FINDING DISTRIBUTION FUNCTION DEFINED BY INTEGRAL EQUATIONS USING FINITE APPROXIMATION *

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Abstract. This paper develops a finite approximation approach to find the solution $F(x)$ of integral equations in the form $F(x) = \xi(x) + \int_{\mathbb{R}} \kappa(x,u)dF(u), \forall x \in \mathbb{R}$, where $F(x)$ may not be continuous, and therefore not have a density function. The integral equations in this form, to the best of our knowledge, have never been studied before. However, such equations arise frequently when modeling stochastic systems. We construct a Banach space of (right-continuous) distribution functions and reformulate the problem into an operator equation. We provide general necessary and sufficient conditions that allow us to show convergence of the approximation approach developed in this paper. We then provide two specific choices of approximation sequences and show that the properties of these sequences are sufficient to generate approximate equation solutions that converge to the true solution assuming solution uniqueness and some additional mild regularity conditions. Our analysis is performed under the supremum norm, allowing wider applicability of our results. Worst-case error bounds are also available from solving a linear program. We demonstrate the viability and computational performance of our approach by constructing two examples. The solution of the first example could be constructed manually, but demonstrates the correctness and convergence properties of our approach. The second example solves a problem involving the Weierstrass function for which no closed-form solution is available.

Key words. Integral Equation, Finite Approximation, Fredholm Equations, Operator Theory, Collective Compactness, Markov Chain, Stationary Distribution.

AMS subject classifications. 45B05, 60H25, 65C30, 65R20

1. Introduction.

1.1. Overview of the problem.

This paper aims to solve a general class of integral equations of the form

$$\text{(IE)} \qquad F(x) = \xi(x) + \int_{\mathbb{R}} \kappa(x,u)dF(u), \quad \forall x \in \mathbb{R}$$

to a desired accuracy. Here $\xi : \mathbb{R} \mapsto \mathbb{R}$ is a given distribution, $\kappa : \mathbb{R}^2 \mapsto \mathbb{R}_+$ is a given kernel, dF is the measure associated with F and $F : \mathbb{R} \mapsto \mathbb{R}$ is to be determined. We analyze (IE) by letting $\mathbb{R}$ as the support. A similar analysis can be developed for support $\Omega \subseteq \mathbb{R}$. Our study of (IE) is motivated by the problem of finding the stationary distribution and associated steady-state performance evaluation measures of stochastic models that can be represented by continuous or mixed state Markov chains. The current paper develops the mathematical foundations and a finite approximation approach for solving such equations. Its application in the context of Markov chains is discussed in [27].

We may regard (IE) as a generalization of Fredholm equations of the second type:

$$\text{(FIE)} \qquad f(x) = g(x) + \int_{\Omega} t(x,u)f(u)du, \qquad x \in \mathbb{R}.$$

The functions in (FIE) are usually assumed to be continuous and thereby integratable *w.r.t.* x [17]. Suppose Ω is a closed interval. Then taking integration *w.r.t.* x on both

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sides of (FIE) yields (IE). However, the opposite direction may not hold as functions in (IE) are not necessarily continuously differentiable; see examples in section 6.

A finite approximation approach. Since exact solutions of (IE) are typically unavailable and known integral equation solution methods are not applicable (see subsection 1.2 for a review), we develop a method which approximates (IE) with a sequence of "discretized" equations of the form

$$\text{(A-IE)} \qquad F(x) = \xi^{(r)}(x) + \sum_{i=1}^{J^{(r)}} \omega_i^{(r)}(x)[F(c_i^{(r)}) - F(c_{i-1}^{(r)})], \quad \forall x \in \mathbb{R}, r \in \mathbb{N},$$

where (r) indexes the approximation sequence and for all $r \in \mathbb{N}$, $-\infty = c_0^{(r)} < c_1^{(r)} < ... < c_{J^{(r)}}^{(r)} < c_{J^{(r)}+1}^{(r)} = +\infty$, $F(c_0^{(r)}) \equiv 0$ and $\xi^{(r)}, \omega_1^{(r)}, \omega_2^{(r)}, ..., \omega_{J^{(r)}}^{(r)} : \mathbb{R} \mapsto \mathbb{R}$ are to be specified depending on an approximation strategy. For any fixed $r \in \mathbb{N}$ and known $\xi^{(r)}, \omega_1^{(r)}, \omega_2^{(r)}, ..., \omega_{J^{(r)}}^{(r)}$, we obtain the solution of (A-IE) by letting $x = c_1^{(r)}, c_2^{(r)}, ..., c_{J^{(r)}}^{(r)}$ into (A-IE) and solving the system of $J^{(r)} \times J^{(r)}$ linear equations to obtain $F(c_1^{(r)}), F(c_2^{(r)}), ..., F(c_{J^{(r)}}^{(r)})$:

$$\sum_{i=1}^{J^{(r)}} [\delta_{ij} - \omega_i^{(r)}(c_j^{(r)}) + \omega_{i+1}^{(r)}(c_j^{(r)})] F(c_i^{(r)}) = \xi(c_j^{(r)}), \quad j = 1, 2, ..., J^{(r)},$$

where $\omega_{J^{(r)}+1}^{(r)} \equiv 0$ and $\delta_{ij} = \mathbf{1}\{i = j\}$. Then the value of $F(x)$ at any arbitrary point x can be obtained by using (A-IE). An illustration of finite approximation can be found in Figure 1 of section 4.

1.2. Related literature. Solution methods for Fredholm integral equations of the second type (FIE) are discussed in [17]. Table 1 provides the known methods with a summary description. To the best of our knowledge, solution methods for problems in the form of (IE) are not known. However, the methods for (IE) presented here are motivated from the approximation methods for (FIE). Depending on the specification of $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$, Type II approach in section 4 uses the concept behind quadrature methods [32, 16, 1], and Type I uses iterated approximation concept [11, 19, 14]. A major challenge in using these concepts is that Fredholm/Volterra integral equations are discussed in a space of continuous functions while (IE) is in a space of right-continuous distribution functions with possible jumps. Our construction of (A-IE) and selection of appropriate knots takes possible jumps into consideration.

As mentioned earlier, transformation from the Fredholm integral equation (FIE) to (IE) is possible if functions in (FIE) are integratable w.r.t. x. Conversely, if functions in (IE) are differentiable w.r.t. x, taking derivatives in x on both sides of (IE) yields the form of (FIE). In the later case, the distribution problem is transformed into a density problem. Stochastic modelling literature has used integral equations as a density problem [9, 10, 26, 35]. However, in many cases it is unrealistic to assume the existence of density, e.g., when a probability mass exists. Other stochastic modelling literature using integral equation approaches include [21, 12, 4].

1.3. Contributions. We provide a general approximation scheme for (IE). The approximate solutions are obtained by solving a linear equation system. We prove the convergence of approximate solutions (Theorem 3.6) as well as their worst-case error bounds (Theorem 3.7). In the process of developing our convergence analysis,

TABLE 1
A summary of solution methods for classical Fredholm/Volterra integral equations in literature.

Methods and source		Description
Kernel method	Degenerate Kernel [23, 24, 30]	1. Approximate kernel $t(x,u)$ with a sequence of $t_n(x,u)$. 2. Assume t_n is degenerate: $t_n(x,u) = \sum_{j=1}^n a_j(x)b_j(u)$.
	Interpolation [29, 28, 6]	1. Approximate kernel $t(x,u)$ with a sequence of $t_n(x,u)$. 2. Interpolation of $t(x,u)$ in x yields following approximation: $t_n(x,u) = \sum_{j=1}^n a_j(x)t(x_j,u)$.
	Tensor approximation [15, 33]	1. Approximate kernel $t(x,u)$ with a sequence of $t_n(x,u)$. 2. Assume the tensor structure: $t_n(x,u) = \sum_i^n \sum_{j=1}^n c_{ij}a_i(x)b_j(u)$.
Projection method	Collocation method [11, 2, 18, 31]	1. An interpolation may be formulated into an projection Π. 2. Solve the approximate equation: $\Pi f(x) = \Pi g(x) + \int_a^b \Pi t(x,u)df(u)$.
	Galerkin method [22, 7, 8, 13]	1. Assume the function space X is a Hilbert space. Let X_n be a sub-space. Let Π_n be the orthogonal projection onto X_n. 2. Solve the approximate equation: $\Pi_n f(x) = \Pi_n g(x) + \int_a^b \Pi_n t(x,u)df(u)$.
Regularisation method [17]		1. Assume the integral operator K has a smoothing effect. 2. Define $f = g + \phi$, $\psi = Kg$. Then the equation is transformed into: $\phi = \psi + K\phi$. 3. The problem is reformulated into a new problem with a regular (sufficiently smooth) solution.
Iterated approximation [11, 19, 14]		1. Obtain an approximate solution $\tilde{f}$ from some other method. 2. Plug it into the right hand side (RHS) of original equation and obtain a new approximate solution: $\tilde{f}' := g(x) + \int_a^b t(x,u)d\tilde{f}(u)$.
Quadrature method (or Nystrom method, discretization) [32, 16, 1]		Solve an approximate equation: $f(x) = g(x) + \sum_{i=1}^n W_i(x)f(s_i)$, where the weight functions $\{W_i\}$ and knots $\{s_i\}$ are to be specified depending on approximation strategies.
Neumann method [20, 25, 3]		1. Assume the original equation can be reformulated into an operator equation $f = g + \mathcal{L}f$. Assume $\mathcal{L}$ has a norm less than 1. 2. The approximate solution is $\sum_{i=0}^n \mathcal{L}^i g$.

we provide necessary and sufficient conditions (Condition 1-Condition 4) for applying operator equation convergence theory. Our analysis is performed under the supremum norm, which requires weaker assumptions than the total variation norm (remark in supplemental material section SM6). Different from many other integral equation analyses which require the transition operator's norm less than one to ensure solution uniqueness, we allow the norm greater than one (Theorem SM6.1). Our finite approximation solution is near-optimal among all discrete approximate solutions under appropriate assumptions (remark in section 5). Moreover, we outline specific strategies on how to construct approximation equations (Condition 5) and provide worst-case error bounds (subsection 5.2). We verify the accuracy and efficiency of our approach via numerical examples.

1.4. Organization. In section 2, we show that (IE) can be written as an operator equation $F = \xi + \mathcal{K}F$ by defining an appropriate Banach space. Here we provide the main definitions and assumptions used in this paper. We also state known results from operator equation convergence theory. In section 3, we use the collective compactness theory to develop a convergence theory and error bounds on the approximate solutions available from (A-IE). In section 4, we provide specific strategies for constructing (A-IE). In section 5, we provide sufficient conditions for solution uniqueness and invertibility assumptions used in our results. We also discuss the computation of errors and argue why our approximation is near-optimal among all discrete approximate solutions. In section 6, we present numerical results for solving two test examples, one with multiple discontinuities and the other with everywhere non-differentiable functions. The multiple discontinuities example shows the accuracy of our method in solving non-continuous problems. The example with non-differentiability shows the applicability of our approach in an extreme case (FIE).

2. Definitions, Known Results, and Assumptions.

2.1. Known operator theory results. We now state key definitions and results from Anselone and Davis' collective compactness theory [5, 17]. This theory proves convergence and error bounds of approximate solutions to operator equations under collective compactness, consistency and solution uniqueness preconditions.

DEFINITION 2.1 (Collective Compactness, Consistency, and Stability [17]). *Consider Banach space $(\mathbf{Y}, ||\cdot||)$ and a linear operator $\mathcal{L}$ on $\mathbf{Y}$. A sequence of linear operators $\{\mathcal{L}^{(r)}\}_{r\in\mathbb{N}}$ are*

- *collectively compact if the set $\{\mathcal{L}^{(r)}F \mid r\in\mathbb{N}, F\in\mathbf{Y}\}$ is relatively compact,*
- *consistent if $\lim_{r\to\infty}||\mathcal{L}^{(r)}F-\mathcal{L}F||=0, \forall F\in\mathbf{Y}$, and*
- *stable if there exists $r_0\in\mathbb{N}$ and $C>0$ such that for all $r>r_0$, $(\mathcal{I}-\mathcal{L}^{(r)})^{-1}$ exists and $||(\mathcal{I}-\mathcal{L}^{(r)})^{-1}||_{O_\mathbf{Y}}\leqslant C$, where $\mathcal{I}$ is the identity operator and $||\cdot||_{O_\mathbf{Y}}$ is the operator norm of $\mathbf{Y}$.*

An error bound and convergence of solution is given in the following results.

LEMMA 2.2 (Theorem 4.7.11 in [17]). *Consider a Banach space $(\mathbf{Y}, ||\cdot||)$, linear operators $\mathcal{L}$ and $\mathcal{L}'$ on $\mathbf{Y}$, inhomogeneous terms ξ and ξ' in $\mathbf{Y}$, and equation solutions F and F' in $\mathbf{Y}$:*

$$F=\xi+\mathcal{L}F, \qquad F'=\xi'+\mathcal{L}'F'.$$

If $(\mathcal{I}-\mathcal{L}')^{-1}$ exists, then $||F'-F||\leqslant||(\mathcal{I}-\mathcal{L}')^{-1}||_{O_\mathbf{Y}}\cdot(||\mathcal{L}'F-\mathcal{L}F||+||\xi'-\xi||)$.

LEMMA 2.3 (Theorem 4.7.11 in [17]). *Consider a Banach space $(\mathbf{Y}, ||\cdot||)$, linear operators $\mathcal{L}$ and $\{\mathcal{L}^{(r)}\}_{r\in\mathbb{N}}$ on $\mathbf{Y}$, inhomogeneous terms ξ and $\{\xi^{(r)}\}_{r\in\mathbb{N}}$ in $\mathbf{Y}$ such that $\lim_{r\to\infty}\xi^{(r)}=\xi$, and equation solutions F and $\{F^{(r)}\}_{r\in\mathbb{N}}$ in $\mathbf{Y}$:*

$$F=\xi+\mathcal{L}F, \qquad F^{(r)}=\xi^{(r)}+\mathcal{L}^{(r)}F^{(r)},\ r\in\mathbb{N}.$$

If $\{\mathcal{L}^{(r)}\}_{r\in\mathbb{N}}$ are collectively compact and consistent to $\mathcal{L}$, and equation $F=\xi+\mathcal{L}F$ has a unique solution F in $\mathbf{Y}$, then (i) $\{\mathcal{L}^{(r)}\}_{r\in\mathbb{N}}$ are stable. (ii) There exists $r_0\in\mathbb{N}$ such that for all $r>r_0$, $F^{(r)}$ uniquely exists and satisfies $||F^{(r)}-F||\leqslant||(\mathcal{I}-\mathcal{L}^{(r)})^{-1}||_{O_\mathbf{Y}}\cdot(||\mathcal{L}^{(r)}F-\mathcal{L}F||+||\xi^{(r)}-\xi||)$. (iii) $\lim_{r\to\infty}F^{(r)}=F$.

LEMMA 2.4 (Theorem 1.3.28, 4.7.7 and 4.7.11 in [17]). *Consider a Banach space $(\mathbf{Y}, ||\cdot||)$, linear operators $\mathcal{L}$ and $\{\mathcal{L}^{(r)}\}_{r\in\mathbb{N}}$ on $\mathbf{Y}$. Suppose $\{\mathcal{L}^{(r)}\}_{r\in\mathbb{N}}$ are collectively compact and consistent to $\mathcal{L}$. The following statements are equivalent:*

- (i) $\mathrm{Ker}(\mathcal{I}-\mathcal{L})=\{0\}$, *i.e., $\forall\xi\in\mathbf{Y}$, $F=\xi+\mathcal{L}F$ has at most one solution $F\in\mathbf{Y}$;*
- (ii) $\mathrm{Im}(\mathcal{I}-\mathcal{L})=\mathbf{Y}$, *i.e., $\forall\xi\in\mathbf{Y}$, $F=\xi+\mathcal{L}F$ has at least one solution $F\in\mathbf{Y}$;*
- (iii) $(\mathcal{I}-\mathcal{L})^{-1}$ *exists, i.e., $\forall\xi\in\mathbf{Y}$, $F=\xi+\mathcal{L}F$ has a unique solution $F\in\mathbf{Y}$;*
- (iv) *There exists $r\in\mathbb{N}$ such that $(1-\mathcal{L}^{(r)})^{-1}$ exists and $||(\mathcal{L}-\mathcal{L}^{(r)})\mathcal{L}||_{O_\mathbf{Y}}\leqslant\frac{1}{||(1-\mathcal{L}^{(r)})^{-1}||_{O_\mathbf{Y}}}$.*
- (v) *There exists $r_0\in\mathbb{N}$ such that $\forall r>r_0$, $(1-\mathcal{L}^{(r)})^{-1}$ exists and $||(\mathcal{L}-\mathcal{L}^{(r)})\mathcal{L}||_{O_\mathbf{Y}}\leqslant\frac{1}{||(1-\mathcal{L}^{(r)})^{-1}||_{O_\mathbf{Y}}}$.*

Particularly, statements $(i)-(v)$ hold if $||\mathcal{L}||_{O_\mathbf{Y}}<1$.

The convergence in Lemma 2.3 is proven by showing that in the error of Lemma 2.2, the first term $||(\mathcal{I}-\mathcal{L}')^{-1}||_{O_\mathbf{Y}}$ is bounded (i.e., stability, indicated by collective compactness) and the second term $(||\mathcal{L}'F-\mathcal{L}F||+||\xi'-\xi||)$ converges to 0 (consistency).

We will prove that for (IE) in a specific space of distributions, these abstract preconditions are ensured by the following conditions that are verifiable:

(i) uniformly bounded variation (Condition 1),
(ii) càdlàg and countable jump discontinuities (Condition 2),
(iii) proper construction of (A-IE) (Condition 3 or Condition 5), and
(iv) operator norm inequality or feasibility on a subset (see section 5).

2.2. A Banach space. Let $\mathbf{D}$ be the collection of probability distribution functions defined on $\mathbb{R}$ and $\mathcal{B}$ be the Borel algebra for $\mathbb{R}$. Let

$$\mathbf{X} := \operatorname{span}(\mathbf{D}) = \left\{ \sum_{k=1}^{n} a_k F_k \,|\, n \in \mathbb{N}_{++}, a_k \in \mathbb{R}, F_k \in \mathbf{D} \right\}$$

be a linear space of distribution functions of all finite signed measures on $(\mathbb{R}, \mathcal{B})$. Let $||\cdot||_\infty$ be the supremum norm on $\mathbf{X}$: $||F||_\infty := \sup_{x\in\mathbb{R}} |F(x)|$, $\forall F \in \mathbf{X}$. Let $V(f;\Omega)$ be the total variation of a function $f : \mathbb{R} \mapsto \mathbb{R}$ on $\Omega \subseteq \mathbb{R}$. For convenience, let $V(f) := V(f;\mathbb{R})$. Moreover, for a multivariate function $f(x,u) : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$, we use $V_u(f(x,u)) \in \mathbb{R}_+$ to denote the total variation of $f(x,u)$ as a single variable function of u with any fixed $x \in \mathbb{R}^n$.

We construct a Banach space of right-continuous distribution functions as follows.

Theorem 2.5 (A Banach Space and Properties). *Let $\bar{\mathbf{X}}$ be the closure of $\mathbf{X} \subseteq \{f : \mathbb{R} \mapsto \mathbb{R}\}$ under the norm $||\cdot||_\infty$. Then $(\bar{\mathbf{X}}, ||\cdot||_\infty)$ is a Banach space. Moreover,*

(i) *(Bound and variation) $||F||_\infty < \infty, \forall F \in \bar{\mathbf{X}}$. $V(F) < \infty, \forall F \in \mathbf{X}$.*
(ii) *(Càdlàg) For any $F \in \bar{\mathbf{X}}$, $F(-\infty) = 0$ and $F(+\infty)$ exists. The left limit $F(x-)$ and the right limit $F(x+)$ exist for all $x \in \mathbb{R}$. F is right-continuous.*
(iii) *(Countable jump discontinuities) For any $F \in \bar{\mathbf{X}}$, the set*

$$J_F(\epsilon) := \{x \in \mathbb{R} \,|\, |F(x+) - F(x-)| > \epsilon\}, \quad \epsilon \in \mathbb{R}_+$$

is finite for any $\epsilon > 0$, and therefore, its jump discontinuities are countable.

Let $||\cdot||_O$ be the operator norm of any linear operator $\mathcal{K}$ on $\bar{\mathbf{X}}$:

$$||\mathcal{K}||_O := \sup_{F\in\bar{\mathbf{X}}, ||F||_\infty = 1} ||\mathcal{K}F||_\infty.$$

We write "F_k converges to F on $\bar{\mathbf{X}}$" as "$F_k \rightrightarrows F$" to emphasize uniform convergence.

2.3. Assumption and reformulation as operator equations. Let us define the collection of finite transition kernels on $(\mathbb{R}, \mathcal{B})$ in the distribution sense.

Definition 2.6 (Finite Transition Kernel). *Let $T(\mathbb{R}, \mathcal{B})$ be the collection of functions $p(x,u) : \mathbb{R}^2 \mapsto \mathbb{R}_+$ such that*

(i) for any fixed $x \in \mathbb{R}$, $p(x,u)$ is $\mathcal{B}$-measurable w.r.t. u, and
(ii) for any fixed $u \in \mathbb{R}$, $p(x,u) \equiv 0$ or there exists $\alpha > 0$ such that $\alpha p(x,u) \in \mathbf{D}$.

We assume the following on (IE) and its finite approximations.

Assumption 1. *For (IE) and a sequence of (A-IE) indexed by $r \in \mathbb{N}$, we have:*

(i) *$\kappa \in T(\mathbb{R}, \mathcal{B})$; $F, \xi \in \bar{\mathbf{X}}$;*
(ii) *$\xi^{(r)} \in \bar{\mathbf{X}}$, $\forall r \in \mathbb{N}$; $\xi^{(r)} \rightrightarrows \xi$; $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}} \subseteq \bar{\mathbf{X}}$, $\forall r \in \mathbb{N}$; and $-\infty = c_0^{(r)} < c_1^{(r)} < \ldots < c_{J^{(r)}}^{(r)} < c_{J^{(r)}+1}^{(r)} = +\infty$, $\forall r \in \mathbb{N}$.*

Assumption 1 is met in the context of stochastic models, which are studied in [27]. The space for ξ and F in (i) of Assumption 1 is relaxed to the closure $\bar{\mathbf{X}}$ because $\mathbf{X}$ is not closed under $||\cdot||_\infty$ and cannot be used as a Banach space in our analysis. We will show later in specific examples the construction of $\xi^{(r)}$, $\{\omega_i^{(r)}\}_{j=1}^{J^{(r)}}$ and $\{c_j^{(r)}\}_{j=1}^{J^{(r)}}$ $(r \in \mathbb{N})$ satisfying (ii) of Assumption 1.

We next reformulate (IE) and (A-IE) into operator equations on $\bar{\mathbf{X}}$. First, we formulate $\kappa \in T(\mathbb{R}, \mathcal{B})$ as a continuous linear operator on $\bar{\mathbf{X}}$.

Definition 2.7 (Transition Operator). *For (IE), its transition operator $\mathcal{K}$ on $\bar{\mathbf{X}}$ is defined as follows: if $F \in \mathbf{X}$, then*

$$\mathcal{K}F(x) := \int_{\mathbb{R}} \kappa(x,u)dF(u), \quad \forall x \in \mathbb{R}. \tag{2.1}$$

Otherwise,

$$\mathcal{K}F(x) := \lim_{k\to\infty} \mathcal{K}F_k(x), \quad \forall x \in \mathbb{R}, \tag{2.2}$$

where $\{F_k\}_{k=1}^{\infty} \subseteq \mathbf{X}$ is any sequence that converges to F on $\bar{\mathbf{X}}$.

The definition in (2.2) is because $F \in \bar{\mathbf{X}}\backslash\mathbf{X}$ as a measure is ill defined and thereby, (2.1) does not apply. Instead, we define via convergence, which is natural because any continuous linear operator on $\bar{\mathbf{X}}$ must have (2.2) as a property. In subsection 3.1, we show that $\mathcal{K}$ is well defined under mild assumptions. With slight abuse of notation, we also write $\mathcal{K}F(x)$ as $\int_{\mathbb{R}} \kappa(x,u)dF(u)$ for all $F \in \bar{\mathbf{X}}$. Now with Definition 2.7, we can write (IE) into an operator equation $F = \xi + \mathcal{K}F$.

Next we consider the sequence of (A-IE), specified by the inhomogeneous terms $\{\xi^{(r)}\}(r \in \mathbb{N})$, weight functions $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}(r \in \mathbb{N})$ and knots $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}(r \in \mathbb{N})$. Similar to $\mathcal{K}$, we construct a sequence of approximation operators $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$, defined by weight functions $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$ and knots $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}$ in (A-IE).

Definition 2.8 (Approximation Operators). *For a sequence of (A-IE) indexed by $r \in \mathbb{N}$, the approximation operators $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ on $\bar{\mathbf{X}}$ are defined as:*

$$\mathcal{K}^{(r)}F(x) := \sum_{i=1}^{J^{(r)}} \omega_i^{(r)}(x)[F(c_i^{(r)}) - F(c_{i-1}^{(r)})], \quad x \in \mathbb{R}, r \in \mathbb{N}, \tag{2.3}$$

where $-\infty = c_0^{(r)} < c_1^{(r)} < ... < c_{J^{(r)}}^{(r)} < c_{J^{(r)}+1}^{(r)} = +\infty$, and $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}} \subseteq \bar{\mathbf{X}}, \forall r \in \mathbb{N}$.

Now we can write a sequence of (A-IE) into operator forms $F = \xi^{(r)} + \mathcal{K}^{(r)}F$, $r \in \mathbb{N}$.

2.4. Verifiable conditions for using operator theory results. We now present a list of conditions, which are necessary and sufficient for applying the collective compactness and operator theory results in subsection 2.1 to solve (IE); see detailed statements as well as their corresponding proof in subsection 3.1. Let us define the following functions:

$$\zeta(x_1,x_2) := \lim_{u\to+\infty} |\kappa(x_2,u) - \kappa(x_1,u)|, \quad \forall x_1, x_2 \in \mathbb{R},$$

$$v_\Lambda(x_1,x_2) := V_u(\kappa(x_2,u) - \kappa(x_1,u)), \quad \forall x_1, x_2 \in \mathbb{R}.$$

Condition 1 (Uniformly Bounded Variation). *For all fixed $x \in \mathbb{R}$, $\kappa(x,u) \in T(\mathbb{R}, \mathcal{B})$ as a single-variable function of u has a uniformly bounded total variation:*

$$v_\kappa := \sup_{x\in\mathbb{R}} V_u(\kappa(x,u)) < \infty.$$

CONDITION 2 (Càdlàg and Countable Jump Discontinuities). *For* $\kappa \in T(\mathbb{R}, \mathcal{B})$,
(i) $\kappa(x, +\infty)$ *is right-continuous, and*
(ii) $\forall \epsilon > 0$, *there exists a finite split of* $\mathbb{R}$ *denoted by knots* $-\infty < s_1 < s_2 < ... < s_{N_\epsilon} < \infty$ *and intervals* $E_0 = (-\infty, s_1), E_1 = [s_1, s_2), ...E_{N_\epsilon} = [s_{N_\epsilon}, +\infty)$, *such that* $\forall x_1, x_2 \in E_i, x_1 < x_2, i = 0, 1, ..., N_\epsilon$, *we have* $v_\Lambda(x_1, x_2) \leqslant \epsilon$.

CONDITION 3. *For* $\kappa \in T(\mathbb{R}, \mathcal{B})$ *satisfying Condition 1 (uniformly bounded variation), operator* $\mathcal{K}$ *in Definition 2.7, and approximation operators* $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ *in Definition 2.8, the quantities satisfy:*
(i) (Point-wise convergence) $\forall F \in \bar{\mathbf{X}}$, $\forall x \in \mathbb{R}$, $\lim_{k\to\infty} \mathcal{K}^{(r)}F(x) = \mathcal{K}F(x)$.
(ii) (Uniform Càdlàg and countable jump discontinuities) for all $\epsilon > 0$, *there exists a finite split of* $\mathbb{R}$ *denoted by knots* $-\infty < s_1 < s_2 < \ldots < s_{N_\epsilon} < \infty$ *and intervals* $E_0 = (-\infty, s_1), E_1 = [s_1, s_2), \ldots, E_{N_\epsilon} = [s_{N_\epsilon}, +\infty)$, *such that* $\forall r \in \mathbb{N}, x_1, x_2 \in E_i, x_1 < x_2, i = 0, 1, \ldots, N_\epsilon$, *we have*

$$\sum_{i=1}^{J^{(r)}} |\Delta\omega_i^{(r)}(x_2) - \Delta\omega_i^{(r)}(x_1)| \leqslant \epsilon, \tag{2.4}$$

where $\omega^{(r)}_{J^{(r)}+1}(x) \equiv 0, \forall r \in \mathbb{N}$, *and*

$$\Delta\omega_i^{(r)}(x) := \omega_{i+1}^{(r)}(x) - \omega_i^{(r)}(x), \quad x \in \mathbb{R}, i = 1, 2, \ldots, J^{(r)}, r \in \mathbb{N}.$$

CONDITION 4 (Solution Uniqueness). *(IE) has a unique solution in* $\bar{\mathbf{X}}$.

Condition 2 is referred to as "càdlàg and countable jump discontinuities" because it describes the property of limit existence, right continuity and countable jumps: κ can be regarded as a map from $x \in \mathbb{R}$ to $\kappa(x, \cdot) \in S_V := \{f : \mathbb{R} \mapsto \mathbb{R} \,|\, V(f) < \infty\}$. When set S_V is equipped with the distance $|f(+\infty)|$ (*resp.* distance $V(f)$), (i) (*resp.* (ii)) of Condition 2 is equivalent to that κ has left limits on $\mathbb{R}$, has limits on $\pm\infty$, is right continuous on $\mathbb{R}$, and has countable jump discontinuities. We can easily verify Condition 1 and Condition 2 via partial derivatives in many cases with bounded supports; see a few examples in [27]. In Condition 3, (ii) describes a uniform "càdlàg and countable jump discontinuities" property across all $r \in \mathbb{N}$. Sufficient conditions which are easier to verify will be given in section 4. Condition 4 is required by Lemma 2.3, and a sufficient condition that ensures Condition 4 is given in section 5.

The results proved in this paper and their required conditions/assumptions are summarized in Table 2 with a brief description.

3. Convergence results. This section presents our convergence results in solving (IE) via finite approximation defined in (A-IE). We show that bounds on errors are computable and approximate solutions are converging to the true solution under Condition 1-Condition 4.

3.1. Operator properties. Theorem 3.1 shows that $\mathcal{K}$ in Definition 2.7 is a well-defined continuous linear operator on $\bar{\mathbf{X}}$ under Condition 1. Moreover, this condition is necessary and sufficient. Proofs for Theorem 3.1, and Lemma 3.2, Lemma 3.3 are given in supplementary material section SM2 and section SM3, respectively.

THEOREM 3.1 (Continuous Linearity). *For* $\kappa \in T(\mathbb{R}, \mathcal{B})$, *operator* $\mathcal{K}$ *in Definition 2.7 is a well defined continuous linear operator on* $\bar{\mathbf{X}}$ *if Condition 1 (uniformly bounded variation) holds. Conversely, if a continuous linear operator* $\mathcal{K}$ *on* $\bar{\mathbf{X}}$ *has the form (2.1), then Condition 1 must hold.*

TABLE 2

A summary of theorems and their required assumptions in all sections.

Theorems	Main preconditions	Summary of results
Section 3. Convergence results.		
Theorem 3.1	-	Condition 1 ⇔ Continuous linearity
Theorem 3.4	Condition 1	Condition 2 ⇔ Compactness
Theorem 3.5	Condition 1	Condition 3 ⇔ Collective compactness & consistency
Theorem 3.6	Assumption 1	Condition 1-Condition 4 ⇒ Solution convergence
Theorem 3.7	Assumption 1	Condition 1, Invertibility ⇒ Error bound
Section 4. Choices of Finite Approximation.		
Theorem 4.2	Condition 1	a. Condition 3 ⇒ Condition 2 b. Condition 5 ⇒ Condition 3
Section 5. Conditions for Solution Uniqueness, Invertibility and Error Computation.		
Theorem 5.1	Condition 1,Condition 3	a. Inequality (5.1) ⇒ Condition 4 b. Assumption 2 ⇒ Condition 4
Theorem 5.2	-	Full rank ⇔ Invertibility
Theorem 5.3	-	a. Computation of inverse norm b. Assumption 2 ⇒ Invertibility
Supplemental material.		
Theorem SM6.1	Condition 1, Condition 2	Subset feasibility ⇔ Condition 4

Interpretations: Condition 1 - $\kappa(x,u)$ in (IE) has uniformly bounded variation.
Condition 2 - Càdlàg and countable jump discontinuities for $\kappa(x,u)$ in (IE).
Condition 3 - Point-wise convergence, uniform càdlàg and countable jump discontinuities for (A-IE).
Condition 4 - (IE) has a unique solution. Condition 5 - Proper construction of (A-IE).
Assumption 1 - A mild equation restriction for (IE). Assumption 2 - Operator norm less than 1.

LEMMA 3.2. *In space $\bar{\mathbf{X}}$, a set $\digamma \subseteq \bar{\mathbf{X}}$ is relatively compact iff*

(i). $\digamma$ is uniformly bounded.

(ii). $\forall \epsilon > 0$, $\exists$ a finite split of $\mathbb{R}$ denoted by knots $-\infty < s_1 < s_2 < \ldots < s_{N_\epsilon} < \infty$ and intervals $E_0 = (-\infty, s_1), E_1 = [s_1, s_2), \ldots, E_{N_\epsilon} = [s_{N_\epsilon}, +\infty)$, such that $\forall F \in \digamma, \forall x_1, x_2 \in E_i, x_1 < x_2, i = 0, 1, \ldots, N_\epsilon$, we have $|F(x_2) - F(x_1)| \leqslant \epsilon$.

LEMMA 3.3. *Consider $\kappa \in T(\mathbb{R}, \mathcal{B})$ satisfying Condition 1 (uniformly bounded variation), operator $\mathcal{K}$ defined in Definition 2.7, and the unit ball $\mathbf{U} := \{F \in \bar{\mathbf{X}}| \ ||F||_\infty \leqslant 1\}$. For any $x_1, x_2 \in \mathbb{R}$, we have*

(i) $\forall F \in \mathbf{U}$, $|\mathcal{K}F(x_2) - \mathcal{K}F(x_1)| \leqslant \zeta(x_1, x_2) + 2v_\Lambda(x_1, x_2)$.

(ii) $\forall \epsilon > 0$, $\exists F \in \mathbf{U} \cap \mathbf{X}$, $|\mathcal{K}F(x_2) - \mathcal{K}F(x_1)| \geqslant \zeta(x_1, x_2) + v_\Lambda(x_1, x_2) - \epsilon$.

The following theorem shows that Condition 2 is necessary and sufficient for the compactness of $\mathcal{K}$.

THEOREM 3.4 (Compactness). *For $\kappa \in T(\mathbb{R}, \mathcal{B})$ satisfying Condition 1 (uniformly bounded variation), operator $\mathcal{K}$ in Definition 2.7 is compact on $\bar{\mathbf{X}}$ iff Condition 2 (càdlàg and countable jump discontinuities) holds.*

Proof of Theorem 3.4. We first note that (i) in Condition 2 can be equivalently written as: $\forall \epsilon > 0$, there exists a finite split of $\mathbb{R}$ denoted by knots $-\infty < s_1 < s_2 < \ldots < s_{N_\epsilon} < \infty$ and intervals $E_0 = (-\infty, s_1), E_1 = [s_1, s_2), \ldots, E_{N_\epsilon} = [s_{N_\epsilon}, +\infty)$, such that $\forall x_1, x_2 \in E_i, x_1 < x_2, i = 0, 1, \ldots, N_\epsilon$, we have $\zeta(x_1, x_2) \leqslant \epsilon$. This can be easily proven by noting the following facts: (a) $\kappa(\cdot, +\infty)$ is bounded by M_κ, where M_κ is a bound of $\kappa(x, u)$ for $x, u \in \mathbb{R}$. (b) $\kappa(\cdot, +\infty)$ is non-decreasing since $\kappa(\cdot, u)$ is non-decreasing for all $u \in \mathbb{R}$.

($\Leftarrow$) Under (i) and (ii) of Condition 2, $\forall \epsilon > 0$, there exists a finite split of $\mathbb{R}$ denoted by knots $-\infty < s_1 < s_2 < \ldots < s_{N_\epsilon} < \infty$ and intervals $E_0 = (-\infty, s_1), E_1 = [s_1, s_2), \ldots, E_{N_\epsilon} = [s_{N_\epsilon}, +\infty)$, such that $\forall x_1, x_2 \in E_i, x_1 < x_2, i = 0, 1, \ldots, N_\epsilon$, we have $\zeta(x_1, x_2) \leqslant \frac{\epsilon}{2}$ and $v_\Lambda(x_1, x_2) \leqslant \frac{\epsilon}{4}$. By Lemma 3.3, $\forall F \in \mathbf{U}, \forall x_1, x_2 \in E_i, x_1 < x_2, i = 0, 1, \ldots, N_\epsilon$, we have $|\mathcal{K}F(x_2) - \mathcal{K}F(x_1)| \leqslant \zeta(x_1, x_2) + 2v_\Lambda(x_1, x_2) \leqslant \epsilon$. By Lemma 3.2, the bounded set $\{\mathcal{K}F \mid F \in \mathbf{U}\}$ is relatively compact. Thus, $\mathcal{K}$ is compact.

($\Rightarrow$) If (i) or (ii) of Condition 2 is not satisfied, then $\exists \epsilon > 0$, for all splits of $\mathbb{R}$ denoted by knots $-\infty < s_1 < s_2 < ... < s_N < \infty$ and intervals $E_0 = (-\infty, s_1), E_1 = [s_1, s_2), \ldots, E_N = [s_N, +\infty)$, there exist $x_1, x_2 \in E_i, x_1 < x_2, i = 0, 1, \ldots, N$ such that $\zeta(x_1, x_2) \geqslant \epsilon$ or $v_\Lambda(x_1, x_2) \geqslant \epsilon$. By Lemma 3.3, $\exists F \in \mathbf{U}$ such that $|\mathcal{K}F(x_2) -$

$\mathcal{K}F(x_1)| \geqslant \zeta(x_1,x_2) + v_\Lambda(x_1,x_2) - \frac{\epsilon}{2} \geqslant \frac{\epsilon}{2}$. By Lemma 3.2, $\{\mathcal{K}F \mid F \in \mathbf{U}\}$ is not relatively compact, i.e., $\mathcal{K}$ is not compact. (i) and (ii) of Condition 2 are necessary.∎

Condition 3 ensures that approximation operators $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ in Definition 2.8 are collective compact and consistent to $\mathcal{K}$. It is also necessary and sufficient.

THEOREM 3.5 (Consistency & Collective Compactness). *For $\kappa \in T(\mathbb{R},\mathcal{B})$ satisfying Condition 1 (uniformly bounded variation), operator $\mathcal{K}$ in Definition 2.7, and approximation operators $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ in Definition 2.8, $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ are collectively compact and consistent (i.e., $\mathcal{K}^{(r)}F \rightrightarrows \mathcal{K}F, \forall F \in \bar{\mathbf{X}}$) on $\bar{\mathbf{X}}$ iff Condition 3 (point-wise convergence, uniform càdlàg and countable jump discontinuities) holds.*

Proof. ($\Leftarrow$) Let Condition 3 hold. We first show $\mathcal{K}^{(r)}$ is a continuous linear operator on $\bar{\mathbf{X}}$ for all $r \in \mathbb{N}$. Definition 2.8 suggests that $\mathcal{K}^{(r)}F \in \bar{\mathbf{X}}$ for all $F \in \bar{\mathbf{X}}$ and $\mathcal{K}^{(r)}$ is linear. For any $F \in \bar{\mathbf{X}}$ such that $||F||_\infty = 1$, we have $||\mathcal{K}^{(r)}F||_\infty \leqslant \sum_{i=1}^{J^{(r)}} |F(c_i^{(r)}) - F(c_{i-1}^{(r)})| \cdot \big|\big|\omega_i^{(r)}\big|\big|_\infty \leqslant 2\sum_{i=1}^{J^{(r)}} \big|\big|\omega_i^{(r)}\big|\big|_\infty < \infty$. Thus, $\mathcal{K}^{(r)}$ is a linear, bounded (i.e., continuous) operator on $\bar{\mathbf{X}}$ for all $r \in \mathbb{N}$.

Next we prove that the set

$$S := \cup_{r\in\mathbb{N}} \{\mathcal{K}^{(r)}F \mid F \in \mathbf{U}\} = \left\{ \sum_{i=1}^{J^{(r)}} \omega_i^{(r)}(x)[F(c_i^{(r)}) - F(c_{i-1}^{(r)})] \,\middle|\, r \in \mathbb{N}, F \in \mathbf{U} \right\}$$
$$= \left\{ \sum_{i=1}^{J^{(r)}} (a_i - a_{i-1})\omega_i^{(r)}(x) \,\middle|\, r \in \mathbb{N}, a_0 = 0, a_i \in [-1,1], i \in \mathbb{N}_{++} \right\}$$

satisfies (ii) of Lemma 3.2. Note that (ii) of Condition 3 suggests that $\forall\epsilon > 0$, there exists a finite split of $\mathbb{R}$ denoted by knots $-\infty < s_1 < s_2 < ... < s_{N_\epsilon} < \infty$ and intervals $E_0 = (-\infty, s_1), E_1 = [s_1, s_2), ...E_{N_\epsilon} = [s_{N_\epsilon}, +\infty)$, such that for all $r \in \mathbb{N}, x_1, x_2 \in E_i, x_1 < x_2, i = 0, 1, ..., N_\epsilon$, we have $\sum_{i=1}^{J^{(r)}} |\Delta\omega_i^{(r)}(x_2) - \Delta\omega_i^{(r)}(x_1)| = |\omega_{J^{(r)}}^{(r)}(x_2) - \omega_{J^{(r)}}^{(r)}(x_1)| + \sum_{i=1}^{J^{(r)}-1} |\Delta\omega_i^{(r)}(x_2) - \Delta\omega_i^{(r)}(x_1)| \leqslant \epsilon$. Therefore,

$$\left| \sum_{i=1}^{J^{(r)}} (a_i - a_{i-1})\omega_i^{(r)}(x_2) - \sum_{i=1}^{J^{(r)}} (a_i - a_{i-1})\omega_i^{(r)}(x_1) \right|$$
$$= \left| a_{J^{(r)}}[\omega_{J^{(r)}}^{(r)}(x_2) - \omega_{J^{(r)}}^{(r)}(x_1)] - \sum_{i=1}^{J^{(r)}-1} a_i[\Delta\omega_i^{(r)}(x_2) - \Delta\omega_i^{(r)}(x_1)] \right|$$
$$\leqslant \left| \omega_{J^{(r)}}^{(r)}(x_2) - \omega_{J^{(r)}}^{(r)}(x_1) \right| + \sum_{i=1}^{J^{(r)}-1} \left| \Delta\omega_i^{(r)}(x_2) - \Delta\omega_i^{(r)}(x_1) \right| \leqslant \epsilon,$$

for all $a_0 = 0, a_i \in [-1,1], i \in \mathbb{N}_{++}$. Thus, (ii) of Lemma 3.2 is satisfied for S.

We next prove the uniform convergence (i.e., consistency) using the point-wise convergence in (i) of Condition 3. For any fixed $F^* \in \bar{\mathbf{X}}$, either $\{\mathcal{K}^{(r)}F^*\}_{r\in\mathbb{N}}$ or $\{\mathcal{K}^{(r)}\frac{F^*}{||F^*||_\infty}\}_{r\in\mathbb{N}}$ belongs to S and thus satisfies (ii) of Lemma 3.2. Therefore, $\{\mathcal{K}^{(r)}F^*\}_{r\in\mathbb{N}}$ satisfies (ii) of Lemma 3.2. Then for any $\epsilon > 0$, we have intervals $E_0 = (-\infty, s_1), E_1 = [s_1, s_2), ...E_N = [s_N, +\infty)$ such that for all $r \in \mathbb{N}, x_1, x_2 \in E_i, x_1 < x_2, i = 0, 1, ..., N$, we have $|\mathcal{K}^{(r)}F^*(x_2) - \mathcal{K}^{(r)}F^*(x_1)| \leqslant \frac{\epsilon}{3}$. Now we arbitrarily select x_i from E_i, $i = 0, ..., N$, and $M \in \mathbb{N}$ such that $\forall r_1, r_2 > M$, $|\mathcal{K}^{(r_1)}F^*(x_i) - \mathcal{K}^{(r_2)}F^*(x_i)| \leqslant \frac{\epsilon}{3}$ for all $i = 0, 1, ..., N$. Therefore, $\forall r_1, r_2 > M$,

$\forall x \in E_i, i = 0, 1, ..., N$, $|\mathcal{K}^{(r_1)}F^*(x) - \mathcal{K}^{(r_2)}F^*(x)| \leqslant |\mathcal{K}^{(r_1)}F^*(x) - \mathcal{K}^{(r_1)}F^*(x_i)| + |\mathcal{K}^{(r_1)}F^*(x_i) - \mathcal{K}^{(r_2)}F^*(x_i)| + |\mathcal{K}^{(r_2)}F^*(x_i) - \mathcal{K}^{(r_2)}F^*(x)| \leqslant \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \leqslant \epsilon$, i.e., $||\mathcal{K}^{(r_1)}F^* - \mathcal{K}^{(r_2)}F^*||_\infty \leqslant \epsilon$ for all $r_1, r_2 \geqslant M$. Thus, $\mathcal{K}^{(r)}F^* \rightrightarrows \mathcal{K}F^*$ for all $F^* \in \bar{\mathbf{X}}$.

Lastly, we prove $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ is collectively compact. Since (ii) of Lemma 3.2 is satisfied for S, we only need to prove that (i) of Lemma 3.2 also holds. i.e., S is uniformly bounded. For any $F \in \bar{\mathbf{X}}$, we have $\mathcal{K}^{(r)}F \rightrightarrows \mathcal{K}F$, $||\mathcal{K}F||_\infty < \infty$ and $||\mathcal{K}^{(r)}F||_\infty < \infty, r \in \mathbb{N}$. Thus, $\sup_{r\in\mathbb{N}} ||\mathcal{K}^{(r)}F||_\infty < \infty$. By the uniform boundedness principle, we have $\sup_{r\in\mathbb{N}} ||\mathcal{K}^{(r)}||_O < \infty$. Thereby, $\sup_{F\in S} ||F||_\infty = \sup_{F\in\mathbf{U}, r\in\mathbb{N}} ||\mathcal{K}^{(r)}F||_\infty \leqslant \sup_{r\in\mathbb{N}} ||\mathcal{K}^{(r)}||_O < \infty$. Thus, (i) of Lemma 3.2 also holds.

$(\Rightarrow)$ Let $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ be collectively compact and consistent. Because the uniform convergence (i.e., consistency) implies point-wise convergence, (i) of Condition 3 holds.

The collective compactness of $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ implies S is relatively compact. By Lemma 3.2, for all $\epsilon > 0$, there exists a finite split of $\mathbb{R}$ denoted by knots $-\infty < s_1 < s_2 < ... < s_{N_\epsilon} < \infty$ and intervals $E_0 = (-\infty, s_1), E_1 = [s_1, s_2), ...E_{N_\epsilon} = [s_{N_\epsilon}, +\infty)$, such that for all $r \in \mathbb{N}, x_1, x_2 \in E_i, x_1 < x_2, i = 0, 1, ..., N_\epsilon$, we have

$$\left|\sum_{i=1}^{J^{(r)}}(a_i - a_{i-1})\omega_i^{(r)}(x_2) - \sum_{i=1}^{J^{(r)}}(a_i - a_{i-1})\omega_i^{(r)}(x_1)\right| \leqslant \epsilon,$$

for all $a_0 = 0, a_i \in [-1, 1], i \in \mathbb{N}_{++}, r \in \mathbb{N}$. For any fixed $r \in \mathbb{N}$, let $a_i = -\text{sign}\big(\Delta\omega_i^{(r)}(x_2) - \Delta\omega_i^{(r)}(x_1)\big), i = 1, 2, ..., J^{(r)}$. Then we have

$$\sum_{i=1}^{J^{(r)}}\left|\Delta\omega_i^{(r)}(x_2) - \Delta\omega_i^{(r)}(x_1)\right| = \sum_{i=1}^{J^{(r)}}(a_i - a_{i-1})\omega_i^{(r)}(x_2) - \sum_{i=1}^{J^{(r)}}(a_i - a_{i-1})\omega_i^{(r)}(x_1) \leqslant \epsilon.$$

Thus, (ii) of Condition 3 holds. ∎

3.2. Convergence of approximate solutions. Recall that we are able to write (IE) into operator equation $F = \xi + \mathcal{K}F$ in space $\bar{\mathbf{X}}$ and a sequence of (A-IE) into $F = \xi^{(r)} + \mathcal{K}^{(r)}F$ $(r \in \mathbb{N})$. Then we have the approximate solution convergence results.

THEOREM 3.6 (Convergence of Finite Approximation). *Consider (IE), a sequence of (A-IE) indexed by $r \in \mathbb{N}$, and their respective solutions $\bar{F}$, $\{\bar{F}^{(r)}\}_{r\in\mathbb{N}}$ in $\bar{\mathbf{X}}$. If Assumption 1 and Condition 1-Condition 4 hold, then there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,*

- *$(\mathcal{I} - \mathcal{K}^{(r)})^{-1}$ exists,*
- *(stability) $\{||(\mathcal{I} - \mathcal{K}^{(r)})^{-1}||_O \,|\, r > r_0\}$ is bounded,*
- *(uniform convergence) $\bar{F}^{(r)}$ uniquely exists, $\bar{F}^{(r)} \rightrightarrows \bar{F}$, and*

$$||\bar{F}^{(r)} - \bar{F}||_\infty \leqslant ||(\mathcal{I} - \mathcal{K}^{(r)})^{-1}||_O \cdot \Big(||\mathcal{K}^{(r)}\bar{F} - \mathcal{K}\bar{F}||_\infty + ||\xi^{(r)} - \xi||_\infty\Big). \tag{3.1}$$

where $\mathcal{I}$ is the identity operator, and operators $\mathcal{K}$ and $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ are as defined in Definition 2.7 and Definition 2.8.

Proof. Using Lemma 2.3, collective compactness and consistency of $\{\mathcal{K}\}_{r\in\mathbb{N}}$ (implied by Condition 1-Condition 3, Theorem 3.5) and solution uniqueness of (IE) (i.e., Condition 4), we directly have Theorem 3.6. ∎

Remark: "equation stability". The assumptions in Theorem 3.6 have an "equation stability" interpretation from the perspective of linear algebra: if (IE) is feasible and has a unique solution, then it remains feasible and has a unique solution for all $\hat{\xi}$ and $\hat{\mathcal{K}}$ in some neighborhood of $\|\hat{\xi}-\xi\|_\infty \leqslant \epsilon$ and $\|\hat{\mathcal{K}}-\mathcal{K}\|_O \leqslant \epsilon$; on the other hand if (IE) is infeasible, then it remains infeasible for all $\hat{\xi}$ in some neighborhood of $\|\hat{\xi}-\xi\|_\infty \leqslant \epsilon$. See detailed proof in section SM4 in supplemental materials.

3.3. Approximation error bound.

THEOREM 3.7 (Error of Finite Approximation). *Consider (IE), an instance of (A-IE), and their respective solutions $\bar{F}$, $\bar{F}^{(r)}$ in $\bar{\mathbf{X}}$. If Assumption 1 and Condition 1 hold and $(\mathcal{I}-\mathcal{K}^{(r)})^{-1}$ exists, then inequality (3.1) holds, where $\mathcal{I}$ is the identity operator, and operators $\mathcal{K}$ and $\mathcal{K}^{(r)}$ are as defined in Definition 2.7 and Definition 2.8.*

Proof. Recall the reformulation of (IE) into operator equation $F=\xi+\mathcal{K}F$ and (A-IE) into $F=\xi^{(r)}+\mathcal{K}^{(r)}F$. Lemma 2.2 directly implies error bounds in Theorem 3.7. ◻

Necessary and sufficient conditions for the existence of $(\mathcal{I}-\mathcal{K}^{(r)})^{-1}$ used in Theorem 3.7 are outlined in subsection 5.2. The difference between the error bound in Theorem 3.6 and that in Theorem 3.7 is that the later does not require Condition 2-Condition 4 or a sufficiently large r, i.e., $r>r_0$ as in Theorem 3.6. Therefore, the later is more suitable for error computation.

4. Choices of Finite Approximation.

Despite the necessity and sufficiency of Condition 3 for consistency and collective compactness, it may be difficult to determine if Condition 3 is satisfied in practice. In this section, we provide tractable sufficient conditions for Condition 3, and discuss how to satisfy Condition 3 by properly choosing the weight functions $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$ and knots $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}$ in (A-IE).

For convenience, we define the approximate transition kernel for $\mathcal{K}^{(r)}$, which is an analog of kernel κ for $\mathcal{K}$:

$$\kappa^{(r)}(x,u) := \sum_{i=1}^{J^{(r)}} \omega_i^{(r)}(x)\cdot \mathbf{1}\{u\in (c_{i-1}^{(r)}, c_i^{(r)}]\}, \quad \forall x,u\in\mathbb{R}, r\in\mathbb{N}. \tag{4.1}$$

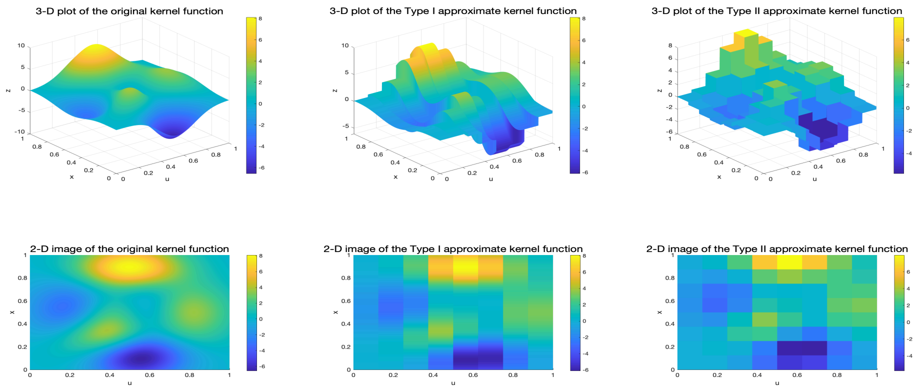
We now provide two types of approximation methods for solving our problem, which can be interpreted as follows (see illustrations in Figure 1). In Type II sequence, the approximate kernel $\kappa^{(r)}$ on a finite-support is obtained by truncating the original kernel κ, and the approximate solution is the stationary distribution under the approximate kernel. For Type I sequence, we use Type II sequence in the first step. After obtaining the approximate distribution, we "plug it back" into the original model, subsequently perform an additional iteration of original transition and use the final distribution as a solution. Therefore, Type I essentially uses the iterated approximation idea briefly described in Table 1.

DEFINITION 4.1. *For $\kappa\in T(\mathbb{R},\mathcal{B})$, define two types of approximation sequences $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$, $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}$ $(r\in\mathbb{N})$ as follows:*

- *(Type I) $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$, $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}$ $(r\in\mathbb{N})$ are exact finite approximation of κ in the sense that $c_i^{(r)}\in\mathbb{R}$, $i=1,2,...,J^{(r)}, r\in\mathbb{N}$ and $\omega_i^{(r)}(x)=\kappa(x,c_i^{(r)}), x\in\mathbb{R}, i=1,2,...,J^{(r)}, r\in\mathbb{N}$.*
- *(Type II) $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$, $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}$ $(r\in\mathbb{N})$ are step-wise finite approximation of κ in the sense: (1) (proper truncations) for all $r\in\mathbb{N}$,*

$$\omega_i^{(r)}(x)=\sum_{j=1}^{J^{(r)}} \kappa(c_j^{(r)}, c_i^{(r)})\mathbf{1}\{x\in[c_j^{(r)}, c_{j+1}^{(r)})\}, \quad x\in\mathbb{R}, i=1,2,...,J^{(r)}.$$

FIG. 1. *Plots of an example kernel function $z = \kappa(x, u)$ on $[0, 1]^2$ and its finite approximations with 9 knots, where the smooth surface of original kernel is approximated by 8 "ribbons" in Type I approximation and by 8×8 "squares" in Type II approximation.*



(2) (Increasing partitions) for all $r_1 < r_2, r_1, r_2 \in \mathbb{N}$, we have $\{c_i^{(r_1)}\}_{i=1}^{J(r_1)} \subseteq \{c_i^{(r_2)}\}_{i=1}^{J(r_2)}$. (3) There exist $\{\epsilon^{(r)}\}_{r\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\lim_{r\to\infty} \epsilon^{(r)} = 0$ such that $\forall r \in \mathbb{N}$, $\forall x_1, x_2 \in [c_{i-1}^{(r)}, c_i^{(r)}) \cap \mathbb{R}$, $x_1 < x_2, i = 1, 2, ..., J^{(r)} + 1$, we have $v_\Lambda(x_1, x_2) + \zeta(x_1, x_2) \leqslant \epsilon^{(r)}$. In other words, $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}$ are defined by a sequence of finite splits for càdlàg and countable jump discontinuities of κ.

The following theorem provides sufficient conditions for Condition 3.

THEOREM 4.2 (Properties of Condition 3). *Consider $\kappa \in T(\mathbb{R}, \mathcal{B})$ satisfying Condition 1 (uniformly bounded variation), operator $\mathcal{K}$ in Definition 2.7, and approximation operators $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ in Definition 2.8. Then*

(i) Condition 3 (point-wise convergence, uniform càdlàg and countable jump discontinuities) implies Condition 2 (càdlàg and countable jump discontinuities).

(ii) Item (i) (point-wise convergence) of Condition 3 holds if $\exists M_\omega, v_\omega < \infty$,

$$|\kappa^{(r)}(x, u)| \leqslant M_\omega, \quad \forall x, u \in \mathbb{R}, r \in \mathbb{N}, \tag{4.2}$$

$$V_u(\kappa^{(r)}(x, u)) \leqslant v_\omega, \quad \forall x \in \mathbb{R}, r \in \mathbb{N}, \tag{4.3}$$

$$\lim_{r\to\infty} \kappa^{(r)}(x, u) = \kappa(x, u), \quad \forall u, x \in \mathbb{R}, \tag{4.4}$$

i.e., approximate transition kernels $\{\kappa^{(r)}\}_{r\in\mathbb{N}}$ defined in (4.1) are uniformly bounded, uniformly variation-bounded and convergent to κ.

(iii) Item (ii) (uniform càdlàg and countable jump discontinuities) of Condition 3 holds if Condition 2 holds, and $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$, $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}$ $(r \in \mathbb{N})$ belong to Type I or Type II approximation sequences given in Definition 4.1.

Proof. (i) is because consistency and collective compactness imply compactness according to [17] (Remark 4.7.9). At the same time, the former is equivalent to Condition 3 (Theorem 3.5), while the later is equivalent to Condition 2 (Theorem 3.4).

(ii) will be proven by dominated convergence. Note that $\{\kappa^{(r)}\}_{r\in\mathbb{N}}$ are bounded, and for any $F \in \mathbf{X}$, $\mathcal{K}^{(r)}F$ can be written as $\mathcal{K}^{(r)}F(x) = \int_\mathbb{R} \kappa^{(r)}(x, u)dF(u), \forall x \in$

$\mathbb{R}, r \in \mathbb{N}$. Thereby, for any $F \in \mathbf{D} \subseteq \mathbf{X}$ and $x \in \mathbb{R}$,the dominated convergence ensures

$$\lim_{r\to\infty} \mathcal{K}^{(r)}F(x) = \int_{\mathbb{R}} \lim_{r\to\infty} \kappa^{(r)}(x,u)dF(u) = \int_{\mathbb{R}} \kappa(x,u)dF(u) = \mathcal{K}F(x).$$

Because $\mathbf{X}$ is a linear span of $\mathbf{D}$, for any $F \in \mathbf{X}$, $\lim_{r\to\infty} \mathcal{K}^{(r)}F(x) = \mathcal{K}F(x), \forall x \in \mathbb{R}$.

We next expand this convergence result to $F \in \bar{\mathbf{X}}$. Consider $F = F' + e$, $F \in \bar{\mathbf{X}}, F' \in \mathbf{X}, ||e||_\infty < \epsilon$. Here F' can be regarded as an approximation to F with a bounded error e in the subspace $\mathbf{X}$. For any $x \in \mathbb{R}$, we have already proven that $\lim_{r\to\infty} \mathcal{K}^{(r)}F'(x) = \mathcal{K}F'(x)$. We only need to prove $\lim_{\epsilon\to 0} \limsup_{r\to\infty} |\mathcal{K}^{(r)}e(x) - \mathcal{K}e(x)| = 0$. Indeed, according to eq(SM2.8) in the proof of Theorem 3.1, we have $||\mathcal{K}e||_\infty \leqslant (2v_\kappa + M_\kappa)\epsilon$, and $||\mathcal{K}^{(r)}e||_\infty \leqslant (2v_\omega + M_\omega)\epsilon, \forall r \in \mathbb{N}$. Therefore, $\lim_{\epsilon\to 0} \limsup_{r\to\infty} |\mathcal{K}^{(r)}e(x) - \mathcal{K}e(x)| = 0$, $\lim_{r\to\infty} \mathcal{K}^{(r)}F(x) = \mathcal{K}F(x)$.

(iii) We will show for Type I and II sequences, $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$ are embedded in κ and inherit the property of càdlàg and countable jump discontinuities. Rewrite (2.4) as

$$\begin{aligned}
&\sum_{i=1}^{J^{(r)}} \left|\Delta\omega_i^{(r)}(x_2) - \Delta\omega_i^{(r)}(x_1)\right| \leqslant \epsilon \\
\Leftrightarrow &\sum_{i=1}^{J^{(r)}} \left| \left[\omega_{i+1}^{(r)}(x_2) - \omega_{i+1}^{(r)}(x_1)\right] - \left[\omega_i^{(r)}(x_2) - \omega_i^{(r)}(x_1)\right] \right| \leqslant \epsilon \\
\Leftrightarrow &V_u(\kappa^{(r)}(x_2,u) - \kappa^{(r)}(x_1,u)) + |\kappa^{(r)}(x_2,+\infty) - \kappa^{(r)}(x_1,+\infty)| \leqslant \epsilon.
\end{aligned}$$

Recall that if Condition 2 holds, $\forall \epsilon > 0$, there exists a finite split of $\mathbb{R}$ denoted by knots $-\infty < s_1 < s_2 < ... < s_{N_\epsilon} < \infty$ and intervals $E_0 = (-\infty, s_1), E_1 = [s_1, s_2), ..., E_{N_\epsilon} = [s_{N_\epsilon}, +\infty)$, such that $\forall x_1, x_2 \in E_i, i = 0, 1, ..., N_\epsilon, x_1 < x_2$, we have $v_\Lambda(x_2, x_1) + \zeta(x_2, x_1) \leqslant \epsilon$. For Type I, by definitions of $\kappa^{(r)}$ in (4.1) and $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}} (r \in \mathbb{N})$ in Definition 4.1, we have

$$\begin{aligned}
&V_u(\kappa^{(r)}(x_2,u) - \kappa^{(r)}(x_1,u)) + |\kappa^{(r)}(x_2,+\infty) - \kappa^{(r)}(x_1,+\infty)| \\
\leqslant &V_u(\kappa(x_2,u) - \kappa(x_1,u)) + |\kappa(x_2,+\infty) - \kappa(x_1,+\infty)| \\
\leqslant &v_\Lambda(x_2,x_1) + \zeta(x_2,x_1) \leqslant \epsilon.
\end{aligned}$$

In other words, the finite splits generated by Condition 2 also apply to that required by *(ii)* of Condition 3. Thus, *(ii)* of Condition 3 holds. The proof for Type II follows similar analysis and is given in the supplemental material section SM5. ∎

Theorem 4.2 shows that we can choose $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$ and knots $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}$ following Definition 4.1. It also naturally introduces the following sufficient condition for Condition 3.

CONDITION 5 (Sufficient Condition for Condition 3). *Consider $\kappa \in T(\mathbb{R}, \mathcal{B})$ satisfying Condition 1 (uniformly bounded variation), and the approximation sequence $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$, $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}$ $(r \in \mathbb{N})$. The quantities satisfy*

(i) Condition 2 (càdlàg and countable jump discontinuities).

(ii) $\exists M_\omega, v_\omega < \infty$ such that (4.2)-(4.4) hold, i.e., $\{\kappa^{(r)}\}_{r\in\mathbb{N}}$ defined in (4.1) are uniformly bounded, uniformly variation bounded and convergent to κ.

(iii) $\{\omega_i^{(r)}\}_{i=1}^{J^{(r)}}$, $\{c_i^{(r)}\}_{i=1}^{J^{(r)}}$ $(r \in \mathbb{N})$ belong to any type in Definition 4.1.

5. Conditions for Solution Uniqueness, Invertibility and Error Computation. In this section, we provide two sufficient conditions for Condition 4 (i.e., the solution uniqueness of (IE)) based on Lemma 2.4. The first condition is based on norm inequality, and the second is based on feasibility. We demonstrate the first condition in the main text. The second is given in supplemental material section SM6, where we also show that a stronger norm than $||\cdot||_\infty$ imposes stronger assumptions to reach the same results. Finally, we show how to determine the invertibility of $\mathcal{I}-\mathcal{K}^{(r)}$ in Theorem 3.7 and a computing approach for errors in Theorem 3.6 and Theorem 3.7.

5.1. Approach I for solution uniqueness: norm inequality. Lemma 2.4 suggests that under Condition 1 and Condition 3 (implying collective compactness and consistency), the following conditions are equivalent: (i) the solution uniqueness of (IE) for all $\xi \in \bar{\mathbf{X}}$, (ii) the solution existence of (IE) for all $\xi \in \bar{\mathbf{X}}$, (iii) the invertibility of $\mathcal{I} - \mathcal{K}$, and (iv) the stability of $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$. In practice, we may prove solution uniqueness by showing (5.1) or (5.2) below. Inequality (5.1) is "almost necessary": it holds as long as the solution is unique and r is sufficiently large.

THEOREM 5.1 (Solution Uniqueness I). *For $\kappa \in T(\mathbb{R}, \mathcal{B})$ satisfying Condition 1, operator $\mathcal{K}$ in Definition 2.7, and approximation operators $\{\mathcal{K}^{(r)}\}_{r\in\mathbb{N}}$ defined in Definition 2.8 and satisfying Condition 3, the following statements are equivalent*

(i) $F = \xi + \mathcal{K}F$ has a unique solution F in $\bar{\mathbf{X}}$ for all $\xi \in \bar{\mathbf{X}}$, (i.e., Condition 4).

(ii) There exists $r \in \mathbb{N}$, $(\mathcal{I} - \mathcal{K}^{(r)})^{-1}$ exists and

$$||(\mathcal{K} - \mathcal{K}^{(r)})\mathcal{K}||_O \leqslant \frac{1}{||(\mathcal{I} - \mathcal{K}^{(r)})^{-1}||_O}. \tag{5.1}$$

(iii) There exists $r_0 \in \mathbb{N}$ such that $\forall r > r_0$, $(\mathcal{I} - \mathcal{K}^{(r)})^{-1}$ exists and (5.1) holds. Finally, statements (i)-(iii) hold if $||\mathcal{K}||_O < 1$ or

$$\sup_{x\in\mathbb{R}} \left\{ \kappa(x, +\infty) + 2V_u(\kappa(x, u)) \right\} < 1. \tag{5.2}$$

Proof. This theorem is mainly direct results of Lemma 2.4: (i), (ii) and (iii) of Theorem 5.1 respectively correspond to (iii), (iv) and (v) of Lemma 2.4. Thus, (i), (ii) and (iii) of Theorem 5.1 are equivalent statements.

Additionally, according to Lemma 2.4, (i)-(iii) of Theorem 5.1 hold if $||\mathcal{K}||_O < 1$.

According to eq(SM2.8), $||\mathcal{K}||_O$ is bounded by $\sup_{x\in\mathbb{R}} \left\{ \kappa(x, +\infty) + 2V_u(\kappa(x, u)) \right\}$. Thus, (i)-(iii) of Theorem 5.1 hold if $\sup_{x\in\mathbb{R}} \left\{ \kappa(x, +\infty) + 2V_u(\kappa(x, u)) \right\} < 1$. ∎

Theorem 5.1 naturally introduces the following assumption, which is easy to verify.

ASSUMPTION 2. *Kernel $\kappa \in T(\mathbb{R}, \mathcal{B})$ satisfies inequality (5.2).*

Assumption 2 is sufficient for (i)-(iii) in Theorem 5.1 as well as Condition 4. In section 6, we will use numerical examples satisfying Assumption 2 to demonstrate the accuracy and efficiency of our solution methods. However, in practice, Assumption 2 can be too strong, particularly for stochastic models. A more practical sufficient condition for Condition 4 is outlined, due to space limitations, in supplemental material section SM6. Demonstrations with stochastic model examples can be found in [27].

5.2. Determining invertibility and computing approximation errors. In this subsection, we simplify our notations. Since we focus on only one instance of (A-IE), we omit the index (r). For example, knot $c_i^{(r)}$ will be denoted by c_i. The only exceptions are that operator $\mathcal{K}^{(r)}$ and kernel $\kappa^{(r)}$ will be denoted by $\mathcal{K}'$ and κ' instead to distinguish from the operator $\mathcal{K}$ in Definition 2.7 and κ in (IE).

The invertibility of $\mathcal{I} - \mathcal{K}'$ in Theorem 3.7 is easy to determine via determinants:

THEOREM 5.2 (Invertibility). *For an instance of operator $\mathcal{K}'$ in Definition 2.8, $(\mathcal{I}-\mathcal{K}')^{-1}$ exists iff matrix H is non-singular, i.e., $\det(H) \neq 0$, where H is defined by $H_{ij} := \delta_{ij} - \omega_i(c_j) + \omega_{i+1}(c_j)$, $i, j = 1, 2, ..., J$.*

Proof. ($\Rightarrow$) Due to the invertibility of $\mathcal{I}-\mathcal{K}'$, for any $\xi \in \bar{\mathbf{X}}$, there exists $F \in \bar{\mathbf{X}}$ such that $F(x) = \xi(x) + \sum_{i=1}^{J} \omega_i(x)[F(c_i) - F(c_{i-1})]$, $\forall x \in \mathbb{R}$, which yields a $J \times J$ linear equation system

$$\sum_{i=1}^{J} [\delta_{ij} - \omega_i(c_j) + \omega_{i+1}(c_j)] F(c_i) = \xi(c_j), \quad j = 1, 2, ..., J. \tag{5.3}$$

The coefficient matrix is exactly the transposed H. Since $(\xi(c_1), \xi(c_2), ..., \xi(c_J))$ can take any values in $\mathbb{R}^J$, H must be non-singular.

($\Leftarrow$) By the non-singularity of H, for any $\xi \in \bar{\mathbf{X}}$, there exists a vector $(F^*(c_1), F^*(c_2), ..., F^*(c_J)) \in \mathbb{R}^J$ that uniquely solves (5.3). This yields $F^*(c_j) = \xi(c_j) + \sum_{i=1}^{J} \omega_i(c_j)[F^*(c_i) - F^*(c_{i-1})]$, $j = 1, 2, ..., J$. Let us define

$$F(x) := \xi(x) + \sum_{i=1}^{J} \omega_i(x)[F^*(c_i) - F^*(c_{i-1})], \quad \forall x \in \mathbb{R}.$$

Then $(F^*(c_1), F^*(c_2), ..., F^*(c_J)) = (F(c_1), F(c_2), ..., F(c_J))$ and $\forall x \in \mathbb{R}$, $F(x) = \xi(x) + \sum_{i=1}^{J} \omega_i(x)[F(c_i) - F(c_{i-1})]$. Thus, $(\mathcal{I}-\mathcal{K}')F = \xi$ and $\mathcal{I}-\mathcal{K}'$ is surjective.

Suppose $F' \in \bar{\mathbf{X}}$ and $(\mathcal{I}-\mathcal{K}')F' = \xi$. Then $(F'(c_1), F'(c_2), ..., F'(c_J)) \in \mathbb{R}^J$ also solves (5.3). Since the solution is unique, we must have $(F'(c_1), F'(c_2), ..., F'(c_J)) = (F^*(c_1), F^*(c_2), ..., F^*(c_J)) = (F(c_1), F(c_2), ..., F(c_J))$. Thus,

$$\begin{aligned} F'(x) &= \xi(x) + \sum_{i=1}^{J} \omega_i(x)[F'(c_i) - F'(c_{i-1})] \\ &= \xi(x) + \sum_{i=1}^{J} \omega_i(x)[F(c_i) - F(c_{i-1})] = F(x), \quad \forall x \in \mathbb{R}. \end{aligned}$$

Therefore, $(\mathcal{I}-\mathcal{K}')F = \xi$ has a unique solution and $\mathcal{I}-\mathcal{K}'$ is injective.

Finally, because $\mathcal{I}-\mathcal{K}'$ is both injective and surjective, $\mathcal{I}-\mathcal{K}'$ is invertible. ◻

We also need to compute errors outlined in (3.1) of Theorem 3.7. For the first factor $||(\mathcal{I}-\mathcal{K}')^{-1}||_O$, we can use the following theorem.

THEOREM 5.3 (Norm of Inverse). *For an operator $\mathcal{K}'$ defined in Definition 2.8,*

(i) $||\mathcal{K}'||_O \leqslant \sup_{x\in\mathbb{R}} \left\{\kappa(x, +\infty) + V_u(\kappa(x, u))\right\}$ if $\mathcal{K}'$ belongs to any type in Definition 4.1. Moreover, if $||\mathcal{K}'||_O < 1$ or Assumption 2 holds, then $(\mathcal{I}-\mathcal{K}')^{-1}$ exists and $||(\mathcal{I}-\mathcal{K}')^{-1}||_O \leqslant \frac{1}{1-||\mathcal{K}'||_O}$.

(ii) If $(\mathcal{I}-\mathcal{K}')^{-1}$ exists and $\{\omega_i\}_{i=1}^J$ are step functions with knots $\{c_i\}_{i=1}^J$, i.e., $\omega_i(x) = \sum_{j=1}^{J} p_{ij} \mathbf{1}\{x \geqslant c_j\}$, $p_{ij} \in \mathbb{R}, i, j = 1, 2, ..., J$, then $||(\mathcal{I}-\mathcal{K}')^{-1}||_O^{-1}$ can be exactly obtained from a series of linear programs:

$$||(\mathcal{I}-\mathcal{K}')^{-1}||_O^{-1} = \min_{k=0,1,...,J} \{y_k^*\}, \tag{5.4}$$

where y_k^* is obtained from the following linear program (where $a_0 \equiv 0$):

$$\min_{y_k\in\mathbb{R}, \{a_j\}_{j=1}^J \in[-1,1]^J} y_k : -y_k \leqslant a_j + \tau\delta_{jk}(1-a_j) - \sum_{i=1}^{J} \omega_i(c_j)[a_i - a_{i-1}] \leqslant y_k,$$

$$j = 0, 1, ..., J, \tau \in \{0, 1\}.$$

Here (5.4) is developed based on the observation that, by definition, $||(\mathcal{I}-\mathcal{K}')^{-1}||_O^{-1} = \inf_{||F||_\infty \leqslant 1} ||(\mathcal{I}-\mathcal{K}')F||$. The decision variables $\{a_j\}_{j=1}^J$ and y_k are used to represent the infimum. A proof is in supplement material section SM7.

For term $||\mathcal{K}'\bar{F} - \mathcal{K}\bar{F}||_\infty$, we discuss its computation when $\bar{F}$ defines a finite measure. By definitions of $\mathcal{K}$, $\mathcal{K}'$ and κ, we have that

$$\begin{aligned}(5.5)\qquad \left|\mathcal{K}'\bar{F}(x) - \mathcal{K}\bar{F}(x)\right| &= \left|\int_{\mathbb{R}} \kappa'(x,u)d\bar{F}(u) - \int_{\mathbb{R}} \kappa(x,u)d\bar{F}(u)\right| \\ &= \left|\int_{\mathbb{R}} \big(\kappa'(x,u) - \kappa(x,u)\big)d\bar{F}(u)\right| \leqslant \bar{F}(+\infty)\cdot \sup_{x,u\in\mathbb{R}} |\kappa'(x,u) - \kappa(x,u)|,\end{aligned}$$

where the approximate kernel κ' is defined based on $\{\omega_i\}_{i=1}^J$, $\{c_i\}_{i=1}^J$ and equation (4.1). Thus, we have $||\mathcal{K}'\bar{F} - \mathcal{K}\bar{F}||_\infty \leqslant \bar{F}(+\infty)\cdot \sup_{x,u\in\mathbb{R}} |\kappa'(x,u) - \kappa(x,u)|$.

Remark: near-optimal approximation. Suppose the support is bounded, $\{\kappa^{(r)}\}_{r\in\mathbb{N}}$ are defined using Type II in Definition 4.1, and $\{\xi^{(r)}\}_{r\in\mathbb{N}}$ are step functions approximating ξ. If both ξ and κ in (IE) are Lipschitz continuous, then both $||\xi-\xi'||_\infty$ and $||\mathcal{K}'\bar{F} - \mathcal{K}\bar{F}||_\infty$ (bounded by $\bar{F}(+\infty)\cdot \sup_{x,u\in\mathbb{R}} |\kappa'(x,u) - \kappa(x,u)|$) can be as small as $O(\frac{1}{J^{(r)}})$, where $J^{(r)}$ is the number of knots used for approximate solution $\bar{F}^{(r)}$. Because $\{\bar{F}^{(r)}\}_{r\in\mathbb{N}}$ are discrete distributions, $J^{(r)}$ is also the number of jumps. Therefore, we can also conclude that the approximation error can be as small as $O(\frac{1}{J})$, where J is the number of jumps in the discrete approximate solution provided by finite approximation. Here, J can be regarded as a measure of complexity for discrete approximate solutions to (IE). We will verify this error in section 6.

Such an error is near-optimal among all methods that use discrete distributions as approximate solutions, as long as the true solution $\bar{F}$ is continuous and strictly increasing on some small interval. For a brief proof, we consider this special small interval. Suppose the true solution increases by δ on this interval. Because the true solution is continuous while the approximate solution is discrete, the error on this small interval is at least $\frac{\delta}{2(J+1)}$. Then the overall error is also at least $\frac{\delta}{2(J+1)}$, which is at least as large as the finite approximation error by a constant multiplier.

6. Numerical Experiments. In this section, we verify the accuracy of our approach by considering two example equations. The first equation has multiple discontinuities and an available solution. The second has "everywhere non-differentiable" elements, and its solution is not available. For each experiment, we evaluate approximation errors and use plots to show the generated solutions. All computations are performed on a 32G memory 3.6GHz 8-Core Intel platform.

6.1. Example I with multiple discontinuities. Let $\Omega = [0, 1)$ and consider

$$(6.1)\qquad \begin{aligned} F(x) &= \xi(x) + \int_\Omega \kappa(x,u)dF(u), \quad x \in \Omega, \\ \xi(x) &= -\frac{2.56}{3\times 10^3} + \frac{1.7257x}{60} + \sum_{k=0}^{2} \mathbf{1}\left\{x \geqslant \frac{k}{3}\right\} \cdot \frac{8^k(2-x)^9}{1.2\times 10^3} + \\ &\quad \sum_{k=0}^{3} \mathbf{1}\left\{x \geqslant \frac{k}{4}\right\} \cdot \frac{3^k(2-x)^9}{6\times 10^3} + \sum_{k=0}^{4} \mathbf{1}\left\{x \geqslant \frac{k}{5}\right\} \cdot \frac{3^k(2-x)^9}{3\times 10^3} + \\ &\quad \sum_{k=0}^{5} \mathbf{1}\left\{x \geqslant \frac{k}{6}\right\} \cdot \frac{1.7257}{3.6\times 10^2}, \quad x \in \Omega, \\ \kappa(x,u) &= \frac{6x + \sum_{k=0}^{5} \mathbf{1}\left\{x \geqslant \frac{k}{6}\right\}}{40(2-u)^8}, \quad x, u \in \Omega. \end{aligned}$$

The solution to (6.1) is $\bar{F}(x) = -\frac{256}{3\times 10^3} + \sum_{k=0}^{2} \mathbf{1}\left\{x \geqslant \frac{k}{3}\right\} \cdot \frac{8^k(2-x)^9}{1.2\times 10^3} + \sum_{k=0}^{3} \mathbf{1}\left\{x \geqslant \frac{k}{4}\right\} \cdot \frac{3^k(2-x)^9}{6\times 10^3} + \sum_{k=0}^{4} \mathbf{1}\left\{x \geqslant \frac{k}{5}\right\} \cdot \frac{3^k(2-x)^9}{3\times 10^3}, \quad \forall x \in \Omega.$

Equation (6.1) is considered in the space $\bar{\mathbf{X}}_\Omega$, defined by the collection of distributions in $\bar{\mathbf{X}}$ with support Ω. Condition 1 is satisfied because $\sup_{x\in\Omega} V_u\left(\kappa(x,u);\Omega\right) \leqslant \sup_{x\in\Omega} \frac{6x+\sum_{k=0}^{5}\mathbf{1}\left\{x\geqslant\frac{k}{6}\right\}}{40} < \frac{1}{3} < \infty$. Condition 2 is also satisfied because $\kappa(\cdot, 1)$ is right-continuous and $v_\Lambda(x_1,x_2) < \frac{3}{20}|x_2 - x_1|$, $\forall (x_1,x_2) \in \left[\frac{k}{6}, \frac{k+1}{6}\right)^2, k = 0, 1, ..., 5$.

Type I approximation. For all $r \in \mathbb{N}_{++}$, define a Type I sequence as $J^{(r)} = 7.5\cdot 2^r; \xi^{(r)}(x) = \xi(x),\ x \in \Omega;\ c_i^{(r)} = \frac{i-1}{7.5\cdot 2^r},\ i = 1, 2, ..., J^{(r)}; \omega_i^{(r)}(x) = \kappa(x, c_i^{(r)}),\ x \in \Omega, i = 1, 2, ..., J^{(r)}$. It is easy to show that $\{\kappa^{(r)}\}_{r\in\mathbb{N}_{++}}$ satisfy uniform boundedness, uniform variation-boundedness and convergence to κ. Thus, Condition 5 is satisfied. Also, (6.1) has a unique solution $\bar{F} \in \bar{\mathbf{X}}_\Omega$ (i.e., Condition 4) due to Theorem 5.1 and

$$\sup_{x\in\Omega}\left\{\kappa(x,1) + 2V_u(\kappa(x,u);\Omega)\right\} \leqslant \sup_{x\in\Omega} \frac{6x + \sum_{k=0}^{5}\mathbf{1}\left\{x \geqslant \frac{k}{6}\right\}}{40} \cdot 3 \leqslant \frac{9}{10} < 1.$$

With all preconditions satisfied, we have the convergence.

For error computation, we have $||\xi^{(r)} - \xi||_\infty = 0, r \in \mathbb{N}_{++}$. By (5.5), $||\mathcal{K}'\bar{F} - \mathcal{K}\bar{F}||_\infty < \frac{12.2}{5*2^r}, r \in \mathbb{N}_{++}$ (note $\bar{F}$ is a linear combination of probability distributions). Additionally, by (i) of Theorem 5.3, $||(\mathcal{I} - \mathcal{K}^{(r)})^{-1}||_O \leqslant \frac{1}{1-\frac{3}{5}} \leqslant 2.5$, $r \in \mathbb{N}_{++}$. Then (3.1) can be written as $||\bar{F}^{(r)} - \bar{F}||_\infty < \frac{6.1}{2^r}$.

Type II approximation. Alternatively, we define approximation equations by: $\omega_i^{(r)}(x) = \sum_{j=1}^{J^{(r)}} \kappa(c_j^{(r)}, c_i^{(r)})\mathbf{1}\{x \in [c_j^{(r)}, c_{j+1}^{(r)})\}, x \in \Omega, i = 1, 2, ..., J^{(r)}, r \in \mathbb{N}_{++}$; $\xi^{(r)}(x) = \sum_{j=1}^{J^{(r)}} \xi(c_j^{(r)})\mathbf{1}\{x \in [c_j^{(r)}, c_{j+1}^{(r)})\}, x \in \Omega, r \in \mathbb{N}_{++}$.

As in Type I case, $\{\kappa^{(r)}\}_{r\in\mathbb{N}_{++}}$ satisfy uniform boundedness, uniform variation boundedness and convergence to κ. Thus, Condition 5 is satisfied. Also, $\xi^{(r)} \rightrightarrows \xi$. With all preconditions satisfied, we have the convergence of finite approximation solutions. For error computation, because $|\xi'(x)| \leqslant 7$ if $60x \notin \mathbb{Z}$, we have $||\xi^{(r)} - \xi||_\infty \leqslant \frac{7}{7.5*2^r}, r \in \mathbb{N}_{++}$. By (5.5), $||\mathcal{K}^{(r)}\bar{F} - \mathcal{K}\bar{F}||_\infty < \frac{12.8}{5*2^r}, r \in \mathbb{N}_{++}$. Additionally, according to (i) of Theorem 5.3, $||(\mathcal{I} - \mathcal{K}^{(r)})^{-1}||_O \leqslant \frac{1}{1-\frac{3}{5}} \leqslant 2.5$, $r \in \mathbb{N}_{++}$. Then (3.1) can be written as $||\bar{F}^{(r)} - \bar{F}||_\infty < \frac{6.4}{2^r}$.

Results. Results for a series of finite approximation solutions ($r = 2, 4, ..., 12$ for Type I and II approximation sequences) are given in Table 3. In Table 3, the “number of knots” refers to the knots used in finite approximation. The “theoretical L^∞ error bounds” refer to the bound obtained from (3.1). The “empirical L^∞ errors” refer to the supremum-normed error between the true and the approximate solutions, computed via discretizing support $[0, 1)$ and enumerating $\{x \in [0, 1) | 10^6 x \in \mathbb{N}\}$. Similarly, the “empirical L^1 errors” refer to the mean absolute error. The true solution and some of the finite approximation solutions are plotted in Figure 2.

FIG. 2. *Plots of the true solution to distribution integral equation (6.1) and finite approximation solutions from (A-IE) ($r = 1$ for Type I and $r = 1, 2, 3, 4$ for Type II), each type with a convergence trend (discontinuities connected for visualization) in Experiment I.*

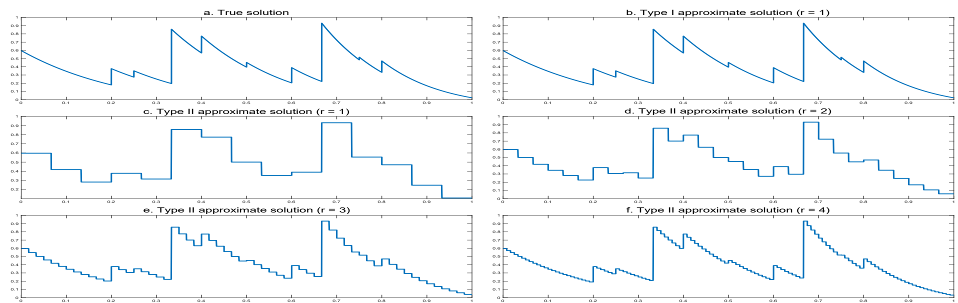


TABLE 3
Comparison of empirical L^1 errors, empirical L^∞ errors, theoretical L^∞ error bounds and computation time (seconds) across a series of finite approximation solutions ($r = 2, 4, ..., 12$ for both Type I and II) in Experiment I.

a. Type I approximate solutions

r	Number of knots	Empirical errors L^1	Empirical errors L^∞	Theoretical error bounds L^∞	Time
2	3.00E+01	5.62E-04	1.04E-03	1.53E+00	2.72E-03
4	1.20E+02	2.26E-04	4.17E-04	3.81E-01	6.62E-03
6	4.80E+02	6.94E-05	1.28E-04	9.53E-02	5.15E-02
8	1.92E+03	2.91E-05	5.38E-05	2.38E-02	8.24E-01
10	7.68E+03	1.90E-05	3.50E-05	5.96E-03	1.94E+01
12	3.07E+04	9.96E-07	1.84E-06	1.49E-03	5.50E+02

b. Type II approximate solutions

r	Number of knots	Empirical errors L^1	Empirical errors L^∞	Theoretical error bounds L^∞	Time
2	3.00E+01	4.73E-02	2.06E-01	1.60E+00	6.45E-02
4	1.20E+02	1.17E-02	5.55E-02	4.00E-01	1.08E-02
6	4.80E+02	2.91E-03	1.41E-02	1.00E-01	6.30E-02
8	1.92E+03	7.27E-04	3.55E-03	2.50E-02	8.27E-01
10	7.68E+03	1.82E-04	8.87E-04	6.25E-03	1.90E+01
12	3.07E+04	4.54E-05	2.21E-04	1.56E-03	5.22E+02

While the solutions generated by both Type I and Type II sequences converge, Type I generated solutions converge faster. With only 15 knots ($r = 1$), Type I generated solution is barely distinguishable from the true solution in Figure 2. In Table 3, the empirical L^1 errors, empirical L^∞ errors, and theoretical L^∞ errors of Type I are all smaller than that for Type II. We observe that the empirical and theoretical errors are in the order of $O(\frac{1}{N})$ (see remark in subsection 5.2). A reason for improved performance of Type I is that it can be regarded as an "iterated" version of Type II (see section 4). The extra iteration increases the "smoothness" of the approximate solutions. This advantage is gained with no significant addition to the computational time, as both approaches solve a system of N linear equations, where N is the number of knots.

With the Banach space and the operator equations constructed in this paper, we also tested a naive adaptation of Neumann method. Recall that the calculation of Neumann series requires computation of $\mathcal{K}^i\xi$. In our adaptation, we performed this calculation by discretizing the support $[0,1)$ and created $\{x \in [0,1) | 10^i x \in \mathbb{N}\} (i = 1, 2, ..., 5)$ equally spaced points to numerically evaluate all integrals. We found that the empirical L^∞ errors of the approach were greater than 0.72 even when using 10^5 discretization points and a similar computational effort to that of Type I or Type II approach. This is because such an approach does not consider possible jumps of the true solution and the error is reflected at these jump points in the final solution.

6.2. Example 2 with everywhere non-differentible functions. We construct the equation by taking the inhomogenenous term $\xi(x) := \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$ to be the Weierstrass function [34]. We use $a = \frac{1}{2}$, $b = 15$ for the numerical testing. The function is continuous but non-differentiable everywhere on $\Omega = [0, 1)$. With $\kappa(x, u) = \frac{xu}{4}$, an analytic solution of the integral equation $F(x) = \xi(x) + \int_\Omega \kappa(x, u) dF(u)$, $x \in \Omega$ is not available. Note that Condition 1 is satisfied because $\sup_{x\in\Omega} V_u\left(\kappa(x, u); \Omega\right) = \sup_{x\in\Omega} \frac{x}{4} = \frac{1}{4} < \infty$. Condition 2 is also satisfied because $\kappa(\cdot, 1)$ is right-continuous and $v_\Lambda(x_1, x_2) = \frac{1}{4}|x_2 - x_1|$, $\forall (x_1, x_2) \in \Omega$. Let us define a Type II sequence: for all $r \in \mathbb{N}$, $J^{(r)} = 2^r; \xi^{(r)}(x) = \sum_{k=0}^{r} a^k \cos(b^k \pi x)$, $x \in \Omega; c_i^{(r)} = \frac{i-1}{2^r}$, $i = 1, 2, ..., J^{(r)}; \omega_i^{(r)}(x) = \sum_{j=1}^{J^{(r)}} \kappa(c_j^{(r)}, c_i^{(r)}) \mathbf{1}\{x \in [c_j^{(r)}, c_{j+1}^{(r)})\}$, $x \in \Omega, i = 1, 2, ..., J^{(r)}$. It is easy to show that $\{\kappa^{(r)}\}_{r\in\mathbb{N}}$ satisfy uniform boundedness, uniform variation-boundedness and convergence to κ. Thus, Condition 5 is satisfied. Additionally, $\xi^{(r)} \rightrightarrows \xi$. The integral equation also has a unique solution $\bar{F} \in \bar{\mathbf{X}}$ (i.e., Condition 4 is satisfied) because $\sup_{x\in\Omega} \left\{\kappa(x, 1) + 2V_u(\kappa(x, u); \Omega)\right\} = \sup_{x\in\Omega} \frac{x \cdot 3}{4} = \frac{3}{4} < 1$ (see Theorem 5.1). Given all preconditions, we have the solution convergence.

Figure 3 plots the generated approximate solutions. Since the true solution is not available, we use concepts of "equation residue" and "error from available solutions" to discuss solution errors. The "L^∞ equation residue" is obtained by evaluating the maximal (across x) absolute difference of right and left side of the equation with each approximate solution, while "L^1 equation residue" is the mean of absolute difference. Since the integration is not computable in closed form, we estimate its value by discretizing support $[0, 1)$ and enumerating $\{x \in [0, 1) | 10^4 x \in \mathbb{N}\}$. The "$L^\infty$ error from available solutions" is the maximal (across x) absolute difference (scaled by 2) of the r-th approximate solution with subsequent solutions (till $r = 15$), while "L^1 error from available solutions" is the mean absolute difference. All maximal/mean errors are obtained by discretizing support $[0, 1)$ and enumerating $\{x \in [0, 1) | 10^6 x \in \mathbb{N}\}$. Statistics of a series of approximate solutions are given in Table 4.

Figure 3 and Table 4 indicate that the approximate solutions are converging despite ξ being an everywhere non-differentiable Weierstrass function. Because of the nature of ξ, the true solution has a huge variation. In Figure 3, the first approximate solution ($r = 1$) is able to roughly describe the "locally-averaged growth trend". Subsequent approximate solutions add local variations and gradually converge.

FIG. 3. *Plot of the approximate solutions obtained from* (A-IE) *(* $r = 1, 3, 7, 9, 11, 13$ *) with a convergence trend in Experiment II.*

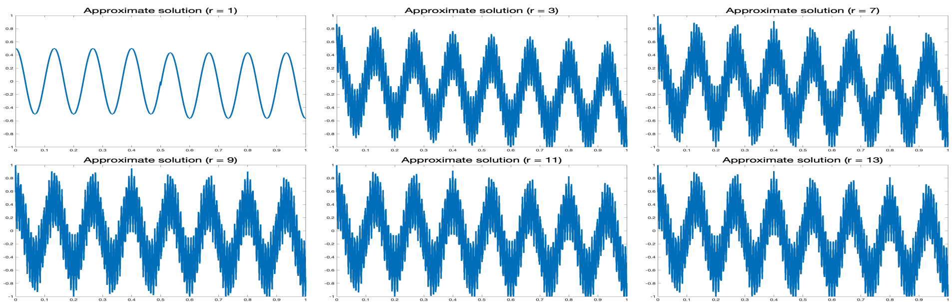


TABLE 4

Comparison of equation residues, errors from available solutions and computation time (seconds) across a sequence of approximation equations ($r = 1, 4, 7, 10, 13$) *in Experiment II.*

r	Number of knots	Equation residue		Error from available solutions		Time
		L^1	L^∞	L^1	L^∞	
1	2.00E+00	7.04E-02	1.70E-01	3.09E-02	1.12E-01	2.43E-03
4	1.60E+01	1.30E-01	2.60E-01	5.57E-02	1.58E-01	1.68E-03
7	1.28E+02	1.58E-02	3.17E-02	1.17E-02	6.49E-02	2.79E-03
10	1.02E+03	2.31E-02	4.62E-02	1.16E-02	3.61E-02	8.65E-02
13	8.19E+03	1.82E-03	3.71E-03	2.34E-03	7.61E-03	1.10E+01

7. Conclusion.

The numerical examples used to demonstrate the effectiveness of our approach satisfy $||\mathcal{K}||_O \leqslant 1$. The developed approach is also valid for $||\mathcal{K}||_O \geqslant 1$. Our method's performance in solving problems with $||\mathcal{K}||_O \geqslant 1$ is discussed in [27], where it is further developed for use to find stationary distributions of continuous-state Markov chains. Since the approximate solutions are obtained via solving a linear equation system, its computational performance can be further improved by taking advantage of the properties of matrices defining the linear system.

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