Gomory-Hu Trees over Wireless

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***Abstract*—The Gomory-Hu tree is a popular optimization algorithm that enables to efficiently find a min-cut (or equivalently, max-flow) for every pair of nodes in a graph. However, graphs cannot capture broadcasting and interference in wireless: over wireless networks we need to resort to the information-theoretical cut-set to bound the max-flow. Leveraging the submodularity of mutual information, we show that the Gomory-Hu algorithm can be used to efficiently find information-theoretic rate characterizations such as the capacity or an approximation to the capacity, over a number of network scenarios including wireless Gaussian networks and deterministic relay networks.**

***Index Terms*—Gomory-Hu trees, min-cut, wireless networks**

I. Introduction

THE Gomory-Hu tree is a popular optimization algorithm that enables to efficiently find the min-cut value (or equivalently, max-flow) for every pair of nodes in a graph [1]. In particular, for an undirected graph with N nodes, the Gomory-Hu algorithm creates a cut tree that captures a min-cut for each of the $N(N-1)/2$ pairs of nodes. The main idea is that, by leveraging submodularity, we can create a cut tree by solving only $N-1$ max-flow problems, thus saving a factor of $N/2$ over the brute-force approach. The algorithm has drawn significant attention since cut trees are widely used in applications such as routing, graph partitioning, graph clustering and scheduling [2]–[5].

Gomory-Hu algorithms have also emerged as an important tool over wireless communications, by using a crude modeling of wireless networks through graphs. Cut trees are used in wireless problems such as online virtual network embedding, resource allocation and scheduling [6]–[8]; in all these cases, wireless networks are abstracted as graphs. However, graph theoretic minimum cuts cannot capture wireless broadcasting and interference that can introduce complex signal interactions between wireless information flows.

In this work, we address the question: can we use the Gomory-Hu algorithm over networks that cannot be abstracted as graphs, such as Gaussian wireless networks with broadcasting and interference?

To do so, we resort to information-theoretical cut-set bounds that provide upper bounds on the maximum rate possible [9]. Leveraging the submodularity of mutual information, we show that the Gomory-Hu tree algorithm can be used to efficiently find information-theoretic rate characterizations such as the capacity or an approximation to the capacity, over a number of network scenarios, that include wireless Gaussian networks and deterministic relay networks.

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Related Work. The Gomory-Hu algorithm was proposed in [1] and was followed by a rich and extensive literature [10]–[12], with research improving aspects such as implementation complexity [13]–[15], and tailoring to applications [2]–[5]. All these works consider networks abstracted through graphs.

The cut-set bound and capacity approximations for wireless networks have also attracted significant research effort. [16] proposed a deterministic model to capture the key wireless signal interactions and [17] provided polynomial time algorithms to calculate the deterministic capacity. [18] proposed a strategy that achieves rate within a constant gap of the cut-set bound in Gaussian relay networks, where the gap is independent of channel gains and power levels, and depends on only the number of nodes in the network. [19] showed that the cut-set bound for any independent input distribution can be expressed as a submodular optimization problem. All these works aim to find the capacity (or the approximate capacity) for a *single* source-destination pair. In this paper, we build on these works and the Gomory-Hu algorithm to efficiently calculate these quantities for all $\binom{N}{2}$ source-destination pairs.
Paper Organization. Section II reviews background on the cut-set bound and presents the modified Gomory-Hu algorithm. Section III introduces our main theorem and Section IV presents its proof. Secion V concludes the paper.

II. System Model and Background

We consider a network with a set of nodes denoted by V. For $i \in \{1, \dots, |V|\}$, the random variables X_i and Y_i denote the channel input and output at node v_i, respectively where $|V|$ is the cardinality of the set V. A cut Ω is a subset of V, that partitions V to two sets, Ω and its complement Ω^c; X_Ω denotes the vector $\left[X_{i_1}, \dots, X_{i_{|\Omega|}}\right]^T$ with $v_{i_j} \in \Omega$, $j \in \{1, \dots, |\Omega|\}$. A similar definition follows for Y_Ω.
Information theoretic cut-set bound. This is an upper bound on the maximum rate (capacity) C that can be sent from a source s to a destination t over a relay network [9]:

$$C \leq \bar{C} = \max_{p\left(X_{[1:|V|]}\right)} \min_{\Omega} I\left(X_\Omega; Y_{\Omega^c} | X_{\Omega^c}\right), \tag{1}$$

where $s \in \Omega$, $t \in \Omega^c$, $p\left(X_{[1:|V|]}\right)$ is the input probability distribution and $I\left(X_\Omega; Y_{\Omega^c} | X_{\Omega^c}\right)$ is the mutual information between X_Ω and Y_{Ω^c} conditioned on X_{Ω^c}. For a fixed input distribution, the bound in (1) can be written as

$$\min_{\Omega} I\left(X_\Omega; Y_{\Omega^c} | X_{\Omega^c}\right). \tag{2}$$

[19] showed that the mutual information in (2) is a submodular function (under some encoding independence assumptions) and if it can be evaluated efficiently, the submodular optimization problem in (2) can be solved in polynomial time.

Throughout the letter, we assume that any cut $\Omega \subset V$ satisfies

$$I\left(X_{\Omega}; Y_{\Omega^c} | X_{\Omega^c}\right) = I\left(X_{\Omega^c}; Y_{\Omega} | X_{\Omega}\right). \tag{3}$$

For brevity, we will use $\gamma(\Omega) = I(X_{\Omega}; Y_{\Omega^c} | X_{\Omega^c})$, and denote the min-cut value in (2) for the $s-t$ pair by

$$\lambda_{s,t} = \min_{\Omega: s \in \Omega, t \in \Omega^c} \gamma(\Omega). \tag{4}$$

A similar definition follows for $\lambda_{u,v}$ with any pair $u, v \in V$.

For general wireless relay networks, it is nontrivial to compute the cut-set bound in (1) even when the input distribution is fixed. However, the capacity $\boldsymbol{C}$ is well characterized for:
(a) Deterministic networks [16]:

$$\boldsymbol{C} = \min_{\Omega} \operatorname{rank}\left(K_{\Omega}\right), \tag{5}$$

where K_{Ω} is the transfer matrix that relates X_{Ω} to Y_{Ω^c}. For i.i.d uniform input distribution over a finite field with p elements, the information theoretic min-cut in (2) is equal to $\boldsymbol{C}$ in (5).
(b) Gaussian relay networks [18]: An approximate capacity characterization $\boldsymbol{C_A}$ was derived as

$$\boldsymbol{C_A} = \min_{\Omega} \log\det\left(I + PK_{\Omega}K_{\Omega}^{\dagger}\right), \quad \bar{C} \leq \boldsymbol{C_A} + O(|V|), \tag{6}$$

where $K_{\Omega}^{\dagger}$ is the conjugate transpose of K_{Ω}, and P is the average power constraint per node. The approximate capacity $\boldsymbol{C_A}$ is equal to the information theoretic min-cut in (2) for i.i.d Gaussian input distribution with mean 0 and variance P.

We note that in Gaussian networks, reciprocal channels (a standard assumption in wireless) are sufficient for condition (3) to be satisfied. Thus, for example, we assume that K is Hermitian, i.e., $K = K^{\dagger}$.

A. Gomory-Hu Algorithm

Consider an undirected edge-weighted graph $\mathcal{G}(V, E)$ with $N = |V|$ nodes and edges with capacities (or weights) $\epsilon : E \to \mathbb{R}^+$. To define a Gomory-Hu tree $\mathcal{T} = (V_{\mathcal{T}}, E_{\mathcal{T}})$ for $(\mathcal{G}, \epsilon)$, we first define fundamental cuts in trees.

Definition 1. *Let $\mathcal{T} = (V_{\mathcal{T}}, E_{\mathcal{T}})$ be a tree and $e \in E_{\mathcal{T}}$. Then, the set of nodes C_e in any of the two connected components in $(V_{\mathcal{T}}, E_{\mathcal{T}} - e)$ is called a fundamental cut for e.*

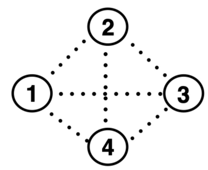
Definition 2. *A Gomory-Hu tree for $\mathcal{G}(V, E)$ is a tree $\mathcal{T} = (V_{\mathcal{T}}, E_{\mathcal{T}})$ with $V_{\mathcal{T}} = V$ such that every $uv \in E_{\mathcal{T}}$, the fundamental cut C_{uv} is a minimum $u-v$ cut in $\mathcal{G}$.*

Definition 2 states that the Gomory-Hu tree gives a min-cut for every node pair that constitutes an edge in the tree. However, [1] proves that the tree also gives a min-cut for any distinct $u, v \in V$. Particularly, if $p_{u,v}$ is the path between u and v on the tree and e is the edge with the minimum weight on $p_{u,v}$, then the fundamental cut C_e is a minimum $u-v$ cut.

In Algorithm 1, we present the modified Gomory-Hu algorithm that fits the wireless setup. The algorithm proceeds in steps, and each step establishes one edge of the tree between two nodes. A node in the tree can be one of the nodes in V or it can be contraction of some nodes in V, i.e., it may be a "contracted" node that represents a group of nodes in V. However, at the end of the algorithm, there are no contracted nodes in the final tree and each node belongs to V. We illustrate the steps of the algorithm with an example.

Algorithm 1 Gomory-Hu Algorithm (Wireless)

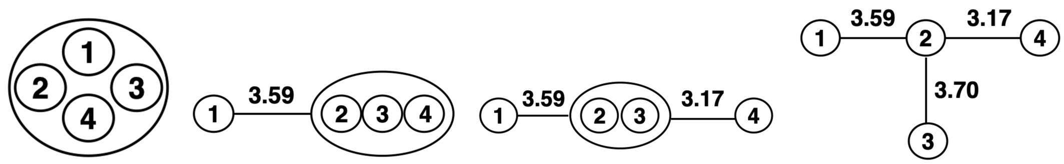
Input: A wireless network with a set of nodes V.
Output: A Gomory-Hu tree $\mathcal{T}$ for the input network.
$V_{\mathcal{T}}$ is a single node contracting all nodes in the networks, $E_{\mathcal{T}} \leftarrow \emptyset$.
while There is some $U \in V_{\mathcal{T}}$ such that $|U| \geq 2$ **do**
  Let s, t be any two distinct nodes in U.
  Let $\beta_1, \beta_2, ..., \beta_k$ be the connected components of $\mathcal{T} - U$.
  For $1 \leq i \leq k$, contract the nodes of V represented in nodes in β_i to a single node α_i, and obtain a new network $\mathcal{H}$ from the input network.
  Find an information theoretic minimum $s-t$ cut Ω in $\mathcal{H}$ by solving (2) and let $A = U \cap \Omega$ and $B = U - \Omega$.
  $V_{\mathcal{T}} \leftarrow (V_{\mathcal{T}} - U) \cup \{A, B\}$.
  for each edge $e = UW \in E_{\mathcal{T}}$ incident with U **do**
    Let i be such that $W \in \beta_i$.
    if $\alpha_i \in \Omega$ **then** $e' \leftarrow AW$, **else** $e' \leftarrow BW$.
    $E_{\mathcal{T}} \leftarrow (E_{\mathcal{T}} - \{e\}) \cup \{e'\}, \quad \epsilon(e') \leftarrow \epsilon(e)$.
  end for
  $E_{\mathcal{T}} \leftarrow E_{\mathcal{T}} \cup \{AB\}, \quad \epsilon(AB) \leftarrow \lambda_{s,t}$.
end while
Return $(V_{\mathcal{T}}; E_{\mathcal{T}})$.



$$K = \begin{bmatrix} 0 & 1-2j & 1 & 2-j \\ 1+2j & 0 & -3j & j \\ 1 & 3j & 0 & 1+j \\ 2+j & -j & 1-j & 0 \end{bmatrix}$$

Fig. 1: A Gaussian wireless network with channel matrix K.

Example 1. We apply Algorithm 1 to the Gaussian network in Fig. 1 for power $P = 1$. The initial tree is a single node contracting all nodes V in the network (Fig. 2a). Thus, **[Iteration 1]** The first iteration chooses $U = \{1, 2, 3, 4\}$ and we arbitrarily choose the nodes $s = 1$ and $t = 3$. In $\mathcal{T} - U$, there are no connected components, thus $\mathcal{H}$ is the same as our initial network. We find the minimum $s-t$ cut $\Omega = \{1\}$ in $\mathcal{H}$ by solving (6) and find $\boldsymbol{C_A} = 3.59$. Then, we construct $A = U \cap \Omega = \{1\}$ and $B = U - \Omega = \{2, 3, 4\}$, and modify $V_{\mathcal{T}}$ such that $V_{\mathcal{T}} \leftarrow (V_{\mathcal{T}} - U) \cup \{A, B\}$. We add a new edge AB with weight 3.59 as in Fig. 2b. **[Iteration 2]** In the next step, the algorithm chooses $U = \{2, 3, 4\}$ from $V_{\mathcal{T}}$, and choose $s = 2$ and $t = 4$. In $\mathcal{T} - U$, we have one connected component $\beta_1 = \{1\}$, thus $\mathcal{H}$ is the same as our initial network. We find the minimum $s-t$ cut $\Omega = \{1, 2, 3\}$ with $\boldsymbol{C_A} = 3.17$. Thus, $A = \{2, 3\}$ and $B = \{4\}$. We modify $V_{\mathcal{T}}$, and add an edge AB with weight 3.17. Since $\alpha_1 \in \Omega$, we place the edge with weight 3.59 (from the previous step) between A and $\{1\}$. The resulting tree is shown in Fig. 2c. **[Iteration 3]** We next choose $U = \{2, 3\}$, and we choose $s = 2$ and $t = 3$. In $\mathcal{T} - U$, we have two connected components: $\beta_1 = \{1\}$ and $\beta_2 = \{4\}$, thus $\mathcal{H}$ is the same as our initial network. We solve (6) to find the minimum $s-t$ cut $\Omega = \{1, 2, 4\}$ with $\boldsymbol{C_A} = 3.70$. Thus, $A = \{2\}$ and $B = \{3\}$. We modify $V_{\mathcal{T}}$ and add an edge AB.



(a) Initial tree $\mathcal{T}$. (b) $\mathcal{T}$ after the first iteration. (c) $\mathcal{T}$ after the second iteration. (d) The final cut tree $\mathcal{T}$.

Fig. 2: The Gomory-Hu tree construction steps.

Since the edge between $\{1\}$ and $\{2,\ 3\}$ in Fig. 2c is incident to U and $\alpha_1 \in \Omega$, we place this edge between $\{1\}$ and A. In the same manner, the edge between $\{4\}$ and $\{2,\ 3\}$ in Fig. 2c is incident to U and $\alpha_2 \in \Omega$, thus we place this edge between $\{4\}$ and A. The resulting tree shown in Fig. 2d is a cut tree.

III. Main Results

Theorem 1 states that a Gomory-Hu tree $\mathcal{T}$ gives an information theoretic min-cut for every node pair that constitutes an edge in the tree. By leveraging Theorem 1 and Lemma 1, Lemma 2 proves that $\mathcal{T}$ gives an information theoretic min-cut for every pair of nodes in the network (not only the ones connected through edges). We prove Theorem 1 in Section IV.

Theorem 1. *The Gomory-Hu tree algorithm can construct a tree $\mathcal{T} = (V_{\mathcal{T}}, E_{\mathcal{T}})$ for a wireless network such that for every $uv \in E_{\mathcal{T}}$, the fundamental cut C_{uv} is the information theoretic minimum $u-v$ cut in (2), i.e., $\gamma(C_{uv}) = \lambda_{u,v}$.*

Remark 1. *Theorem 1 presents our results for the information theoretic min-cut given in (2). The results still hold for the capacity characterization in (5), and the approximate capacity characterization in (6) since these are computed for specific input distributions in (2) as discussed in Section II.*

Lemma 1. *Let $v_1, \dots, v_k$ be any set of $k \geq 2$ distinct nodes. Then, we have the following relationship*

$$\lambda_{v_1,v_k} \geq \min_{1 \leq i \leq k-1} \lambda_{v_i,v_{i+1}}. \tag{7}$$

Proof. The proof of Lemma 1 is delegated to Appendix A. □

Lemma 2. *For any distinct $u, v \in V$, let $p_{u,v}$ be the unique path between u and v on a Gomory-Hu tree $\mathcal{T}$. Let edge $ab \in p_{u,v}$ give $\min_{ij \in p_{u,v}} \lambda_{i,j}$. Then, we have $\lambda_{u,v} = \lambda_{a,b}$ and C_{ab} gives an information theoretic minimum $u-v$ cut.*

Proof. The proof of Lemma 2 is delegated to Appendix B. □

Together, these results imply that we can use Algorithm 1 to find an information theoretic min-cut for every pair of nodes in a wireless network by solving only $|V|-1$ min-cut problems.

IV. Proof of Theorem 1

We prove Theorem 1 by following similar steps to the proof of the original Gomory-Hu algorithm [20]. The main difference is that we use the submodularity of mutual information in the following lemma that plays a central role in the proof.

Lemma 3. *Let $s, t \in V$ be distinct nodes and set Ω_1 be an information theoretic minimum $s-t$ cut. Let u, v be distinct nodes such that $u, v \notin \Omega_1$ (we allow that either u or v is either s or t). Then, there exists an information theoretic minimum $u-v$ cut Ω_2 such that $\Omega_1 \subseteq \Omega_2$ or $\Omega_1 \cap \Omega_2 = \emptyset$.*

Proof. The proof of Lemma 3 is delegated to Appendix C. □

The proof of Theorem 1 uses the loop invariant in Lemma 4.

Lemma 4. *Initially and after each iteration, for any edge $WZ \in E_{\mathcal{T}}$ (W, Z are potentially contracted nodes), there is some $s \in W$, $t \in Z$ such that the set of nodes of V represented in nodes in the fundamental cut C_{WZ} is an information theoretic minimum $s-t$ cut (i.e., $\epsilon(WZ) = \lambda_{s,t}$).*

Proof. In the beginning of the algorithm, the tree is a single node without any edges, thus the statement is trivial. We show that it is not violated in other iterations. We look at a specific iteration i of Algorithm 1 for fixed U, s, t, Ω, A and B. We assume that Lemma 4 holds before starting the iteration i, and $s \in \Omega$ (otherwise, rename s and t). Thus, $s \in A = U \cap \Omega$. We first show that Lemma 4 holds for the new edge AB.

Claim 1. *If we expand the contracted nodes that reside in Ω, we obtain an information theoretic minimum $s-t$ cut.*

Since $s \in A$, $t \in B$ and the set mentioned in Claim 1 is the set of nodes of V represented in nodes in C_{AB} (at the end of iteration i), the statement in Lemma 4 holds for the edge AB.

Proof of Claim 1. For $0 \leq j \leq k$, let $\mathcal{H}_j$ be the network arise from the original network by constructing each of $\alpha_1, \alpha_2, \dots, \alpha_k$ one by one. $\mathcal{H}_k$ corresponds to $\mathcal{H}$ in the algorithm and $\mathcal{H}_0$ is the original network. The proof by induction uses Lemma 3 and the fact that the nodes of V represented in each α_j is an information theoretic min-cut for some $s_j - t_j$ pair (by loop invariant). Thus, $\mathcal{H}$ contains an $s-t$ cut with capacity $\lambda_{s,t}$, and expanding the contracted nodes in Ω gives an information theoretic minimum $s-t$ cut. □

In Algorithm 1, we also replace the edges of the form UW with AW or BW. We assume UW is replaced by AW (the other case can be proven in the same manner). Due to Lemma 4, there is some $p \in U$ and $q \in W$ such that the set of nodes of V represented in nodes in C_{UW} (before the modification of $\mathcal{T}$) is a minimum $p-q$ cut. This set is also the set of nodes of V represented in nodes in C_{AW} (after the modification). Thus, if $p \in A$, Lemma 4 holds for edge AW. Hence, suppose $p \in B$.

Claim 2. $\lambda_{s,q} = \lambda_{p,q}$.

This completes the analysis because $s \in A, q \in W$ and minimum cut value given by the set of nodes of V represented in nodes in C_{AW} is equal to $\lambda_{p,q} = \lambda_{s,q}$. Hence, the set gives an information theoretic minimum $s-q$ cut.

Proof of Claim 2. Let $\overline{\Omega} \subseteq V$ be the set obtained from Ω by expanding the contracted nodes in Ω. Claim 1 states that $\bar{\Omega}$ is an information theoretic minimum $s-t$ cut. Moreover, since $s,\ q \in \bar{\Omega}$, there is an information theoretic minimum $s-q$ cut which contains $V-\bar{\Omega}$ or it is disjoint from $V-\bar{\Omega}$ due to Lemma 3. If the min-cut contains $V-\bar{\Omega}$, it is also a cut for $p-q$ pair since $t,\ p \in B \subseteq V-\bar{\Omega}$. Thus, we have $\lambda_{s,q} \geq \lambda_{p,q}$. If the min-cut is disjoint from $V-\bar{\Omega}$, it is a cut for $s-t$ pair and we have $\lambda_{s,q} \geq \lambda_{s,t}$. Since the minimum $s-t$ cut $\bar{\Omega}$ separates p and q, it is also a cut for $p-q$ pair and we have $\lambda_{s,t} \geq \lambda_{p,q}$. If we combine these two results: $\lambda_{s,q} \geq \lambda_{p,q}$. Thus, in both cases, we have $\lambda_{s,q} \geq \lambda_{p,q}$. Moreover, $\epsilon(UW)$ is the value of an $s-q$ cut, thus we have $\lambda_{s,q} \leq \epsilon(UW) = \lambda_{p,q}$. If we combine the two inequalities, we have $\lambda_{s,q} = \lambda_{p,q}$. □

The remaining edges and their fundamental cuts do not change, thus Lemma 4 holds for them. □

Lemma 4 holds at the end of the last iteration and there are no contracted nodes in the final tree, thus for every edge $uv \in E_{\mathcal{T}}$, C_{uv} is an information theoretic minimum $u-v$ cut.

We can illustrate Lemma 4 in *Example 1*. The edge between $W = \{1\}$ and $Z = \{2,\ 3,\ 4\}$ in Fig. 2b has a fundamental cut $C_{WZ} = \{1\}$ which is the information theoretic min-cut for $s = 1 \in W$ and $t = 3 \in Z$. In the second step, Lemma 4 holds for the edge between $W_1 = \{1\}$, $Z_1 = \{2,\ 3\}$ due to the previous step. For the edge between $W_2 = \{2,\ 3\}$ and $Z_2 = \{4\}$, the nodes of V represented in $C_{W_2Z_2}$ is $\{1,\ 2,\ 3\}$ which is the min-cut for pair $s = 2 \in W_2,\ t = 4 \in Z_2$. It's not difficult to see Lemma 4 holds at the last iteration as well.

V. Conclusion

In this letter, we showed that the Gomory-Hu algorithm that efficiently finds a min-cut for every pair of nodes in a graph can be extended to wireless networks that cannot be abstracted as graphs. In particular, we showed that the Gomory-Hu algorithm can find the information-theoretic rate characterizations such as the capacity or an approximation to the capacity in a number of wireless network classes.

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Appendix A
Proof of Lemma 1

The proof of Lemma 1 is similar to in [21]. Let Ω be an information theoretic minimum $v_1 - v_k$ cut and without loss of generality $v_1 \in \Omega$ and $v_k \notin \Omega$. There is some $1 \leq j \leq k-1$ such that $v_j \in \Omega$ and $v_{j+1} \notin \Omega$. Thus, the set Ω is also a $v_j - v_{j+1}$ cut and $\lambda_{v_j,v_{j+1}} \leq \lambda_{v_1,v_k}$. Since $\min_{1\leq i\leq k-1} \lambda_{v_i,v_{i+1}} \leq \lambda_{v_j,v_{j+1}}$, this concludes the proof of Lemma 1.

Appendix B
Proof of Lemma 2

For the sequence of nodes on $p_{u,v}$, due to Lemma 1, we have $\lambda_{u,v} \geq \lambda_{a,b}$. Moreover, due to Theorem 1, the capacity of C_{ab} is equal to $\lambda_{a,b}$ and C_{ab} is also a $u-v$ cut. Thus, we have $\lambda_{u,v} \leq \lambda_{a,b}$. When we combine these two inequalities, we obtain $\lambda_{u,v} = \lambda_{a,b}$. Hence, C_{ab} is an information theoretic minimum $u-v$ cut. This concludes the proof of Lemma 2.

Appendix C
Proof of Lemma 3

Let Ω_2 be an information theoretic minimum $u-v$ cut. We assume $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_1 \cap \Omega_2 \neq \emptyset$ (otherwise, Lemma 3 holds). Without loss of generality, we suppose $s \in \Omega_1$ (rename it if necessary) and $s \in \Omega_2$ (otherwise, replace Ω_2 with $V-\Omega_2$). Again by renaming u and v if necessary, we assume $u \in \Omega_2$. By the submodularity of mutual information [19], we have $\gamma(\Omega_1)+\gamma(\Omega_2) \geq \gamma(\Omega_1\cap\Omega_2)+\gamma(\Omega_1\cup\Omega_2)$. Since $s \in \Omega_1\cap\Omega_2$ and $t \notin \Omega_1\cap\Omega_2$, $\Omega_1\cap\Omega_2$ is an $s-t$ cut. Since Ω_1 is a minimum $s-t$ cut, we have $\gamma(\Omega_1) \leq \gamma(\Omega_1\cap\Omega_2)$. Due to this result and the submodularity, we have $\gamma(\Omega_1\cup\Omega_2) \leq \gamma(\Omega_2)$. Moreover, $u \in \Omega_1\cup\Omega_2$ and $v \notin \Omega_1\cup\Omega_2$, thus $\Omega_1\cup\Omega_2$ is a $u-v$ cut. Since $\gamma(\Omega_1\cup\Omega_2) \leq \gamma(\Omega_2)$, $\Omega_1\cup\Omega_2$ is an information theoretic minimum $u-v$ cut that contains Ω_1. This concludes the proof.