This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3321901

Momentum-Based Nash Set-Seeking Over Networks via Multi-Time Scale Hybrid Dynamic Inclusions

Daniel E. Ochoa and Jorge I. Poveda

***Abstract*—Multi-time scale techniques, such as singular perturbations and averaging theory, have played an important role in the development of distributed Nash equilibrium seeking algorithms for network systems. Such techniques rely on the *uniform asymptotic stability* properties of the dynamics that evolve in each of the time scales of the closed-loop system. When such properties are absent, the synthesis of multi-time scale Nash equilibrium-seeking algorithms is more challenging and it requires additional regularization mechanisms. In this paper, we investigate the synthesis and analysis of these mechanisms in the context of *accelerated pseudogradient flows* with time-varying damping in non-cooperative games. Specifically, we introduce a new class of *distributed and hybrid* Nash set-seeking (NSS) algorithms that synergistically combine dynamic momentum-based flows with *coordinated* discrete-time resets. The reset mechanisms can be seen as restarting techniques that allow individual players to choose their own momentum restarting policy to potentially achieve better transient performance. The resulting closed-loop system is modeled as a hybrid dynamic inclusion, which is analyzed using tools from hybrid dynamical system's theory. Our algorithms are developed for potential games, as well as for monotone games for which a potential function does not exist. They can be implemented in games where players have access to gradient Oracles with full or partial information of the multi-agent system, as well as in games where players have access only to *measurements* of their costs. In the latter case, we use tools from hybrid extremum seeking control.**

***Index Terms*—Learning in Games, Nash equilibria, Noncooperative games, Hybrid Dynamical Systems.**

I. Introduction

AMONG the different notions of equilibria related to game-theoretic models, the notion of Nash equilibrium (NE), introduced in [1], has become ubiquitous in many engineering and socio-technical systems such as transportation networks [2], energy systems [3], [4], and robotic networks [5], to name just a few. To converge to this equilibrium, different Nash equilibrium-seeking (NES) algorithms have been developed during the last decades, see [6]–[9]. In the context of game-theoretic control system design, many results in the literature are somehow inspired or related to the time-invariant pseudogradient (PSG) flows studied by Rosen in [10], which take the form $\dot{q} = -\mathcal{G}(q)$, where $\mathcal{G}$ is the *pseudogradient* vector of the game [10, Eq. (3.9)], and $q \in \mathbb{R}^n$ is the vector of actions of the players. For example, it is well-known that in potential games PSG flows can robustly minimize the potential function at a rate of order $\mathcal{O}(1/t)$. Additionally, for strongly monotone games, pseudogradient flows can achieve NES with exponential rates of convergence of order $\mathcal{O}(e^{-\kappa t})$, with κ being the strong monotonicity coefficient of $\mathcal{G}$. These stability and convergence results have become instrumental for the design of extended NES algorithms that incorporate additional mechanisms based on fast consensus dynamics [7], projections [11], tracking terms [12], adaptive dynamics [13], etc. See also the recent work [8] and references therein. However, while these results have provided significant insight into the design of NES dynamics, existing results still suffer from transient limitations inherited from PSG flows, which can be further exacerbated in games with shallow monotonicity properties.

On the other hand, compared to PSG flows, time-varying momentum-based dynamics, which are common in the optimization and machine learning literature [14]–[18], have not received as much attention in the context of games. In particular, in this paper we are interested in studying the Nash equilibria learning capabilities of the second-order dynamics

$$\dot{q} = \frac{2}{\tau}(p - q), \quad \dot{p} = -2\tau\mathcal{G}(q), \quad \dot{\tau} = \eta, \tag{1}$$

with $\tau(0){=}T_0{>}0$ and $\eta > 0$, which are related to the continuous-time approximation of Nesterov's optimization algorithm [14] via the transformation $p = \frac{\tau}{2}\dot{q} + q$ when $\mathcal{G}$ is a gradient operator. Such systems are particularly useful for optimization problems with cost functions having vanishing curvature at the optimal points, since they exhibit a geometric property, termed *acceleration*, able to minimize smooth convex functions at a rate of order $\mathcal{O}(1/t^2)$. Moreover, in strongly convex optimization problems, systems of the form (1), combined with suitable "restarting" heuristics, can achieve exponential rates of convergence [14], [19]. Indeed, dynamics of the form (1) have been shown to accelerate convergence in adaptive estimation problems [20], extremum seeking control [17], and concurrent learning techniques [21]. Hence, in light of intriguing numerical results in the context of games, it is natural to ask whether systems of the form (1) are also suitable for the *robust* and *efficient* solution of NES problems in noncooperative games, and whether these dynamics can be extended to network games and model-free settings. In fact, most of the existing results in the literature have focused only on (non-uniform) convergence results in centralized potential games [22], or in momentum-based dynamics with maximally monotone operators $\mathcal{G}$ via Yosida regularizations of the form

The authors are with the Department of Electrical and Computer Engineering at the University of California San Diego, CA, USA. E-mails: `dochoatamayo,poveda@ucsd.edu`. Work supported in part by AFOSR grant FA9550-22-1-0211 and NSF grant CAREER ECCS 2305756. Corresponding author: Jorge I. Poveda.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3321901

$\frac{1}{\lambda}(I-(I+\lambda\mathcal{G})^{-1})$, which are usually not suitable for distributed implementations [23].

Main Results: In this paper, we provide answers to the above questions by using tools from nonlinear control theory. First, we show that the direct implementation of (1) is, in general, not suitable for the efficient distributed solution of Nash set-seeking (NSS) problems in non-cooperative games, even when the game is strongly monotone and there exists a potential function. The limitations arise from three main structural issues: First, the dependence on a "centralized" momentum coefficient τ that precludes distributed implementations, and that can also lead to *uncoordinated* algorithms with poor transient performance. Second, the unbounded grow of τ in system (1) makes them prone to instability under arbitrarily small additive disturbances, unavoidable in feedback-based implementations. Third, in non-potential games, traditional Lyapunov functions used in optimization are not applicable, and solutions to (1) may even diverge, despite game monotonicity and bounded states τ.

While the above features might suggest that momentum-based dynamics are problematic in the context of games, it turns out that for suitable classes of non-cooperative games, systems of the form (1) *can* be used to efficiently and robustly find NE in a decentralized way, whenever they are combined with suitable distributed discrete-time dynamics that persistently reset some of the states of the players in a *coordinated* way. However, in contrast to results in the optimization literature [14], [17], [19], our results suggest that for general (non-potential) noncooperative games the frequency of the resets must occur in a certain frequency band to simultaneously achieve stability and suitable convergence properties. We establish these results using tools from hybrid dynamical systems (HDS) theory [24], and we leverage their intrinsic robustness properties to extend the algorithms to decentralized network games and model-free settings via multi-time scale hybrid control theoretic tools. Based on this, the following original contributions are presented in the paper:

i) We propose the first NSS algorithms with continuous-time *dynamic* momentum and *robust* asymptotic stability properties in non-cooperative games with n players. The algorithms incorporate three main elements: a) a class of distributed *continuous-time* pseudogradient-based dynamics with *time-varying* momentum coefficients inspired by (1); b) distributed periodic *discrete-time* resets implemented by the players, which incorporate *heterogeneous* reset policies that allow players to decide whether or not to restart their own momentum; c) a robust *set-valued* distributed coordination mechanism that synchronizes the reset times of the players to induce suitable network-wide acceleration properties.

ii) To accommodate situations where players do not have access to full-information Oracles that provide evaluations of their pseudogradients, we introduce a new *distributed* momentum-based hybrid NSS algorithm for games with partial information, where players leverage communication with neighbors to estimate their actions on-the-fly. The design of these dynamics follows similar multi-time scale ideas used for ODEs in the literature [7], but which are not directly applicable to systems of the form (1). Indeed, unlike existing results based on fast consensus dynamics and "reduced" pseudogradient flows, our reduced dynamics are hybrid and set-valued, which prevents the direct application of standard tools for ODEs.

iii) We present payoff-based versions of all our hybrid NSS algorithms, suitable for *model-free* learning in non-cooperative games where players have access only to measurements of their cost. These dynamics exploit recent tools developed in the context of averaging-based hybrid extremum seeking control [17], and their analysis is fundamentally different from other model-free non-hybrid algorithms studied in the literature, e.g. [11], [25]. In particular, the dynamics considered in this paper have set-valued jump maps that lead to non-unique solutions with non-trivial hybrid time domains having multiple simultaneous jumps in the continuous time domain, a behavior that is unavoidable in decentralized multi-agent HDS. We also show that these adaptive dynamics can approximately recover the convergence bounds of the model-based algorithms.

To the best of our knowledge, the algorithms presented in this paper are the first in the literature to implement dynamic momentum and distributed restarting techniques in n-player noncooperative games.

The rest of this paper is organized as follows. Section II presents preliminaries. Section III presents the problem statement. Section IV presents the hybrid NSS dynamics for games with full-information. Section V relaxes this assumption using multi-time estimation techniques, and Section VI presents the model-free results. Section VII presents the analysis, and Section VIII presents the conclusions.

II. Preliminaries

1) Notation: Given a compact set $\mathcal{A}\subset\mathbb{R}^n$ and a vector $z\in\mathbb{R}^n$, we use $|z|_{\mathcal{A}}:=\min_{s\in\mathcal{A}}\|z-s\|_2$. We use $\mathbf{1}_n$ to represent an n-dimensional vector with 1 in all its entries, and define $\mathbf{1}_n\cdot A:=\{x\in\mathbb{R}^n\ :\ x_1=x_2=\ldots=x_n=a,\ a\in A\}$, for any set $A\subset\mathbb{R}$. We use $\mathbb{S}^1:=\{z\in\mathbb{R}^2 : z_1^2+z_2^2=1\}$ to denote the unit circle in $\mathbb{R}^2$, and $\mathbb{T}^n$ to denote the n^{th} Cartesian product of $\mathbb{S}^1$. We also use $r\mathbb{B}$ to denote a closed ball in the Euclidean space, of radius $r>0$, and centered at the origin. We use $I_n\in\mathbb{R}^{n\times n}$ to denote the identity matrix, and $(x,y):=[x^\top,y^\top]^\top$ for the concatenation of the vectors x and y. Also, we use $\mathcal{D}(k)$ to represent a diagonal matrix of appropriate dimension whose diagonal is given by the entries of a vector k. We also use $\overline{k}$ (resp. $\underline{k}$) to denote the largest (resp. smallest) entry of k. A function $\beta:\mathbb{R}_{\geq0}\times\mathbb{R}_{\geq0}\to\mathbb{R}_{\geq0}$ is said to be of class $\mathcal{KL}$ if it is non-decreasing in its first argument, non-increasing in its second argument, $\lim_{r\to0^+}\beta(r,s)=0$ for each $s\in\mathbb{R}_{\geq0}$, and $\lim_{s\to\infty}\beta(r,s)=0$ for each $r\in\mathbb{R}_{\geq0}$.

2) Games: In this paper, we consider noncooperative games with $n\in\mathbb{Z}_{\geq2}$ players, where each player i can control its own action q_i, and has access to the actions q_j of neighboring players $j\in\mathcal{N}_i:=\{j\in\mathcal{V}:(i,j)\in\mathcal{E}\}$, who are characterized by an undirected, connected, and time-invariant graph $\mathbb{G}=\{\mathcal{E},\mathcal{V}\}$, where $\mathcal{V}=\{1,2,\ldots,n\}$ is the set of players and $\mathcal{E}$ is the set of edges between players. We use $\mathcal{L}$ to denote the Laplacian matrix of the graph $\mathbb{G}$. The main goal of each player i is to minimize its own cost function $\phi_i:\mathbb{R}^n\to\mathbb{R}$ by controlling its own action q_i. We assume that the costs ϕ_i are *twice continuously differentiable*, and we use $q = (q_1, q_2, \ldots, q_n)$ to denote the overall vector of actions of the game. We also use q_{-i} to denote the vector of all actions with the action of player i removed. To simplify our exposition, we assume that the actions q_i are scalars. However, all our results also hold for vectorial actions by using suitable Kronecker products. We use $\mathcal{G}$ to denote the pseudogradient of the game, where $q \mapsto \mathcal{G}(q) := \left(\frac{\partial \phi_1(q)}{\partial q_1}, \frac{\partial \phi_2(q)}{\partial q_2}, \ldots, \frac{\partial \phi_n(q)}{\partial q_n}\right) \in \mathbb{R}^n$. Following standard assumptions in the literature of fast NES [9] and accelerated optimization [14]–[18], in this paper we will work with the following assumptions.

Assumption 1: The mapping $\mathcal{G}$ is ℓ-globally Lipschitz, i.e., there exists a constant $\ell > 0$ such that $|\mathcal{G}(q) - \mathcal{G}(q')| \leq \ell |q - q'|$, for all $q, q' \in \mathbb{R}^n$. □

Assumption 2: The mapping $\mathcal{G}$ is $1/\ell$-cocoercive, i.e., there exists ℓ such that $\big(\mathcal{G}(q) - \mathcal{G}(q')\big)^\top (q - q') \geq \frac{1}{\ell}|\mathcal{G}(q) - \mathcal{G}(q')|^2$ for all $q, q' \in \mathbb{R}^n$. Moreover, $|\mathcal{G}(\cdot)|^2$ is radially unbounded. □

The first property of Assumption 2 implies Assumption 1, but the converse is not necessarily true in non-potential games [26]. We will also use the following definition to characterize the monotonicity properties of the games.

Definition 1: A game with pseudogradient $\mathcal{G}$ is said to be:

1) *Monotone* if $\big(\mathcal{G}(q) - \mathcal{G}(q')\big)^\top (q - q') \geq 0$, for all $q,\ q' \in \mathbb{R}^n$.
2) *Strictly monotone* if $\big(\mathcal{G}(q) - \mathcal{G}(q')\big)^\top (q - q') > 0$, for all $q \neq q' \in \mathbb{R}^n$.
3) *κ-Strongly Monotone* with $\kappa > 0$, if $\big(\mathcal{G}(q) - \mathcal{G}(q')\big)^\top (q - q') \geq \kappa |q - q'|^2$, for all $q, q' \in \mathbb{R}^n$.
4) *κ-Strongly Monotone quadratic* if it is a κ-Strongly Monotone game with $\mathcal{G}(q) = Aq + b$ for some $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$.
5) *Potential* if there exists a continuously differentiable and radially unbounded function $P : \mathbb{R}^n \to \mathbb{R}$, such that $\mathcal{G}(q) = \nabla P(q)$, for all $q \in \mathbb{R}^n$. □

Remark 1: Cocoercive maps are monotone but not necessarily strongly monotone. Games that are κ-strongly monotone and ℓ-Lipschitz are κ/ℓ^2-cocoercive [26, Prop. 2.1]. □

We will also work with the following assumption.

Assumption 3: The function $\phi_i : \mathbb{R} \to \mathbb{R}$ is radially unbounded in q_i for ever $q_{-i} \in \mathbb{R}^{n-1}$ and all $i \in \mathcal{V}$. □

3) Hybrid Dynamical Systems: To study our algorithms, in this paper we consider HDS with state $x \in \mathbb{R}^n$, and dynamics

$$x \in C,\ \dot{x} = F(x), \quad \text{and} \quad x \in D,\ \ x^+ \in G(x), \tag{2}$$

where $x \in \mathbb{R}^n$ is the state of the system, $F : \mathbb{R}^n \to \mathbb{R}^n$ is called the flow map, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping called the jump map, and $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ are closed sets, called the flow set and the jump set, respectively [24]. We use $\mathcal{H} = (C, F, D, G)$ to denote the data of the HDS $\mathcal{H}$. For a precise definition of *hybrid time domains* and *solutions* to HDS of the form (2), we refer the reader to [24, Ch.2]. The following definitions will be used throughout the paper.

Definition 2: The compact set $\mathcal{A} \subset C \cup D$ is said to be *uniformly globally asymptotically stable* (UGAS) for system (2) if $\exists\ \beta \in \mathcal{KL}$ such that every solution x satisfies:

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t + j),\ \forall\ (t,j) \in \mathrm{dom}(x). \tag{3}$$

for all $x(0,0) \in \mathbb{R}^n$. When $\beta(r,s) = c_1 r e^{-c_2 s}$ for some $c_1, c_2 > 0$, the set $\mathcal{A}$ is *uniformly globally exponentially stable* (UGES). When $\exists\ T^* > 0$ such that $\beta(r,s) = 0$, $\forall\ s \geq T^*, r > 0$, the set $\mathcal{A}$ is said to be *uniformly globally fixed-time stable* (UGFxS). □

We will also consider ε-parameterized HDS of the form:

$$x \in C_\varepsilon,\ \ \dot{x} = F_\varepsilon(x), \quad \text{and} \quad x \in D_\varepsilon,\ \ x^+ \in G_\varepsilon(x), \tag{4}$$

where $\varepsilon > 0$. For these systems we will study *semi-global practical stability* properties as $\varepsilon \to 0^+$.

Definition 3: The compact set $\mathcal{A} \subset C \cup D$ is said to be *Semi-Globally Practically Asymptotically Stable* (SGP-AS) as $\varepsilon \to 0^+$ for system (4) if $\exists\ \beta \in \mathcal{KL}$ such that for each pair $\delta > \nu > 0$ there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ every solution of (4) with $|x(0,0)|_{\mathcal{A}} \leq \delta$ satisfies

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t + j) + \nu, \tag{5}$$

$\forall\ (t,j) \in \mathrm{dom}(x)$. When β is exponential, we say that $\mathcal{A}$ is *semi-globally practically exponentially stable* (SGP-ES). □

The notions of SGP-AS (-ES) can be extended to systems that depend on multiple parameters $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell)$. In this case, we say that $\mathcal{A}$ is SGP-AS as $(\varepsilon_\ell, \ldots, \varepsilon_2, \varepsilon_1) \to 0^+$ where the parameters are tuned in order starting from ε_1.

Definition 4 (Robustness): Consider the perturbed HDS

$$x + e \in C, \qquad \dot{x} = F(x + e) + e, \tag{6a}$$

$$x + e \in D, \quad x^+ \in G(x + e) + e, \tag{6b}$$

where e is measurable, and $\sup_{(t,j)\in \mathrm{dom}(e)} |e(t,j)| \leq \varepsilon$ with $\varepsilon > 0$. System (6) is said to be R-UGAS (resp. R-UGES) if: 1) it is UGAS (resp. ES) when $\varepsilon = 0$; and 2) it is SGP-AS (resp. SGP-ES) as $\varepsilon \to 0^+$. □

Definition 5: Two hybrid signals $x_1 : \mathrm{dom}(x_1) \to \mathbb{R}^n$ and $x_2 : \mathrm{dom}(x_2) \to \mathbb{R}^n$ are said to be (T, J, ε)-close if: (1) for each $(t,j) \in \mathrm{dom}(x_1)$ with $t \leq T$ and $j \leq J$ there exists s such that $(s,j) \in \mathrm{dom}(x_2)$, with $|t - s| \leq \varepsilon$ and $|x_1(t,j) - x_2(t,j)| \leq \varepsilon$; (2) for each $(t,j) \in \mathrm{dom}(x_2)$ with $t \leq T$ and $j \leq J$ there exists s such that $(s,j) \in \mathrm{dom}(x_1)$, with $|t - s| \leq \varepsilon$ and $|x_2(t,j) - x_1(t,j)| \leq \varepsilon$. □

III. Problem Statement and Motivation

A NE is defined as an action profile $q^* \in \mathbb{R}^n$ that satisfies

$$\phi_i(q_i^*, q_{-i}^*) = \inf_{q_i \in \mathbb{R}} \phi_i(q_i, q_{-i}^*), \ \ \forall\ i \in \mathcal{V}. \tag{7}$$

When the game is monotone, q^* is a NE if and only if $\mathcal{G}(q^*) = 0$ [27, Prop. 2.1]. Moreover, strict monotonicity of $\mathcal{G}$ implies that there is exactly one NE, if it exists. For κ-strongly monotone games and monotone potential games, existence is always guaranteed [28, Thm. 2.3.3].

Our goal is to *efficiently* and *robustly* find the set of points q^* that satisfy (7), denoted $\mathcal{A}_{\text{NE}}$, using algorithms with dynamic momentum. However, as the following example shows, this task is not trivial, even for potential games.

Example 1: (*Instability Under Small Disturbances*) Consider a duopoly game with pseudogradient $\mathcal{G}(q) = Aq + b$, where $A = [10, -5; -5, 10]$, and $b = [-250, -150]$. This is a κ-strongly monotone potential game, studied in [11, Sec. II] using PSG flows. The unique NE is $q^* = (130/3, 110/3)$, and since A is symmetric, the game has a (quadratic) potential

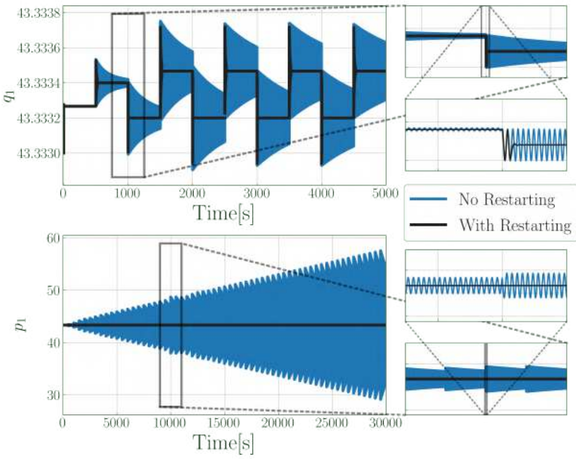

Fig. 1. Instability of (1) in a duopoly game with perturbed gradients and $T_0 = 2\sqrt{2} \times 10^{-3}$. The instability can be removed by incorporating resets, which generate the stable trajectories shown in black.

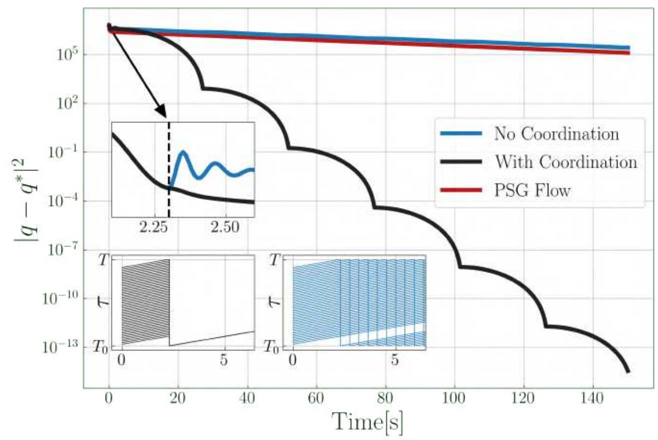

Fig. 2. Coordinated vs non-coordinated resets in a quadratic κ-strongly monotone potential-game with $\kappa = 0.01, \ell = 100$ and $n = 30$. The insets show the evolution of the states τ_i with and without coordination mechanisms.

function, which permits the direct application of [14, Thm. 3] to conclude convergence of all functions q generated by (1) towards the NE q^*. Nevertheless, if (1) is implemented with a perturbed gradient $\mathcal{G}(q) + e(t)$, where e is an arbitrarily small bounded disturbance, the highly oscillatory unstable behavior, shown in blue in Figure 1, emerges. In this case, there is no $\beta \in \mathcal{KL}$ such that the bound (5) holds for the solutions of (1), [29, Thm.1]. However, we will show that this bound actually exists when the signals $(\tau, \dot{q})$ are reset, generating the stable behavior shown in black color in Figure 1. □

The robustness issues illustrated in Example 1 prevent the direct implementation of the momentum-based dynamics (1) in noisy environments, or in settings where some of the states or gradients are computed on-the-fly using multi-time scale techniques such as singular perturbations or averaging theory. In fact, such techniques usually require "reduced" or "average" systems with stability properties characterized by $\mathcal{KL}$ bounds of the form (3); see [30, Assumption 4.].

While the incorporation of resets can help stabilize system (1), as the following example illustrates, *uncoordinated* resets can eventually impede the potential advantages of using algorithms with dynamic momentum.

Example 2: (Slow Convergence and Uncoordinated Resets) Consider a *distributed* implementation of system (1) in a κ-strongly monotone potential-game with 30 players and $\kappa = 0.01$. Each player i implements its own states (q_i, p_i, τ_i), with dynamics $\dot{q}_i = \frac{2}{\tau_i}(p_i - q_i)$, $\dot{p}_i = -2\tau_i \frac{\partial \phi_i}{\partial q_i}$, and $\dot{\tau}_i = \eta$, with $\eta = \frac{1}{2}$. Also, players implement periodic resets of $(\tau_i, \dot{q}_i)$ every 25 seconds (in their own local time reference frame) via the individual jump maps $\tau_i^+ = 0.1$ and $p_i^+ = q_i$. While this periodic reset strategy has been shown to guarantee fast convergence in *centralized* optimization problems, e.g., [31, Thm. 1], Figure 2 shows the behavior in noncooperative games with distributed and uncoordinated resets. As shown in blue, the solutions of (1) actually converge to the NE, but at a slower rate compared to the PSG flow. On the other hand, the trajectory corresponding to players implementing *coordinated* resets, shown in black color, achieves much faster performance by exploiting momentum, c.f. Theorem 1. □

The following example shows that even when resets are implemented in a centralized manner, in non-potential games, the solutions to (1) may not converge to the NE.

Example 3: (*Instability in Non-Potential Games*) We consider a non-potential κ-strongly monotone quadratic game with 30 players and $\kappa = 0.02$. For this game, the standard PSG flow guarantees exponential convergence via [10, Thm. 1]. However, as shown in color blue in Figure 3, system (1) generates trajectories that diverge, even when resets are (slowly) implemented in a centralized manner. The same plot shows in black color a trajectory that rapidly converges to the unique NE of the game. We will show that this stable and fast behavior can be guaranteed using a *hybrid* algorithm with distributed coordinated resets that dissipate energy "sufficiently often" via suitable contraction properties; c.f., Theorem 3. □

IV. Distributed Hybrid NSS Dynamics with Coordinated Restarting

To achieve robust NSS with dynamic momentum, we start by endowing each player $i \in \mathcal{V}$ with a state $x_i = (q_i, p_i, \tau_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$, and a gradient Oracle that provides real-time measurements of the partial derivative $\frac{\partial \phi_i(q)}{\partial q_i}$ at the overall action state $q \in \mathbb{R}^n$. The reset mechanisms of the players make use of three positive tunable parameters (η, T_0, T), which satisfy $T > T_0 > 0$ and $1/2 \geq \eta > 0$, and which are selected *a priori* by the system designer. The state x_i evolves according to hybrid dynamics that are coordinated by a local timer τ_i. In particular, the continuous-time dynamics of each player are

$$\tau_i \in [T_0, T) \Longrightarrow \begin{pmatrix} \dot{q}_i \\ \dot{p}_i \\ \dot{\tau}_i \end{pmatrix} = F_i(x) := \begin{pmatrix} \frac{2}{\tau_i}(p_i - q_i) \\ -2\tau_i \frac{\partial \phi_i(q)}{\partial q_i} \\ \eta \end{pmatrix}, \quad (8)$$

and the discrete-time dynamics are given by

$$\tau_i = T \Longrightarrow \begin{pmatrix} q_i^+ \\ p_i^+ \\ \tau_i^+ \end{pmatrix} = R_i(x_i) := \begin{pmatrix} q_i \\ \alpha_i p_i + (1-\alpha_i) q_i \\ T_0 \end{pmatrix}. \quad (9)$$

In (9), the parameters $\alpha_i \in \{0, 1\}$ model the different individual *reset policies* of the players. The choice $\alpha_i = 0$ leads to resets of the form $p_i^+ = q_i$, which corresponds to

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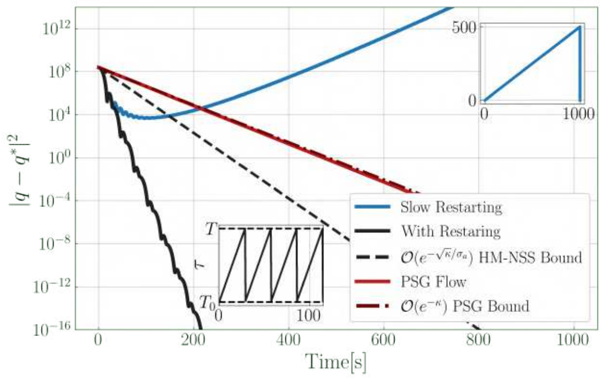

Fig. 3. Lack of convergence of trajectories of (1) in a non-potential κ-strongly monotone quadratic game with $\boldsymbol{\kappa = 0.02,\ \ell = 0.0214,\ n = 30,\ T_0 = 0.1,\ T = 3.74}$. The black line shows the trajectory of the proposed hybrid controller with resets.

$\dot{q}_i^+ = 0$, i.e., the momentum of player i is reset. On the other hand, $\alpha_i = 1$ corresponds to keeping p_i constant.

Since players have access to Oracles that provide real-time evaluations of their gradient, they can implement the hybrid dynamics (8)-(9) in a fully decentralized way by running their own timers τ_i to coordinate the flows (8) and the jumps (9). However, as shown in Example 2, lack of coordination between the resets of the players can hinder the acceleration properties expected from using dynamic momentum. To address this issue, we proceed to endow each player with a distributed hybrid coordination mechanism for the resets.

A. Coordinated Distributed Resets

The coordination mechanism of each player $j \in \mathcal{V}$ uses a set-valued coordination mapping $\mathcal{C}_j : \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}_{\geq 0}$, defined as

$$\mathcal{C}_j(\tau_j) := \begin{cases} T & \text{if } \tau_j \in (T_0 + r_j, T] \\ \{T_0, T\} & \text{if } \tau_j = T_0 + r_j \\ T_0 & \text{if } \tau_j \in [T_0, T_0 + r_j) \end{cases}, \tag{10}$$

where the individual parameter $r_j > 0$ satisfies $r_j \in \left(0, \frac{T-T_0}{n}\right)$. Using $\mathcal{C}_j$, the coordination mechanism works as follows: whenever the timer of player i satisfies $\tau_i = T$, the following two events happen: 1) player i resets its own state x_i using the dynamics (9), and 2) player i sends a pulse to its neighbors $j \in \mathcal{N}_i$, who proceed to update their state x_j as follows:

$$q_j^+ = q_j, \qquad p_j^+ = p_j, \qquad \tau_j^+ \in \mathcal{C}_j(\tau_j). \tag{11}$$

Since player i can only signal its neighbors, the rest of the players $j \notin \mathcal{N}_i$ will keep their states constant after the above two events, i.e., $x_j^+ = x_j$, for all $j \notin \mathcal{N}_i$.

The combination of continuous-time dynamics with momentum (8), and the set-valued discrete-time dynamics that model the coordinated resets leads to a HDS of the form (2), where multiple resets can happen simultaneously (in the continuous-time domain) when more than two players satisfy the condition $\tau_i = T$. To ensure that this system has suitable robustness properties we need to guarantee that small disturbances in the states do not lead to drastic changes in the behavior of the players. This property can be ensured by working with *well-posed* HDS in the sense of [24, Ch. 7]. Roughly speaking, for a HDS to be well-posed, a suitable (graphically) convergent sequence of solutions of the overall system must also converge (in a graphical sense) to another solution of the hybrid system. In the context of (8)-(11), we need to guarantee, among other properties, that for each $\tau_0 \in [T_0, T]$ and each graphically convergent sequence of solutions $\{\tau_k\}_{k\in\mathbb{N}}$ with components $\tau_{i,k}$ satisfying

$$0 \leq \tau_{1,k}(0,0) \leq \ldots \leq \tau_{n,k}(0,0) < \tau_0, \quad \forall\, k \in \mathbb{N}, \tag{12a}$$

$$\text{and} \lim_{k\to\infty} \tau_{1,k}(0,0) = \ldots = \lim_{k\to\infty} \tau_{n,k}(0,0) = \tau_0, \tag{12b}$$

the sequence $\{\tau_k\}_{k\in\mathbb{N}}$ converges (graphically) to a mapping $\tilde{\tau}$ that is also a solution starting from the initial condition $\tilde{\tau}_1(0,0) = \ldots = \tilde{\tau}_n(0,0) = \tau_0$. Thus, when $\tau_0 = T$, the above conditions imply that players will reset their timers $\tau_{i,k}$ sequentially with smaller and smaller times between resets as $k \to \infty$. Thus, in the limit, resets must also be sequential with no time between resets. Since the sequence depends on the initial conditions, a well-posed model of the coordination mechanism must take into account *every* possible order of sequential resets. In other words, if multiple players simultaneously satisfy $\tau_i = T$, then we need to consider all possible sequential resets induced by such players. As discussed in [32], this behavior is unavoidable in well-posed multi-agent HDS with decentralized discrete-time dynamics.

B. Well-Posed Hybrid NSS Dynamics

To formalize the above discussion, we proceed to construct a suitable jump map and a jump set that describe the behavior of the overall NSS dynamics. Specifically, we introduce a new set-valued mapping $G^0 : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^{3n}$, which is defined to be non-empty only when $\tau_i = T$ and $\tau_j \in [T_0, T)$ with $j \neq i$, for each $i \in \mathcal{V}$, and has elements given by

$$\begin{aligned} G^0(x) := \Big\{ &(v_1, v_2, v_3) \in \mathbb{R}^{3n} : (v_{1,i}, v_{2,i}, v_{3,i}) = R_i(x_i), \\ &v_{1,j} = q_j, v_{2,j} = p_j, v_{3,j} \in \mathcal{C}_j(\tau_j),\ \forall\, j \in \mathcal{N}_i, \\ &v_j = x_j, \forall\, j \notin \mathcal{N}_i \Big\}, \end{aligned} \tag{13}$$

where $x := (x_1, x_2, \cdots, x_n)$, and where the *reset map* R_i and the *coordination mapping* $\mathcal{C}_j$ are defined in (9) and (10), respectively. Using the construction (13), the jump map of the overall hybrid system is defined as

$$x^+ \in G_1(x) := \overline{G^0}(x), \tag{14}$$

where $\overline{G^0}$ is the outer-semicontinuous hull of G^0, [33, pp. 154], i.e., the unique set-valued mapping that satisfies $\text{graph}(G_1) = \text{cl}(\text{graph}(G^0))$. By construction, the mapping G_1 is locally bounded and outer-semicontinuous in $\mathbb{R}^n \times \mathbb{R}^n \times [T_0, T]^n$. Moreover, it preserves the sparsity properties of the graph $\mathbb{G}$, and guarantees that any pair of resets of the form (9) satisfy condition (12).

Using the jump map (14), we can now define the *hybrid momentum-based*-NSS (HM-NSS) dynamics $\mathcal{H}_1 := (C_1, F_1, D_1, G_1)$, with overall state $x = (p, q, \tau) \in \mathbb{R}^{3n}$, and vectorial continuous-time dynamics:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{\tau} \end{pmatrix} = F_1(x) = \begin{pmatrix} 2\mathcal{D}(\tau)^{-1}(p-q) \\ -2\mathcal{D}(\tau)\mathcal{G}(q) \\ \eta \mathbf{1}_n \end{pmatrix}, \tag{15}$$

where $p := (p_1, p_2, \ldots, p_n)$ and $\tau := (\tau_1, \tau_2, \ldots, \tau_n)$. The flow set C_1 is defined as:

$$C_1 := \Big\{ x \in \mathbb{R}^{3n} : q \in \mathbb{R}^n,\ p \in \mathbb{R}^n,\ \tau \in [T_0, T]^n \Big\}, \tag{16}$$

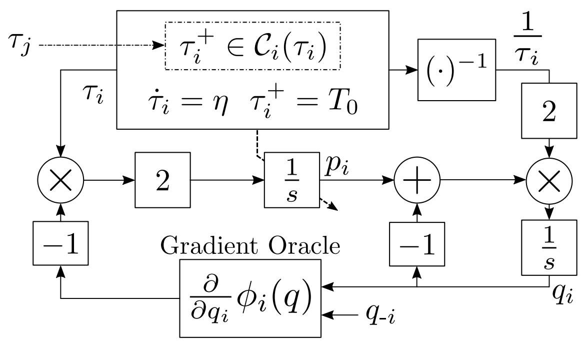

Fig. 4. Scheme of Individual HM-NSS dynamics. Periodic coordinated resets restart the state p_i and the timer τ_i, where $j \in \mathcal{N}_i$.

the jump map G_1 is given by (14), and the jump set is

$$D_1 := \left\{ x \in \mathbb{R}^{3n} : \ x \in C_1, \quad \max_{i \in \mathcal{V}} \tau_i = T \right\}. \tag{17}$$

Figure 4 shows a block-diagram representation of the hybrid dynamics of each player.

The next lemma is instrumental for our results. All proofs are presented in Section VII.

Lemma 1: The HDS (14)-(17) is well-posed in the sense of [24, Def. 6.29]. Moreover, under Assumption 1, every maximal solution of $\mathcal{H}_1$ is complete, and there are at most n jumps in any continuous time interval of length $\frac{1}{\eta}(T - T_0)$. Furthermore, for each solution x and for all $(t, j) \in \mathcal{T}(x)$, we have that $x(t, j) \in \mathcal{A}_{\text{sync}} := (\{T_0, T\}^n) \cup (\mathbf{1}_n \cdot [T_0, T])$, where

$$\mathcal{T}(x) := \{(t, j) \in \text{dom}(x) : t + j \geq T^*\}, \tag{18}$$

and $T^* := (T - T_0)/\eta + n$. □

The qualitative behavior of system $\mathcal{H}_1$ will depend on the choice of parameters (η, T, T_0), which characterize the frequency and the minimum and maximum values of the momentum coefficient τ. Different choices of (η, T, T_0) will lead to different *reset conditions* (RCs). In turn, as hinted in Example 3, and in contrast to standard optimization [17], different types of games will require different RCs to guarantee convergence to the set of NE. The RCs will be defined in terms of the following *condition numbers* of the game, the reset mechanism, and the graph:

$$\sigma_\phi := \frac{\ell}{\kappa}, \qquad \sigma_r := \frac{T}{T_0}, \qquad \sigma_{\mathcal{L}} = \frac{\lambda_{\max}(\mathcal{L})}{\lambda_2(\mathcal{L})}, \tag{19}$$

where ℓ is given by Assumptions 1 or 2, κ is given in Definition 1, and $\lambda_2(\mathcal{L}), \lambda_{\max}(\mathcal{L})$ are the smallest positive and the largest eigenvalues, respectively, of the Laplacian $\mathcal{L}$.

C. Main Stability Results

We study the stability and convergence properties of the dynamics $\mathcal{H}_1$ with respect to the compact set

$$\mathcal{A} := \mathcal{A}_{qp} \times \mathcal{A}_{\text{sync}}, \tag{20}$$

where $\mathcal{A}_{qp} := \{(q, p) \in \mathbb{R}^{2n} : p = q, \ q \in \mathcal{A}_{\text{NE}}\}$. The first RC that we consider is given by

$$T^2 - T_0^2 > \frac{\rho_J}{2} \cdot (1 - \underline{\alpha}), \tag{RC$_1$}$$

where $\rho_J \in \mathbb{R}_{\geq 0}$ is a parameter to be determined and $\underline{\alpha} = \min_{i \in \mathcal{V}} \alpha_i$. This condition will regulate how frequently players reset their states. Also, let

$$\gamma(\rho_J) := \left(1 - \frac{1}{\sigma_r^2} - \frac{\rho_J}{2T^2}\right), \tag{21}$$

where σ_r is defined in (19). This quantity will be instrumental to characterize the rates of convergence of the algorithms.

1) Results for Potential Games: Our first result focuses on monotone potential games and κ-strongly potential games.

Theorem 1: Let $\mathcal{G}$ describe a monotone potential game. Suppose that Assumption 1 holds, and consider the HDS $\mathcal{H}_1$ under (RC$_1$). Then, the following holds:

(i$_1$) If $\alpha = \mathbf{1}_n$ and $\rho_J \geq 0$ then the set $\mathcal{A}$ is R-UGAS. Moreover, and during flows, for any $i \in \mathcal{V}$ the potential function satisfies the bound

$$P(q(t, j)) - P(\mathcal{A}_{\text{NE}}) \leq \frac{c_j}{\tau_i^2(t, j)}, \quad \forall\, (t, j) \in \mathcal{T}(x), \tag{22}$$

where $\{c_j\}_{j=0}^{\infty} \searrow 0^+$ depends on $x(0, 0)$.

(i$_2$) If $\alpha \in \{0, 1\}^n$, $\mathcal{G}$ describes a κ-strongly monotone potential game and $\rho_J = \kappa^{-1}$, then the set $\mathcal{A}$ is R-UGES, and there exists $\lambda > 0$ such that for each compact set $K_0 \subset C_1 \cup D_1$ there exists $M_0 > 0$ such that for all solutions x with $x(0, 0) \in K_0$, and for all $(t, j) \in \text{dom}(x)$ the following bound holds:

$$|q(t, j) - q^*| \leq M_0 e^{-\lambda(t+j)}. \tag{23}$$

(i$_3$) If $\alpha = \mathbf{0}_n$, $\mathcal{G}$ describes a κ-strongly monotone potential game and $\rho_J = \kappa^{-1}$, then the set $\mathcal{A}$ is R-UGES, and for each compact set $K_0 \subset C_1 \cup D_1$ there exists $M_0 > 0$ such that all solutions x with $x(0, 0) \in K_0$, and for all $(t, j) \in \text{dom}\,(x)$ the following bound holds:

$$|q(t, j) - q^*| \leq \sigma_r \sqrt{\sigma_\phi}\,(1 - \gamma(\rho_J))^{\frac{\alpha(j)}{2}} M_0,$$

where $\alpha(j) := \max\{0, \lfloor \frac{j-n}{n} \rfloor\}$ and $\gamma(\rho_J) \in (0, 1)$. □

The results of Theorem 1 establish robust NSS for $\mathcal{H}_1$ in monotone and strongly monotone potential games. Thus, unlike system (1), for the hybrid dynamics $\mathcal{H}_1$ there exists a class $\mathcal{KL}$ function β such that a bound of the form (5) holds under small bounded additive disturbances on the dynamics. This bound effectively rules out the instability observed in Figure 1. The bounds of Theorem 1 also establish suitable semi-acceleration properties. Such bounds will eventually hold since the UGAS result also implies that for all times $(t, j) \notin \mathcal{T}(x)$, the trajectories remain (uniformly) bounded, and Lemma 1 guarantees completeness of solutions. Indeed, solutions of $\mathcal{H}_1$ exhibit a "transient phase", where the momentum coefficients synchronize to each other, followed by a "semi-acceleration phase" where the system behaves as having a global momentum coefficient coordinating the network. Figures 1 and 2 illustrate the advantages of using the hybrid dynamics compared to the ODE (1).

Remark 2: When all players implement the reset protocol $\alpha_i = 1$, item (i$_1$) establishes a *semi-acceleration* property of order $\mathcal{O}(1/\tau^2)$ that holds during intervals of flow in $\mathcal{T}(x)$. Since intervals of flow in $\mathcal{T}(x)$ have a length proportional to $T - T_0$, they can be made arbitrarily large by increasing T. Moreover, if all players initialize their coefficients as $\tau_i(0, 0) = T_0$, then during the first interval of flow we have

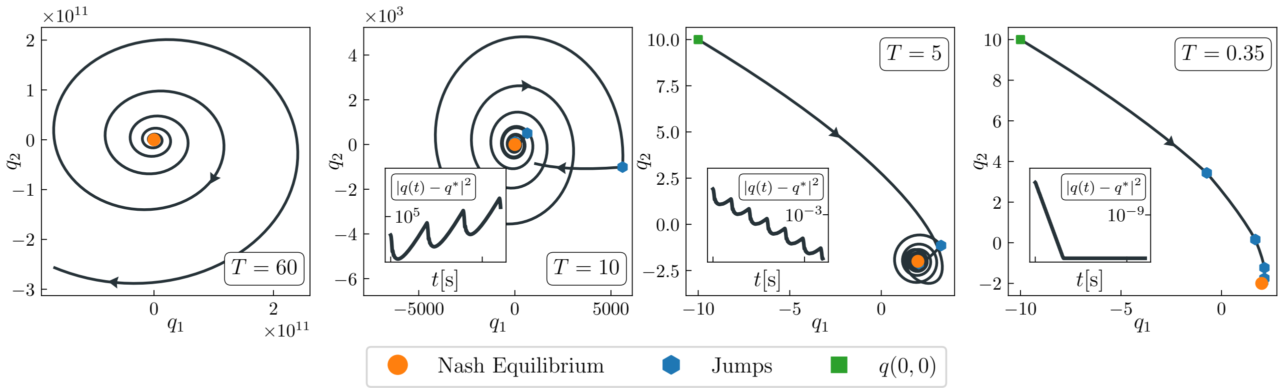

Fig. 5. Phase planes showing the trajectories of the actions of the players resulting from the HM-NSS dynamics in a non-potential 2-player $\boldsymbol{\kappa}$-*strongly monotone quadratic game* with $\boldsymbol{\kappa = 6,\ \ell = 6.2}$ and $\boldsymbol{\tau(0,0) = 0.1 \cdot 1_2}$. Asymptotic stability is achieved by reducing $\boldsymbol{T}$.

that $P(q(t,0)) - P(q^*) \leq \frac{d_0}{t^2}$, for all $(t,0) \in \text{dom}(x)$, where $d_0 > 0$ is determined by the initial conditions of the system and the properties of the pseudogradient $\mathcal{G}$. To the best knowledge of the authors, the result of Theorem 1-(i$_1$) is the first in the literature that establishes R-UGAS *and* this type of acceleration property in *distributed* NES dynamics. Centralized convergence results without resets were recently studied independently in [8]. □

Remark 3: For *κ-strongly monotone potential games*, the reset policy $\alpha_i = 0$, $\forall i \in \mathcal{V}$, guarantees exponential NSS with rate of convergence dictated by $1 - \gamma\left(\kappa^{-1}\right)$. In this case, by borrowing results from the literature on centralized accelerated optimization [17], [19], we can consider a "quasi-optimal" restarting parameter $T = e\sqrt{\frac{1}{2\kappa} + T_0^2}$, which guarantees acceleration-like exponential convergence of order $\mathcal{O}(e^{-\sqrt{\kappa}t})$ whenever $T_0 \ll 1$. Finally, the result of item (i$_2$) shows that the stability and convergence properties of $\mathcal{H}_1$ are robust to heterogeneous reset policies in the game. □

2) Results for Non-Potential Games: When a potential function does not exist, the analysis of the HDS $\mathcal{H}_1$ is more challenging. To study this case, we introduce the following matrix parameterized by $(\rho_F, \delta) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$:

$$\mathcal{M}_\delta(q, \rho_F) := I_n - \mathcal{S}_\delta\left(q, \rho_F\right)\mathcal{S}_\delta\left(q, \rho_F\right)^\top, \qquad (24)$$

with $\mathcal{S}_\delta : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}^{n\times n}$ given by the scaled matrix

$$\mathcal{S}_\delta(q, \rho_F) := \chi(\rho_F, \delta)^{\frac{1}{2}}\left(\rho_F I_n - \partial\mathcal{G}(q)\right),$$

where $\partial\mathcal{G}$ is the Jacobian of $\mathcal{G}$, and where the mapping $\chi : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is given by

$$\chi(\rho_F, \delta) = \frac{T^2}{1 - \delta T^2} \cdot \frac{1}{\rho_F(1-\eta) - \delta\rho_F^2},$$

which is defined for all arguments such that $\delta T^2 < 1$ and $1 - \eta > \delta\rho_F$. We use the following definition to extend [34, Def. 4.1.2] to matrices of the form (24).

Definition 6: The mapping $q \mapsto \mathcal{S}_\delta(q, \rho_F)$ is *ρ_F-Globally Contractive* $(\rho_F\text{-GC})$ if $\mathcal{M}_\delta(q, \rho_F) \succ 0$ for all $q \notin \mathcal{A}_{\text{NE}}$. □

Note that when $\mathcal{M}_\delta \succ 0$, the coefficient χ characterizes the *level of contraction* of $\mathcal{S}_\delta$. Indeed, $\mathcal{M}_\delta \succ 0$ if and only if

$$\frac{1}{\chi(\rho_F, \delta)} \geq \sigma_{\max}\left(\rho_F I_n - \partial\mathcal{G}(q)\right)^2, \qquad (25)$$

where $\sigma_{\max}(\cdot)$ is the maximum singular value of its argument [35, Thm. 7.7.2]. Using the definition of χ, and inequality (25), it can be observed that in order to ensure that $\mathcal{S}_\delta$ is ρ_F-GC for some pair (δ, ρ_F), the resetting parameter T cannot be chosen arbitrarily large. Example 4 illustrates this point.

Example 4: Consider a κ-strongly monotone quadratic game with $\kappa = 6$, and

$$\mathcal{G}(q) = \begin{pmatrix} 6 & 1.5 \\ -1.5 & 6 \end{pmatrix}(q - q^*), \qquad (26)$$

where $q^* = (2, -2)$. First, let $\delta = 0$, and note that for this game $\mathcal{M}_0(q, \rho_F) = \mathcal{D}\left(m_0(\rho_F)\mathbf{1}_2\right) \in \mathbb{R}^{2\times 2}$, where

$$m_0(\rho_F) = 1 - T^2\frac{4(\rho_F - 12)\rho_F + 153}{4(1-\eta)\rho_F}.$$

Notice that $4(\rho_F - 12)\rho_F + 153 > 0$ for all $\rho_F \in \mathbb{R}_{>0}$, and recall that $\eta \leq \frac{1}{2}$ by assumption. Thus, for all $\rho_F > 0$ there exists $\bar{T} \in \mathbb{R}_{>0}$ such that $\mathcal{M}_0(q, \rho_F) \succ 0$ for all $T \in (0, \bar{T})$, and $\mathcal{M}_0(q, \rho_F) \preceq 0$ for $T \geq \bar{T}$. Similarly, when $\delta > 0$ we have that if $\mathcal{S}_\delta$ is ρ_F-GC, then $\mathcal{S}_0$ is also ρ_F-GC. Thus, we can conclude that for every ρ_F and $\delta \geq 0$ there exists $\bar{T}$ such that $\mathcal{S}_\delta$ is not ρ_F-GC for any $T \geq \bar{T}$. □

By using the global contractivity property of Definition 6 to inform the tuning of the resetting parameters of $\mathcal{H}_1$, we can achieve NSS in non-potential games.

Theorem 2: Let $\mathcal{G}$ describe a strictly monotone game, and suppose that Assumptions 2 and 3 hold. Consider the HDS $\mathcal{H}_1$ under (RC$_1$) with $\rho_J \geq 0$ and with reset policy $\alpha = \mathbf{1}_n$. If $\mathcal{S}_0$ is ℓ-GC then the set $\mathcal{A}$ is R-UGAS, for every $i \in \mathcal{V}$, and for all solutions x the following bound holds during flows

$$|\mathcal{G}(q(t,j))|^2 \leq \frac{\tilde{c}_j}{\tau_i^2(t,j)}, \quad \forall\ (t,j) \in \mathcal{T}(x), \qquad (27)$$

where $\{\tilde{c}_j\} \searrow 0^+$ is a sequence parameterized by $x(0,0)$. □

Unlike Theorem 1, in non-potential games the ρ_F-global-contractivity of $\mathcal{S}_\delta$ plays a fundamental role in the stability analysis of $\mathcal{H}_1$. In particular, the ℓ-GC property of $\mathcal{S}_\delta$ will guarantee a suitable dissipativity property during flows via Lyapunov-based tools. Note that, while in Theorem 2 this is only a sufficient condition, the plots of Figure 5 indicate that keeping T "sufficiently small" is also a necessary condition to preserve stability in non-potential games. In this figure, we show the phase plane of solutions to $\mathcal{H}_1$ with different values of T, in a game with pseudogradient given by (26).

Next, we provide a sufficient condition on the parameter T such that $\mathcal{S}_0$ is ℓ-GC in cocoercive strictly monotone games.

Lemma 2: Suppose that Assumption 2 holds, and $\mathcal{G}$ describes a strictly-monotone game. Let (η, T, ℓ) satisfy:

$$0 < T^2 < \frac{1-\eta}{2\ell}. \qquad (\mathrm{RC}_2)$$

Then, $\mathcal{S}_0$ is ℓ-GC. □

We now turn our attention to games that are κ-strongly monotone and ℓ-Lipschitz. For these games, we ask that the contractivity properties of $\mathcal{S}_\delta$ hold with $\delta > 0$, and that (RC$_1$) holds with a particular value of ρ_J.

Theorem 3: Suppose that Assumption 1 holds and let $\mathcal{G}$ describe a κ-strongly monotone game. Consider the HDS $\mathcal{H}_1$ under (RC$_1$), and suppose that $\mathcal{S}_\delta$ is $(\sigma_\phi \ell)$-GC with $0 < \delta < (1-\eta)/(\sigma_\phi \ell)$. Then, the following holds:

(i$_4$) If $\alpha \in \{0,1\}^n$ and $\rho_J = 0$, then $\mathcal{A}$ is R-UGES, and there exists $\lambda > 0$ such that for each compact set $K \subset C_1 \cup D_1$ there exists $M_0 > 0$ such that for all solutions x, with $x(0,0) \in K_0$, the bound (23) holds.

(i$_5$) If $\alpha = \mathbf{0}_n$ and $\rho_J = \sigma_\phi^2 \kappa^{-1}$, then $\mathcal{A}$ is R-UGES and for each compact set $K_0 \subset C_1 \cup D_1$ there exists $M_0 > 0$ such that for all solutions x, with $x(0,0) \in K_0$, and for all $(t,j) \in \mathrm{dom}\,(x)$, the following bound holds:

$$|q(t,j) - q^*| \leq \sigma_r \sigma_\phi \left(1 - \gamma\left(\rho_J\right)\right)^{\frac{\alpha(j)}{2}} M_0,$$

where $\alpha(j) := \max\{0, \lfloor \frac{j-n}{n} \rfloor\}$, and $\gamma\left(\rho_J\right) \in (0,1)$. □

Before commenting on the implications of Theorem 3, we present a reset condition for κ-strongly monotone games that is analogous to the one of Lemma 2.

Lemma 3: Suppose that Assumption 1 holds and that $\mathcal{G}$ describes a κ-strongly monotone game. Let $(\eta, T, \sigma_\phi \ell)$ satisfy:

$$0 < T^2 < \frac{1 - \eta - \delta\sigma_\phi \ell}{\sigma_\phi \ell - \kappa + \delta(1 - \eta - \delta \sigma_\phi \ell)}, \qquad (\mathrm{RC}_3)$$

with $0 \leq \delta < (1-\eta)/(\sigma_\phi \ell)$. Then $\mathcal{S}_\delta$ is $(\sigma_\phi \ell)$-GC. □

Remark 4: When $\rho_J = \sigma_\phi^2 \kappa^{-1}$, the conjunction of (RC$_1$) and (RC$_3$) imposes upper *and* lower bounds for the reset times of the HDS $\mathcal{H}_1$ for all times $(t,j) \in \mathcal{T}(x)$. This result is in contrast to the case of potential games (and standard convex optimization problems) with periodic restarting where only a lower bound between resets is usually needed to achieve exponential convergence [17], [19]. Instead, Theorem 3 asks for the resets to occur in a particular frequency band: they should not occur too frequently (i.e., T should not be too small) such that (RC$_1$) holds and the distance $|q - q^*|$ shrinks by a constant quantity after each interval of flow; however, resets should also happen frequently enough (i.e., T should not be too large) such that $\mathcal{S}_\delta$ remains $(\sigma_\phi \ell)$-GC. □

The next lemma provides a sufficient condition to guarantee feasibility of the reset conditions of Theorem 3.

Lemma 4: For any $\kappa > 0$, $\eta \leq 1/2$ and σ_ϕ such that $\sigma_\phi^4 - \sigma_\phi^2 < 2(1-\eta)$, there exists (T, T_0) such that (RC$_1$) and (RC$_3$) hold with $\rho_J = \sigma_\phi^2 \kappa^{-1}$, provided δ is sufficiently small. □

In Theorem 3, the restarting policy $\alpha = \mathbf{0}_n$ leads to exponential NSS with rate of convergence characterized by $(1 - \gamma(\sigma_\phi^2/\kappa))$. For this coefficient, one can choose a "quasi-optimal" restarting parameter T to induce an acceleration-like property in κ-strongly monotone games:

Lemma 5: Under the Assumptions of Theorem 3-(i$_5$), and for any $\nu > 0$, the choice $T = T^{\mathrm{opt}} := e\sigma_\phi \sqrt{\frac{1}{2\kappa} + \frac{T_0^2}{\sigma_\phi^2}}$ guarantees that $|q(t,j) - q^*| \leq \nu$ for all $t \geq t_{\nu^{\mathrm{opt}}}$, where

$$t_\nu^{\mathrm{opt}} = \frac{1}{\eta}\left(e\sigma_\phi \sqrt{\frac{1}{2\kappa} + \frac{T_0^2}{\sigma_\phi^2}} - T_0\right) \ln\left(\frac{\sigma_\phi \sigma_r M_0}{\nu}\right),$$

and M_0 is a constant that depends on $|q(0,0) - q^*|$. Moreover, the convergence is of order $\mathcal{O}(e^{-\sqrt{\kappa} t/\sigma_\phi})$ as $T_0 \to 0^+$. □

Remark 5: The result of Lemma 5 showcases the exponential bound induced by the HM-NSS dynamics: as $\sigma_\phi \to 1$, the convergence is of order $\mathcal{O}(e^{-\sqrt{\kappa} t})$, which, compared to PSG flows, is advantageous in games with low curvature and moderate condition number, see Figure 3. However, as σ_ϕ increases, the theoretical convergence rate decreases. Whether or not a small σ_ϕ is a *necessary* condition to achieve acceleration in games with dynamic momentum remains an open question. Additional numerical experiments that explore this question can be found in the extended manuscript [36]. □

It is possible to find additional conditions on the game and the parameters of $\mathcal{H}_1$ such that T^{opt} satisfies (RC$_1$) and (RC$_3$). However, such conditions are rather involved and unintuitive, and therefore are omitted for brevity. Yet, we note that in Example 3 the quasi-optimal restarting T^{opt} can be verified to be feasible. We also note that, based on numerical experiments, our theoretical bounds are conservative, see Figure 3. Indeed, for κ-strongly monotone quadratic games, it is possible to obtain less conservative reset conditions (RC$_2$) and (RC$_3$) by using a different Lyapunov function that leverages the affine structure of the pseudogradient. See the extended manuscript [36] for more details.

Remark 6: The results of Theorems 2 and 3 can also be applied to games with a potential function P and a vector of weights $\omega \in \mathbb{R}^n$ such that $D(\omega)\nabla P$ is strictly or κ-strongly monotone, provided $\mathcal{G} := D(\omega)\nabla P$ satisfies the required conditions in S_δ. Such weighted potential games have been recently studied in [37] in the context of congestion games. □

V. Hybrid Momentum-based NSS with Partial Information

In the previous section, we assumed that players had access to individual Oracles able to generate measurements of $\frac{\partial \phi_i(\cdot)}{\partial q_i}$ at the *overall* state q. In this section, we relax this assumption by considering Oracles that provide *evaluations* of these functions. Thus, to perform gradient evaluations, players need to estimate on-the-fly the overall state q.

A. Individual Multi-Time Scale Hybrid Dynamics

To achieve distributed NSS over graphs with partial information, we proceed to endow each player $i \in \mathcal{V}$ with an auxiliary state $\mathbf{e}^i$ that serves as an individual estimation of the actions of the other players: $\mathbf{e}^i := (\mathbf{e}_1^i, \mathbf{e}_2^i, \ldots, q_i, \ldots, \mathbf{e}_{n-1}^i, \mathbf{e}_n^i) \in \mathbb{R}^n$. Since players do not need to estimate their own action, it is also convenient to introduce the auxiliary state $\mathbf{e}_{-i}^i \in \mathbb{R}^{n-1}$ which contains

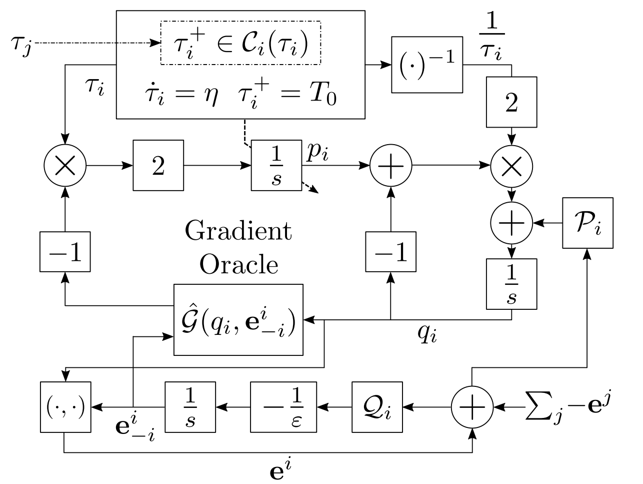

Fig. 6. Scheme of Individual HM-NSS dynamics for games with partial information. Consensus dynamics are implemented to estimate the actions of other players. In the figure, $j \in \mathcal{N}_i$.

the same entries of $\mathbf{e}^i$ with the exception of q_i, which is removed. Using this notation, we now assume that each player i has access to individual *gradient Oracles* characterized by mappings of the form $(q_i, \mathbf{e}_{-i}^i) \mapsto \hat{\mathcal{G}}_i(q_i, \mathbf{e}_{-i}^i)$, which satisfy $\hat{\mathcal{G}}_i(q_i, q_{-i}) = \frac{\partial \phi_i(q)}{\partial q_i}$. Following similar notation used in the literature of network games [7], we define the matrices

$$\mathcal{Q}_i := \begin{pmatrix} I_{(i-1)} & 0_{(i-1)\times 1} & 0_{(i-1)\times(n-i)} \\ 0_{(n-i)\times(i-1)} & 0_{(n-i)\times 1} & I_{(n-i)} \end{pmatrix},$$
$$\mathcal{P}_i := \begin{pmatrix} 0_{1\times(i-1)} & 1 & 0_{1\times(n-i)} \end{pmatrix}.$$

By using these definitions, each player i now implements the following momentum-based augmented continuous-time NSS dynamics:

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \\ \dot{\tau}_i \\ \dot{\mathbf{e}}_{-i}^i \end{pmatrix} = \begin{pmatrix} \frac{2}{\tau_i}(p_i - q_i) + \mathcal{P}_i \sum_{j\in\mathcal{N}_i}(\mathbf{e}^i - \mathbf{e}^j) \\ -2\tau_i \hat{\mathcal{G}}_i(q_i, \mathbf{e}_{-i}^i) \\ \eta \\ -\frac{1}{\varepsilon}\mathcal{Q}_i \sum_{j\in\mathcal{N}_i}(\mathbf{e}^i - \mathbf{e}^j) \end{pmatrix}, \quad (29)$$

where $\varepsilon > 0$ is a new tunable parameter. These dynamics are implemented whenever the state τ_i satisfies $\tau_i \in [T_0, T)$. The momentum-based dynamics (29) implement a dynamic consensus mechanism with state $\mathbf{e}_{-i}^i$. This mechanism uses a high gain $\frac{1}{\varepsilon}$ to induce a time-scale separation in the flows of the hybrid algorithm. In particular, if the states $\mathbf{e}^i$ were to instantaneously achieve their steady state value, the flows (29) would reduced to the flows (15). When players are uncoordinated, the individual resets are triggered by the condition $\tau_i = T$, and are given by $x_i^+ = R_i(x_i)$, $\mathbf{e}_{-i}^{i+} = \mathbf{e}_{-i}^i$, where R_i is defined in (9). However, lack of coordination between resets can induced the same issues discussed in Example 2. To avoid this issue, we will incorporate the hybrid coordinated restarting mechanism described in Section IV-A. Figure 6 shows a block-diagram of the multi-time scale hybrid dynamics of each player.

B. Well-Posed Coordinated HDS with Partial Information

To write the coordinated HDS in vectorial form, we introduce the matrices $\mathcal{Q} := D(\mathcal{Q}_i) \in \mathbb{R}^{(n^2-n)\times n^2}$ and $\mathcal{P} := D(\mathcal{P}_i) \in \mathbb{R}^{n\times n^2}$, and note that $q = \mathcal{P}\mathbf{e} \in \mathbb{R}^{n^2-n}$. Additionally, we define the state $\hat{q} := \mathcal{Q}\mathbf{e}$, such that using $\mathcal{P}\mathcal{P}^\top = I_n$, $\mathcal{Q}\mathcal{Q}^\top = I_{n^2-n}$, and $\mathcal{P}\mathcal{Q}^\top = 0$, we can write $\mathbf{e} = \psi(q, \hat{q}) := \mathcal{P}^\top q + \mathcal{Q}^\top \hat{q}$, where $\mathbf{e} = (\mathbf{e}^1, \cdots, \mathbf{e}^n)$, and express the overall hybrid NSS dynamics as a HDS (2) with data $\mathcal{H}_2 = (C_2, F_2, D_2, G_2)$ and state $(x, \hat{q})$, where $x := (q, p, \tau) \in \mathbb{R}^{3n}$. The flow map is given by

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{\tau} \\ \dot{\hat{q}} \end{pmatrix} = F_2(x, \hat{q}) := \begin{pmatrix} 2D(\tau)^{-1}(p - q) - \mathcal{P}\mathbf{L}\psi(q, \hat{q}) \\ -2D(\tau)\hat{\mathcal{G}}(\psi(q, \hat{q})) \\ \eta \mathbf{1}_n \\ -\frac{1}{\varepsilon}\mathcal{Q}\mathbf{L}\psi(q, \hat{q}) \end{pmatrix}, \quad (30)$$

where $\mathbf{L} := \mathcal{L} \otimes I_n$ denotes the communication matrix of the graph $\mathbb{G}$. The continuous-time dynamics in (30) are allowed to evolve whenever $(x, \hat{q})$ belongs to the flow set:

$$C_2 := \Big\{(x, \hat{q}) \in \mathbb{R}^{n^2+2n} : \ q \in \mathbb{R}^n, p \in \mathbb{R}^n, \\ \tau \in [T_0, T]^n, \ \hat{q} \in \mathbb{R}^{n^2-n}\Big\}. \quad (31)$$

On the other hand, the jump set is defined as:

$$D_2 := \Big\{(x, \hat{q}) \in \mathbb{R}^{n^2+2n} : x \in C, \ \max_{i\in\mathcal{V}} \tau_i = T\Big\}, \quad (32)$$

and the discrete-time dynamics of the algorithm are given by:

$$(x^+, \hat{q}^+) \in G_2(x, \hat{q}) := G_1(q, p, \tau) \times \{\hat{q}\}, \quad (33)$$

where G_1 is defined as in (14). Similar to Lemma 1, the next lemma follows directly by construction of the HDS.

Lemma 6: For the HDS $\mathcal{H}_2 := (C_2, F_2, D_2, G_2)$, all the properties of Lemma 1 still hold. □

We will study the stability properties of the HDS $\mathcal{H}_2$ with respect to the following compact set:

$$\mathcal{A}_{\mathbb{G}} := \mathcal{A} \times \{\mathcal{Q}(\mathbf{1}_n \otimes q^*)\}, \quad (34)$$

where $\mathcal{A}$ was defined in (20). In this case, we will use the following restricted reverse-Lipschitz assumption, also used in [22] for NES with static inertia.

Assumption 4: There exists $\zeta > 0$ such that $|\mathcal{G}(q) - \mathcal{G}(q^*)| \geq \zeta |q - q^*|$, for all $q \in \mathbb{R}^n$. □

The next result leverages items (i_1)-(i_5) of Theorems 1-3.

Theorem 4: Let $\mathcal{G}$ describe a strictly monotone game. Suppose that Assumptions 2, 3 and 4 hold, and consider the HDS $\mathcal{H}_2$ under (RC_1). If $\mathcal{S}_\delta$ is $\ell-$GC with $0 < \delta < (1-\eta)/\ell$, then under any of the conditions (i_1)-(i_5) the following holds:

(a) For all $\varepsilon \in (0, \varepsilon_\delta^*)$, where ε_δ^* is given by (28), the set $\mathcal{A}_{\mathbb{G}}$ is R-UGAS.
(b) For each $(\hat{t}, \hat{j}, \nu) \in \mathbb{R}^3_{>0}$ and each compact set $K_x \times K_{\hat{q}} \subset C_2 \cup D_2$, there exists ε^{**} such that for each $\varepsilon \in (0, \varepsilon^{**})$ and each solution of $\mathcal{H}_2$ with $x(0,0) \in K_x$ and $\hat{q}(0,0) \in K_{\hat{q}}$, there exists a solution $\tilde{x}$ of system $\mathcal{H}_1$ with $\tilde{x} \in K_x$ such that x and $\tilde{x}$ are $(\hat{t}, \hat{j}, \nu)$-close. □

$$\varepsilon_\delta^* := \frac{1}{2\sigma_{\mathcal{L}}\sqrt{n}}\left(1 + \sigma_r^2 \frac{\max\left\{\frac{1}{T^2} + 4\frac{\ell}{T\lambda_{\max}(\mathcal{L})}, \ 2 + 2\frac{\ell}{T\lambda_{\max}(\mathcal{L})}\right\}}{\delta \min\{1, \zeta^2\}}\right)^{-1} \quad (28)$$

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3321901

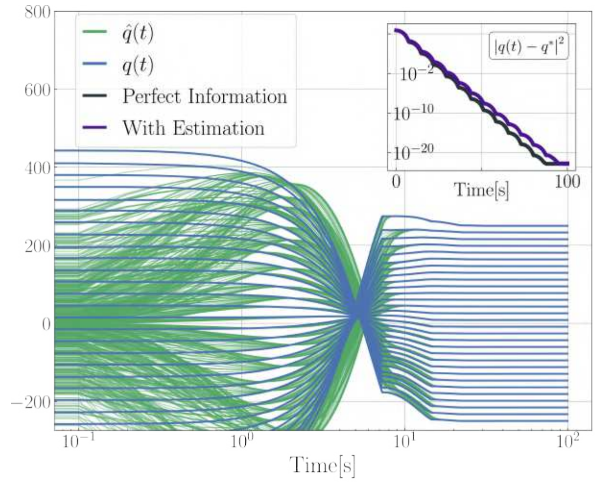

Fig. 7. Trajectories of $\boldsymbol{q}, \hat{\boldsymbol{q}}$ in a non-potential $\boldsymbol{\kappa}$-strongly monotone quadratic game with $\boldsymbol{n = 30}$, $\boldsymbol{\kappa = 0.01}$, $\boldsymbol{\ell = 0.1}$, $\boldsymbol{\tau_s(0,0) = 0.1 \cdot 1_n}$, and $\boldsymbol{\varepsilon = 5 \times 10^{-3}}$. The inset shows the distance to $\boldsymbol{q^*}$.

Item (a) of Theorem 4 establishes robust stability and convergence properties for the hybrid NSS dynamics $\mathcal{H}_2$. On the other hand, item (b) establishes that, on compact sets of initial conditions and on compact time domains, the trajectories x will behave as the trajectories of the "full-information" system $\mathcal{H}_1$ as $\varepsilon \to 0^+$ in (30). In particular, by combining items (a) and (b), we recover the convergence bounds of Theorems 1, 2, and 3, now in a semi-global practical sense as $\varepsilon \to 0^+$. This behavior is illustrated in Figure 7, which shows the trajectories q and $\hat{q}$ in a κ-strongly monotone game. As observed, the solutions of $\mathcal{H}_2$ approximate those of $\mathcal{H}_1$ as $\varepsilon \to 0^+$.

Remark 7: Assumption 4 always holds for *κ-strongly monotone games* with $\zeta = \kappa$. Hence, for these games one can compute an alternative expression of ε_δ^* by substituting Assumptions 2-4 in Theorem 4 by Assumption 1 when $\mathcal{S}_\delta$ is $(\sigma_\phi \ell)$-GC. Moreover, to guarantee that $\mathcal{S}_\delta$ is ℓ-GC, a suitable upper bound for T can be obtained by mirroring the derivations of Lemma 3, which we omit here due to space limitations. □

To our best knowledge, Theorem 4 is the first result in the literature that establishes robust convergence and stability properties for decentralized momentum-based NSS algorithms over graphs. Note that the stable incorporation of the multi-time scale consensus mechanism is enabled by the use of resets, since otherwise no $\mathcal{KL}$ bound (or strong Lyapunov function) would exist for the reduced dynamics of the flows.

VI. Model-Free NSS with Momentum

We now dispense with the gradient Oracles considered in the previous sections, and we design momentum-based *model-free* hybrid NSS dynamics, suitable for applications where players have access only to real-time *measurements* of the signals that correspond to their cost functions ϕ_i (e.g., the difference between the individual cost and revenue in a market), which are generated by the game. Such algorithms can be designed via tools recently developed in the context of hybrid equilibrium seeking control [17].

A. Model-Free NSS Dynamics

To achieve model-free NSS, each player i generates an individual probing signal $t \mapsto \mu_i(t)$, obtained as the solution of a dynamic oscillator with state $\mu_i := (\tilde{\mu}_i, \hat{\mu}_i) \in \mathbb{R}^2$, evolving on the unit circle $\mathbb{S}^1$ according to

$$\dot{\mu}_i = \frac{1}{\varepsilon_p}\mathcal{R}_i\mu_i, \quad \mu_i \in \mathbb{S}^1, \quad \mathcal{R}_i := 2\pi\varsigma_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{35}$$

where ε_p and ς_i are positive tunable parameters. Note that $\mathbb{S}^1$ is forward invariant under the dynamics of μ_i. Using this probing signal, each player implements the flows:

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \\ \dot{\tau}_i \end{pmatrix} = \begin{pmatrix} \frac{2}{\tau_i}(p_i - q_i) \\ -\frac{4}{\varepsilon_a}\tau_i \phi_i(q + \varepsilon_a \tilde{\mu}) e_1^\top \mu_i \\ \eta \end{pmatrix}, \tag{36}$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^{2n}$, and where $\tilde{\mu}$ is the vector that contains the odd components of μ. The dynamics (36) use real-time *measurements* of the cost ϕ_i, and are implemented whenever $\tau_i \in [T_0, T)$. Conversely, when $\tau_i = T$ and players are uncoordinated, they reset their states according to the dynamics $x_i^+ = R_i(x_i)$, $\mu_i^+ = \mu_i$, where R_i is defined as in (9). The constant $\varepsilon_a > 0$ is also a tunable parameter.

We impose the following assumption on the parameters ς_i of (35), which is standard in the literature [11], [25].

Assumption 5: For all i, ς_i is a positive rational number, $\varsigma_i \neq \varsigma_j$, $\varsigma_i \neq 2\varsigma_j$, $\varsigma_i \neq 3\varsigma_j$, for all $i \neq j \in \mathcal{V}$. □

As in the model-based case, an uncoordinated implementation of the model-free hybrid dynamics can be detrimental to the stability and/or transient performance of the algorithm. Thus, we incorporate the hybrid coordination mechanism described in Section IV-A to coordinate the resets of the players, which results in the following discrete-time dynamics

$$(x^+, \mu^+) \in G_3(x, \mu) := G_1(x) \times \{\mu\}, \tag{37}$$

where G_1 is given by (14). This jump map will preserve the sequential nature of the resets needed to guarantee a well-posed HDS that satisfies (12). Using $\bar{\phi} := (\phi_1, \phi_2, \ldots, \phi_n)$, the continuous-time dynamics of the model-free hybrid NSS algorithm can be written in vector form as:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{\tau} \\ \dot{\mu} \end{pmatrix} = F_3(x, \mu) := \begin{pmatrix} 2D(\tau)^{-1}(p - q) \\ -\frac{4}{\varepsilon_a} D(\tau)\bar{\phi}(q + \varepsilon_a \tilde{\mu})\tilde{\mu} \\ \eta \\ \frac{1}{\varepsilon_p} D(\mathcal{R}_i)\mu \end{pmatrix}, \tag{38}$$

and the flow and jump sets are defined as:

$$C_3 := C_1 \times \mathbb{T}^n, \quad \text{and} \quad D_3 := D_1 \times \mathbb{T}^n. \tag{39}$$

Figure 8 shows a scheme of the proposed algorithm.

B. Semi-Global Practical Stability Results

The data $\mathcal{H}_3 = (C_3, F_3, D_3, G_3)$ defines the third hybrid NSS dynamics considered in this paper. The stability and convergence properties of $\mathcal{H}_3$ are given in the following theorem, which also leverages items (i_1)-(i_5) of Theorems 1-3.

Theorem 5: Let $\mathcal{G}$ describe a strictly monotone game, and consider the HDS $\mathcal{H}_3$ under (RC_1). Then, under any of the conditions (i_1)-(i_5) the following holds:

(a) The set $\mathcal{A} \times \mathbb{T}^n$ is SGPAS as $(\varepsilon_p, \varepsilon_a) \to 0^+$.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3321901

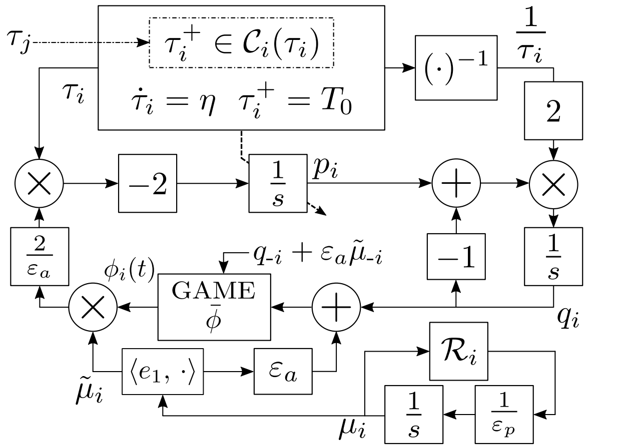

Fig. 8. Scheme of Individual Model-Free HM-NSS dynamics with real-time measurements of the cost. In the figure, $j \in \mathcal{N}_i$.

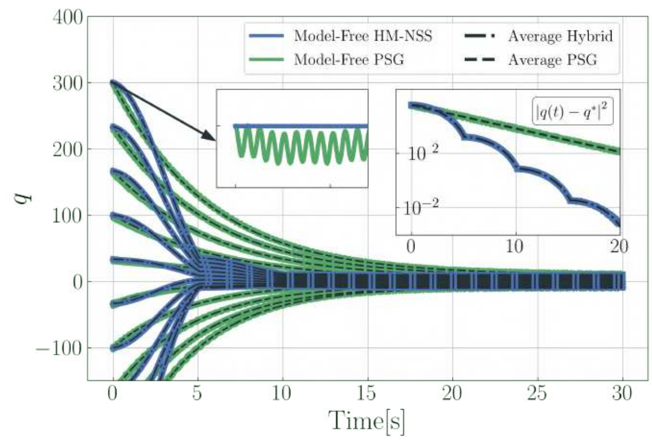

Fig. 9. Trajectories of $\mathcal{H}_1$ and $\mathcal{H}_3$ in a non-potential κ-strongly monotone quadratic game with $\kappa = 0.197$, $\ell = 0.2$ and $n = 10$.

(b) For each $(\hat{t}, \hat{j}, \nu) \in \mathbb{R}^3$ and each compact set $K_x \subset C_1 \cup D_1$, $\exists\ \varepsilon_a^* > 0$ s.t. $\forall\ \varepsilon_a \in (0, \varepsilon_a^*)$ $\exists\ \varepsilon_p^* > 0$ s.t. $\forall\ \varepsilon_p \in (0, \varepsilon_p^*)$, and for each trajectory x of system $\mathcal{H}_3$ with $x(0,0) \in K_x$ there exists a solution $\tilde{x}$ of system $\mathcal{H}_1$ such that x and $\tilde{x}$ are $(\hat{t}, \hat{j}, \nu)$-close. □

The result of Theorem 5 establishes two key properties: First, for any desired precision $\nu > 0$, and any compact set of initial conditions K_x, every solution of the HDS $\mathcal{H}_3$ initialized in K_x will satisfy a bound of the form[1]

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) + \frac{\nu}{2}, \tag{40}$$

with $\beta \in \mathcal{KL}$, provided the parameters ε_a and ε_p are sufficiently small. Second, selecting ε_a and ε_p sufficiently small leads to trajectories x of $\mathcal{H}_3$ with approximately the same fast convergence bounds established in Section IV-C.

Remark 8: The model-free dynamics $\mathcal{H}_3$ are based on averaging theory for (perturbed) hybrid systems [17], [30]. Thus, as $\varepsilon_a, \varepsilon_p \to 0^+$ the trajectories of $\mathcal{H}_3$ behave as their average hybrid dynamics (modulo a small perturbation), which are precisely given by $\mathcal{H}_1$. Both dynamics are set-valued, which differs from existing results in the literature of model-free Nash set-seeking [11]. Figure 9 compares a solution to $\mathcal{H}_3$ and a solution to the model-free dynamics of [11] based on PSG flows, in a κ-strongly monotone quadratic game. □

We finish this section by commenting on the extensions of system $\mathcal{H}_3$ to applications where players could have access to an individual "Black-Box Oracle" that allows them to *evaluate* (as opposed to measure) their local cost ϕ_i at their current state q_i, using estimations of the actions of the other players and without knowledge of the mathematical form of ϕ_i (e.g., using dynamic simulators). In this case, we can follow the same approach of Section V, by incorporating an auxiliary estimation state $\hat{q}$. In this case, the hybrid system $\mathcal{H}_4 = (C_4, F_4, D_4, G_4)$ will have a flow map given by

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{\tau} \\ \dot{\mu} \\ \dot{\hat{q}} \end{pmatrix} = F_4(\zeta) := \begin{pmatrix} 2D(\tau)^{-1}(p-q) - \mathcal{P}\mathbf{L}\psi(q,\hat{q}) \\ -\frac{4}{\varepsilon_a} D(\tau)\bar{\phi}(\psi(q+\varepsilon_a\tilde{\mu},\hat{q}))\tilde{\mu} \\ \eta \\ \frac{1}{\varepsilon_p} D(\mathcal{R}_i)\mu \\ -\frac{1}{\varepsilon_c}\mathcal{Q}\mathbf{L}\psi(q,\hat{q}) \end{pmatrix}, \tag{41}$$

[1] We note that $|\mu(t,j)|_{\mathbb{T}^n} = 0$ for all (t,j) in the domain of the solutions.

a jump map $G_4(x,\mu,\hat{q}) := G_1(x) \times \{\mu\} \times \{\hat{q}\}$, flow set $C_4 := C_1 \times \mathbb{T}^n \times \mathbb{R}^{n^2-n}$ and jump set $D_4 := D_1 \times \mathbb{T}^n \times \mathbb{R}^{n^2-n}$. For this hybrid system, a result like Theorem 5-(a) also holds, now with respect to the set $\mathcal{A} \times \mathbb{T}^n \times \{\mathcal{Q}(\mathbf{1}_n \otimes q^*)\}$ and as $(\varepsilon_p, \varepsilon_a, \varepsilon_c) \to 0^+$. Similarly, a result like Theorem 5-(b) holds by noting that the average hybrid dynamics of $\mathcal{H}_4$ are precisely given by the HDS $\mathcal{H}_2$ (modulo a small perturbation on the gradient), whose robust stability properties were already established in Section V. Thus, we can follow exactly the same steps of the proof of Theorem 4 to obtain an equivalent result.

VII. Analysis and Proofs

In this section, we present the main proofs of our results.

A. Proofs of Section IV

Proof of Lemma 1: Well-posedness follows directly by [24, Thm. 6.30], since F_1 is continuous, C_1 and D_1 are closed sets, and G_1 is outer-semicontinuous (OSC) and locally bounded (LB) in D_1. To rule out finite escape times it suffices to study the behavior of the states (q,p). Using Assumption 1, the form of (15), and the fact that $\mathcal{G}(q^*) = 0$, we have that $|\dot{q}| \leq \frac{2}{T_0}|p-q|$, $|\dot{p}| \leq 2T\ell|q-q^*|$, which implies that $|(\dot{q},\dot{p})| \leq \tilde{\ell}|(q,p)-(q^*,q^*)|$, with $\tilde{\ell} := 2\sqrt{3}\max\{\frac{1}{T_0}, T\ell\}$. Thus, by the Gronwall-Bellman inequality, the flow map (15) does not generate finite escape times. Moreover, since $\tau^+ \in \{T_0, T\}^n$, we have that $G_1(D) \subset C_1 \cup D_1$, which implies that solutions do not stop due to jumps leaving the set $C_1 \cup D_1$. The dynamics of τ are decoupled and can be written as:

$$\tau \in C_\tau := [T_0, T]^n, \qquad \dot{\tau} = \eta \mathbf{1}_n, \tag{42a}$$

$$\tau \in D_\tau := \left\{\tau \in C_\tau : \max_i \tau_i = T\right\}, \quad \tau^+ \in G_\tau(\tau), \tag{42b}$$

where $G_\tau(\tau)$ is the projection of G_1 into the τ-component, which is independent of (p,q). This hybrid system is well-posed by construction, and by [32, Thm. 1] it renders $\mathcal{A}_{\text{sync}}$ UGFxS, with a convergence bound T^* given by $T^* := \frac{1}{\eta}(T - T_0) + n$, $\forall\ \tau(0,0) \in [T_0, T]^n$. Moreover, by [32, Thm. 1], each solution has at most n jumps in any interval of length $L := \frac{1}{\eta}(T - T_0)$, and, for any pair of hybrid times $(t,j), (s,i) \in \operatorname{dom}(\tau)$ with $t + j \geq s + i \geq T^*$ the following dwell-time condition holds $L + t - s \geq \left\lfloor \frac{j-i}{n} \right\rfloor L$, where $\lfloor \cdot \rfloor$

denotes the floor function. Thus, any solution τ of system (42a) is complete and also satisfies $|\tau(t,j)|_{\mathcal{A}_{\text{sync}}} = 0$ for all $t+j \geq T^*$ such that $(t,j) \in \text{dom}(\tau)$. Since the states (q,p) of $\mathcal{H}_1$ evolve in $\mathbb{R}^n \times \mathbb{R}^n$, for each $\tau(0,0) \in [T_0,T]^n$ the hybrid time domains of system (14)-(17) are the same hybrid time domains of system (42a). This equivalence, plus the above properties, establish the result. ■

The previous Lemma directly implies the following:

Lemma 7: Let $\nu > 0$ and consider the HDS $\mathcal{H}_1$ with restricted flow and jump sets given by:

$$C_\nu := \Big\{ x \in \mathbb{R}^{3n} : (p,q) \in \{(q^*,q^*)\} + \nu\mathbb{B},\ \tau \in [T_0,T]^n \Big\},$$

$$D_\nu := \Big\{ x \in \mathbb{R}^{3n} :\ x \in C_\nu,\quad \max_{i\in\mathcal{V}} \tau_i = T \Big\},$$

and jump map G_1 with values intersected with the set C_ν. Then, the restricted system $\mathcal{H}_\nu = \{F_1, C_\nu, G_\nu, D_\nu\}$ renders UGFxS the set $\mathcal{A}_\nu := (\{(q^*,q^*)\} + \nu\mathbb{B}) \times \mathcal{A}_{\text{sync}}$. □

With Lemmas 1 and 7 at hand, we proceed to analyze the HDS $\mathcal{H}_1$ by studying the HDS $\mathcal{H}_\nu$ with data intersected with the set $\mathcal{A}_\nu$. We denote this new HDS as $\mathcal{H}_s := \{F_s, C_s, G_s, D_s\}$, and we note that any compact set $\mathcal{A}' \subset \mathbb{R}^n \times \mathbb{R}^n$ such that $\mathcal{A}' \times \mathcal{A}_{\text{sync}}$ is UGAS for this system will also be UGAS for $\mathcal{H}_\nu$ thanks to the hybrid reduction principle [24, Cor. 7.24]. Moreover, since ν is arbitrary and independent of any parameter, and $\mathcal{H}_1$ has no finite escape times, the set $\mathcal{A}' \times \mathcal{A}_{\text{sync}}$ will also be UGAS for $\mathcal{H}_1$. Thus, in the following we focus on studying the stability properties $\mathcal{H}_s$.

1) Proofs for Potential-games: For simplicity, we first present the proofs for potential games.

Lemma 8: Under the conditions of Theorem 1-(i$_1$), system $\mathcal{H}_s$ renders UGAS the set $\mathcal{A}$ given by (20). □

Proof: Using the potential P, we define the error $\tilde{P}(q) := P(q) - P(\mathcal{A}_{\text{NE}})$, and we consider the Lyapunov function

$$V(x) = V_1(x) + V_2(x) + V_3(x), \tag{43}$$

where the smooth functions V_i are defined as follows:

$$V_1(x) := \frac{1}{4}|p-q|^2,\ V_2(x) := \frac{1}{4}|p|^2_{\mathcal{A}_{NE}}, \tag{44a}$$

$$V_3(x) := \frac{|\tau|^2}{n}\tilde{P}(q), \tag{44b}$$

where $|z|^2_{D(\omega),\mathcal{A}_{NE}} = \min_{s\in\mathcal{A}_{NE}} |z-s|^2_{D(\omega)}$ and $|z|^2_{D(\omega)} = z^\top D(\omega) z$. By our definition of potential-games, and the construction of V_1 and V_2, the function V is radially unbounded and positive definite with respect to the compact set $\mathcal{A} \cap (C_s \cup D_s)$. During flows in C_s, we have:

$$\dot{V}(x) \leq -\frac{1}{\tau_s}|p-q|^2 - \tau_s \left(\left(q - \Pi_{\mathcal{A}_{NE}}(p)\right)^\top \mathcal{G}(q) - \tilde{P}(q) \right), \tag{45}$$

where $\Pi_{\mathcal{A}_{NE}}(p)$ is the projection of p on $\mathcal{A}_{NE}$. Since $\mathcal{G}$ is Lipschitz and $\tilde{P}$ is convex (implied by the monotonicity of $\mathcal{G}$ [33, Thm. 12.17]), it follows that [38, Thm. 5.8] $\frac{1}{2\ell}\left|\mathcal{G}(q)\right|^2 \leq \left(q - \Pi_{\mathcal{A}_{NE}}(p)\right)^\top \mathcal{G}(q) - \tilde{P}(q)$, and thus, from (45), we obtain during flows that

$$\dot{V}(x) \leq -\frac{1}{\tau_s}|p-q|^2 - \frac{\tau_s}{2\ell}|\mathcal{G}(q)|^2. \tag{46}$$

Since $|\mathcal{G}(q)| = 0$ if and only if $q \in \mathcal{A}_{\text{NE}}$, during flows we have $\dot{V}(x) < 0$ for all $x \in C_s \backslash \mathcal{A}$. On the other hand, during jumps, we have that $\Delta_j^{j+1} V(x) := V(x(t,j+1)) - V(x(t,j))$ satisfies $\Delta_j^{j+1} V(x) = \Delta_j^{j+1} V_3(x)$. Additionally, by the definition of G_1 in (14) the following two facts hold: first, if $x \in D_s$, we have two possible cases for all players $i \in \mathcal{V}$: a) if $\tau_i = T_0$, then $\tau_i^+ = T_0$; b) if $\tau_i = T$ then $\tau_i^+ \in \{T_0, T\}$; second, if $x \in D_s$, we have that in each jump one and only one player i satisfies $\tau_i = T$ and $\tau_i^+ = T_0$. Therefore, since $T > T_0$ there exists $\tilde{\varepsilon} > 0$ such that $T_0^2 - T^2 = -\tilde{\varepsilon}$. Hence, it follows that $\Delta_j^{j+1} V_3(x) = \frac{\tilde{P}(q)}{n}\sum_{i=1}^n (\tau_i^{2+} - \tau_i^2) = -\frac{\tilde{\varepsilon}}{n}\tilde{P}(q) \leq 0$. This implies that V does not increase during each reset triggered by a player. Given that the hybrid time domains of $\mathcal{H}_s$ are intervals of flow of duration $\frac{1}{\eta}(T - T_0)$, followed by n consecutive jumps, we can apply the previous inequality n times to obtain: $\Delta_j^{j+n} V(z) = \sum_{k=1}^n \Delta_{j+k-1}^{j+k} V(z) = -\varepsilon\tilde{P}(q) \leq 0, \quad \forall\ x \in D_s$. By [24, Prop. 3.27], the periodic strong decrease of V during flows, and its non-increase during jumps, imply that $\mathcal{H}_s$ renders UGAS the set $\mathcal{A}$. ■

Lemma 9: Under the conditions of Theorem 1-(i$_2$), system $\mathcal{H}_s$ renders UGES the set $\mathcal{A}$.

Proof: Let V given by (43), where $\mathcal{A}_{\text{NE}} = \{q^*\}$ due to strong monotonicity. During flows, we have (45) with $\Pi_{\mathcal{A}_{NE}}(p) = q^*$, which using the strong monotonicity of $\mathcal{G}$ leads to

$$\dot{V}(x) \leq -\frac{1}{\tau_s}|p-q|^2 - \tau_s\frac{\kappa}{2}|q-q^*|^2 \leq -\lambda V(x), \tag{47}$$

where we used the global Lipschitz property of $\mathcal{G}$, and the quadratic upper bound of (43), with

$$\lambda := \frac{2}{3\Delta}\frac{\min\{1, 0.25 T_0 T\kappa\}}{\max\{1, 2T^2\ell\}} \approx \frac{1}{12T}\frac{1}{\sigma_r}\frac{1}{\sigma_\phi}, \tag{48}$$

where the approximation holds when T_0 is sufficiently small, and T is sufficiently large (but finite). Thus, during each interval of flow, V satisfies the t-time bound

$$V(t,j) \leq V(t_j,j)e^{-\lambda(t-t_j)}, \tag{49}$$

for all $(t,j) \in \text{dom}\,(x)$ such that $j = kn$ for some $k \in \mathbb{N}$. To study V during jumps, let Θ and I be the set of indices of players who implement $\alpha_i = 0$, and $\alpha_i = 1$, respectively. After the n consecutive jumps that proceed the flows:

$$\begin{aligned}\Delta_j^{j+n} V(x) \leq& -\frac{1}{4}\sum_{i\in\Theta}\omega_i\left((p_i-q_i)^2 + (p_i - q_i^*)^2\right)\dots\\ &-\frac{1}{2}\left(\kappa(\tau_s^2 - T_0^2) - \frac{1}{2}\right)\sum_{i\in\Theta}(q_i - q_i^*)^2\dots\\ &-\frac{\kappa}{2}(\tau_s^2 - T_0^2)\sum_{i\in I}(q_i - q_i^*)^2 \leq 0,\end{aligned}$$

where we used the strong monotonicity of $\mathcal{G}$, which implies strong convexity of $\tilde{P}$ [33, Thm. 12.17], and the condition $T^2 - T_0^2 > \frac{1}{2\kappa}$ implied by (RC$_1$) with $\rho_J = \kappa^{-1}$. Therefore, it follows that $V(t_j,j) \leq V(t_j, j-n)e^{-\lambda(t-t_j)}$ for all $j \geq n$ and $t \geq t_j$. Since each interval of flow has length $L = (T-T_0)/\eta$, it follows that $\tilde{V}(t_j,j) \leq \tilde{V}(t_{j-n} + L, j-n)e^{-\lambda L}e^{-\lambda(t-t_j)}$. Iterating, and using (49):

$$V(t,j) \leq V(0,0)e^{-\lambda\left(\lfloor\frac{j}{n}\rfloor - 1\right)L}e^{-\lambda(t-t_j)}. \tag{50}$$

By κ-strong convexity of $\tilde{P}$ and the ℓ-Lipschitz of $\mathcal{G}$:

$$\min\left\{\frac{1}{4}, \frac{\kappa T_0^2}{2}\right\}|x|^2_{\mathcal{A}} \leq V(x) \leq \frac{1}{2}\max\left\{1 + \frac{\ell^2}{\kappa},\ \frac{3}{2}\right\}|x|^2_{\mathcal{A}}, \tag{51}$$

and thus, from (50), we obtain:

$$|x(t,j)|_{\mathcal{A}} \leq c\,|x(0,0)|_{\mathcal{A}}\, e^{-\frac{\lambda}{2}(t-(T-\max\tau(0,0))/\eta)}, \quad (52)$$

with $c>0$. Moreover, using the structure of the hybrid time domains, all hybrid times $(t,j)\in\mathrm{dom}\,(x)$ satisfy

$$-\frac{\lambda}{2}t \leq -\frac{1}{3n}\lambda\,(t+j)+\frac{\lambda L}{3}, \quad (53)$$

for all $\lambda>0$. Hence, we obtain:

$$|x(t,j)|_{\mathcal{A}} \leq \hat{c}_s\,|x(0,0)|_{\mathcal{A}}\, e^{-\frac{\lambda}{3n}(t+j)}, \quad (54)$$

where $\hat{c}_s := ce^{\lambda L\left(\frac{1}{3}+\frac{1}{2\eta}\right)}$. ■

Lemma 10: Under the conditions of Theorem 1-(i_3), system $\mathcal{H}_s$ renders UGES the set $\mathcal{A}$.

Proof: Using the Lyapunov function V given by (43), and the fact that $\mathcal{A}_{\mathrm{NE}}=\{q^*\}$, we obtain again inequality (47) during flows. Since now $\alpha=\mathbf{0}_n$, during jumps we have

$$\Delta_j^{j+n}V(x) \leq -V_1-V_2-\frac{1}{2}(\tau_s^2-T_0^2)\tilde{P}(q)+\frac{\bar{\omega}}{4}|q-q^*|^2. \quad (55)$$

By strong convexity of $\tilde{P}$, we can further bound (55) as:

$$\Delta_j^{j+n}V(x) \leq -\gamma\left(\kappa^{-1}\right)V(x), \quad (56)$$

where $\gamma(\cdot)$ is given by (21), which under (RC_1) satisfies $\gamma\left(\kappa^{-1}\right)\in(0,1)$. Thus, by [39, Thm. 1], inequalities (47) and (56), and the quadratic upper and lower bounds of V, we obtain that $\mathcal{H}_s$ renders UGES the set $\mathcal{A}$. ■

With Lemmas 8-10 at hand for system $\mathcal{H}_s$, we can now proceed to proof the three main items of Theorem 1.

Proof of Theorem 1: *(a) Stability:* By the hybrid reduction principle [24, Cor. 7.24], UGAS of $\mathcal{A}$ for system $\mathcal{H}_s$ (established in Lemmas 8, 9 and 10), and UGFxS of $\mathcal{A}_\nu$ for system $\mathcal{H}_\nu$, imply that $\mathcal{A}$ is UGAS for system $\mathcal{H}_\nu$. Moreover, since the choice of $\nu>0$ is arbitrary, and has no effect on the dynamics of the system, and since the trajectories of the original HDS $\mathcal{H}_1$ are complete and bounded, the compact set $\mathcal{A}$ is also UGAS for system $\mathcal{H}_1$. This establishes UGAS of $\mathcal{A}$ under the conditions of items (i_1), (i_2) and (i_3). For items (i_2) and (i_3), UGES follows by the exponential convergence bounds of Lemmas 9-10 and the fixed-time synchronization of τ. R-UGAS and R-UGES follow directly by robustness results of well-posed HDS, specifically by [24, Thm. 7.21].

(b) Convergence Bounds: For any solution x and all $(t,j)\in\mathcal{T}(x)$ we have that $|\tau(t,j)|_{\mathcal{A}_{\text{sync}}}=0$. Thus, for such times the trajectories of $\mathcal{H}_1$ satisfy the Lyapunov inequalities established in Lemmas 8-10. To establish (22), we use inequality (46), which implies that for each $(t,j),(s,j)\in\mathcal{T}(x)$, such that $t>s$, we have $V(t,j)\leq V(s,j)$. Since $V_3\leq V$, and using $s_j:=\min\ \{t\in\mathbb{R}_{\geq0},\ (t,j)\in\mathcal{T}(x)\}$, we obtain

$$\tilde{P}(q(t,j)) \leq \frac{2n}{\tau^\top\tau}V(s_j,j)=\frac{c_j}{\tau_s^2}, \quad \forall t\ > s_j, \quad (57)$$

where $c_j := 2V(s_j,j)$. Using the fact V is non-increasing during flows and jumps, and also converges to zero, we obtain that $\{c_j\}_{j=0}^{\infty}\searrow 0^+$. To obtain the convergence bound of item (i_2), we first note that from the proof of Lemma 1 it follows that $\frac{d|x|_{\mathcal{A}}}{dt}\leq\tilde{\ell}\,|x|_{\mathcal{A}}$ for all $(t,j)\in\mathrm{dom}\,(x)$, where $\tilde{\ell}=2\sqrt{2}\max\left\{\frac{1}{T_0},T\ell\right\}$. In particular, this implies that

$$|x(t_s,j_s)|_{\mathcal{A}} \leq |x(0,0)|_{\mathcal{A}}\, e^{\tilde{l}(T-\max\tau(0,0))/\eta}, \quad (58)$$

where t_s,j_s are the smallest times for which $|\tau(t,j)|_{\mathcal{A}_{sync}}=0$ for all $t+j\geq t_s+j_s$. Note that $x(t_s,j_s)\in C_s\cup D_s$, and hence (52) holds with $|x(0,0)|_{\mathcal{A}}$ replaced by $|x(t_s,j_s)|_{\mathcal{A}}$, i.e., $|x(t,j)|_{\mathcal{A}}$ satisfies:

$$|x(t,j)|_{\mathcal{A}} \leq c\,|x(t_s,j_s)|_{\mathcal{A}}\, e^{-\frac{\lambda}{2}(t-(T-\max\tau(0,0))/\eta)}, \quad (59)$$

for all $t+j\geq t_s+j_s$. Using (59), (58), and the structure of the hybrid time domains in (53):

$$|x(t,j)|_{\mathcal{A}} \leq \hat{c}\,|x(0,0)|_{\mathcal{A}}\, e^{-\frac{\lambda}{3n}(t+j)}, \quad (60)$$

with $\hat{c}=ce^{\left(\lambda\left(\frac{1}{2}+\frac{L}{3}\right)+\tilde{l}L\right)}$ which establishes the bound in (23). This also implies that $\mathcal{H}_1$ renders $\mathcal{A}$ UGES under the conditions of Theorem 1-(i_2). Finally, to establish the convergence bound of item (i_3), we note that (56) implies $V(x(t,j+n))\leq(1-\gamma\left(\kappa^{-1}\right))V_3(x(t,j))$. Since $V_3(x)\leq V(x)$ for all $(t,j)\in\mathcal{T}(x)$, V does not increase during flows, and using the periodicity of the hybrid time domains:

$$V_3(t,j_s+kn) \leq (1-\gamma\left(\kappa^{-1}\right))^k V_3(t_s,j_s),\ \ \forall\, k\in\mathbb{Z}_{\geq0}, \quad (61)$$

for all $t\in(t_s+(k-1)L,t_s+kL)$, where (t_s,j_s) denotes the first hybrid time after which the timers τ flow synchronized. By Lemma 1, such times are uniformly bounded as $0\leq t_s+j_s\leq 2T^*$. Using (61), the definition of V_3, as well as strong convexity and smoothness of $\tilde{P}$, we obtain:

$$|q(t,j_s+kn)-q^*| \leq \sigma_r\sqrt{\frac{\ell}{\kappa}}(1-\gamma\left(\kappa^{-1}\right))^{\frac{k}{2}}|q(t_s,j_s)-q^*|, \quad (62)$$

for all $k\in\mathbb{Z}_{\geq0}$. Finally, since by Lemma 1 all solutions are bounded, for each compact set of initial conditions K_0 there exists $M_0>0$ such that $|x(t,j)|_{\mathcal{A}}\leq M_0$ for all $(t,j)\in\mathrm{dom}(x)$ such that $0\leq t\leq t_s$ and $0\leq j\leq j_s$. This bound and (62), implies the bound of the theorem via the change of variable $j=j_s+kn$ and the upper bound $n\leq j_s\leq 2n$. ■

2) Proofs for Non-Potential Games: As before, we divide the proof in different lemmas.

Lemma 11: Consider the HDS $\mathcal{H}_s$ under the Assumptions of Theorem 2. Then, the set $\mathcal{A}$ is UGAS. □

Proof: By Assumption 3 and the strict monotonicity of the pseudo-gradient, existence of the NE is guaranteed via [6, Cor 4.2]. Let $\tilde{V}=V_1+V_2+\tilde{V}_3$, where V_1 and V_2 are defined in (44), and $\tilde{V}_3$ is now:

$$\tilde{V}_3(x) := c_o\frac{|\tau|^2\,|\mathcal{G}(q)|^2}{2n}, \quad (63)$$

where c_o corresponds to the cocoercivity constant of $\mathcal{G}$. By construction and Assumption 2, V is radially unbounded, and also positive definite with respect to $\mathcal{A}\cap(C_s\cup D_s)$. Using c_o-cocoercivity of $\mathcal{G}$, inequality (45) becomes

$$\dot{\tilde{V}}(x) \leq -\tau_s\tilde{x}^\top M_{1/c_o}(q,\tau_s)\tilde{x}, \quad (64)$$

where $\tilde{x}:=\big((p-q),\mathcal{G}(q)\big)$, and

$$M_{1/c_o}(q,\tau_s) := \begin{pmatrix} \frac{1}{\tau_s^2}I_n & I_n-c_o\partial\mathcal{G}(q)^\top \\ I_n-c_o\partial\mathcal{G}(q) & c_o(1-\eta)I_n \end{pmatrix}. \quad (65)$$

Since $\eta\leq\frac{1}{2}$ by design, $c_o=1/\ell$, and $\tau_s\in[T_0,T]$, under the conditions of Theorem 2, we have that $M_\ell(q,\tau_s)\succ 0$ for all $\tau_s\in[T_0,T]$ and $q\neq q^*$ whenever

$$0 \prec I_n-\frac{T^2}{\ell(1-\eta)}\big(\ell I_n-\partial\mathcal{G}(q)^\top\big)\big(\ell I_n-\partial\mathcal{G}(q)\big). \quad (66)$$

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3321901

The expression in (66) is precisely (24) with $\rho_F = \ell$ and $\delta = 0$. Thus, since by assumption $\mathcal{S}_0$ is ℓ-GC, it follows that (66) holds. Also, note that when $q = q^*$ inequality (64) reduces to $\dot{\tilde{V}}(x) \leq -\frac{1}{\tau_s}|p-q|^2 \leq 0$.

On the other hand, after the n consecutive jumps that proceed each interval of flow, the change of V is $\Delta_j^{j+n}\tilde{V}(z) = \frac{c_o}{2}|\mathcal{G}(q)|^2\left(T_0^2 - T^2\right) \leq 0$. Now, we show that no complete solution keeps $\tilde{V}$ in a non-zero level set. In particular, since for all $(q,p,\tau) \in \mathbb{R}^n\backslash\{q^*\}\times\mathbb{R}^n\times[T_0,T]$ we have that $\dot{V} < 0$, it suffices to consider the case $q = q^*$, which leads to $\dot{V} = 0$ only when $p = q$, i.e., when $(p,q) \in \mathcal{A}$. Since the flows are periodic, we obtain UGAS of $\mathcal{A}$ by [24, Thm. 8.8]. ■

Proof of Theorem 2: *(a) Stability Properties:* Follows by the same ideas used in the proof of the stability properties of Theorem 1-(i_1), but using Lemma 11 instead of Lemma 8.

(b) Convergence Bounds: Follows by the same steps used in the proof of Theorem 1-(i_1), substituting (57) by

$$|\mathcal{G}(q)|^2 \leq \frac{2\ell n}{\tau^\top\tau}\tilde{V}_3(s_j,j) = \frac{\tilde{c}_j}{\tau_s^2}, \quad \tilde{c}_j := 2\ell\tilde{V}_3(s_j,j). \quad ■$$

Lemma 12: Consider the HDS $\mathcal{H}_s$ under the Assumptions of Theorem 3-(i_4). Then, the set $\mathcal{A}$ is UGES. □

Proof: $\tilde{V} = V_1 + V_2 + \tilde{V}_3$, where V_1 and V_2 are defined in (44), and $\tilde{V}_3$ is given by (63) with $c_o = \kappa/\ell^2$. The time derivative of $\tilde{V}$ now satisfies $\dot{\tilde{V}}(x) \leq -\tau_s\tilde{x}^\top M_{\sigma_\phi\ell}(q,\tau_s)\tilde{x}$, with $\tilde{x} := \big((p-q),\mathcal{G}(q)\big)$. By assumption we know that $\mathcal{S}_\delta$ is $(\sigma_\phi\ell)$-GC, which is equivalent to:

$$0 \prec I_n - \left(\frac{T^2}{1-T^2\delta}\right)\frac{\left(\sigma_\phi\ell I_n - \partial\mathcal{G}(q)^\top\right)\left(\sigma_\phi\ell I_n - \partial\mathcal{G}(q)\right)}{\sigma_\phi\ell(1-\eta) - \sigma_\phi^2\ell^2\delta}.$$

In turn, when $0 < \delta < (1-\eta)/\sigma_\phi\ell$ and $0 < \eta \leq 1/2$, the above inequality directly implies that $M_{\sigma_\phi\ell}(q,\tau_s) \succ \delta I_n$, for all $\tau_s \in [T_0,T]$ and all $q \neq q^*$. Thus, for such points, and during flows, we have $\dot{\tilde{V}} \leq -\delta(|p-q|^2 + |\mathcal{G}|(q))$. Using κ-strong-monotonicity and κ/ℓ^2-cocoercivity of $\mathcal{G}$ we conclude

$$\dot{\tilde{V}}(x) \leq -\lambda\tilde{V}(x), \text{ with } \lambda = \frac{4T_0\delta}{\max\left\{3, 2(\frac{1}{\kappa^2} + \frac{\kappa}{\ell^2}T^2)\right\}}. \quad (67)$$

On the other hand, during jumps, using (RC_1), the definition of $\tilde{V}_3$, and the Reset Policy $\alpha \in \{0,1\}^n$, the change of $\tilde{V}$ is

$$\Delta_j^{j+n}\tilde{V} \leq -\frac{1}{4}\sum_{i\in\Theta}\left((p_i-q_i)^2 + (p_i-q_i^*)^2\right) - \gamma\left(\sigma_\phi^2\kappa^{-1}\right)\tilde{V}_3(x), \quad (68)$$

where $\gamma(\sigma_\phi^2\kappa^{-1}) \in (0,1)$ is given by (21), and Θ is defined in the proof of Lemma 9. Thus, it follows that $\Delta_j^{j+n}\tilde{V} \leq 0$. Moreover, by the κ-strong monotonicity and ℓ-Lipschitz continuity of $\mathcal{G}$, $\tilde{V}$ satisfies the quadratic bounds $\underline{c}\,|x|_{\mathcal{A}}^2 \leq \tilde{V}(x) \leq \overline{c}\,|x|_{\mathcal{A}}^2$, where: $\underline{c} := \min\left\{\frac{1}{4}, \frac{\kappa T_0^2}{2\sigma_a^2}\right\}$ and $\overline{c} := \max\left\{\frac{3}{4}, \frac{1}{2} + \frac{\kappa T^2\ell^2}{2}\right\}$. The exponential decrease of V during the periodic flows, the non-increase of V during the jumps, and the quadratic upper and lower bounds of $\tilde{V}$, imply that $\mathcal{H}_s$ renders UGES the set $\mathcal{A}$. ■

Lemma 13: Consider the HDS $\mathcal{H}_s$ under the Assumptions of Theorem 3-(i_5). Then, the set $\mathcal{A}$ is UGES. □

Proof: Consider the Lyapunov function $\tilde{V}$ used in the proof of Lemma 11, which still satisfies (67). During jumps, the reset policy $\alpha = \mathbf{0}_n$ implies that $\Theta = \mathcal{V}$ in (68), leading to $\Delta_j^{j+n}\tilde{V}(x) \leq -V_1(x) - V_2(x) - \gamma(\sigma_\phi^2\kappa^{-1})V_3(x) \leq -\gamma(\sigma_\phi^2\kappa^{-1})\tilde{V}(x)$. The result follows by [39, Thm. 1] and the quadratic upper and lower bounds of $\tilde{V}$. ■

Proof of Theorem 3: *(a) Stability Properties:* Follows by using using Lemmas 12 and 13 in conjunction with the same ideas used in the proof of Theorem 1.

(b) Convergence Bounds: We follow the same steps of the proof of Theorem 1, using now $\tilde{V}_3$ instead of V_3. For item (i_4), this leads to the following bound instead of (60):

$$|x(t,j)|_{\mathcal{A}} \leq \hat{c}\,|x(0,0)|_{\mathcal{A}}\, e^{-\frac{\lambda}{3n}(t+j)},$$

where λ are defined in (67), $\hat{c} := \sqrt{\overline{c}/\underline{c}}\cdot e^{\left(\frac{5}{6}\lambda+\tilde{l}\right)L}$, and $\underline{c}$ and $\overline{c}$ are as defined in the proof of Lemma 12. Finally, for item (i_5), we obtain the following bound instead of (62):

$$|q(t,j_s+kn) - q^*| \leq \sigma_r\sigma_\phi\left(1-\gamma\left(\sigma_\phi^2\kappa^{-1}\right)\right)^{\frac{k}{2}}|q(t_s,j_s) - q^*|,$$

from here, the proof follows the exact same steps. ■

Proof of Lemmas 2 and 3: We first show Lemma 3. Using $c_o = \kappa/\ell^2$ we have that (RC_3) can be equivalently written as $\tilde{\alpha} > 1 - 2c_o\kappa + c_o^2\ell^2$, with $\tilde{\alpha} := \left(\frac{1}{T^2} - \delta\right)\left(c_o(1-\eta) - \delta\right)$. Since $\mathcal{G}$ is ℓ-Lipschitz continuous, we have that $\partial\mathcal{G}(q)^\top\partial\mathcal{G}(q) \prec \ell^2 I_n$ [22]. Additionally, given the monotonicity properties of $\mathcal{G}$, it follows that $\partial\mathcal{G}(q) + \partial\mathcal{G}(q)^\top \succ 2\kappa I_n$ [28, Prop 2.3.2 c)]. Using these facts, the above condition on $\tilde{\alpha}$ implies that

$$0 \prec I - \left(\frac{T^2}{1-T^2\delta}\right)\frac{\left(I_n - c_o\partial\mathcal{G}(q)^\top\right)\left(I_n - c_o\partial\mathcal{G}(q)\right)}{c_o(1-\eta) - \delta},$$

which means, whenever $0 \leq \delta < c_0(1-\eta)$, that $\mathcal{S}_\delta$ is $(1/c_o)$-GC. Lemma 2 follows by the same arguments, using $c_o = 1/\ell$, and letting $\kappa \to 0^+$. ■

Proof of Lemma 4: The result follows by direct computation considering (RC_1) and (RC_3) simultaneously. A step-by-step computation can be found in the extended manuscript [36]. ■

Proof of Lemma 5: The result is obtained from the convergence bound of Theorem 3-(i_5) by leveraging the periodicity of the hybrid time domains, and optimizing with respect to T. A step-by-step computation can be found in the extended manuscript [36]. ■

B. Proofs of Section 3

To prove Theorem 4, we present two auxiliary lemmas:

Lemma 14: Consider the assumptions of Theorem 4, and let $\mathcal{H}_{2,s} = \{C_{2,s}, F_{2,s}, D_{2,s}, G_{2,s}\}$ be obtained by intersecting the data of $\mathcal{H}_2$ with $\mathcal{A}_{2,\nu} := \mathcal{A}_\nu \times (\mathcal{Q}(\mathbf{1}_n \otimes q^*) + \nu\mathbb{B})$, where $\mathcal{A}_\nu = (\{(q^*,q^*)\} + \nu\mathbb{B}) \times \mathcal{A}_{\text{sync}}$. Then $\mathcal{H}_{2,s}$ renders UGAS the set $\mathcal{A} \times \{\mathcal{Q}(\mathbf{1}_n \otimes q^*)\}$.

Proof: Consider the change of variable $\theta = \hat{q} - h(q)$, with $h(q) := \mathcal{Q}(\mathbf{1}_n \otimes q)$, and let

$$\begin{aligned} W(q,\theta,\varepsilon) := &-\mathcal{Q}\mathbf{L}\mathcal{Q}^\top\theta - \varepsilon\mathcal{Q}\left(\mathbf{1}_n \otimes 2D(\tau)^{-1}(p-q)\right) \\ &+ \varepsilon\mathcal{Q}\left(\mathbf{1}_n \otimes \mathcal{P}\mathbf{L}\mathcal{Q}^\top\theta\right). \end{aligned} \quad (69)$$

This change of coordinates leads to a HDS $\mathcal{H}_\vartheta$ with state $\vartheta := (x,\theta)$, where $x = (q,p,\tau)$, and data $\mathcal{H}_\vartheta = (C_{2,\vartheta}, F_{2,\vartheta}, D_{2,\vartheta}, G_{2,\vartheta})$, where $C_{2,\vartheta}, D_{2,\vartheta}$ and $G_{2,\vartheta}$ are obtained directly from (31), (32), and (33) respectively via the change of coordinates, and where the flow map is defined by

$F_{2,\vartheta}(\vartheta) := (U(x, \theta + h(q)),\ W(q, \theta, \varepsilon)/\varepsilon)$ where:

$$U(x, \theta + h(q)) = \begin{pmatrix} 2D(\tau)^{-1}(p-q) - \mathcal{P}\mathbf{L}\mathcal{Q}^\top\theta \\ -2D(\tau)\hat{\mathcal{G}}(\mathbf{1}_n \otimes q + \mathcal{Q}^\top\theta) \\ \eta\mathbf{1}_n \end{pmatrix}. \quad (70)$$

Let $\mathcal{H}_{\vartheta,s}$ be the HDS that results from intersecting the data of $\mathcal{H}_\vartheta$ with $\mathcal{A}_\nu \times (\nu\mathbb{B})$, with $\nu > 0$. Studying the stability of $\mathcal{A} \times \{\mathcal{Q}(\mathbf{1}_n \otimes q^*\}$ under $\mathcal{H}_{2,s}$, is equivalent to analyzing the stability of the compact set $\mathcal{A}_{\mathbb{G},\theta} = \mathcal{A} \times \{0\}^{n^2-n}$ under $\mathcal{H}_{\vartheta,s}$. For this last system, we consider the Lyapunov function

$$V_{\mathbb{G}}(\vartheta) = (1-d)\tilde{V}(x) + d \cdot V_\theta(\theta),\ \text{with } d \in (0,1), \quad (71)$$

where $\tilde{V}$ is defined as in Lemma 11, and $V_\theta(\theta) := \frac{1}{2}|\theta|^2$. Using the proof of Lemma 11, and the equality $\hat{\mathcal{G}}(\mathbf{1} \otimes q) = \mathcal{G}(q)$, it follows that $\frac{\partial \tilde{V}(x)}{\partial x} U(x, h(q)) \leq -\tau_s \tilde{x}^\top M_\ell(q, \tau_s)\tilde{x}$, with $\tilde{x} := \big((p-q), \mathcal{G}(q)\big)$ and M_ℓ given by (65) with $c_o = \frac{1}{\ell}$. Under the assumptions of Theorem 4 we know that

$$0 \prec I_n - \left(\frac{T^2}{1 - T^2\delta}\right) \frac{(\ell I_n - \partial\mathcal{G}(q)^\top)(\ell I_n - \partial\mathcal{G}(q))}{\ell(1-\eta) - \ell^2\delta},$$

and thus that $M_\ell(q, \tau_s) \succ \delta I_n\ \forall \tau_s \in [T_0, T]$. Hence, letting $\xi(x) := \left(|p-q|^2 + |q-q^*|^2\right)^{1/2}$ we obtain that

$$\frac{\partial \tilde{V}(x)}{\partial x}\dot{x} \leq -T_0\delta \min\{1, \zeta\}\xi^2(x), \quad (72)$$

where we used the bound of Assumption 4. Also,

$$\frac{\partial \tilde{V}}{\partial x}\left(U(x, \theta + h(q)) - U(x)\right) \leq c_1\left(|p-q| + |q-q^*|\right)|\theta|, \quad (73)$$

$$c_1 := \frac{T^2\lambda_{\max}(\mathcal{L})}{\sqrt{2}} \max\left\{\frac{1}{T^2} + \frac{4\ell}{T\lambda_{\max}(\mathcal{L})}, 2 + \frac{2\ell}{T\lambda_{\max}(\mathcal{L})}\right\}.$$

On the other hand, by the fact that the underlying communication graph is undirected and connected, it follows that $\mathcal{Q}\mathbf{L}\mathcal{Q}^\top$ is positive definite [7, Lemma 6], and, moreover that

$$\frac{\partial V_\theta}{\partial \theta} W(q, \theta, 0) \leq -\frac{\lambda_2(\mathcal{L})}{n}|\theta|^2, \quad (74)$$

$$\left(\frac{\partial V_\theta}{\partial x} - \frac{\partial V_\theta}{\partial \theta}\frac{\partial h}{\partial x}\right) U(x, \theta + h(q)) \leq c_2\psi(x)|\theta| + c_3|\theta|^2, \quad (75)$$

where $c_2 := 2\sqrt{2n}/T_0$ and $c_3 := 2\sqrt{n}\lambda_{\max}(\mathcal{L})$. Hence, using (72)-(75) it follows that the time derivative of $V_{\mathbb{G}}$ satisfies $\dot{V}_{\mathbb{G}} \leq -(\xi(x), \theta)^\top \Lambda_\varepsilon(\xi(x), \theta)$ with

$$\Lambda_\varepsilon := \begin{pmatrix} (1-d)T_0\epsilon\min\{1, \zeta^2\} & -\frac{1}{2}(1-d)c_1 - \frac{1}{2}c_2 \\ -\frac{1}{2}(1-d)c_1 - \frac{1}{2}c_2 & d\left(\frac{\lambda_2(\mathcal{L})}{\varepsilon n} - c_3\right) \end{pmatrix},$$

which is positive definite provided that $\varepsilon \in (0, \varepsilon^*_\delta)$ where ε^*_δ is as defined in (28). Note moreover, that if ε satisfies this condition there exists $k_\varepsilon > 0$ such that

$$\dot{V}_{\mathbb{G}} \leq -k_\varepsilon\left(|p-q|^2 + |q-q^*|^2 + |\theta|^2\right). \quad (76)$$

Using the results regarding the change of the Lyapunov function $\tilde{V}$ during jumps presented in the proofs of Lemmas 11, 12 and 13, given that (RC_1) is satisfied with $\rho_J = 0$ by assumption, and since $V_\theta^+(\theta) = V_\theta(\theta)$ for all θ whenever $\vartheta \in D_{2,\vartheta}$, it follows that $\Delta_j^{j+n} V_{\mathbb{G}}(\vartheta) \leq 0$ for any resetting policy $\alpha \in \{0,1\}^n$. This inequality and (76) imply that $\mathcal{H}_{\vartheta,s}$ renders the set $\mathcal{A}_{\mathbb{G},\theta}$ UGAS via [24, Prop. 3.27]. The stability results for $\mathcal{H}_{2,s}$ follow directly by the change of cooordinates $\hat{q} = \theta + h(q)$ and the described result for $\mathcal{H}_{\vartheta,s}$. ■

Lemma 15: Every solution of $\mathcal{H}_2$ is complete.

Proof: Follows by using the Lipschitz continuity of the flow-map F_2 and the Gronwall-Bellman inequality. Step-by-step derivations are presented in the extended manuscript [36]. ■

Proof of Theorem 4: (a) Let $\mathcal{H}_{2,\nu}$ be defined from $\mathcal{H}_2$ by following the same procedure described in the statement of Lemma 7. Since the addition of the state $\hat{q}$ and its associated dynamics do not affect the synchronization dynamics, $\mathcal{H}_{2,\nu}$ renders UGFxS the set $\mathcal{A}_{2,\nu}$, where $\mathcal{A}_{2,\nu}$ is as defined in Lemma 14. Therefore, by the hybrid reduction principle [24, Cor. 7.24], UGAS of $\mathcal{A} \times \{\mathcal{Q}(\mathbf{1}_n \otimes q^*)\}$ for system $\mathcal{H}_{2,s}$, established in Lemma 14, implies that $\mathcal{A} \times \{\mathcal{Q}(\mathbf{1}_n \otimes q^*)\}$ is UGAS for $\mathcal{H}_{2,\nu}$. Since the choice of $\nu > 0$ is arbitrary and since solutions of $\mathcal{H}_2$ are complete and bounded, using Lemma 15, we have that $\mathcal{A} \times \{\mathcal{Q}(\mathbf{1}_n \otimes q^*)\}$ is UGAS for $\mathcal{H}_2$.

(b) Let $\nu > 0$, and $K_0 := K_x \times K_{\hat{q}} \subset \mathbb{R}^{3n} \times \mathbb{R}^{n^2-n}$ be an arbitrary compact set. Define $\overline{v} := \max_{\vartheta \in K_0} V_{\mathbb{G}}(\vartheta)$, where $V_{\mathbb{G}}$ is as given in (71). Notice that $\overline{v}$ exists since $V_{\mathbb{G}}$ is continuous and K_0 is compact by assumption. It follows that $K_0 \subseteq L_{V_{\mathbb{G}}}(\overline{v})$, where $L_{V_{\mathbb{G}}}(c)$ represents the c-sublevel set of $V_{\mathbb{G}}$. Since $V_{\mathbb{G}}$ is radially unbounded by construction and Assumption 2, $L_{V_{\mathbb{G}}}(\overline{v})$ is compact. Let $K^V := L_{V_{\mathbb{G}}}(\overline{v})$ and define the HDS $\mathcal{H}_{2,K} = (F_2,\ C_2 \cap K^V,\ G_2,\ D_2 \cap K^V)$. Notice that under $\mathcal{H}_{2,K}$, $\hat{q}$ evolves in a compact set. Moreover, by the arguments presented in the proof of item (a), $\mathcal{H}_{2,K}$ renders K^V strongly forward invariant for any $\varepsilon \in (0, \varepsilon^*_\delta)$. Hence, using Lemma 15, it follows that, given any arbitrary compact set $\tilde{K}_x \times \tilde{K}_{\hat{q}} \subset K^V$, every solution to $\mathcal{H}_{2,K}$ with $(x(0,0), \hat{q}(0,0)) \in \tilde{K}_x \times \tilde{K}_{\hat{q}}$ is complete. Therefore, by [30, Thm. 1], for any pair $\hat{t}, \hat{j} > 0$ there exists $\tilde{\varepsilon} \in (0, \varepsilon^*_\delta)$ such that for each $\varepsilon \in (0, \tilde{\varepsilon}]$ and each solution z to $\mathcal{H}_{2,K}$, with $z(0,0) \in K_x \times K_{\hat{q}}$, there exists a solution x to $\mathcal{H}_1$such that x and z are $(\hat{t}, \hat{j}, \nu)-$close. The result follows with $\varepsilon^{**} = \min\{\tilde{\varepsilon}, \varepsilon^*_\delta\}$. ■

Proof of Theorem 5: First, using a Taylor expansion of the form $\phi_i(q + \varepsilon_a\tilde{\mu})\tilde{\mu}_i = \tilde{\mu}_i\phi_i(q) + \varepsilon_a\tilde{\mu}_i\tilde{\mu}^\top\nabla\phi_i(q) + \tilde{\mu}_i\mathcal{O}(\varepsilon_a^2)$, and the fact that $|\tilde{\mu}_i| \leq 1$ for all $i \in \mathcal{V}$, and that $\frac{1}{\tilde{L}}\int_0^{\tilde{L}} \tilde{\mu}_i(t)\tilde{\mu}(t)^\top dt = e_i$, where $\tilde{L} = 2\pi\text{LCM}\{1/\varsigma_1, \ldots, 1/\varsigma_n\}$ and LCM denotes the least common multiple, the average dynamics of $\mathcal{H}_3$ are given by $\mathcal{H}_3^A = (C_1, F_1^A, D_1, G_1)$, where G_1, C_1 and D_1 are given by (14), (16), and (17), respectively, and

$$F_1^A(x) = \begin{pmatrix} 2\mathcal{D}(\tau)^{-1}(p-q) \\ -2\mathcal{D}(\tau)\left(\mathcal{G}(q) + \mathcal{O}(\varepsilon_a)\right) \\ \eta\mathbf{1}_n \end{pmatrix}. \quad (77)$$

It follows that, on compact sets, we have $F_1^A(x) \in \overline{\text{con}}F_1(x + k\varepsilon_a\mathbb{B}) + k\varepsilon_a\mathbb{B}$, for some $k > 0$, where F_1 was defined in (15). Thus, any solution of the average dynamics $\mathcal{H}_3^A$ is also a solution of an inflated HDS generated from $\mathcal{H}_1$. By [24, Thm. 7.21], we conclude that, under the Assumptions of Theorems 1-3, system $\mathcal{H}_3^A$ renders SGPAS as $\varepsilon_a \to 0^+$ the compact set $\mathcal{A}$. Since $\mathcal{H}_3^A$ and $\mathcal{H}_1$ are nominally well-posed, all assumptions of [17, Thm.7] are satisfied, and we conclude that $\mathcal{H}_3$ renders SGPAS as $(\varepsilon_p, \varepsilon_a) \to 0^+$ the compact set $\mathcal{A} \times \mathbb{T}^n$. Item (b) follows directly by [17, Prop. 6]. ■

VIII. Conclusions

We introduced a class of hybrid Nash set-seeking algorithms with dynamic momentum for the efficient solution of noncooperative games with finitely many players. The algorithms incorporate continuous-time dynamics with momentum and discrete-time decentralized coordinated resets that model restarting mechanisms. By using tools from hybrid dynamical systems theory, we developed model-based algorithms that rely on full-information Oracles, as well as algorithms suitable for games with partial information and model-free settings. In the latter cases, we established robust stability and convergence properties using multi-time scale techniques based on singular perturbations and averaging theory.

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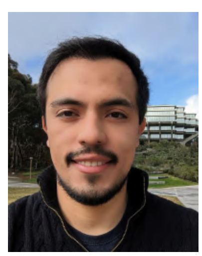

Daniel E. Ochoa is a Ph.D. candidate in the Department of Electrical and Computer Engineering at the University of California, San Diego. He received his M.Sc. degree in Electrical Engineering from the University of Colorado, Boulder in 2022, and a M.Sc. degree in Electronics and Computer engineering from the University of Los Andes, Colombia in 2019. He holds double Bachelor's degrees in Electronics Engineering (cum laude) and Physics from the University of Los Andes, Colombia, with a minor in Computational Mathematics.

Jorge I. Poveda is an Assistant Professor in the Electrical and Computer Engineering Department at UC San Diego. He received his M.Sc. and Ph.D. degrees in Electrical and Computer Engineering from UC Santa Barbara in 2016 and 2018, respectively. Before joining UCSD, he was an Assistant Professor at the University of Colorado, Boulder, and a Postdoctoral Fellow at Harvard University. He has received the NSF 2020 Research Initiation (CRII) and 2022 Early Career (CAREER) awards, the 2022 AFOSR Young Investigator Award, the UCSB-CCDC Outstanding Scholar Fellowship (2013) and Best Ph.D. Thesis award (2020), and the 2023 AACC Donald P. Eckman award. His research interests include hybrid control, adaptive and network systems, and model-free control and optimization.