix} V_j^n \\ V_k^n \end{bmatrix} \right) \tag{3a}$$

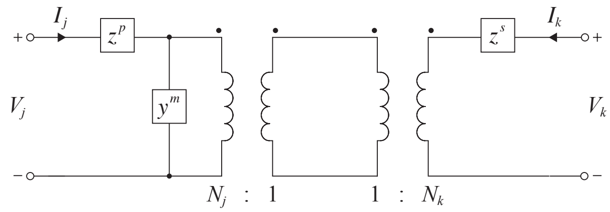
We can also represent (3a) by a two-wire model by adding current injections from the neutrals, $I_j^n = -I_j$ and $I_k^n = -I_k$, yielding

$$\begin{bmatrix} I_j \\ I_k \\ I_j^n \\ I_k^n \end{bmatrix} \;=\; \begin{bmatrix} Y & -Y \\ -Y & Y \end{bmatrix} \begin{bmatrix} V_j \\ V_k \\ V_j^n \\ V_k^n \end{bmatrix} \tag{3b}$$

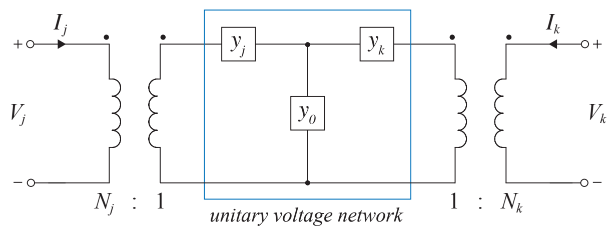
C. Unitary voltage network

As far as the end-to-end behavior is concerned, the transformer model in Figure 1 is equivalent to the model in Figure 3(a) where the ideal transformer with turns ratio N_j/N_k is replaced by two ideal transformers in series with turns ratios N_j and $1/N_k$. Referring the series impedances (z^p, z^s) and shunt admittance y^m to the other sides of the ideal transformers using

$$y_0 \;:=\; N_j^2\, y^m, \qquad y_j \;:=\; \frac{N_j^2}{z^p}, \qquad y_k \;:=\; \frac{N_k^2}{z^s}$$



(a) Equivalent model



(b) Unitary voltage network

Fig. 3: Models of nonideal transformer with unitary voltage network.

this model is equivalent to the one in Figure 3(b). The network between the two ideal transformers is sometimes referred to as a unitary voltage network. Note that no nodes in the transformer models may be grounded. The main advantage of this approach is that the unitary voltage network can be generalized and used to model nonstandard transformers with multiple windings [18–21].

To derive the admittance matrix that maps (V_j, V_k) to (I_j, I_k), we start with the unitary voltage network. Referring

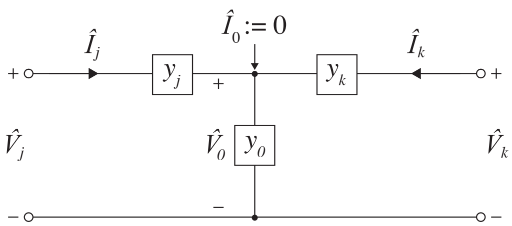


Fig. 4: Unitary voltage network of the model in Figure 3(b).

to the internal variables defined in Figure 3(b), the voltage drops is mapped to the current injections into the unitary voltage network by an admittance matrix:

$$\begin{bmatrix} \hat{I}_0 \\ \hat{I}_j \\ \hat{I}_k \end{bmatrix} = \begin{bmatrix} y_0 + y_j + y_k & -y_j & -y_k \\ -y_j & y_j & 0 \\ -y_k & 0 & y_k \end{bmatrix} \begin{bmatrix} \hat{V}_0 \\ \hat{V}_j \\ \hat{V}_k \end{bmatrix}$$

Since $\hat{I}_0 = 0$ we can eliminate $\hat{V}_0$ and derive the Kron-reduced admittance matrix Y_{uvn} that maps $\hat{V} := (\hat{V}_j, \hat{V}_k)$ to $\hat{I} := (\hat{I}_j, \hat{I}_k)$. This yields the *internal model* $\hat{I} = Y_{\text{uvn}} \hat{V}$ where

$$Y_{\text{uvn}} \; := \; \frac{1}{\sum_i y_i} \begin{bmatrix} y_j(y_0 + y_k) & -y_j y_k \\ -y_j y_k & y_k(y_0 + y_j) \end{bmatrix} \tag{4a}$$

Let the terminal current and voltage be $I := (I_j, I_k)$ and $V := (V_j, V_k)$. The *conversion rule* that relates the internal variables $(\hat{V}, \hat{I})$ and the terminal variables (V, I) is $\hat{V} = MV$ and $\hat{I} = M^{-1} I$ where

$$M \; := \; \begin{bmatrix} 1/N_j & 0 \\ 0 & 1/N_k \end{bmatrix} \tag{4b}$$

Substituting into the internal model $\hat{I} = Y_{\text{uvn}} \hat{V}$ we obtain the admittance matrix $M Y_{\text{uvn}} M$ (external model) of the transformer that relates the terminal variables (V, I):

$$I \; = \; (M Y_{\text{uvn}} M) \, V \tag{4c}$$

Since the model (4) based on unitary voltage network is equivalent to the T equivalent circuit, the simplified model (3) is also an approximation of (4). If the shunt admittance matrix $y^m = 0$ then all these models are equivalent.

We now extend the simplified model (3) and the unitary voltage network model (4) to three-phase transformers in an unbalanced setting.

III. Three-phase transformers: simplified circuit

The notations and derivation of the simplified model (3) extend directly to the three-phase setting.

The external model of a three-phase transformer depends on the models of its constituent single-phase transformers and their configuration. In particular each of the primary and secondary sides can be in Y or Δ configuration, giving four configurations for a standard three-phase transformer. The external model can be derived in four simple steps, similar to the derivation of (3):

1) For the primary side, define the internal variables $(\hat{V}_j, \hat{I}_j)$ and external variables (V_j, V_j^n, I_j) (defined precisely below) and relate them through a conversion rule.
2) For the secondary side, define the internal variables $(\hat{V}_k, \hat{I}_k)$ and external variables (V_k, V_k^n, I_k) and relate them through a conversion rule.
3) Couple these relations through an internal model (the transformer gains) on $(\hat{V}_j, \hat{I}_j)$, $(\hat{V}_k, \hat{I}_k)$ for each of the single-phase transformers.
4) Derive the external model, a relation between external variables (V_j, I_j) and (V_k, I_k), by eliminating the internal variables.

We now describe these steps in detail. Define *conversion matrices* $\Gamma, \Gamma^{\mathsf{T}}$ that maps between internal and external variables in a Δ configuration:

$$\Gamma \; := \; \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \Gamma^{\mathsf{T}} \; := \; \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

It turns out that the spectral properties of $\Gamma, \Gamma^{\mathsf{T}}$ underlie much of the behavior of three-phase systems, balanced or unbalanced [5].

1) Primary side: Consider the primary circuit of a three-phase transformer in Y or Δ configuration in Figure 5. The *internal* voltages and currents associated with the ideal transformer are denoted by $\hat{V}_j^Y \; := \; \left(\hat{V}_j^{an}, \hat{V}_j^{bn}, \hat{V}_j^{cn}\right)$, $\hat{I}_j^Y \; := \; \left(\hat{I}_j^{an}, \hat{I}_j^{bn}, \hat{I}_j^{cn}\right)$ for Y configuration and $\hat{V}_j^{\Delta} \; :=$

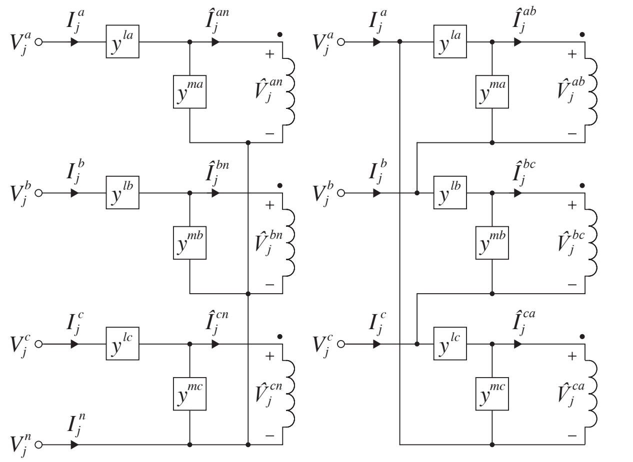


Fig. 5: Primary side of a three-phase transformer in Y (left) or Δ (right) configuration.

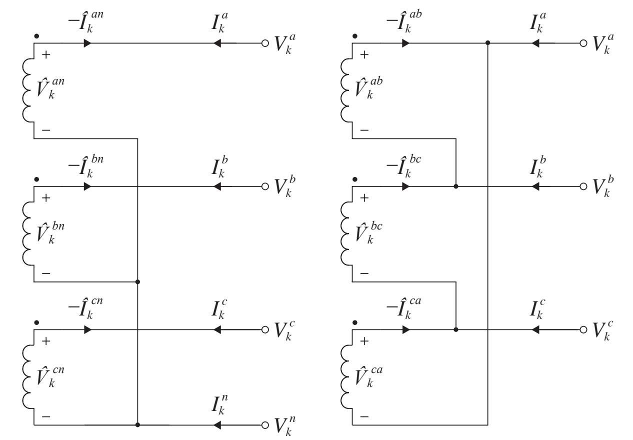


Fig. 6: Secondary side of a three-phase transformer in Y (left) or Δ (right) configuration.

$\left(\hat V_j^{ab}, \hat V_j^{bc}, \hat V_j^{ca}\right)$, $\hat I_j^{\Delta} := \left(\hat I_j^{ab}, \hat I_j^{bc}, \hat I_j^{ca}\right)$ for Δ configuration. The *terminal* voltages and currents are denoted by (without hat) $V_j := \left(V_j^a, V_j^b, V_j^c\right)$, $I_j := \left(I_j^a, I_j^b, I_j^c\right)$ regardless of the configuration. For Y configuration the (terminal) neutral voltage and current are denoted by $\left(V_j^n, I_j^n\right)$ in the direction shown in Figure 5. As for the single-phase model, the terminal voltages $\left(V_j, V_j^n, V_k, V_k^n\right)$ are defined with respect to a common reference point (the ground); in particular the neutrals are not assumed to be grounded. If the neutral is solidly grounded, then $V_j^n = 0$.

The leakage admittances of the transformer are denoted by the diagonal matrix $y^l := \text{diag}\left(y^{la}, y^{lb}, y^{lc}\right)$ and the shunt admittances are denoted by $y^m := \text{diag}\left(y^{ma}, y^{mb}, y^{mc}\right)$. From (2b) for each single-phase transformer, the terminal variables are related to the internal variables according to the *conversion rule*:

$$\begin{aligned}
Y \text{ configuration:} \quad I_j &= y^l\left(V_j - V_j^n\,\mathbf{1} - \hat V_j^Y\right) && \text{(5a)}\\
I_j &= y^m \hat V_j + \hat I_j^Y, \quad I_j^n = -\mathbf{1}^\mathsf{T} \hat I_j^Y && \text{(5b)}\\
\Delta \text{ configuration:} \quad \hat I_j^{\Delta} &= y^l \Gamma V_j - (y^l + y^m)\hat V_j^{\Delta} && \text{(5c)}\\
I_j &= \Gamma^\mathsf{T}\left(\hat I_j^{\Delta} + y^m \hat V_j^{\Delta}\right) && \text{(5d)}
\end{aligned}$$

where $\mathbf{1} \in \mathbb{R}^3$ denotes the column vector with three 1s. Here (5c) follows from Ohm's law on each branch, e.g., $\hat I_j^{ab} + y^{ma}\hat V_j^{ab} = y^{sa}\left(V_j^a - V_j^b - \hat V_j^{ab}\right)$.

2) Secondary side: Consider the secondary side of a three-phase transformer in Y or Δ configuration in Figure 6. The internal voltages and currents are similarly denoted by $\hat V_k^Y := \left(\hat V_k^{an}, \hat V k j^{bn}, \hat V_k^{cn}\right)$, $\hat I_k^Y := \left(\hat I_k^{an}, \hat I_k^{bn}, \hat I_k^{cn}\right)$ for Y configuration and $\hat V_k^{\Delta} := \left(\hat V_k^{ab}, \hat V_k^{bc}, \hat V_k^{ca}\right)$, $\hat I_k^{\Delta} := \left(\hat I_k^{ab}, \hat I_k^{bc}, \hat I_k^{ca}\right)$ for Δ configuration. The terminal voltages and currents are denoted by $V_k := \left(V_k^a, V_k^b, V_k^c\right)$, $I_k := \left(I_k^a, I_k^b, I_k^c\right)$ regardless of the configuration. For Y configuration the neutral voltage and current are denoted by (V_k^n, I_k^n); see Figure 6. If the neutral is solidly grounded, then $V_k^n = 0$.

From (2c) for each single-phase transformer, the terminal variables are related to the internal variables according to the *conversion rule*:

$$\begin{aligned}
Y \text{ configuration:} \quad V_k &= \hat V_k^Y + V_k^n\,\mathbf{1} && \text{(6a)}\\
I_k &= \hat I_k^Y, \quad I_k^n = -\mathbf{1}^\mathsf{T}\hat I_k^Y && \text{(6b)}\\
\Delta \text{ configuration:} \quad \hat V_k^{\Delta} &= \Gamma V_k && \text{(6c)}\\
I_k &= \Gamma^\mathsf{T}\hat I_k^{\Delta} && \text{(6d)}
\end{aligned}$$

3) Internal model: The voltage and current gains across the ideal transformer define an *internal model* whcih couples the internal variables in the primary and secondary circuits and connects the relations (5) and (6). These gains are determined by the turns ratios of the constituent single-phase ideal transformers according to (2a), but tailored for different configurations. Denote the voltage gain of the ideal three-phase transformer by a real diagonal matrix $n := \text{diag}\left(n^a, n^b, n^c\right) \in \mathbb{R}^{3\times 3}$ and its turns ratio by $a := n^{-1} \in \mathbb{R}^{3\times 3}$. Then

$$\begin{aligned}
YY: \quad \hat V_k^Y &= n\,\hat V_j^Y, & -\hat I_k^Y &= a\,\hat I_j^Y && \text{(7a)}\\
\Delta\Delta: \quad \hat V_k^{\Delta} &= n\,\hat V_j^{\Delta}, & -\hat I_k^{\Delta} &= a\,\hat I_j^{\Delta} && \text{(7b)}\\
\Delta Y: \quad \hat V_k^Y &= n\,\hat V_j^{\Delta}, & -\hat I_k^Y &= a\,\hat I_j^{\Delta} && \text{(7c)}\\
Y\Delta: \quad \hat V_k^{\Delta} &= n\,\hat V_j^Y, & -\hat I_k^{\Delta} &= a\,\hat I_j^Y && \text{(7d)}
\end{aligned}$$

These are internal models of a three-phase (ideal) transformer.

4) External model: The external model of a three-phase transformer relates the terminal variables (V_j, V_j^n, I_j) and (V_k, V_k^n, I_k) on both sides of the transformer in terms of the leakage admittance y^s, the shunt admittance y^m, and the turns ratio a. It can be derived by eliminating the internal variables $\left(\hat V_j^{Y/\Delta}, \hat I_j^{Y/\Delta}\right)$ and $\left(\hat V_k^{Y/\Delta}, \hat I_k^{Y/\Delta}\right)$ from the conversion rules (5) (6) and the internal model (7).

Let $V := (V_j, V_k) \in \mathbb{C}^6$ and $I := (I_j, I_k) \in \mathbb{C}^6$ denote the terminal voltages and currents of a three-phase transformer.

Define a 6×6 admittance matrix Y_{YY} and a column vector $\gamma \in \mathbb{C}^6$:

$$Y_{YY} \;:=\; \begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2(y^l + y^m) \end{bmatrix} \tag{8a}$$

$$\gamma \;:=\; \left(V_j^n \mathbf{1}, V_k^n \mathbf{1}\right) \tag{8b}$$

where $\mathbf{1}$ is the 3-dimensional vectors of all 1s. As we will see below Y_{YY} is the admittance matrix of a transformer in YY configuration. It is the same as that in (3a) for a single-phase transformer, except that a, y^l, y^m are now 3×3 diagonal matrices rather than scalars. The vector γ is the neutral voltages of a transformer in YY configuration. We present a unified representation of standard transformer models as a theorem.

Theorem 1. The external models of three-phase transformers in YY, $\Delta\Delta$, ΔY and $Y\Delta$ configurations take the form

$$I \;=\; D^{\mathsf{T}} Y_{YY} D \left(V - \gamma\right) \tag{8c}$$

where

$$YY: \qquad D \;:=\; \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix} \tag{8d}$$

$$\Delta\Delta: \qquad D \;:=\; \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \tag{8e}$$

$$\Delta Y: \qquad D \;:=\; \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix} \tag{8f}$$

$$Y\Delta: \qquad D \;:=\; \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix} \tag{8g}$$

where $\mathbb{I}$ is the identity matrix of size 3. □

For $\Delta\Delta$ configuration, $D\gamma = 0 \in \mathbb{C}^6$ in (8c), reflecting that a Δ configuration contains no neutral voltage; similarly for ΔY and $Y\Delta$ configurations. The result in Theorem 2 agrees with those in [10]. The 6×6 block diagonal matrix D is called a connection matrix in the literature, e.g. [13, 18].

Remark 2.
1) Neither the voltage gains $\left(n^a, n^b, n^c\right)$ nor the admittances $\left(y^{sa}, y^{sb}, y^{sc}\right)$ and $\left(y^{ma}, y^{mb}, y^{mc}\right)$ may be equal across phases a, b, c.
2) The admittance matrices of $\Delta\Delta$, ΔY, $Y\Delta$ configurations can be obtained by pre-multiplying the admittance matrix Y_{YY} by Γ^{T} and post-multiplying it by Γ for a (primary or secondary) circuit that is in Δ configuration and setting its neutral voltage to zero.
3) When neutrals are solidly earthed for YY configuration so that $V_j^n = V_k^n = 0$, three-phase transformers in YY and $\Delta\Delta$ configurations have a three-phase Π circuit representation because $D^{\mathsf{T}} Y_{YY} D$ is block-symmetric. Transformers in ΔY and $Y\Delta$ configurations do not.
4) The derivation method is modular and general. For instance, the neutrals of Y configurations may or may not be connected to the other side, may or may not be grounded, with zero or nonzero grounding impedances. It can be extended to non-standard transformers such as open transformers. For example an open $\Delta\Delta$ transformer with an open ca leg has the same model (8c) (8e) as a (closed) $\Delta\Delta$ transformer, except that the diagonal admittance matrices y^l, y^m in Y_{YY} in (8b) are modified to $\tilde{y}^l := \text{diag}\left(y^{la}, y^{lb}, 0\right)$ and $\tilde{y}^m := \text{diag}\left(y^{ma}, y^{mb}, 0\right)$ respectively, i.e., with zero admittances for the missing third leg; see [5] for details.

IV. Three-phase transformers: unitary voltage network

In this section we extend the single-phase model in Section II-C with unitary voltage network to three-phase transformers. Multiple copies of the single-phase circuit in Figure 3(b) can be connected in Δ or Y configuration on each side of the unitary voltage network, per phase, to create three-phase transformers. The derivation of their external models follows a similar method as that in Section III: (i) define internal variables for the unitary voltage network in each phase; (ii) derive the internal model that relate these internal variables; (iii) the transformer gains across the two ideal transformers define the conversion between the internal and terminal variables; and finally (iv) eliminate the internal variables to arrive at the external model.

A. Internal model: unitary voltage network per phase

The internal variables on the unitary voltage network in *each phase* $\phi \in \{a, b, c\}$ are defined in Figure 4. Note that the voltages $(\hat{V}_0^\phi, \hat{V}_j^\phi, \hat{V}_k^\phi)$ are defined to be the voltage drops, whether the unitary voltage network is grounded or not. Define the internal variables:

$$\hat{I}_i \;:=\; \left(\hat{I}_i^a, \hat{I}_i^b, \hat{I}_i^c\right), \quad \hat{V}_i \;:=\; \left(\hat{V}_i^a, \hat{V}_i^b, \hat{V}_i^c\right), \quad i = 0, j, k$$

and admittance matrices:

$$y_i \;:=\; \text{diag}\left(y_i^a, y_i^b, y_i^c\right), \qquad i = 0, j, k$$

Then the internal currents $\hat{I}_i$ are related to the internal voltages $\hat{V}_i$ through a 9×9 admittance matrix:

$$\begin{bmatrix} \hat{I}_0 \\ \hat{I}_j \\ \hat{I}_k \end{bmatrix} \;=\; \begin{bmatrix} \sum_i y_i & -y_j & -y_k \\ -y_j & y_j & 0 \\ -y_k & 0 & y_k \end{bmatrix} \begin{bmatrix} \hat{V}_0 \\ \hat{V}_j \\ \hat{V}_k \end{bmatrix}$$

where $\sum_i y_i = y_0 + y_j + y_k$ is a diagonal matrix of all admittances. Since $\hat{I}_0 = 0 \in \mathbb{C}^3$ we can eliminate $\hat{V}_0$ and derive the 6×6 Kron-reduced admittance matrix Y_{uvn} that maps $\hat{V} := (\hat{V}_j, \hat{V}_k) \in \mathbb{C}^6$ to $\hat{I} := (\hat{I}_j, \hat{I}_k) \in \mathbb{C}^6$:

$$\hat{I} \;=\; Y_{\text{uvn}} \hat{V} \tag{9a}$$

$$Y_{\text{uvn}} \;:=\; \left(\mathbb{I}_2 \otimes \left(\sum_i y_i\right)^{-1}\right) \begin{bmatrix} y_j(y_0 + y_k) & -y_j y_k \\ -y_j y_k & y_k(y_0 + y_j) \end{bmatrix} \tag{9b}$$

where $\mathbb{I}_2$ is the identity matrix of size 2 and $\otimes$ denotes the Kronecker product. This defines the *internal model* that relates $\hat{I}$ and $\hat{V}$. Note that the phases of these internal variables are decoupled in (9) since the admittance matrices $y_i \in \mathbb{C}^{3\times 3}$ are diagonal. The phases will be coupled in the external model that relates the terminal variables (V_j, V_k) and (I_j, I_k) through Y or Δ configuration, as we now explain.

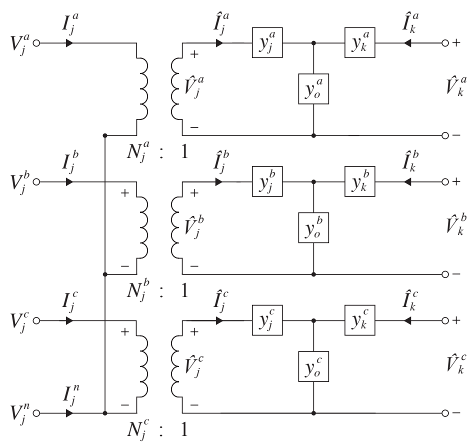
B. Conversion rules

Let the terminal currents and terminal voltages of the three-phase transformer be respectively

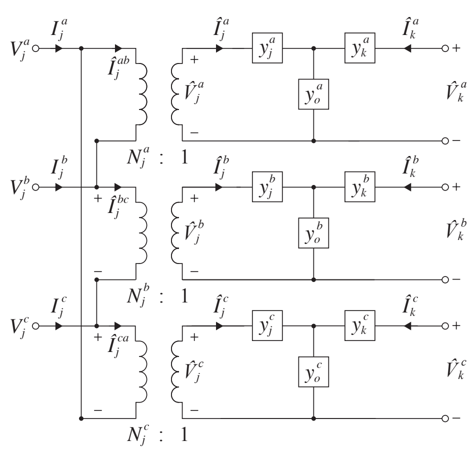
$$I_i \;:=\; (I_i^a, I_i^b, I_i^c), \quad V_i \;:=\; (V_i^a, V_i^b, V_i^c), \quad i = j, k$$

and the terminal neutral voltages of Y configurations be V_i^n, $i = j, k$. The primary side is illustrated in Figure 7. These voltages are defined respect to an arbitrary and common reference point, e.g., the ground. Let $M_j := \operatorname{diag}\left(1/N_j^a, 1/N_j^b, 1/N_j^c\right)$ and $M_k := \operatorname{diag}\left(1/N_k^a, 1/N_k^b, 1/N_k^c\right)$ be the transformer gain matrices of the ideal transformers on each side of the unitary voltage network.

To derive the conversion rule between internal and terminal variables, consider first the primary side where three single-phase ideal transformers are connected to the left end of the unitary voltage network. Figure 7(a) shows the primary side in Y configuration. The *conversion rule* between



(a) Y configuration



(b) Δ configuration

Fig. 7: Primary side of a three-phase transformer with unitary voltage networks.

the internal variables $(\hat{V}_j, \hat{I}_j)$ and the terminal variables (V_j, I_j, V_j^n) is:

$$Y \text{ config.:} \qquad \hat{V}_j \;=\; M_j\left(V_j - V_j^n \mathbf{1}\right) \tag{10a}$$

$$\hat{I}_j \;=\; M_j^{-1} I_j \tag{10b}$$

where $\mathbf{1} := (1, 1, 1)$. Figure 7(b) shows the primary side in Δ configuration. Let $\hat{I}_j^\Delta := (\hat{I}_j^{ab}, \hat{I}_j^{bc}, \hat{I}_j^{ca})$ denote the internal currents entering the primary side of the ideal transformer as indicated in Figure 7(b). The internal variables $(\hat{V}_j, \hat{I}_j, \hat{I}_j^\Delta)$ are related to the terminal variables (V_j, I_j) according to the *conversion rule*:

$$\Delta \text{ config.:} \qquad \hat{V}_j \;=\; M_j \Gamma V_j \tag{10c}$$

$$\hat{I}_j \;=\; M_j^{-1} \hat{I}_j^\Delta \tag{10d}$$

$$I_j \;=\; \Gamma^\mathsf{T} \hat{I}_j^\Delta \tag{10e}$$

where $\Gamma, \Gamma^\mathsf{T}$ are conversion matrices. Similarly on the secondary side we have the conversion rule (see Figure 8):

$$Y \text{ config.:} \qquad \hat{V}_k \;=\; M_k\left(V_k - V_k^n \mathbf{1}\right) \tag{10f}$$

$$\hat{I}_k \;=\; M_k^{-1} I_k \tag{10g}$$

$$\Delta \text{ config.:} \qquad \hat{V}_k \;=\; M_k \Gamma V_k \tag{10h}$$

$$\hat{I}_k \;=\; M_k^{-1} \hat{I}_k^\Delta \tag{10i}$$

$$I_k \;=\; \Gamma^\mathsf{T} \hat{I}_k^\Delta \tag{10j}$$

C. External model

Let $V := (V_j, V_k) \in \mathbb{C}^6$ and $I := (I_j, I_k) \in \mathbb{C}^6$ denote the vectors of terminal voltages and currents respectively. Let $M := \operatorname{diag}(M_j, M_k) \in \mathbb{C}^{6\times 6}$ be the transformer gain matrices. Eliminating the internal variables $(\hat{V}, \hat{I})$ from the internal model (9) and the conversion rules (10) yields the following *external model* of three-phase transformers:

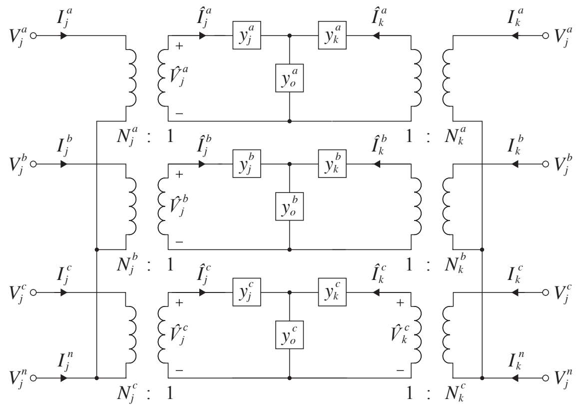
$$I \;=\; D^\mathsf{T}(M Y_{\text{uvn}} M) D\,(V - \gamma)$$

where Y_{uvn} is defined in (9), and $D \in \mathbb{C}^{6\times 6}$ and $\gamma \in \mathbb{C}^6$ in (8).

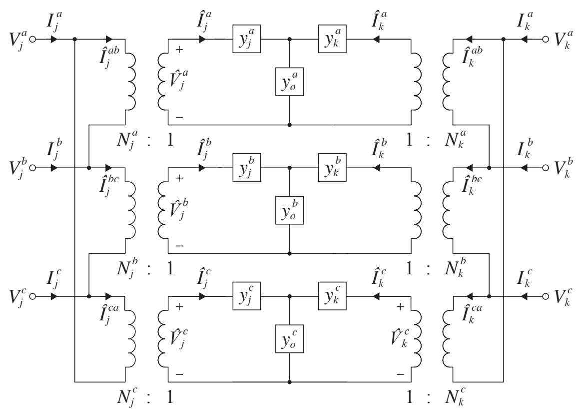
We often do not know the numbers N_j^ϕ, N_k^ϕ of turns of the primary and secondary windings respectively and hence cannot determine the matrices M_j, M_k, but we can always determine the turns ratio matrix $a := M_j^{-1} M_k$ from the specified rated voltages. Then the matrix $Y_{YY} := M Y_{\text{uvn}} M$ in (12) can also be written in terms of the 3×3 turns ratio and admittance matrices a, y^p, y^s, y^m (see [5] for details):

$$Y_{YY} \;=\; y^p y^s \left(a^2 y^m + a^2 y^p + y^s\right)^{-1} \begin{bmatrix} \mathbb{I} + a^2 y^m (y^s)^{-1} & -a \\ -a & a^2\left(\mathbb{I} + y^m (y^p)^{-1}\right) \end{bmatrix} \tag{11}$$

where $y^p := \operatorname{diag}\left(1/z^{p\phi}\right)$, $y^s := \operatorname{diag}\left(1/z^{s\phi}\right)$, $y^m := \operatorname{diag}\left(y^{m\phi}\right)$. Here $(z^{p\phi}, z^{s\phi})$ are the series impedances per phase ϕ in the primary and secondary circuits respectively and $y^{m\phi}$ are the shunt admittances per phase ϕ; see Figure 1. We hence have the following.



(a) YY configuration



(b) $\Delta\Delta$ configuration

Fig. 8: Three-phase transformer models.

Theorem 2. The external models of three-phase transformers in YY, $\Delta\Delta$, ΔY and $Y\Delta$ configurations are

$$I \;=\; D^{\mathsf{T}} Y_{YY} D \,(V - \gamma) \tag{12}$$

where Y_{YY} is defined in (11), and $D \in \mathbb{C}^{6\times 6}$ and $\gamma \in \mathbb{C}^6$ are defined in (8). $\square$

Theorem 2 has been verified numerically to agree with the model in [20] for a ΔY_g transformer with zero shunt admittance.

Remark 3.
1) This model is generally different from the simplified model of Section III. From (12) and (8), these models however have the same structure. They differ only in the admittance matrix Y_{YY} for the YY configuration and the difference is due to different models for single-phase nonideal transformers.
2) When the shunt admittances are assumed zero in both models, i.e., $y_0^\phi = y^{m\phi} = 0$ for $\phi \in \{a, b, c\}$, these two models are equivalent, as in the single-phase case.

V. Conclusion

We have presented a simple and transparent method to derive the admittance matrices of three-phase banks of single-phase transformers in an unbalanced setting. The basic idea is to explicitly separate a transformer model into an *internal model* that specifies the relation between internal variables of each single-phase transformer, and a *conversion rule* that maps its internal variables to its terminal variables. Its admittance matrix is then derived by eliminating the internal variables from this set of equations. The method is general and provides rigor and clarity, facilitating extensions. It also reveals a transformer model that is unified and modular.

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