A Stochastic Binary Vertex-Triggering Resetting Algorithm for Global Synchronization of Pulse-Coupled Oscillators

Muhammad Umar Javed, Jorge I. Poveda, and Xudong Chen, *Member, IEEE*

***Abstract*—In this article, we propose a novel stochastic binary resetting algorithm for networks of pulse-coupled oscillators (or, simply, agents) to reach global synchronization. The algorithm is simple to state: Every agent in a network oscillates at a common frequency. Upon completing an oscillation, an agent generates a Bernoulli random variable to decide whether it sends pulses to all of its out-neighbors or it stays quiet. Upon receiving a pulse, an agent resets its state by following a binary phase update rule. We show that such an algorithm can guarantee *global* synchronization of the agents almost surely as long as the underlying information flow topology is a rooted directed graph. The proof of the result relies on the use of a stochastic hybrid dynamical system approach. Toward the end of this article, we present numerical demonstrations for the validity of the result and numerical studies about the units of time needed to reach synchronization for networks with various information flow topologies.**

***Index Terms*—Hybrid dynamical systems (HDSs), networked systems, stochastic processes, synchronization of multiagent systems.**

I. Introduction

IN THIS article, we consider a network of N pulse-coupled oscillators (PCOs), characterized by periodic resetting dynamics, sharing information with their neighbors where the neighboring relations are described by a directed graph (digraph). Each agent has an individual state $\tau_i \in \mathbb{R}$, which evolves according to the following continuous-time dynamics:

$$\tau_i \in [0, 1) \Rightarrow \dot{\tau}_i = \frac{1}{T} \; \forall i \in \{1, 2, \ldots, N\} \tag{1}$$

where $T > 0$ is the period of oscillation, and [0,1) is a normalized unit interval. When the state of an agent i finishes an oscillation, i.e., $\tau_i = 1$, it will instantaneously *reset* its individual state back to zero

$$\tau_i = 1 \;\Rightarrow\; \tau_i^+ = 0. \tag{2}$$

Simultaneously, the agent sends a pulse, with a certain probability $p \in (0, 1)$ to trigger *all* of its (out-)neighbors j. Each out-neighbor j of agent i, upon receiving the pulse, instantaneously updates its state τ_j using a set-valued *binary* phase update rule

$$\tau_j^+ \in \mathcal{R}_j(\tau_j) = \begin{cases} \{0\}, & \tau_j \in [0, r_j) \\ \{0, 1\}, & \tau_j = r_j \\ \{1\}, & \tau_j \in (r_j, 1] \end{cases} \tag{3}$$

where the constant $r_j \in (0, 1)$ partitions the unit interval.

Among others, in this article, we show that if the underlying information flow topology is a rooted directed graph, then for any p and any $r := [r_1, \ldots, r_N]^\top$, the network of PCOs will reach synchronization almost surely from all initial conditions. Since each individual state τ_i is confined to evolve in the normalized interval [0,1], one can view the state as flowing in a unit circle (that is formed by identifying the two endpoints 0 and 1 with each other), in the counterclockwise direction, with frequency $1/T$. In this way, global synchronization of PCOs can be cast as a consensus problem on the N-torus (e.g., see [1], [2], and [3]). See Fig. 1 for an illustration of our algorithm on the 2-torus.

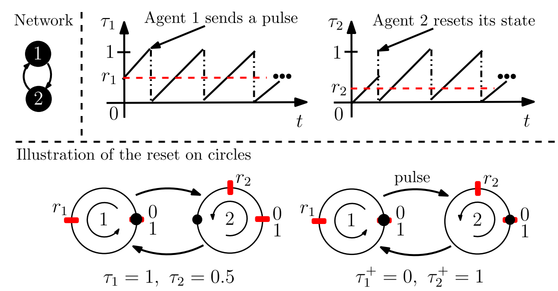

Fig. 1. Illustration of our stochastic binary resetting algorithm: When agent 1 satisfies $\tau_1 = 1$, it updates its state by $\tau_1^+ = 0$. Meanwhile, it draws a Bernoulli random variable to decide whether to send a pulse to its out-neighbors (in this case, only agent 2). If the pulse is sent, then upon receiving the pulse, agent 2 will update its state by following the binary update rule described in (3). In this example, since $\tau_2 > r_2$, agent 2 updates its state by $\tau_2^+ = 1$. Note that once all agents are synchronized, they remain synchronized under our algorithm (1)–(3).

Synchronization of PCOs using deterministic resetting algorithms has been widely investigated in the literature, and we refer the reader to [2], [4], [5], [6], [7], [8], [9], [10], [11],

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Muhammad Umar Javed is with the Amazon Web Services, Amazon, Seattle, WA 80302 USA (e-mail: mumarjav@amazon.com).

Jorge I. Poveda is with the ECE Department, University of California San Diego, San Diego, CA 92093 USA (e-mail: poveda@ucsd.edu).

Xudong Chen is with the ECEE Department, University of Colorado Boulder, Boulder, CO 80302 USA (e-mail: xudong79@gmail.com).

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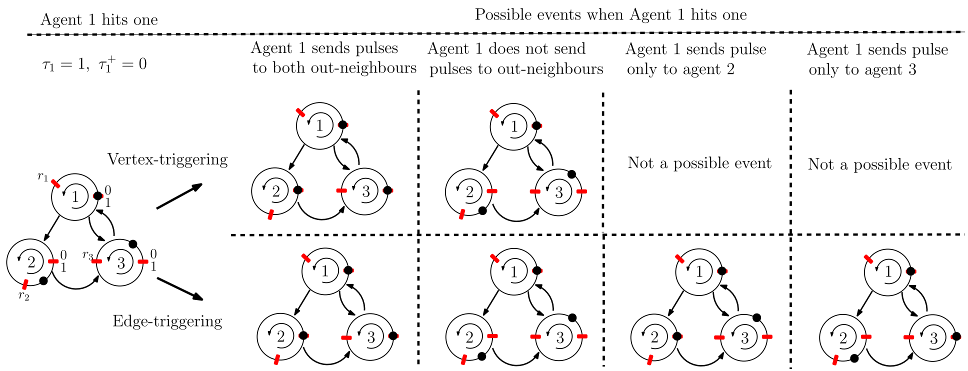

Fig. 2. Two different random triggering models "vertex-triggering" and "edge-triggering." We present possible events when agent 1 hits value 1 (i.e., the agent satisfies $\tau_1 = 1$, $\tau_1^+ = 0$) for both models. An out-neighbor of agent 1 updates its state to either 0 or 1 when it receives a pulse from agent 1; otherwise, the out-neighbor will retain its state.

[12], [13], [14], [15], and [16]. However, in none of these works, has global synchronization been shown to achieve overall rooted digraphs using deterministic resetting algorithms. Some works have relaxed the global convergence requirement to either *local* convergence (e.g., [4], [14], and [15]) or *almost global* convergence (e.g., [12], [13], and [16]) while other works have restrictions on the underlying information flow topologies [2], [5], [6], [7], [8], [9], [10], [11]. Recently, we have shown in [2] that a certain deterministic binary resetting algorithm cannot achieve global synchronization over all rooted digraphs. Whether or not there exists a deterministic resetting algorithm that can achieve global synchronization of PCOs over all rooted digraphs still remains open.

The problem of global synchronization of PCOs using stochastic resetting algorithms has also been investigated in the literature [2], [3], [17], [18], [19]. Our study on the problem, as well as the main results established in the article, is different from the ones in those existing works, as we elaborate ahead.

First, we mention the works [3] and [17]. In these works, the authors have considered a similar stochastic resetting algorithm. A key difference is that their phase update rule is described by a piecewise continuous function (the only discontinuity is at r_j, which is set to be 0.5 for all the agents), with each piece being *strictly monotonically increasing*, whereas ours is piecewise *constant*. Although the difference in the phase update rule seems to be moderate, the analyses of the two resulting systems differ significantly. In particular, the arguments developed in [3] and [17] do not apply to our case; certain key results, such as [17, Lemma 8], do not hold anymore. For example, the authors there have considered the arc of minimum length that covers all the agents on the unit circle and shown that the number of agents on the boundary points of the arc cannot increase over time. This is not true if one uses the binary phase update rule. Besides the difference in the phase update rule, there is also a difference in the underlying information flow topology. Using their resetting algorithm, the authors have established almost sure global synchronization over undirected connected graphs (i.e., communications between agents are reciprocal) in [3] and over strongly connected digraphs in [17]. The class of rooted digraphs considered in this article is more general.

Next, in the work [18], Pagliari and Scaglione have considered a different type of triggering: Upon hitting 1, an agent i will generate *multiple* independent, identically distributed (i.i.d.) Bernoulli random variables, with the number of random variables matching the number of its neighbors (that is, the underlying information flow topology is undirected), so as to decide *individually* whether or not it sends a pulse to each of its neighbors. This is in contrast to the triggering model considered in this article where an agent, upon hitting 1, draws only a *single* Bernoulli random variable and *broadcasts* to all of its (out-)neighbors. Because of this, we call our triggering model *vertex-triggering* and theirs *edge-triggering*. See Fig. 2 for an illustration of both models. Note that our previous work [2] has also considered *edge-triggering*. An advantage of "vertex-triggering" over "edge-triggering" is that the former requires fewer Bernoulli random variables drawn at a time, making it easier for the agents to implement the resetting algorithm. The difference between the two algorithms will also be carried over to the analysis: For edge-triggering, the underlying information flow topology can be viewed as an Erdős–Rényi type random graph whenever an agent hits 1 (since the edges are drawn independently). In [2], we relied on such a probability model to establish almost sure global synchronization. However, this probability model cannot be used here to describe the information flow topology for the case of vertex-triggering. Due to the difference between the two probability models, we will have different sample paths of random graphs along the dynamics of the two systems. Consequently, the characterizations of the so-called "synchronization strings" (roughly speaking, these are the strings in a sample path that can lead to global synchronization as we will introduce in Definition 3.2) will also be different.

We further mention the work of Hartman et al. [19] where they have considered a completely different stochastic resetting algorithm. There, the dynamics of the agents are not pulse-coupled; instead, the authors have assumed that every agent can access the mean of the states and uses that information to make decisions and to take actions.

Our method to establish almost sure global synchronization relies on the use of stochastic hybrid dynamical systems (SHDSs) [20], where the set-valued binary update rule will be used to define the jump maps of the system. Indeed, the combination of continuous-time dynamics, describing the continuous evolution of the PCOs, and discrete-time dynamics, describing the resets, naturally lead to a hybrid dynamical system (HDS). Moreover, since the pulse-triggering of an agent (upon hitting 1) is at random and since only the (out-)neighbors of the agent could receive the pulse (if the pulse is generated and sent), the jump maps of the SHDS are stochastic and depend on the underlying information flow topology. Formally, to establish the SHDS, we will first introduce a family of infinite sequences of i.i.d random digraphs, sampled from a finite set, called the set of *feasible digraphs*. Roughly speaking, a digraph is feasible if every agent is connected to either all or none of its out-neighbors. Every such random digraph corresponds to an occurrence of an agent hitting 1, and it indicates whether the agent sends a pulse or not. We then use such a sequence of random digraphs to define the sequence of jump maps of the SHDS. We analyze random solutions of the SHDS by analyzing solutions of an HDS over a fixed, but arbitrary, infinite sequence of feasible digraphs. We present a novel condition on the sequence that can guarantee global synchronization of the HDS. We then establish almost sure global synchronization of the SHDS by showing that the condition can be satisfied almost surely. Toward the end of this article, we have conducted numerical studies for validation of the main result and for comparison of our algorithm with an existing vertex-triggering algorithm [17].

The rest of this article is organized as follows. Section II presents some preliminaries. The main results for the deterministic and stochastic settings are presented and established in Sections III and IV, respectively. Section V is about numerical studies. Finally, Section VI concludes this article.

Notations: Given a vector x in $\mathbb{R}^n$, let $|x|$ be the standard Euclidean norm of x. For a compact set $\mathcal{A} \subset \mathbb{R}^n$, let $|x|_{\mathcal{A}} := \min_{y\in\mathcal{A}} |x-y|$. We also use $|\cdot|$ to denote the cardinality of a finite set. We use $\mathbf{c}_n \in \mathbb{R}^n$ to denote a constant vector with all entries equal to $c \in \mathbb{R}$. We use $(a_1, a_2) \succeq (b_1, b_2)$ to denote $a_1 \geq b_1$ and $a_2 \geq b_2$. The floor function is denoted by $\lfloor\cdot\rfloor$. Given a set B, we use B^N to denote the N-Cartesian product of B, i.e., $B^N := B \times B \times \cdots \times B$ (N times). A function α is said to be of class-$\mathcal{K}$ if it is strictly increasing in its argument and $\alpha(0) = 0$. Additionally, if $\alpha(r) \to \infty$ as $r \to \infty$, then α belongs to class-$\mathcal{K}_\infty$. We denote by $\mathbb{B}$ (respectively, $\mathbb{B}^\circ$) the closed (respectively, open) ball of radius one centered at zero. A set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be locally bounded (LB) at $x \in \mathbb{R}^m$ if there exists a neighborhood K_x of x such that $M(K_x)$ is bounded. Given a set $\mathcal{X} \subset \mathbb{R}^m$, the mapping M is LB relative to $\mathcal{X}$ if the set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by M for $x \in \mathcal{X}$, and by $\varnothing$ for $x \notin \mathcal{X}$, is LB at each $x \in \mathcal{X}$. The graph of a set-valued mapping G is defined as $\text{graph}(G) := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : y \in G(x)\}$. Given a measurable space $(\Omega, \mathcal{F})$, a set-valued map $G : \Omega \rightrightarrows \mathbb{R}^n$ is said to be $\mathcal{F}$*-measurable* [21, Def. 14.1], if for each open set $\mathcal{O} \subset \mathbb{R}^n$, the set $G^{-1}(\mathcal{O}) := \{\omega \in \Omega : G(\omega) \cap \mathcal{O} = \varnothing\} \in \mathcal{F}$.

II. Preliminaries

In this section, we present basic notions from graph theory, and deterministic and SHDSs.

A. Graph Theory

A directed graph, or digraph, is denoted by $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} := \{1, 2, \ldots, N\}$ the set of vertices and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ the set of edges. In this article, we consider only simple digraphs, i.e., digraphs without self-arcs. We denote by (i, j) an edge of $\mathcal{G}$; we call i an in-neighbor of j, and j an out-neighbor of i. We denote the set of out-edges of vertex i as $\mathcal{E}_i^-$. A path from a vertex i to a vertex j is a sequence $\{i_0, i_1, \ldots, i_m\}$, with $i_0 = i$ and $i_m = j$, in which each pair $(i_l, i_{l+1}) \in \mathcal{E}$ for all $l \in \{0, 1, \ldots, m-1\}$ and all the vertices are pairwise distinct. The length of a path is defined to be the number of edges in that path. A vertex $i \in \mathcal{V}$ is said to be a root of $\mathcal{G}$ if for any other vertex $j \in \mathcal{V}$, there exists a path from i to j. A digraph $\mathcal{G}$ with at least one root is a *rooted digraph*. We denote the set of all the root vertices of $\mathcal{G}$ as $\mathcal{V}_R$. In a rooted digraph $\mathcal{G}$, the *depth of a vertex* j with respect to a given root vertex i^* is defined to be the length of the minimum path from i^* to j. We denote by $\mathcal{V}_q(i^*)$ the vertices at depth q, and q^* the maximum depth. The *depth of a rooted digraph* $\mathcal{G}$ is defined to be $\text{dep}(\mathcal{G}) := \max_{i^*\in\mathcal{V}_R} q^*$. A d*-regular digraph* $\mathcal{G}(\mathcal{V}, \mathcal{E}')$, for $d < N$, is a digraph where each vertex i has d out-neighbors $((i+j) \mod N) + 1$, for $j = 0, \ldots, d-1$. Note, in particular, 1- and $(N-1)$-regular digraphs are *cycle* and *complete* digraphs, respectively.

B. HDS With Random Inputs

An SHDS with state $x \in \mathbb{R}^n$ and random input $v \in \mathbb{R}^m$ is characterized by the following set of equations [20], [22]:

$$x \in C, \qquad \dot{x} = f(x) \tag{4a}$$

$$x \in D, \quad x^+ \in G(x, v^+), \;\; v \sim \mu(\cdot) \tag{4b}$$

where the function $f : \mathbb{R}^n \to \mathbb{R}^n$, called the *flow map*, describes the continuous-time dynamics of the system; the set $C \subset \mathbb{R}^n$, called the *flow set*, describes the points in the space where x is allowed to evolve according to the differential equation (4a); $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, called the *jump map*, is a set-valued mapping that characterizes the discrete-time dynamics of the system; and $D \subset \mathbb{R}^n$, called the *jump set*, describes the points in the space where x is allowed to evolve according to the stochastic difference inclusion (4b). We use v^+ as a place holder for a sequence of i.i.d. input random variables $\{\mathbf{v_k}\}_{k=1}^\infty$ with probability distribution μ, derived from an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1: An SHDS (4) is said to satisfy the *Basic Conditions* if the following holds.

1) The sets C and D are closed, $C \subset \text{dom}(f)$, and $D \subset \text{dom}(G)$.
2) The function f is continuous.
3) The set-valued mapping $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is LB and the mapping $v \mapsto \text{graph}(G(\cdot, v)) := \{(x, y) \in$

$\mathbb{R}^n \times \mathbb{R}^n : y \in G(x,v)\}$ is measurable with closed values.

For further details of SHDS (4) (concept of solution, causality assumption, etc), we refer the reader to Appendix B.

When the discrete-time dynamics (4b) does not depend on random inputs, the SHDS (4) is reduced to a standard HDS [23]

$$x \in C, \qquad \dot{x} = f(x) \tag{5a}$$

$$x \in D, \qquad x^+ \in G(x). \tag{5b}$$

Solutions of hybrid systems are parameterized by both continuous- and discrete-time indices $t \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$. The index t increases continuously during flows (4a) or (5a), and the index k increases by one when a jump occurs via (4b) or (5b). Solutions to (5) are defined on hybrid time domains, which are characterized by a pair of time indices (t,k). For further details on the concept of solution to (5) and hybrid time domains, we refer the reader to Appendix A. A solution is said to be 1) *maximal* if its time domain is not a proper subset of the domain of other solution; 2) *complete* if its time domain is unbounded; 3) *uniformly non-Zeno* if there exists $\tilde{T}, \tilde{K} \in \mathbb{R}_{>0}$ such that for every $(t_1,k_1),(t_2,k_2) \in \mathrm{dom}(x)$, $t_2 - t_1 \leq \tilde{T}$ implies that $k_2 - k_1 \leq \tilde{K}$.

C. Stability and Convergence Notions

In this article, we will consider the following properties for the solutions of the SHDS (4) and the HDS (5).

Definition 2.2: The HDS (5) is said to render a closed set $\mathcal{A}$ *strongly forward invariant* [23] if every complete solution x, with $x(0,0) \in \mathcal{A}$, satisfies $x(t,k) \in \mathcal{A}$ for all $(t,k) \in \mathrm{dom}(x)$. Similarly, the SHDS (4) is said to render the set $\mathcal{A}$ *surely strongly forward invariant* if every random solution x_ω to (4) with $x_\omega(0,0) \in \mathcal{A}$ stays surely in $\mathcal{A}$.

We next introduce the following notions from [23] and [24]

Definition 2.3: The HDS (5) renders a closed set $\mathcal{A}$.

1) *Uniformly Globally Stable* (UGS) if there exists a class-$\mathcal{K}_\infty$ function α such that any solution x to (5) satisfies $|x(t,k)|_{\mathcal{A}} \leq \alpha(|x(0,0)|_{\mathcal{A}})$ for all $(t,k) \in \mathrm{dom}(x)$;
2) *Globally Finite-Time Attractive* (GFTA) if for each solution x of (5) there exists $\bar{T}(x(0,0)) > 0$ such that $|x(t,k)|_{\mathcal{A}} = 0$ for all $(t,k) \in \mathrm{dom}(x)$ and $t + k \geq \bar{T}(x(0,0))$;
3) *Globally Fixed-Time Attractive* (GFxTA) if $\mathcal{A}$ is GFTA and, additionally, $\bar{T} > 0$ is a constant independent of $x(0,0)$.

We further have the following definition from [20], which applies to systems of the form (4).

Definition 2.4: The SHDS (4) renders a compact set $\mathcal{A}$.

1) *Uniformly Lyapunov stable in probability* if for each $\varepsilon > 0$ and $\rho > 0$ there exists a $\delta > 0$ such that for all $\mathbf{x}_\omega(0,0) \in \mathcal{A} + \delta\mathbb{B}$, every maximal random solution $\mathbf{x}_\omega$ from $\mathbf{x}_\omega(0,0)$ satisfies the inequality
$$\mathbb{P}\left(\mathbf{x}_\omega(t,k) \in \mathcal{A} + \varepsilon\mathbb{B}^\circ \quad \forall (t,k) \in \mathrm{dom}(\mathbf{x}_\omega)\right) \geq 1 - \rho. \tag{6}$$
2) *Uniformly Lagrange stable in probability* if for each $\delta > 0$ and $\rho > 0$, there exists $\varepsilon > 0$ such that the inequality (6) holds.
3) *Uniformly globally attractive in probability* if for each $\varepsilon > 0, \rho > 0$, and $R > 0$, there exists $\gamma \geq 0$ such that for all random solutions $\mathbf{x}_\omega$ with $\mathbf{x}_\omega(0,0) \in \mathcal{A} + R\mathbb{B}$ the following holds:
$$\mathbb{P}\left(\mathbf{x}_\omega(t,k) \in \mathcal{A} + \varepsilon\mathbb{B}^\circ \quad \forall t + k \geq \gamma, (t,k) \in \mathrm{dom}(\mathbf{x}_\omega)\right) \geq 1 - \rho.$$

System (4) is said to render a compact set $\mathcal{A} \subset \mathbb{R}^n$ *Uniformly Globally Asymptotically Stable in Probability* (UGASp) if it satisfies conditions (1), (2), and (3).

For an SHDS of the form (4), UGASp of a compact set can be established via the stochastic hybrid invariance principle [22, Th. 8], see Theorem 1.1.

III. Deterministic Resetting Algorithm With Time-Varying Jump Maps

In this section, we first introduce a deterministic HDS, with time-varying jump maps, for analyzing the asymptotic behavior of a typical random solution of the networked system of PCOs described in Section I. The results of this section will be used later to establish almost sure global synchronization in Section IV.

A. Well-Posed HDS

To formalize the HDS, we start by introducing a notion about feasible subgraphs of a given digraph.

Definition 3.1: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an arbitrary digraph. A subgraph $\phi = (\mathcal{V}, \mathcal{E}')$, on the same vertex set $\mathcal{V}$, is *feasible* if the edge set $\mathcal{E}'$ satisfies the following condition: For any vertex $i \in \mathcal{V}$, either $\mathcal{E}_i^- \subseteq \mathcal{E}'$ or $\mathcal{E}_i^- \cap \mathcal{E}' = \varnothing$.

Let Φ be the collection of feasible subgraphs of $\mathcal{G}$ and Ξ be the collection of infinite sequences of the feasible subgraphs, i.e., any element $\xi \in \Xi$ is given by $\xi := \phi_1\phi_2\phi_3\cdots$, where each $\phi_i \in \Phi$ is feasible. We note that both Φ and Ξ are implicitly dependent of $\mathcal{G}$. Now, for a given $\xi \in \Xi$, we define a corresponding HDS

$$\mathcal{H}_\xi := (C, f, D, G). \tag{7}$$

The state of the HDS is $x := (\tau, \lambda) \in \mathbb{R}^N_{\geq 0} \times \mathbb{Z}_{\geq 0}$. The continuous-time dynamics of this system are given by

$$\dot{x} = f(x) := \begin{bmatrix} f_\tau(\tau) \\ f_\lambda(\lambda) \end{bmatrix} = \begin{bmatrix} \frac{1}{T}\mathbf{1}_N \\ 0 \end{bmatrix} \tag{8}$$

where the state x evolves in the set C defined as

$$C := C_\tau \times \mathbb{Z}_{\geq 0}, \quad C_\tau := [0,1]^N. \tag{9}$$

The discrete-time dynamics are given by

$$x^+ \in G(x) := \begin{bmatrix} G_{\lambda+1}(\tau) \\ \lambda + 1 \end{bmatrix} \tag{10}$$

where the state x evolves in the set D defined as

$$D := D_\tau \times \mathbb{Z}_{\geq 0},\ D_\tau := \{\tau \in C_\tau : \max_{i\in\mathcal{V}} \tau_i = 1\}. \quad (11)$$

In other words, the set D_τ comprises all points $\tau \in C_\tau$, such that $\tau_i = 1$ for at least one $i \in \{1, \ldots, N\}$.

Note that in (8)–(11), the substate λ can be viewed as a discrete-time counter that increases by one every time there is a jump in the system. For each λ, the set-valued map G_λ in (10) is defined as the outer-semicontinuous hull of the mapping

$$G^0_\lambda(\tau) = \{g \in \mathbb{R}^N : g_i = 0,\ g_j \in \mathcal{R}_{j,\lambda}(\tau) \quad \forall j \neq i\} \quad (12)$$

where the set-valued map $\mathcal{R}_{j,\lambda}$ is defined over the feasible digraph $\phi_\lambda := (\mathcal{V}, \mathcal{E}_\lambda)$ as

$$\tau_j^+ \in \mathcal{R}_{j,\lambda}(\tau) := \begin{cases} 0, & \tau_j \in [0, r_j) \quad (i,j) \in \mathcal{E}_\lambda \\ \{0,1\}, & \tau_j = r_j \quad (i,j) \in \mathcal{E}_\lambda \\ 1, & \tau_j \in (r_j, 1] \quad (i,j) \in \mathcal{E}_\lambda \\ \tau_j, & \quad (i,j) \notin \mathcal{E}_\lambda. \end{cases}$$

For any $r \in (0,1)^N$, and for any $\xi \in \Xi$, every solution of the HDS (7) is complete and uniformly non-Zeno. The completeness follows from the following facts.

1) By construction, the HDS (7) satisfies the Basic Conditions of Definition 2.1 (noting that G is independent of v).
2) $f_\tau(\tau) = 1/T\mathbf{1}_N > 0$ for all $x \in C \setminus D$, which guarantees the existence of nontrivial solutions from $C \setminus D$.
3) The HDS has no finite escape times.
4) $G(D) \subset C \cup D$, so solutions of the HDS cannot stop due to jumps leaving $C \cup D$, [23, Prop. 6.10].

Lack of Zeno behavior follows from the next result.

Lemma 3.1: Consider an HDS $\mathcal{H}_\xi$ as in (7). Let $\underline{r} := \min_{i\in\mathcal{V}} r_i$. Then, the number of jumps in any period of length T is bounded below and above by 1 and $N(\lfloor 1/\underline{r}\rfloor + 1)$, respectively.

Proof: We first establish the lower bound. Pick an arbitrary agent i and let $\tau_i(t,k)$ be its state at time (t,k). We consider the period $[t, t+T]$. There are two cases: 1) During this period, no in-neighbor of agent i hits 1 and triggers it. In this case, by the continuous-time dynamics of the HDS (8), agent i will reach the value 1 in at most T seconds and, then, jumps to 0; (2) during that period, there exists at least one in-neighbor of agent i that hits 1 and jumps. In either case, the total number of jumps of the entire network that occur during the period $[t, t+T]$ is bounded below by 1.

We now establish the upper bound. For each individual agent, we will evaluate an upper bound for the number of times it can hit 1. To do so, we note that if an agent i hits 1 and jumps at a certain time (t,k) (so that $\tau_i(t,k+1) = 0$), then for the next r_iT seconds, the agent cannot hit 1. This holds because the least time for the agent i to hit value 1 is to first flow for r_iT seconds and, then, to have one of its in-neighbors to hit 1 and trigger it. The above arguments then show that the number of times the agent i can hit 1 during the period $[t, t+T]$ is bounded above by $(1/r_i + 1)$ and, hence, $\lfloor 1/r_i\rfloor + 1$. Finally, because the number of jumps of the entire network that occur during the period $[t, t+T]$ is equal to the number of times the agents hit 1 during the same period, we conclude that the number of jumps is bounded above by $\sum_{i=1}^N(\lfloor 1/r_i\rfloor + 1) \leq N(\lfloor 1/\underline{r}\rfloor + 1)$. ■

B. Stability Analysis

We study the stability properties of the HDS, introduced in (7), with respect to the closed set $\mathcal{A}$ defined as follows:

$$\mathcal{A} := \mathcal{A}_s \times \mathbb{Z}_{\geq 0},\ \mathcal{A}_s := \{\mu\mathbf{1}_N : \mu \in [0,1]\} \cup \{0,1\}^N. \quad (13)$$

It should be clear that $x \in \mathcal{A}$ if and only if $\tau \in \mathcal{A}_s$. We say that the HDS (7) reaches synchronization if $x(t,k)$ enters $\mathcal{A}$ (or, equivalently, $\tau(t,k)$ enters $\mathcal{A}_s$) for some hybrid time (t,k). To proceed, we first have the following result, which says that the system remains synchronized once it achieves synchronization. The proof can be found in [25, Appendix A].

Lemma 3.2: For any $\xi \in \Xi$, the HDS $\mathcal{H}_\xi$, introduced in (7), renders the set $\mathcal{A}_s$ (resp. $\mathcal{A}$) strongly forward-invariant for the substate τ (resp. the state x).

Recall that for a root i^* of $\mathcal{G}$, the set of vertices at depth q is denoted by $\mathcal{V}_q(i^*)$. We will now introduce the notion of *synchronization string*:

Definition 3.2: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a rooted digraph and $i^* \in \mathcal{V}_R$ be a root. Let q^* be the depth of $\mathcal{G}$ with respect to i^*. For any $q = 0, \ldots, q^*$, we let $\mathcal{G}_q := (\mathcal{V}, \mathcal{E}_q)$ be a feasible subgraph of $\mathcal{G}$ with the edge set $\mathcal{E}_q := \cup_{i\in\mathcal{V}_q(i^*)}\mathcal{E}_i^-$. Then, the *synchronization string* ζ with respect to i^* is a finite string of feasible subgraphs of $\mathcal{G}$

$$\zeta := \mathcal{G}_0 \cdots \mathcal{G}_0\mathcal{G}_1 \cdots \mathcal{G}_1 \cdots \mathcal{G}_{q^*-1}, \cdots \mathcal{G}_{q^*-1} \quad (14)$$

where each subgraph $\mathcal{G}_q$ is repeated contiguously in the string for ℓ^* times, where $\ell^* := N(\lfloor 1/\underline{r}\rfloor + 1)$. Correspondingly, the length of the string ζ is $L^* := \ell^* q^*$.

With the definition above, we will now state the first main result of the article:

Theorem 3.3: Let $\mathcal{G}$ be a rooted digraph and $r \in (0,1)^N$. Suppose that $\xi \in \Xi$ contains a synchronization string ζ, defined in (14), with respect to a root $i^* \in \mathcal{V}_R$; then, for every maximal solution x of the corresponding HDS $\mathcal{H}_\xi$, defined in (7), with $\lambda(0,0) = 0$ and $\tau(0,0) \in [0,1]^N$, there exists a hybrid time $(t^*, k^*) \in \mathrm{dom}(x)$, uniformly bounded above, such that $\tau(t,k) \in \mathcal{A}_s$ for all $(t,k) \succeq (t^*, k^*)$.

We establish Theorem 3.3. To proceed, we will first introduce a few subsets $\mathcal{A}_q(i^*)$, for $q = 0, \ldots, q^*$, that describe partial synchronizations in the network. For each q, let $\tau^{(q)} \in [0,1]^{|\mathcal{V}_q(i^*)|}$ be the vector that collects the states of vertices of depth q, with respect to i^* (so that $\tau^{(0)} = \tau_{i^*}$). Next, we relabel (if necessary) the vertices in the rooted digraph $\mathcal{G}$ so that

$$\tau = [\tau^{(0)}; \tau^{(1)}; \cdots; \tau^{(q^*)}]. \quad (15)$$

In the sequel, we fix the root i^* and the corresponding labeling. Let $\mathcal{U}_q(i^*) := \cup_{\ell=0}^q \mathcal{V}_\ell(i^*)$ and $M_q := |\mathcal{U}_q(i^*)|$. Then, the subsets $\mathcal{A}_q(i^*)$ are defined as follows:

$$\mathcal{A}_q(i^*) := \left(\{\mu\mathbf{1}_{M_q} : \mu \in [0,1]\} \cup \{0,1\}^{M_q}\right) \times [0,1]^{N-M_q}. \quad (16)$$

Note, in particular, that if $q = 0$, then $\mathcal{A}_0(i^*) = C_\tau$ and if $q = q^*$, then $\mathcal{A}_{q^*}(i^*) = \mathcal{A}_s$, where $\mathcal{A}_s$ is defined in (13). It should be

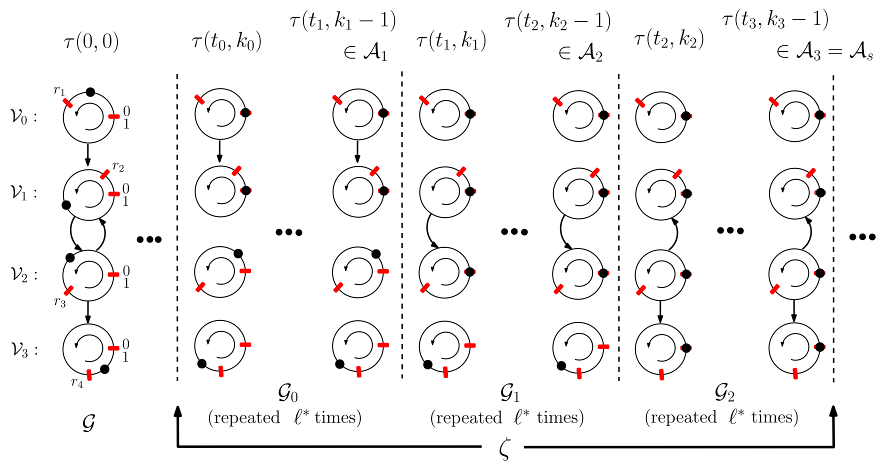

Fig. 3. To illustrate Theorem 3.3, we consider a network of four PCOs of depth $q^*=3$ with vertices $\mathcal{V}_0=\{1\},\mathcal{V}_1=\{2\},\mathcal{V}_2=\{3\},\mathcal{V}_3=\{4\}$, edge set $\{(1,2),(2,3),(3,2),(3,4)\}$, and $\underline{r}=1/8$. Correspondingly, $\ell^*=36$ and the length of the synchronization string ζ becomes 108. For the initial condition $\tau(0,0)$ and infinite sequence of graphs ξ, (t_{q-1},k_{q-1}) and (t_q,k_q-1), for $q=1,2,3$, are, respectively, the hybrid time of the first and the last appearance of $\mathcal{G}_q$ in a synchronization string ζ contained in ξ. Since $\mathcal{G}_q$ only contains the out-edges of vertices $\mathcal{V}_q$, this leads to $\tau(t_q,k_q-1)\in\mathcal{A}_q$ for every q. Because, for such a ξ, each $\mathcal{A}_q$ is strongly forward-invariant, we have that $\tau(t_3,k_3-1)\in\mathcal{A}_s$.

clear that if the sub-state τ belongs to one of these compact sets, say $\mathcal{A}_q(i^*)$ at a certain time, then the vertices in $\mathcal{U}_q(i^*)$ are synchronized at that time. By definition, we have the following chain of inclusions:

$$\mathcal{A}_{q^*}(i^*)\subsetneq\mathcal{A}_{q^*-1}(i^*)\subsetneq\cdots\subsetneq\mathcal{A}_1(i^*)\subsetneq\mathcal{A}_0(i^*). \tag{17}$$

Let $\xi=\phi_1\phi_2\cdots$ be given in the statement of Theorem 3.3. Because ξ contains the synchronization string ζ, there exists a $k_0\in\mathbb{Z}_{\geq 1}$ such that $\zeta=\phi_{k_0}\cdots\phi_{k_{q^*}-1}$, where $(k_{q^*}-k_0)$ is the length of ζ. Furthermore, for each $q=1,\ldots,q^*-1$, we let

$$k_q:=k_0+q\ell^* \tag{18}$$

where q^* is the depth of $\mathcal{G}$ with respect to the root i^* and ℓ^* is defined in Definition 3.2. It should be clear that k_{q^*} can be expressed as $k_{q^*}=k_0+q^*\ell^*$. From the definition of synchronization string, we have that for any $q=0,\ldots,q^*-1$, the digraphs $\phi_{k_q},\ldots,\phi_{k_{q+1}-1}$ are the same given by $\mathcal{G}_q$.

Now, let x be an arbitrary maximal solution of the HDS $\mathcal{H}_\xi$ and we fix this solution in the sequel. Since $\lambda(0,0)=0$, we have that $\lambda(\cdot,k)=k$, for all $(t,k)\in\operatorname{dom}(x)$.

Let $t_0:=\min\{t\in\mathbb{R}_{\geq 0}:(t,k_0)\in\operatorname{dom}(x)\}$, i.e., t_0 is the continuous time-instant that corresponds to the occurrence of the k_0th jump of the HDS $\mathcal{H}_\xi$. Note that such t_0 is well-defined and, in fact, uniformly bounded above. Indeed, to see this, note that by Lemma 3.1, the number of jumps in any period of T is lower bound by 1. It thus implies that $t_0\leq k_0T$. Similarly, for any other $q=1,\ldots,q^*$, we let

$$t_q:=\min\{t\in\mathbb{R}_{\geq 0}:(t,k_q)\in\operatorname{dom}(x)\}. \tag{19}$$

Again, from Lemma 3.1, there is a uniform upper bound for t_q as $t_q\leq t_0+q\ell^*T$. See Fig. 3 for an illustration of (t_q,k_q).

With the sets $\mathcal{A}_q(i^*)$ and the hybrid times (t_q,k_q) defined above, we establish the following result:

Proposition 3.4: Let x be a maximal solution of the HDS $\mathcal{H}_\xi$ (7) and (t_q,k_q), for $q=0,\ldots,q^*$, be the hybrid times defined as above. Then, under the assumption of Theorem 3.3, the following holds: For each $q=0,\ldots,q^*-1$, there exists a hybrid time $(t'_q,k'_q)\in\operatorname{dom}(x)$, with $(t_q,k_q)\preceq(t'_q,k'_q)\preceq(t_{q+1},k_{q+1}-1)$, such that $\tau(t'_q,k'_q)\in\mathcal{A}_{q+1}(i^*)$, where $\mathcal{A}_{q+1}(i^*)$ is defined in (16). Moreover, $\tau(t,k)\in\mathcal{A}_{q+1}(i^*)$ for all $(t,k)\in\operatorname{dom}(x)$ with $(t,k)\succeq(t'_q,k'_q)$.

Proof: We will show that there exist hybrid times (t'_q,k'_q), for $q=0,\ldots,q^*-1$, with $(t_q,k_q)\preceq(t'_q,k'_q)\preceq(t_{q+1},k_{q+1}-1)$, such that if $(t'_q,k'_q)\preceq(t,k)\preceq(t'_{q+1},k'_{q+1})$, for $q=0,\ldots,q^*-2$, then $\tau(t,k)\in\mathcal{A}_{q+1}(i^*)$. Moreover, $\tau(t'_{q^*-1},k'_{q^*-1})\in\mathcal{A}_{q^*}(i^*)=\mathcal{A}_s$. Note that if this holds, then by the chain of inclusion (17) and the strong forward invariance of $\mathcal{A}_s$ established in Lemma 3.2, we have that for any $(t,k)\succeq(t'_q,k'_q)$, $\tau(t,k)\in\mathcal{A}_{q+1}$ for all $q=0,\ldots,q^*-1$.

Starting from the hybrid time (t_0,k_0), all elements in the string $\phi_{k_0}\cdots\phi_{k_1-1}$ are the same given by the digraph $\mathcal{G}_0$, which induce the set-valued mappings $G_{k_0},\ldots,G_{k_1-1}$ defined in (12). Since the root vertex i^* has no in-neighbor in $\mathcal{G}_0$, for any state $\tau(t_0,k_0)$, i^* will hit 1 in less than or equal to T seconds (in continuous-time) after t_0, i.e., there exists a hybrid time (t'_0,k'_0-1) such that $\tau^{(0)}(t'_0,k'_0-1)=1$ with $0\leq t'_0-t_0\leq T$ and $\tau^{(0)}(t'_0,k'_0)=0$. Furthermore, by Lemma 3.1, the number of jumps over any period of length T is bounded above by ℓ^* (defined in Definition 3.2), we have that $k_0\leq k'_0\leq k_1-1$. Because $\mathcal{G}_0$ contains only the out-edges of the root vertex i^*, each of the set-valued mapping $G_{k_0},\ldots,G_{k_1-1}$ maps $\tau(t'_0,k'_0)$ to $\{0,1\}^{M_1}\times[0,1]^{N-M_1}$, where we recall that M_1 is the cardinality of the set $\mathcal{U}_1(i^*)=\{i^*\}\cup\mathcal{V}_1(i^*)$. Thus, we have that $\tau(t'_0,k'_0)\in\mathcal{A}_1(i^*)$.

Since $\tau(t'_0,k'_0)\in\mathcal{A}_1(i^*)$ and since $\phi_{k'_0+1},\ldots,\phi_{k_1-1}$ contain only the out-edges of the root i^*, we have that

$$\tau(t,k)\in\mathcal{A}_1(i^*),\text{ for all }(t,k)\in\operatorname{dom}(\tau)\text{ such that } (t'_0,k'_0)\preceq(t,k)\preceq(t_1,k_1-1). \tag{20}$$

Starting from the hybrid time (t_1, k_1), all elements in the string $\phi_{k_1} \cdots \phi_{k_2-1}$ are the same given by the digraph $\mathcal{G}_1$, which induce the set-valued mappings $G_{k_1}, \ldots, G_{k_2-1}$ defined in (12). By construction of $\mathcal{G}_1$, only vertices in $\mathcal{V}_1(i^*)$ have out-neighbors. Thus, if there is a jump at a hybrid time (t, k), with $(t_1, k_1) \preceq (t, k) \preceq (t_2, k_2 - 1)$, then only the vertices in $\mathcal{V}_1(i^*)$ can trigger. On one hand, by Lemma 3.1, the number of jumps that can occur during any period of length T is bounded above by $\ell^* = k_2 - k_1$. On the other hand, since the vertices in $\mathcal{U}_1(i^*)$ are synchronized at $(t_1, k_1 - 1)$, starting from t_1 these vertices will hit 1 simultaneously (in continuous-time) in less than or equal to T seconds. The above arguments imply that there exists a hybrid time (t'_1, k'_1), with $t_1 \le t'_1 \le t'_1 + T$ and $k_1 \le k'_1 \le k_2 - 1$, such that $[\tau^{(0)}, \tau^{(1)}](t'_1, k'_1) = \mathbf{0}_{M_1}$. Furthermore, each of the set-valued mappings $G_{k_1}, \ldots, G_{k_2-1}$ maps $\tau(t'_1, k'_1)$ to $\{0,1\}^{M_2} \times [0,1]^{N-M_2}$, where M_2 is the cardinality of the set $\mathcal{U}_2(i^*) = \{i^*\} \cup \mathcal{V}_1(i^*) \cup \mathcal{V}_2(i^*)$. Thus, we have that $\tau(t'_1, k'_1) \in \mathcal{A}_2(i^*)$. Note that by the above arguments, we have also shown that $\tau(t, k) \in \mathcal{A}_1(i^*)$ for $(t'_0, k'_0) \preceq (t, k) \preceq (t'_1, k'_1)$.

One can iterate the above arguments to obtain sequentially the hybrid times (t'_q, k'_q), for all $q = 0, \ldots, q^* - 1$, as described in the beginning of the proof. ■

Theorem 3.3 follows immediately from Proposition 3.4.

Proof of Theorem 3.3: Let $x = (\tau, \lambda)$ be a maximal solution. Then, by Proposition 3.4, there is a hybrid time (t'_{q^*-1}, k'_{q^*-1}) such that $\tau(t, k) \in \mathcal{A}_s$ for all $(t, k) \succeq (t'_{q^*-1}, k'_{q^*-1})$. We then set $(t^*, k^*) := (t'_{q^*-1}, k'_{q^*-1})$. Furthermore, since $(t^*, k^*) \preceq (t_{q^*}, k_{q^*})$ and since (t_{q^*}, k_{q^*}) is uniformly bounded above, (t^*, k^*) is uniformly bounded above as well. ■

Toward the end of the section, we consider the scenario where ξ contains the synchronization string infinitely often. Precisely, we have the following definition.

Definition 3.3: An infinite sequence $\xi \in \Xi$ contains a synchronization string *infinitely often* if it has infinitely many disjoint finite strings that are the synchronization string. The infinite sequence ξ contains a synchronization string *uniformly infinitely often* if there exists a positive integer n such that every string of length n in ξ contains a synchronization string.

With Definition 3.3, we will now strengthen Theorem 3.3:

Theorem 3.5: Let $\mathcal{G}$ be a rooted digraph and $r \in (0,1)^N$. Let the HDS $\mathcal{H}_\xi$ be given as in (7) and the set $\mathcal{A}$ be defined in (13). Suppose that ξ contains the synchronization string ζ infinitely often (resp. uniformly infinitely often); then, $\mathcal{H}_\xi$ renders the set $\mathcal{A}$ UGS and GFTA (resp. UGS and GFxTA).

Proof: We show that $\mathcal{H}_\xi$ renders the set $\mathcal{A}$ uniformly globally stable and globally finite-time attractive (resp. globally fixed-time attractive) when ξ contains ζ infinitely often (resp. uniformly infinitely often):

Proof of uniform global stability of $\mathcal{A}$: Consider the function $V : [0,1]^N \to \mathbb{R}_{\ge 0}$ defined as the infimum of all the arcs that cover all agents on the unit circle, where the points 0 and 1 are identified to be the same. The mathematical expression of V can be given as follows [17]:

$$V(\tau) := 1 - \max_{1 \le i \le N} \begin{cases} \tau_{\gamma_{i+1}} - \tau_{\gamma_i} & \text{for } i < N \\ 1 - \tau_{\gamma_i} + \tau_{\gamma_1} & \text{for } i = N \end{cases} \tag{21}$$

where γ_i, for $i \in \{1, \ldots, N\}$, is an index permutation such that $\tau_{\gamma_i} \le \tau_{\gamma_{i+1}}$ for all i. This function satisfies the following properties.

a) It is positive definite with respect to the compact set $\mathcal{A}_s$ defined in (13).
b) It remains constant during flows because all the oscillators have the same frequency $\frac{1}{T}$.
c) It does not increase at jumps since jumps never increase the number of distinct positions of the agents.

Next, we define a Lyapunov function candidate $W : [0,1]^N \times \mathbb{Z}_{\ge 0} \to \mathbb{R}_{\ge 0}$ for the HDS $\mathcal{H}_\xi$ as follows: For any $x = (\tau, \lambda)$, let $W(x) := V(\tau) + |\lambda|_{\mathbb{Z}_{\ge 0}}$. Since $\lambda \in \mathbb{Z}_{\ge 0}$, we have that $W(x) = V(\tau)$ and $|x|_{\mathcal{A}} = |\tau|_{\mathcal{A}_s}$ for all $x \in C \cup D$. Thus, it suffices to show that the HDS $\mathcal{H}_\xi$ renders the set $\mathcal{A}_s$ uniformly globally stable for the substate $\tau \in C_\tau$. We establish this fact below. Since C_τ is compact and since V is continuous, V is bounded; in fact, it is known [2] that $V(\tau) \le 1 - \frac{1}{N}$. Because V is positive definite with respect to compact set $\mathcal{A}_s$, there exists two class $\mathcal{K}$ functions α_1, α_2 such that the condition $\alpha_1(|\tau|_{\mathcal{A}_s}) \le V(\tau) \le \alpha_2(|\tau|_{\mathcal{A}_s})$ holds for all $\tau \in C_\tau$ [23, Ch. 3]. Next, we know from properties (b) and (c) above that V is nonincreasing, i.e., $V(\tau(t,k)) \le V(\tau(0,0))$ for all $(t,k) \in \operatorname{dom}(x)$. Thus, we have from above inequalities that $|\tau(t,k)|_{\mathcal{A}_s} \le \alpha_1^{-1}(\alpha_2(|\tau(0,0)|_{\mathcal{A}_s}))$ for all $(t,k) \in \operatorname{dom}(x)$. Since $\alpha_1^{-1}\alpha_2$ is a $\mathcal{K}$-function, we have that the HDS $\mathcal{H}_\xi$ renders the set $\mathcal{A}$ uniformly globally stable [23].

Proof of global finite-time attractivity of $\mathcal{A}$: Let $x = (\tau, \lambda)$ be an arbitrary maximal solution of $\mathcal{H}_\xi$. We first establish global finite-time attractivity under the assumption that ξ contains the synchronization string ζ infinitely often. Specifically, we need to show that there exists a $\bar{T}(x(0,0))$ such that

$$|x(t,k)|_{\mathcal{A}} = 0 \quad \forall\, t + k \ge \bar{T}(x(0,0)),\ (t,k) \in \operatorname{dom}(x). \tag{22}$$

Given ξ, let σ_i be the index of ξ corresponding to the first digraph of the ith appearance of ζ. Since ξ contains ζ infinitely often, there exists an integer $i > 0$ such that $\sigma_i > \lambda(0,0)$ and $\sigma_j \le \lambda(0,0)$ for all $j < i$. In words, the integer i is such that the ith appearance of ζ in the sequence ξ is its first appearance after index $\lambda(0,0)$. If, further, ξ contains ζ uniformly infinitely often, then, by Definition 3.3, $\sigma_i - \lambda(0,0) \le n$. Now, let

$$\bar{k}_0 := \sigma_i - \lambda(0,0),\ \bar{t}_0 := \min\{t \in \mathbb{R}_{\ge 0} : (t, \bar{k}_0) \in \operatorname{dom}(x)\}.$$

Then, $(\bar{t}_0, \bar{k}_0)$ is the hybrid time of the solution x corresponding to the first digraph of the first appearance of ζ. By Lemma 3.1, we have that $\bar{t}_0 \le \bar{k}_0 T$. Next, similar to (t_q, k_q) defined in (18) and (19), we let

$$\bar{k}_q := \bar{k}_0 + q\ell^*,\ \bar{t}_q := \min\{t \in \mathbb{R}_{\ge 0} : (t, \bar{k}_q) \in \operatorname{dom}(x)\}.$$

Using again Lemma 3.1, we have that $\bar{t}_q - \bar{t}_0 \le q\ell^* T$. Thus, all the hybrid times $(\bar{t}_q, \bar{k}_q)$ are bounded above. Furthermore, by the same arguments of Proposition 3.4, we have that $|x(t,k)|_{\mathcal{A}} = 0$ for all $(t,k) \succeq (\bar{t}_{q^*}, \bar{k}_{q^*})$, with $(t,k) \in \operatorname{dom}(x)$. The proof of global finite-time attractivity is then done by setting

$$\bar{T}(x(0,0)) := \bar{t}_{q^*} + \bar{k}_{q^*}. \tag{23}$$

Finally, we assume that ξ contains ζ *uniformly* infinitely often and establish global fixed-time attractivity. We do so by showing that the quantity $\bar{T}(x(0,0))$ in (23) is uniformly bounded above. By the above arguments, we have the following two facts: 1) Since $\bar{k}_0 = \sigma_i - \lambda(0,0) \leq n$ and $\bar{k}_{q^*} = \bar{k}_0 + q^*\ell^*$, we have that $\bar{k}_{q^*} \leq n + q^*\ell^*$; 2) since $\bar{t}_0 \leq \bar{k}_0 T \leq nT$ and since $\bar{t}_{q^*} \leq \bar{t}_0 + q^*\ell^* T$, we have that $\bar{t}_{q^*} \leq (n + q^*\ell^*)T$. Thus, for any $x(0,0)$, $\bar{T}(x(0,0)) \leq (n + q^*\ell^*)(T+1)$. ■

IV. Stochastic Resetting Algorithm

In this section, we consider networks of PCOs with the underlying information flow topology being a random digraph: Every time an agent hits 1, it will generate a *single* Bernoulli random variable, independent of others, to decide whether or not to send pulses to *all* of its out-neighbors. This stochastic model differs from the one in our previous work [2]; there, whenever an agent hits 1, it will generate *multiple* i.i.d. Bernoulli random one for *each* of its out-neighbors.

A. Well-Posed SHDS

We start by showing that the feasible subgraphs of a given digraph $\mathcal{G}$ can be mapped one-to-one to certain binary sequences. To that end, let $N^* \leq N$ be the number of vertices in $\mathcal{G}$ with at least one out-neighbor. Without loss of generality, we will label these vertices as $1, \ldots, N^*$ and let $\mathcal{V}^* := \{1, \ldots, N^*\}$. Consider binary sequences $v_1 \ldots v_{N^*}$ of length N^*, with each $v_i \in \{0,1\}$. One can assign to each feasible digraph $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ such a binary sequence: For each $i = 1, \ldots, N^*$, set

$$v_i := \begin{cases} 1, & \text{if } \mathcal{E}_i \subseteq \mathcal{E}' \\ 0, & \text{otherwise.} \end{cases}$$

Conversely, each binary sequence gives rise to a feasible digraph. Thus, with the labeling of the vertices in $\mathcal{V}^*$ and the above correspondence, we can use a binary sequence $v_1 \ldots v_{N^*}$ to represent a feasible digraph. Consequently, the set Φ can be realized as the collection of all binary sequences of length N^*, denoted as $\Psi := \{0,1\}^{N^*}$.

We next introduce a simple model that can generate a random feasible digraph. Let the digits $v_1, \ldots, v_{N^*}$ of a binary sequence v be i.i.d. Bernoulli (p) random variables, i.e., the probability that v_i takes value 1 (resp. 0) is p (resp. $(1-p)$). We denote by μ the corresponding probability measure on Φ. It follows that for any feasible digraph $\phi \in \Phi$ represented by a binary sequence $v := v_1 \ldots v_{N^*}$

$$\mu(\phi) = p^{N'}(1-p)^{N^*-N'} \tag{24}$$

where N' is the total number of 1's in the binary sequence.

With the above random model, we can now construct an SHDS. First, we consider set-valued mappings $S_{ij} : [0,1] \times \Psi \rightrightarrows [0,1]$, defined for each edge $(i,j) \in \mathcal{E}$ of $\mathcal{G}$ as follows:

$$S_{ij}(\tau_j, v) = v_i \bar{\mathcal{R}}_j(\tau_j) + (1 - v_i)\tau_j \tag{25}$$

where $\bar{\mathcal{R}}_j$ is the outer semicontinuous hull of mapping $\mathcal{R}_j$, defined in (3), and v_i is the digit in a binary sequence that corresponds to the vertex i in $\mathcal{V}^*$. Next, using (25), we define a new set-valued mapping $G_S^0 : [0,1]^N \times \Psi \rightrightarrows \mathbb{R}^N$ as follows:

$$\left\{ g \in \mathbb{R}^N : g_i = 0, g_j \in \begin{Bmatrix} S_{ij}(\tau_j, v), & (i,j) \in \mathcal{E} \\ \{\tau_j\}, & (i,j) \notin \mathcal{E} \end{Bmatrix} \right\} \tag{26}$$

where g_j is defined for all $j \neq i$ and the mapping $G_S^0(\tau, v)$ is nonempty only when $\tau_i = 1$ for some $i \in \mathcal{V}$ and $\tau_j \in [0,1)$ for $j \neq i$. We used the subindex S to indicate that the mapping G_S^0 is stochastic. Finally, the jump map for the SHDS is defined as the outer-semicontinuous hull of G_S^0, i.e.,

$$G_S(\tau, v) := \overline{G_S^0(\tau, v)}. \tag{27}$$

Note that when a jump occurs and a random digraph $\phi_k \in \Phi$ is drawn, not every edge of ϕ_k plays a role in the jump map G_S. Only the edges (i,j) with $\tau_i = 1$ for some $i \in \mathcal{V}$, matter.

Let $\omega := \omega_1\omega_2\omega_3 \cdots$ be a sequence of i.i.d. random variables, with each $\omega_i \sim \mu(\cdot)$ a feasible digraph. We denote by Ω the collection of sample paths ω. Each sample path ω will be used to determine the sequence of jump maps at all discrete times through (27).

It follows that the resulting SHDS depends on three parameters, namely, the parameter p associated with the Bernoulli random variable, the partition vector r, and the digraph $\mathcal{G}$. We will thus write the SHDS with state τ as

$$\mathcal{H}_S(p, r, \mathcal{G}) := (C_\tau, f_\tau, D_\tau, G_S) \tag{28}$$

where f_τ, C_τ, D_τ are defined in (8), (9), and (11) and again the subindex S indicates that the overall system is stochastic.

Note that the HDS in (7), parameterized by an infinite string of feasible digraphs ξ, is the deterministic counterpart of the SHDS (28) where the sample path ω is realized as ξ. It should be clear from our probability model that ω contains the synchronization string ζ infinitely often almost surely.

To proceed, we first have the following fact.

Lemma 4.1: For any $p \in (0,1)$, any $r \in (0,1)^N$, and any digraph $\mathcal{G}$, the SHDS $\mathcal{H}_S(p, r, \mathcal{G})$ satisfies the basic conditions. Every maximal random solution of $\mathcal{H}_S$ is surely complete and uniformly non-Zeno and, moreover, the number of jumps in any period of length T is surely bounded below and above by 1 and $N(\lfloor 1/\underline{r} \rfloor + 1)$, respectively.

Proof: The fact that $\mathcal{H}_S$ satisfies the basic conditions follows directly from construction. Completeness of solutions (surely) follows by the same arguments presented before Lemma 3.1, and the fact that $G_S(\tau, v) \subset C_\tau \cup D_\tau$ for all $\tau \in D_\tau$ and for all $v \in \Psi$, which guarantees that all random solutions cannot stop due to jumps. Finally, note that each realization of ω in $\mathcal{H}_S$ corresponds to a sequence of subgraphs of $\mathcal{G}$ in the deterministic HDS $\mathcal{H}_\xi$, defined in (7). By Lemma 3.1, any such sequence leads to solutions of (7) that are uniformly non-Zeno hybrid arcs, with a number of jumps in any period of length T bounded below and above by 1 and $N(\lfloor 1/\underline{r} \rfloor + 1)$, respectively. Since for every $\tau_\omega(0,0) \in C_\tau \cup D_\tau$ and every random solution of $\mathcal{H}_S$ there exists a solution of $\mathcal{H}_\xi$ such that their hybrid time domains are identical and their τ-components agree with each other, it follows that the above properties hold surely for every random solution of $\mathcal{H}_S$. ■

B. Global Synchronization With Probability One

We start by noting the following fact: For any $p \in (0,1)$, $r \in (0,1)^N$, and any digraph $\mathcal{G}$, the SHDS $\mathcal{H}_S(p,r,\mathcal{G})$ renders the set $\mathcal{A}_s$ surely strongly forward invariant. We omit the proof of such a fact as it uses the same arguments as the ones in the proof of Lemma 3.2.

Next, similar to our earlier work in [2], we introduce the following definition of the *sync-triplet* for the SHDS $\mathcal{H}_S$.

Definition 4.1: Let $\mathcal{A}_s$ be given in (13). Let $p \in (0,1)$, $r \in (0,1)^N$, and $\mathcal{G}$ be a digraph of N vertices. Then, $(p,r,\mathcal{G})$ is a *sync-triplet* if the following two items hold.

1) For every initial condition in $C \cup D$, there exist nontrivial random solutions almost surely, and every maximal random solution of $\mathcal{H}_S(p,r,\mathcal{G})$ is complete and uniformly non-Zeno almost surely.
2) The SHDS $\mathcal{H}_S(p,r,\mathcal{G})$ renders $\mathcal{A}_s$ UGASp (see Definition 2.4).

A random solution of $\mathcal{H}_S(p,r,\mathcal{G})$ depends on ω and we denote it by $\boldsymbol{\tau}_\omega$. Note that there may exist multiple random solutions even if we fix ω and the initial condition, which is due to the set-valued nature of the jump map in the SHDS (28). For each random solution $\boldsymbol{\tau}_\omega$, we define

$$\boldsymbol{T}^*(\boldsymbol{\tau}_\omega) = \inf\{t \mid \boldsymbol{\tau}_\omega(t_\omega,k_\omega) \in \mathcal{A}_s,\ (t_\omega,k_\omega) \in \mathrm{dom}(\boldsymbol{\tau}_\omega)\} \tag{29}$$

which is the first (continuous) time that the random solution $\boldsymbol{\tau}_\omega$ enters the compact set $\mathcal{A}_s$ defined in (13). We call $\boldsymbol{T}^*(\boldsymbol{\tau}_\omega)$ the *sync-time* of the random solution.

Now, we will present the main result of this section.

Theorem 4.2: For any $p \in (0,1)$, any $r \in (0,1)^N$, and any rooted digraph $\mathcal{G}$, $(p,r,\mathcal{G})$ is a sync-triplet. Moreover, for any initial condition $\boldsymbol{\tau}_\omega(0,0)$, the following holds for all positive integers n and all random solutions $\boldsymbol{\tau}_\omega$ of the SHDS:

$$\mathbb{P}(\boldsymbol{T}^*(\boldsymbol{\tau}_\omega) > nT^*) \le \rho^n \tag{30}$$

where $T^* := (\mathrm{dep}(\mathcal{G})\ell^* + 1)T$, with ℓ^* defined in Definition 3.2, and $\rho \in (0,1)$ is a constant given by

$$\rho := 1 - (p\,(1-p)^{\mathrm{dep}(\mathcal{G})-1})^{N\ell^*}. \tag{31}$$

Remark 4.1: Theorem 4.2 can further be generalized to the case where agents have heterogeneous probabilities $p_i \in (0,1)$. Correspondingly, the constant ρ on the right-hand side of (30) changes to $1 - (\underline{p}\,(1-\overline{p})^{\mathrm{dep}(\mathcal{G})-1})^{N\ell^*}$ where $\underline{p} := \min_{i\in\mathcal{V}^*} p_i$ and $\overline{p} := \max_{i\in\mathcal{V}^*} p_i$. With slight modification, the arguments ahead can be used to establish the heterogeneous case.

Before presenting the proof of Theorem 4.2, we need a few preliminary results. First, we let $\mathcal{S}(\boldsymbol{\tau}_\omega(0,0))$ be the set of all maximal random solutions of (28) from the initial condition $\boldsymbol{\tau}_\omega(0,0) \in [0,1]^N$. For each initial condition, we define the following event:

$$\Omega_1(\boldsymbol{\tau}_\omega(0,0)) := \{\omega \in \Omega \mid \forall \boldsymbol{\tau}_\omega \in \mathcal{S}(\boldsymbol{\tau}_\omega(0,0)), \exists i^* \in \mathcal{V}_R,$$
$$\exists\, (t^*_\omega, k^*_\omega) \in \mathrm{dom}(\boldsymbol{\tau}_\omega) \text{ with } t^*_\omega \le T \text{ s.t. } \boldsymbol{\tau}_{\omega,i^*}(t^*_\omega,k^*_\omega) = 1\}.$$

In other words, the above event is about having a certain root vertex i^* in the network hitting 1 before continuous-time T. Similar to [2, Lemma 6], the following result holds.

Lemma 4.3: For any $\boldsymbol{\tau}_\omega(0,0) \in [0,1]^N, \Omega_1(\boldsymbol{\tau}_\omega(0,0)) = \Omega$.

For a positive integer ℓ and a root vertex i^* of $\mathcal{G}$, we define an event $\Omega_2(\ell,i^*)$, by using the synchronization string ζ from Definition 3.2, as follows:

$$\Omega_2(\ell,i^*) := \{\omega \in \Omega \mid \omega_{\ell+1}\cdots\omega_{\ell+L^*} = \zeta\} \tag{32}$$

where L^* is the length of the string ζ. We compute ahead the probability of this event.

Lemma 4.4: Let ℓ^* be given in Definition 3.2. Then

$$\mathbb{P}(\Omega_2(\ell,i^*)) \ge (p\,(1-p)^{\mathrm{dep}(\mathcal{G})-1})^{N\ell^*}. \tag{33}$$

Proof: Recall from Definition 3.2 that each digraph $\mathcal{G}_q$ in ζ is a subgraph of the rooted digraph $\mathcal{G}$ with the same vertex set but contains only the out-edges of the vertices at depth k with respect to the root vertex i^*. Also, each $\mathcal{G}_q$ is a feasible digraph and it follows from (24) that

$$\mu(\mathcal{G}_q) = p^{|\mathcal{V}_q(i^*)|}(1-p)^{N^*-|\mathcal{V}_q(i^*)|}$$

where we recall that $\mathcal{V}_q(i^*)$ is the set of vertices at depth q with respect to the root vertex i^*. Using the fact that the random variables ω_q, for $q \ge 1$, are i.i.d., we evaluate the probability of the event $\Omega_2(\ell,i^*)$ as follows:

$$\begin{aligned}\mathbb{P}\left(\Omega_2\left(\ell,i^*\right)\right) &= \prod_{q=0}^{q^*-1}\left(p^{|\mathcal{V}_q(i^*)|}\left(1-p\right)^{N^*-|\mathcal{V}_q(i^*)|}\right)^{\ell^*}\\ &= \left(p^{\sum_{q=0}^{q^*-1}|\mathcal{V}_q(i^*)|}\left(1-p\right)^{N^*q^*-\sum_{q=0}^{q^*-1}|\mathcal{V}_q(i^*)|}\right)^{\ell^*}\\ &= \left(p^{N^*}\left(1-p\right)^{N^*q^*-N^*}\right)^{\ell^*}\\ &\ge \left(p\left(1-p\right)^{\mathrm{dep}(\mathcal{G})-1}\right)^{N\ell^*}\end{aligned}$$

where the third equality follows from the fact that $\sum_{q=0}^{q^*-1}|\mathcal{V}_q(i^*)| = N^*$ and the last inequality follows from the fact that $N^* \le N$ and $q^* \le \mathrm{dep}(\mathcal{G})$. ■

With the above preliminary results, we prove Theorem 4.2.

Proof of Theorem 4.2: We again consider the function $V : [0,1]^N \to \mathbb{R}_{\ge 0}$ as introduced in (21), i.e., the infimum of all arcs that cover all agents on the unit circle. Using three properties described in the proof of Theorem 3.5, we have that V (positive definite w.r.t. $\mathcal{A}_s$) is nonincreasing on average along the random solutions of (28) and, hence, serves as a valid Lyapunov function for the SHDS (c.f. Appendix B).

By Lemma 4.1, the SHDS (28) satisfies the basic conditions and every maximal random solution $\boldsymbol{\tau}_\omega$ of the SHDS is surely complete and uniformly non-Zeno. Thus, by the stochastic hybrid invariance principle (c.f. Theorem 1.1), in order to show that the set $\mathcal{A}_s$ is UGASp, it suffices to show that there does not exist a complete random solution $\boldsymbol{\tau}_\omega$ that remains in a nonzero level set of the Lyapunov function almost surely.

To establish the above fact, we will show that there exists positive constants η and T^* such that for any sample path ω and for any initial condition $\boldsymbol{\tau}_\omega(0,0)$, the following holds:

$$\mathbb{P}(\Omega_3(\boldsymbol{\tau}_\omega(0,0))) > \eta \tag{34}$$

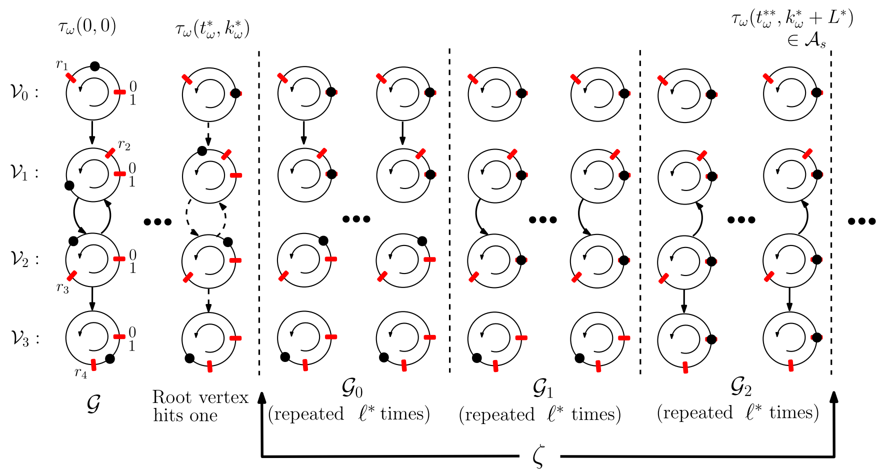

Fig. 4. To illustrate Theorem 4.2, we consider the same network of PCOs from Fig. 3. For the initial condition $\tau_\omega(0,0)$, the root vertex hits 1 at (t^*_ω, k^*_ω) where $t^*_\omega \leq T$. Followed by that, the synchronization string ζ appears in ω from indices $k^*_\omega + 1$ to $k^*_\omega + L^*$ where $L^* = 108$. This leads to $\tau_\omega(t^{**}_\omega, k^*_\omega + L^*) \in \mathcal{A}_s$ where intermediate steps are shown in Fig. 3.

where the event $\Omega_3(\boldsymbol{\tau}_\omega(0,0))$ is given by

$$\Omega_3(\boldsymbol{\tau}_\omega(0,0)) := \{\omega \in \Omega \mid \forall \boldsymbol{\tau}_\omega \in \mathcal{S}(\boldsymbol{\tau}_\omega(0,0)) \quad \forall t_\omega \geq T^* \\ \text{s.t. } (t_\omega, k_\omega) \in \operatorname{dom}(\boldsymbol{\tau}_\omega),\ V(\boldsymbol{\tau}_\omega(t_\omega, k_\omega)) = 0\}\,.$$

We show ahead that η and T^* can be chosen to be the following values $\eta := 1 - \rho$, where ρ is defined in (31), and $T^* = (\operatorname{dep}(\mathcal{G})\ell^* + 1)T$.

By Lemma 4.3, for any random solution $\boldsymbol{\tau}_\omega$, there exists a hybrid time (t^*_ω, k^*_ω), with $t^*_\omega \leq T$, and a root i^* of $\mathcal{G}$ such that $\boldsymbol{\tau}_{\omega,i^*}(t^*_\omega, k^*_\omega) = 1$. Conditioning on the fact that $\boldsymbol{\tau}_{\omega,i^*}(t^*_\omega, k^*_\omega) = 1$, we consider the event $\Omega_2(k^*_\omega, i^*)$. For convenience, we let t^{**}_ω be the continuous-time instant corresponding to the $(k^*_\omega + L^*)$th jump. By Lemma 4.1, the number of jumps in a period of length T is surely bounded below by 1. Then, for the discrete-time to increase from k^*_ω to $k^*_\omega + L^*$, the continuous-time will increase by at most L^*T surely, i.e., we have that $t^{**}_\omega - t^*_\omega \leq L^*T$ for every sample path ω and every solution $\boldsymbol{\tau}_\omega$. Next, by definition of the event $\Omega_2(k^*_\omega, i^*)$, the underlying digraphs between hybrid times (t^*_ω, k^*_ω) and $(t^{**}_\omega, k^*_\omega + L^*)$ are given by the synchronization string ζ (see Fig. 4 for an illustration). Thus, by the same arguments of Theorem 3.3, the random solution $\boldsymbol{\tau}_\omega$ will reach synchronization before $(t^{**}_\omega, k^*_\omega + L^*)$ provided that event $\Omega_2(k^*_\omega, i^*)$ is true. Since $t^*_\omega \leq T$ and $t^{**}_\omega - t^*_\omega \leq L^*T \leq \operatorname{dep}(\mathcal{G})\ell^*T$ and since $\mathcal{A}_s$ is forward invariant, we have that $V(\boldsymbol{\tau}_\omega(t_\omega, k_\omega)) = 0$, for all $t_\omega \geq T^*$. Thus, to establish (34), it now remains to show that the probability of the event $\Omega_2(j^*_\omega, i^*)$ is nonzero; by Lemma 4.4, $\mathbb{P}(\Omega_2(j^*_\omega, i^*)) = \eta$. Thus, the triplet $(p, r, \mathcal{G})$ is a sync-triplet.

Finally, we show that (30) holds. First, by the Bayes rule

$$\mathbb{P}\left(\boldsymbol{T^*}(\boldsymbol{\tau}_\omega) > nT^*\right) = \mathbb{P}\left(\boldsymbol{T^*}(\boldsymbol{\tau}_\omega) > (n-1)T^*\right)\ldots \\ \ldots \times \mathbb{P}\left(\boldsymbol{T^*}(\boldsymbol{\tau}_\omega) > nT^* \,|\, \boldsymbol{T^*}(\boldsymbol{\tau}_\omega) > (n-1)T^*\right).$$

The conditional probability on the right-hand side of the above expression can further be simplified as $\mathbb{P}(\boldsymbol{T^*}(\boldsymbol{\tau'}_{\omega'}) > T^*)$, where $\boldsymbol{\tau}'_{\omega'}$ is a new random solution with the initial condition $\tau'_{\omega'}(0,0)$ given by $\tau'_{\omega'}(0,0) = \tau_\omega((n-1)T^*, k_\omega)$, for some k_ω and $\omega' := \omega_{k_\omega+1}\omega_{k_\omega+2}\cdots$. Note that by definition of $\Omega_3(\tau'_{\omega'}(0,0))$ and (34), we have that

$$\mathbb{P}\left(\boldsymbol{T^*}(\boldsymbol{\tau'}_{\omega'}) > T^*\right) = 1 - \mathbb{P}(\Omega_3(\tau'_{\omega'}(0,0))) < 1 - \eta = \rho.$$

It then follows that:

$$\mathbb{P}\left(\boldsymbol{T^*}(\boldsymbol{\tau}_\omega) \geq nT^*\right) < \rho\mathbb{P}\left(\boldsymbol{T^*}(\boldsymbol{\tau}_\omega) \geq (n-1)T^*\right).$$

The above recursive formula then implies that (30) holds. ∎

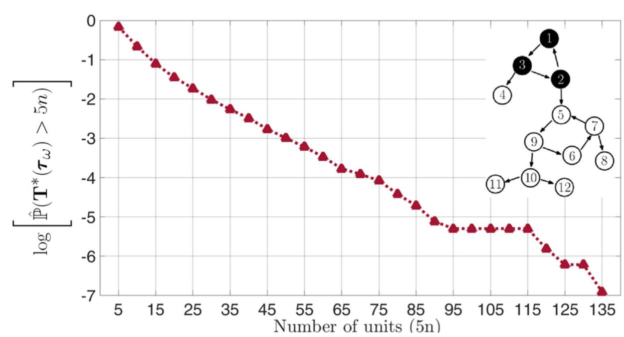

Fig. 5. Plot of $\log\left[\hat{\mathbb{P}}(\mathbf{T}^*(\boldsymbol{\tau}_\omega) > 5n)\right]$ versus the number of units $5n$ needed for a sample path to achieve synchronization. There are 1000 random solutions simulated.

V. Simulation Results

In this section, we present numerical studies of the proposed algorithm (28). We set $T = 1$ and $p = 0.5$.

First, we verify the validity of Theorem 4.2 and investigate the sync-time $\boldsymbol{T}^*(\boldsymbol{\tau}_\omega)$ defined in (29). For this purpose, we consider a rooted network of $N = 12$ PCOs, as shown in Fig. 5. Next, we let the parameters r_i be chosen uniformly randomly from $(0,1)^N$ and then choose 1000 random initial conditions uniformly from $(0,1)^N$. For each initial condition, we simulate the SHDS (28) and let $(5(n-1), 5n]$, for $n \geq 1$, be the interval that contains

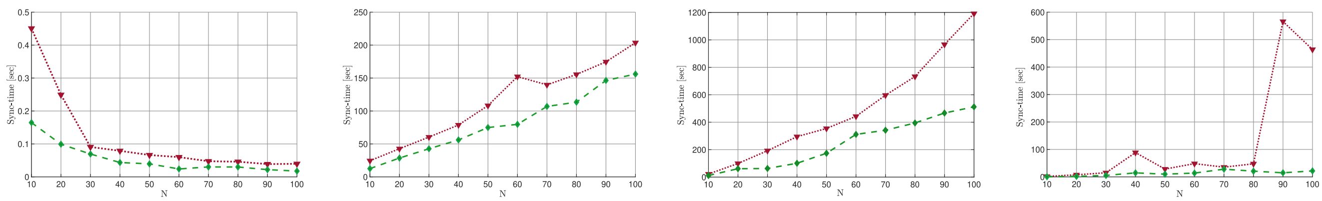

Fig. 6. We plot the averages of sync-times of our algorithm with binary jump map (28) (depicted by the dashed, green curves) and the algorithm in [17] with piecewise linear jump map (35) (depicted by the dotted, red curves) as functions of the number N of agents. The information flow topologies, from left to right, are chosen to be complete-, path-, cycle-, and 5-regular digraphs. For every such information flow topology and for every number N, the simulation results demonstrate that our algorithm synchronizes faster than the one proposed in [17].

the sync-time. In Fig. 5, we plot (in log scale) the empirical version of $\mathbb{P}(\boldsymbol{T}^*(\boldsymbol{\tau}_\omega) > 5n)$ for different units $5n$, $n \geq 1$, i.e., we plot $\hat{\mathbb{P}}(\boldsymbol{T}^*(\boldsymbol{\tau}_\omega) > 5n) := 1 - \sum_{k=1}^{n} \frac{\mathrm{Freq}(k)}{1000}$ where $\mathrm{Freq}(k)$ is the total number of times that $\boldsymbol{T}^*(\boldsymbol{\tau}_\omega)$ belongs to $(5(n-1), 5n]$.

Next, we compare the performance of our binary resetting algorithm (28) with the algorithm considered in [17] (where the authors use a piece-wise linear jump map for numerical studies). In the absence of delays, we reproduce their piecewise linear jump map $H(z)$ as follows:

$$H(z) = \begin{cases} h_1(z) = m_1 z, & 0 \leq z \leq 0.5 \\ h_2(z) = m_2 z + 1 - m_2, & 0.5 < z \leq 1 \end{cases} \tag{35}$$

where m_1 and m_2 are tuning parameters with $0 < m_1 \leq 0.5$ and $0 < m_2 \leq 0.5$. To be consistent with their algorithm, we let the parameters r_i of our algorithm be 0.5. The metric of performance is chosen to be the *sync-time* (29). Note that if one uses the algorithm in [17], then reaching synchronization is only *asymptotic* with probability one. Thus, we relax the criterion of reaching synchronization such that the Lyapunov function V defined in (21) only needs to satisfy $V(\boldsymbol{\tau}_\omega) \leq 0.05$. Correspondingly, we modify the sync-time $\boldsymbol{T}^*(\boldsymbol{\tau}_\omega)$ to be $\boldsymbol{T}^*_{0.05}(\boldsymbol{\tau}_\omega) := \min_{t \geq 0}\{t : V(\boldsymbol{\tau}_\omega) \leq 0.05\}$.

We first set $m_1 = 0.3261$ and $m_2 = 0.46$ as was done in the numerical studies in [17]. We run simulations for both algorithms for four different classes of information flow topologies: 1) complete digraphs, 2) path digraphs, 3) cycle digraphs, and 4) 5-regular digraphs (see Section II for definition). For each class of digraphs, we increase the number N of agents from 10 to 100, with the step of increment being 10. Then, for each N, we generate 50 initial conditions uniformly randomly from $(0, 1)^N$ used for both algorithms. In Fig. 6, we plot the averaged sync-time for comparison.

Next, inspired by the use of piecewise linear jump map in [17], we investigate via simulations how the slopes m_1, m_2 of the linear maps affect the sync-time. Note that the binary jump map can be viewed as an extremum case of the piecewise linear map in a sense that the slopes of the two linear functions in (35) are 0, i.e. $m_1 = m_2 = 0$. Now, we set $m_1 = m_2 =: m$ and study the average sync-time as a function of m. To this end, we fix $N = 50$ vertices and consider again path-, cycle-, complete-, 5-regular digraphs. We increase m from 0 to 0.5 with the step of increment being 0.05. For each digraph and for each m, we generate 50 random initial conditions and run the simulations.

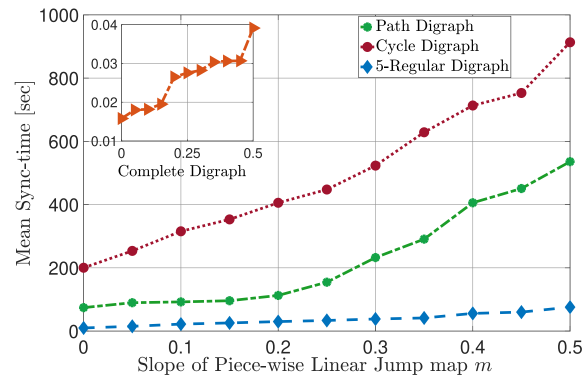

Fig. 7. Averaged sync-times versus the slope $m_1 = m_2 = m$ of linear jump map of vertex-triggering algorithm in [17] for $N = 50$.

We plot the averaged sync-time as a function of m in Fig. 7 for each digraph. It is observed that the averaged sync-time is the least when $m = 0$.

VI. Conclusion

In this article, we have presented a stochastic *binary, vertex-triggering* resetting algorithm by which networks of PCOs can achieve global synchronization over rooted digraphs almost surely. The result is stated in Theorem 4.2. Its proof relies on the use of a hybrid-system machinery and the analysis of the asymptotic behavior of a typical random solution of an associated SHDS. Numerical studies have shown that our algorithm outperforms (in terms of the time needed for synchronization) an existing vertex-triggering algorithm over several different classes of information flow topologies.

Appendix

A. Hybrid Dynamical Systems

Solutions of (5) are parameterized by both continuous- and discrete-time indices $t \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$. A compact hybrid time domain is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ of the form $\cup_{k=0}^{K}([t_k, t_{k+1}] \times \{k\})$ for some $K \in \mathbb{Z}_{\geq 0}$ and real numbers $0 = t_0 \leq t_1 \leq \cdots \leq t_{K+1}$. A hybrid time domain is a set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that for each T, K, the set $E \cap ([0, T] \times \{0, 1, 2, \ldots, K\})$ is a compact hybrid time domain. A function

$x : E \to \mathbb{R}^n$ is said to be a hybrid arc if E is a hybrid time domain, and for each k such that the interval $I_k = \{t \geq 0 : (t,k) \in \text{dom}(x)\}$ has nonempty interior, the function $t \mapsto x(t,k)$ is locally absolutely continuous. A hybrid arc x is said to be a solution to (5) if the following statements hold.

1) $x(0,0) \in C \cup D$.
2) If $(t_1,k),(t_2,k) \in \text{dom}(x)$ with $t_1 < t_2$, then for almost every $t \in [t_1,t_2]$, $x(t,k) \in C$ and $\dot{x}(t,k) = f(x(t,k))$.
3) If $(t,k),(t,k+1) \in \text{dom}(x)$, then $x(t,k) \in D$ and $x(t,k+1) \in G(x(t,k))$.

B. Stochastic HDSs

Random solutions to SHDS (4) are functions of $\omega \in \Omega$ denoted $\mathbf{x}(\omega)$, such that 1) $\omega \mapsto \mathbf{x}(\omega)$ has measurability properties that are adapted to the minimal filtration of $\mathbf{v}$; 2) for each $\omega \in \Omega$, the sample path $\mathbf{x}(\omega)$ is a standard solution to the HDS (5) with the appropriate causal dependence on the random input $\mathbf{v}(\omega)$ through the jumps. To formally define these mappings, for $\ell \in \mathbb{Z}_{\geq 1}$, let $\mathcal{F}_\ell$ denote the collection of sets $\{\omega \in \Omega : (\mathbf{v}_1(\omega), \mathbf{v}_2(\omega), \dots, \mathbf{v}_\ell(\omega)) \in F\}$, $F \in \mathbf{B}(\mathbb{R}^m)^\ell)$, which are the sub-$\sigma$-fields of $\mathcal{F}$ that form the minimal filtration of $\mathbf{v} = \{\mathbf{v}_\ell\}_{\ell=1}^\infty$, which is the smallest σ-algebra on $(\Omega, \mathcal{F})$ that contains the preimages of $\mathbf{B}(\mathbb{R}^m)$-measurable subsets on $\mathbb{R}^m$ for times up to ℓ. A stochastic hybrid arc is a mapping $\mathbf{x}$ from Ω to the set of hybrid arcs, such that the set-valued mapping from Ω to $\mathbb{R}^{n+2}$, given by $\omega \mapsto \text{graph}(\mathbf{x}(\omega)) := \{(t,k,z) : \tilde{x} = \mathbf{x}(\omega), (t,k) \in \text{dom}(\tilde{x}), z = \tilde{x}(t,k)\}$, is $\mathcal{F}$-measurable with closed-values. Let $\text{graph}(\mathbf{x}(\omega))_{\leq \ell} := \text{graph}(\mathbf{x}(\omega)) \cap (\mathbb{R}_{\geq 0} \times \{0,1,\dots,\ell\} \times \mathbb{R}^n)$. An $\{\mathcal{F}_\ell\}_{\ell=0}^\infty$ adapted stochastic hybrid arc is a stochastic hybrid arc $\mathbf{x}$ such that the mapping $\omega \mapsto \text{graph}(\mathbf{x}(\omega))_{\leq \ell}$ is $\mathcal{F}_\ell$ measurable for each $\ell \in \mathbb{N}$. An adapted stochastic hybrid arc $\mathbf{x}(\omega)$, or simply $\mathbf{x}_\omega$, is a solution to SHDS (4), satisfying the basic conditions of Definition 2.1, starting from x_0 denoted $\mathbf{x}_\omega \in \mathcal{S}_r(x_0)$ if: (1) $\mathbf{x}_\omega(0,0) = x_0$; (2) if $(t_1,k),(t_2,k) \in \text{dom}(\mathbf{x}_\omega)$ with $t_1 < t_2$, then for all $t \in [t_1,t_2]$, $\mathbf{x}_\omega(t,k) \in C$ and $\dot{\mathbf{x}}_\omega(t,k) = f(\mathbf{x}_\omega(t,k))$; (3) if $(t,k),(t,k+1) \in \text{dom}(\mathbf{x}_\omega)$, then $\mathbf{x}_\omega(t,k) \in D$ and $\mathbf{x}_\omega(t,k+1) \in G(\mathbf{x}_\omega(t,k), \mathbf{v}_{k+1}(\omega))$. A random solution $\mathbf{x}_\omega$ is said to be a) almost surely *nontrivial* if its hybrid time domain contains at least two points almost surely; b) almost surely *complete* if for almost every sample path $\omega \in \Omega$ the hybrid arc $\mathbf{x}_\omega$ has an unbounded time domain; and almost surely eventually discrete if for almost every sample path $\omega \in \Omega$, the hybrid arc $\mathbf{x}_\omega$ is eventually discrete. A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lyapunov function relative to a compact set $\mathcal{A} \subset \mathbb{R}^n$ for the SHDS (4) if $V(\mathbf{x}_\omega) = 0 \iff \mathbf{x}_\omega \in \mathcal{A}$, V is radially unbounded with respect to set $\mathcal{A}$, nonincreasing during flows, and $\int_{R^m} \max_{g \in G(\mathbf{x}_\omega, v)} V(g)\mu(dv) \leq V(\mathbf{x}_\omega)\ \forall \mathbf{x}_\omega \in D$. The following stochastic hybrid invariance principle [22, Th. 8] is instrumental for our analysis of Theorem 4.2.

Theorem B.1: Let V be a Lyapunov function relative to a compact set $\mathcal{A} \subset \mathbb{R}^n$ for the SHDS system $\mathcal{H}$. Then, $\mathcal{A}$ is UGASp if and only if there does not exist an almost surely complete solution $\mathbf{x}_\omega$ that remains in a nonzero level set of the Lyapunov function almost surely.

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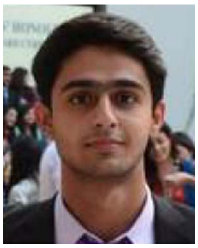


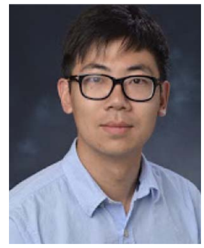
Muhammad Umar Javed received the B.S. degree in electrical engineering from the Lahore University of Management Sciences, Lahore, Pakistan, in 2017, and the M.S. and Ph.D. degrees in electrical engineering from the University of Colorado Boulder, Boulder, CO, USA, in 2019 and 2022, respectively.

He is currently an Applied Scientist with Amazon Web Services, Amazon, Seattle, WA, USA.

Jorge I. Poveda received the M.Sc. and Ph.D. degrees in electrical and computer engineering from UC Santa Barbara, Santa Barbara, CA, USA, in 2016 and 2018, respectively.

He is currently an Assistant Professor with the Electrical and Computer Engineering Department, University of California, San Diego (UCSD), San Diego, CA, USA. Before joining UCSD, he was an Assistant Professor with the University of Colorado Boulder, Boulder, CO, USA, from 2019 to 2022, and a Postdoctoral Fellow with Harvard University, Cambridge, MA, USA, in 2018. His research interests include hybrid control, networked systems, and decision-making algorithms.

Dr. Poveda is a recipient of the Air Force's YIP award, NSF's CRII and CAREER awards, UCSB's CCDC Outstanding Scholar Fellowship and Best Ph.D. Thesis awards, and the campus-wide RIO Faculty Fellowship at CU Boulder.



Xudong Chen (Member, IEEE) received the B.S. degree in electronics engineering from Tsinghua University, Beijing, China, in 2009, and the Ph.D. degree in electrical engineering from Harvard University, Cambridge, MA, USA, in 2014.

He is currently an Assistant Professor with ECEE Department, University of Colorado Boulder, Boulder, CO, USA. He was a Postdoctoral Fellow with Coordinated Science Laboratory, University of Illinois, Urbana-Champaign, IL, USA.

Dr. Chen is an awardee of the 2020 Air Force's YIP award, the 2021 NSF Career award, and the 2021 Donald P. Eckman award.