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High-Order Decentralized Pricing Dynamics for Congestion Games: Harnessing Coordination to Achieve Acceleration

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Abstract— **We introduce a class of decentralized high-order pricing dynamics (HOPD) for the solution of optimal incentive problems in affine congestion games with full resource utilization. The dynamics incorporate momentum and decentralized coordinated resets to achieve better transient performance compared to traditional first-order gradient-based pricing algorithms. The proposed dynamics are studied using tools from graph theory, game theory, and hybrid dynamical systems theory. Our main results establish suitable stability and convergence properties with respect to the set of incentives that generate Nash flows that also maximize the social welfare function of the game. The theoretical results are illustrated via numerical examples in two different types of communication graphs, highlighting the effect of the communication topology and the coordination between players on the transient performance of the HOPD.**

I. INTRODUCTION

In recent years, there has been a growing interest in the study of decentralized resource allocation problems with competitive users in large-scale network systems, including transportation networks, power grids, and the internet [1]–[3]. The growth in the scale of such networks difficulties the implementation of centralized solutions to problems that involve trade-offs between social system-level efficiency and selfish individual performance. Many efforts have been devoted to address these issues, and to the design of localized control laws that ensure desirable global system performance under the presence of self-interested agents [4]–[7]. Game theory provides a collection of mathematical tools that are instrumental for the analysis and design of such systems [2], [8]–[10]. In games, self-interested rational decision-makers are usually referred to as *players* with a *strategy set*, or *resource set*, and an individual *payoff function*. Players often use a distributed *learning algorithm* to iteratively update their actions until convergence to a suitable equilibrium point is achieved. The most common equilibrium of interest is the so-called *Nash Equilibrium* (NE) [8], which describes a profile of actions where players have no unilateral incentive to deviate. However, it is well-known that while NEs can provide a notion of individual optimality for the players, they can lead to poor social outcomes as measured by networked-wide welfare functions. Therefore, to maximize the performance of multi-agent engineering systems controlled via local feedback laws, it is essential to align the emerging Nash equilibria with the socially optimal point of the system. In the static scenario (i.e., no iterative learning dynamics), this methodology is referred to in the literature as *mechanism design* [9], [11].

In this paper, we focus on a particular class of games referred to as *congestion games*, where a fixed amount of resources must be allocated among n different strategies, and the payoff related to each strategy depends on the total allocation. In such types of games, the notion of Nash flow, or Wardrop equilibria, has been used to characterize resource allocations that are optimal from the individual strategy point of view. However, since Wardrop equilibria might not be socially optimal, social planners are faced with the challenge of designing suitable incentives (e.g., tolls in transportation systems, prices in power systems, etc) such that the emerging Nash flows are also socially optimal. To solve this challenge, different types of dynamic pricing algorithms have been considered in the literature [12]–[16]. To guarantee that the system continuously operates at its optimal point, pricing algorithms must react quickly to changes in traffic demand, weather conditions, road accidents, etc. This adaptability requirement, similar to the "alertness" property of feedback control systems, has motivated the development of different recursive algorithms that iteratively update the incentives as the system operates [17]–[22].

Recently, in [23] the authors introduced a class of decentralized gradient-based pricing dynamics (G-PD) that achieve global convergence to socially optimal incentives in a class of affine congestion games. As shown in [23], these distribued welfare gradient dynamics guarantee exponential convergence, with a rate of convergence of order $\mathcal{O}(\kappa)$, where κ defines the strong monotonicity properties of the flow map. However, this convergence rate can be quite slow in problems where $\kappa \ll 1$. In such situations, one may hypothesize that *high-order dynamics* that incorporate momentum might achieve a better transient performance compared to first-order gradient-based algorithms. However, to the best of our knowledge, this conjecture has not been fully explored in decentralized incentive dynamics, let alone when stability and robustness properties are sought after.

In this paper, we provide a positive answer to the above thesis. Our main contribution is the introduction of a class

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of high-order decentralized pricing dynamics that incorporate momentum to achieve better transient performance, and which are also coupled with a suitable coordinated restarting mechanism to guarantee robust stability properties. The proposed dynamics build on recent results developed in the context of accelerated optimization and learning using hybrid control theoretic tools [24]–[27], but specialized here to the setting of congestion games over graphs. The stability properties of the dynamics are studied using tools from hybrid dynamical systems, game theory, and graph theory. Illustrative numerical examples are presented to showcase the advantages of the proposed method compared to existing first-order gradient-based pricing dynamics.

The rest of this paper is organized as follows: Section II presents the preliminaries. Section III describes the pricing problem in congestion games. Section IV presents the high-order pricing mechanism and their stability certificates. Section V presents simulation examples, and finally Section VI ends with the conclusions.

II. PRELIMINARIES

In this section, we introduce the notation used throughout this paper, as well as some preliminaries on hybrid dynamical systems.

A. Notation

Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $z \in \mathbb{R}^n$, we use $|z|_{\mathcal{A}} := \inf_{s\in\mathcal{A}} \|z - s\|_2$ to denote the minimum distance of z to $\mathcal{A}$. We use $\mathcal{A}^\circ$ to denote the interior of the set $\mathcal{A}$, and $r\mathbb{B}$ to denote a closed ball in the Euclidean space, of radius $r > 0$, and centered at the origin. The vector of ones in $\mathbb{R}^n$ is denoted by $\mathbf{1}_n$, and the identity matrix in $\mathbb{R}^{n\times n}$ is denoted by I_n. Given $x, y \in \mathbb{R}^n$, we denote their concatenation by $(x, y) := [x^\top, y^\top]^\top$, where $x^\top$ and $y^\top$ are the transpose of x and y, respectively. The function $\text{diag}(k)$ represents a diagonal matrix with diagonal given by the entries of a vector $k \in \mathbb{R}^n$. We use k_i to refer to the i-th component of a vector $k \in \mathbb{R}^n$, and let $\overline{k} := \max_i k_i$ and $\underline{k} := \min_i k_i$. The set of singular values of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\{\sigma_i(A)\}_{i=1}^n$; we always assume that $i \leq j$ implies $\sigma_i(A) \leq \sigma_j(A)$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if it is non-decreasing in its first argument, decreasing in its second argument, and $\lim_{s\to\infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$.

B. Hybrid Dynamical Systems

To study the different dynamics present in this paper, we consider hybrid dynamical systems (HDS) with state $x \in \mathbb{R}^n$, whose evolution in time is described by

$$x \in C, \qquad \dot{x} = F(x) \tag{1a}$$

$$x \in D, \quad x^+ \in G(x), \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state of the system, $F : \mathbb{R}^n \to \mathbb{R}^n$ is called the flow map, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping called the jump map, and $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ are closed sets, called the flow set and the jump set, respectivel. We use $\mathcal{H} = (C, F, D, G)$ to denote the data of the HDS. Solutions $x : \text{dom}(x) \to \mathbb{R}^n$ to system (1) are indexed by a continuous time parameter t, which increases continuously during flows, and a discrete-time index j, which increases by one during jumps. Therefore, solutions $x : \text{dom}(x) \to \mathbb{R}^n$ to system (1) are defined on *hybrid time domains*. Solutions with an unbounded time domain are said to be *complete*. For a precise definition of hybrid time domains and solutions to HDS of the form (1), we refer the reader to [28, Ch.2]. The following definitions will be instrumental to study the stability properties of systems of the form (1).

Definition 2.1: The compact set $\mathcal{A} \subset C \cup D$ is said to be *uniformly globally asymptotically stable* (UGAS) for system (1) if $\exists\ \beta \in \mathcal{KL}$ such that every solution x satisfies:

$$|x(t, j)|_{\mathcal{A}} \leq \beta(|x(0, 0)|_{\mathcal{A}}, t + j),\ \forall\ (t, j) \in \text{dom}(x). \tag{2}$$

When $\beta(r, s) = c_1 r e^{-c_2 s}$ for some $c_1, c_2 > 0$, the set $\mathcal{A}$ is *uniformly globally exponentially stable* (UGES). When $\exists$ $T^* > 0$ such that $\beta(r, s) = 0$, $\forall\ s \geq T^*, r > 0$, the set $\mathcal{A}$ is said to be *uniformly globally fixed-time stable* (UGFxS). □

III. PROBLEM STATEMENT

In this paper, we consider a congestion game [29] with n possible strategies, where $i \in \mathcal{V} := \{1, \ldots, n\}$ denotes the i^{th} strategy of the game. Additionally, we assign to each strategy i a node in a graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is the set of edges or links. These links capture information restrictions in problems where each node implements an algorithm that makes use of information from other strategies $j \in \mathcal{N}$. We let $z_i \in [0, 1]$ be the proportion of a fixed resource that is allocated to the i^{th} strategy. The vector of allocations is then defined as $z := (z_1, \ldots, z_n)$, which belongs to the simplex $\Delta = \left\{z \in \mathbb{R}^n\ :\ \mathbf{1}_n^\top z = 1,\ z_i \geq 0\right\}$. Moreover, we consider that each strategy has an associated cost of the form:

$$\tilde{c}_i(z_i, q_i) = c_i(z_i) + \delta \cdot q_i, \tag{3}$$

with $(z_i, q_i) \in \Delta \times \mathbb{R}$, and where c_i represents the incentive-free cost of choosing strategy i, the scalar $q_i \in \mathbb{R}$ denotes an external incentive input, and $\delta \in \mathbb{R}_{>0}$ is a sensitivity parameter. We define the vector of incentives as $q := (q_1, \cdots, q_n) \in \mathbb{R}^n$, and we use $\tilde{c}(z, q) = (\tilde{c}_1(z_1, q_1), \ldots, \tilde{c}_n(z_n, q_n))$ to denote the vector of costs of the congestion game under the influence of the external input q. Under suitable monotonicity properties on the cost functions, congestion games are potential games with potential function [30, Sec 2.4]

$$P_q(z) = \sum_{i=1}^{n} \int_0^{z_i} \tilde{c}_i(\zeta, q_i) d\zeta. \tag{4}$$

For every fixed exogenous value $q \in \mathbb{R}^n$, a *Nash flow* of the congestion game corresponds to a particular resource allocation $z_q^f \in \Delta$ that minimizes P_q. Hence, by the KKT conditions, a Nash flow must satisfy

$$-\tilde{c}_i\left(z_{i,q}^f, q\right) + \mu + \lambda_i = 0 \qquad \forall i \in \mathcal{V} \tag{5a}$$

$$z_q^f \in \Delta, \quad \lambda_i z_{i,q}^{Nf} = 0, \quad \lambda_i \geq 0 \qquad \forall i \in \mathcal{V}. \tag{5b}$$

In general, Nash flows might not be socially optimal. To formally quantify the social optimality of a given allocation z, the concept of social welfare is introduced.

Social welfare: The social welfare is defined as:

$$W(z) := -\sum_{i=1}^{n} c_i(z_i) z_i. \tag{6}$$

Consequently, a *socially optimal* flow z^* corresponds to the social state that maximizes (6). Thus, by the KKT conditions, z^* satisfies:

$$-\frac{\partial c_i}{\partial z_i}(z^*) z_i^* - c_i(z_i^*) + \tilde{\mu} - \tilde{\lambda}_i = 0, \qquad \forall i \in \mathcal{V}, \tag{7a}$$

$$z^* \in \Delta, \quad \tilde{\lambda}_i z_i^* = 0, \quad \tilde{\lambda}_i \geq 0, \qquad \forall i \in \mathcal{V}. \tag{7b}$$

In this paper, we design a class of dynamic pricing mechanism to render Nash flows socially optimal with fast transient-performance. To this end, we consider a subclass of congestion games that satisfy the following assumption.

Assumption 3.1: For the congestion game with $n \in \mathbb{Z}_{>0}$ strategies, the following conditions are satisfied:

(i_1) *Affine costs:* There exists a positive definite diagonal matrix $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ such that $c(z) = Az + b$ for all $z \in \mathbb{R}^n$.

(i_2) *Full utilization:* For every incentive $q \in \mathbb{R}^n$, the corresponding Nash flow $z_q^f \in \Delta^\circ$, i.e, $\left(z_q^f\right)_i \neq 0$ for all $i \in \mathcal{V}$.

□

Remark 3.1: While Assumption 3.1 is conservative, affine congestion games are commonly found in various societal systems, including parallel network routing and traffic problems [11]. Moreover, since sets of incentives are bounded in most practical applications, it is reasonable to assume that every strategy $i \in V$ will receive a positive allocation, resulting in the full utilization scenario. □

Assumption 3.1 ensures the strong convexity of the potential function in (4), which implies that conditions (5) hold for a unique Nash Flow z_q^f for every $q \in \mathbb{R}^n$. We refer to the mapping $\mathcal{O}(q) := z_q^f$ as the *oracle mapping*. The following lemma provides a characterization of the oracle mapping for the types of games that satisfy Assumption 3.1. Proofs are omitted due to lack of space.

Lemma 3.1: Under Assumption 3.1 the oracle mapping $\mathcal{O}(\cdot)$ satisfies:

$$\mathcal{O}(q) = -Q(b + \delta \cdot q) + \alpha, \tag{8}$$

where

$$\alpha := \frac{A^{-1}\mathbf{1}_n}{\mathbf{1}_n^\top A^{-1} \mathbf{1}_n},$$

and where $Q \in \mathbb{R}^{n \times n}$ is the Laplacian matrix of a graph with adjacency matrix

$$\mathbb{A} := \frac{A^{-1}\mathbf{1}_n \mathbf{1}_n^\top A^{-1}}{\mathbf{1}_n^\top A^{-1} \mathbf{1}_n},$$

that is

$$Q := \left(I - A^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^\top}{\mathbf{1}_n^\top A^{-1} \mathbf{1}_n} \right) A^{-1}. \tag{9}$$

□

In this paper, we assume that the dynamics describing the convergence to the Nash flow under a given incentive q are instantaneous, and therefore can be omitted. This assumption can be justified using singular perturbation techniques for multi-time scale dynamical systems. Now, note that Assumption 3.1-(i_1) guarantees strong concavity of the welfare function W, which implies that the system of equations (7) are satisfied for a unique socially optimal flow z^*. However, according to the following lemma, the incentives q that generate this resource allocation state may not be unique, see also [11], [23].

Lemma 3.2: Suppose that Assumption 3.1 holds. Then, the set of incentives that generate socially optimal Nash flows via the costs (3) is given by

$$\mathcal{A}_q := \{ q \in \mathbb{R}^n \ : \ q = q^* + \mu \mathbf{1}_n, \ \mu \in \mathbb{R} \}, \tag{10}$$

where $q^* = \frac{-b}{2\delta}$. □

IV. HIGH-ORDER PRICING DYNAMICS FOR CONGESTION GAMES

To control the incentives, we assume that the dynamic pricing mechanism has access to *on-the-fly* measurements of the information tuple

$$\mathcal{I} := \left\{ \mathcal{O}(q), c\left(\mathcal{O}(q)\right), \frac{\partial c}{\partial z} \mathcal{O}(q) \right\},$$

where $\mathcal{O}(\cdot)$ was defined in (8). Due to the large-scale nature of congestion games, it may not be practical to measure $\mathcal{I}$ centrally. Instead, we assume that each node i has access to its own information and to the information of the neighboring nodes characterized by the communication graph $\mathbb{G}$. In this way, each node will implement individual dynamics based on the received information to influence the Nash Flow of the congestion game. To simplify our presentation, we make the following assumption on the graph $\mathbb{G}$. However, we stress that this assumption can be relaxed.

Assumption 4.1: $\mathbb{G}$ is connected and undirected. □

For games satisfying Assumption 3.1 and a graph satisfying Assumption 4.1, the work in [23] introduced the so-called *distributed welfare-gradient dynamics*, given by

$$\dot{q} = \gamma \mathcal{L} \mathcal{G}^{\mathcal{O}}(q), \tag{11}$$

where $\gamma > 0$ is a scalar gain, and where $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{G}^{\mathcal{O}} : \mathbb{R}^n \to \mathbb{R}^n$ are defined as:

$$\mathcal{G}(z) := c(z) + \frac{\partial c(z)}{\partial z} z \tag{12}$$

$$\mathcal{G}^{\mathcal{O}}(q) := (\mathcal{G} \circ \mathcal{O})(q). \tag{13}$$

As shown in [23], these dynamics render the set $\mathcal{A}_q$ exponentially stable. However, in some cases, the exponential convergence can be prohibitively slow since it is dictated by the strong monotonicity constant of the mapping $q \mapsto \mathcal{L}\mathcal{G}^{\mathcal{O}}(q)$. To address this issue, and to achieve better transient performance in a decentralized fashion, we take inspiration from the hybrid regularized version of Nesterov's ODE studied in [25]. In particular, we introduce the *high-order pricing dynamics* (HOPD), described by a hybrid dynamical system with data $\mathcal{H}_1 := (C_1, F_1, D_1, G_1)$ and state $x = (q, p, \tau) \in \mathbb{R}^{3n}$, where $p \in \mathbb{R}^n$ is a momentum state, and $\tau \in \mathbb{R}^n$ corresponds to a set of timers $\{\tau_i\}_{i\mathcal{V}}$ which coordinate the evolution of the distributed dynamics. Specifically, the flow map F_1 of the proposed HOPD is given by

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{\tau} \end{pmatrix} = F_1(x) = \begin{pmatrix} 2\mathcal{T}^{-1}(p-q) \\ 2\gamma\mathcal{L}\mathcal{T}\mathcal{G}^{\mathcal{O}}(q) \\ \frac{1}{2}\mathbf{1}_n \end{pmatrix}, \tag{14}$$

where $\gamma \in \mathbb{R}_{>0}$ is again a tunable gain, and $\mathcal{T} := \text{diag}(\tau)$. Note that the p-dynamics maintain the sparsity of the communication infrastructure imposed by the graph $\mathbb{G}$. The flow set C_1 is defined as

$$C_1 := \mathbb{R}^n \times \ker(\mathcal{L})^{\perp} \times [T_0, T]^n, \tag{15}$$

where (T_0, T) are tunable parameters which satisfy $T > T_0 > 0$. The proposed HOPD algorithm also employs a restarting mechanism given by

$$\tau_i = T \Longrightarrow \begin{pmatrix} q_i^+ \\ p_i^+ \\ \tau_i^+ \end{pmatrix} = R_i(x_i) := \begin{pmatrix} p_i \\ p_i \\ T_0 \end{pmatrix}, \tag{16}$$

which is triggered by the condition $\tau_i = T$. While a fully decentralized implementation of these dynamics might seem appealing due to its simplicity, as demonstrated in Section V, ensuring coordinated and synchronized restarting is crucial to achieve good transient performance and to fully exploit the advantages of incorporating momentum. To achieve this coordinated behavior, we adopt the synchronization dynamics of [31] and we employ a coordination mechanism where the updates of node $j \in \mathcal{V}$ make use of a set-valued mapping $\mathcal{C}_j : \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}_{\geq 0}$, defined as:

$$\mathcal{C}_j(\tau_j) := \begin{cases} T & \text{if } \tau_j \in (T_0 + r_j, T] \\ \{T_0, T\} & \text{if } \tau_j = T_0 + r_j \\ T_0 & \text{if } \tau_j \in [T_0, T_0 + r_j) \end{cases}, \tag{17}$$

where the individual parameter $r_j > 0$ satisfies $r_j \in \left(0, \frac{T-T_0}{n}\right)$. Using $\mathcal{C}_j$, the coordination mechanism works as follows: whenever the timer of the i^{th} node satisfies $\tau_i = T$, the following two events occur: 1) Node i resets its own state x_i using the dynamics (16), and 2) node i sends a pulse to their neighbors $j \in \mathcal{N}_i$, who proceed to update their state $x_j = (q_j, p_j, \tau_j)$ as follows:

$$q_j^+ = q_j, \qquad p_j^+ = p_j, \qquad \tau_j^+ \in \mathcal{C}_j(\tau_j). \tag{18}$$

Since node i can only signal their neighbors, the rest of the nodes $j \notin \mathcal{N}_i$ will keep their states constant after the above two events, i.e., $x_j^+ = x_j$, for all $j \notin \mathcal{N}_i$. To formally describe this behavior, we introduce the set-valued mapping $G^0 : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^{3n}$, which is defined to be non-empty only when $\tau_i = T$ and $\tau_j \in [T_0, T)$ with $j \neq i$, for each $i \in \mathcal{V}$, and satisfies

$$\begin{aligned} G^0(x) := \Big\{&(v_1, v_2, v_3) \in \mathbb{R}^{3n} : (v_{1,i}, v_{2,i}, v_{3,i}) = R_i(x_i), \\ &v_{1,j} = q_j, v_{2,j} = p_j, v_{3,j} \in \mathcal{C}_j(\tau_j), \ \forall\ j \in \mathcal{N}_i, \\ &v_j = x_j, \forall\ j \notin \mathcal{N}_i\Big\}, \end{aligned} \tag{19}$$

where $x := (x_1, x_2, \cdots, x_n)$, and where the *reset map* R_i and the *coordination mapping* $\mathcal{C}_j$ are defined in (16) and (17), respectively. Using the construction (19), the overall coordination mechanism is characterized by the the jump map

$$x^+ \in G_1(x) := \overline{G^0}(x), \tag{20}$$

where $\overline{G^0}(x)$ represents the outer-semi-continuous hull of the map G^0. Finally, the jump set is defined as

$$D_1 := \mathbb{R}^n \times \ker(\mathcal{L})^{\perp} \times D_\tau, \tag{21}$$

where $D_\tau := \{\tau \in \mathbb{R}^n \ :\ \max_{i \in \mathcal{V}} \tau_i = T\}$.

Remark 4.1: Distributed momentum-based hybrid dynamics with flow and jump maps resembling (14) and (20) have been recently studied in distributed optimization problems [32], and non-cooperative games [27]. However, they have remained unexplored in the context of congestion games and dynamic pricing. □

Remark 4.2: The use of the resetting parameters r_i, as well as the construction of the jump map $\overline{G^0}(x)$ from the individual coordination mapping $\mathcal{C}_j$ and the reset map $\mathcal{R}_i$, are *fundamental* for the formulation of well-posed hybrid dynamical systems [28, Ch. 6] with suitable *robust stability* certificates for the synchronization of timers [33]. □

A. Main Results

The following Lemma characterizes the synchronization certificates for the timer variable τ under the HOPD.

Lemma 4.1: Let

$$\mathcal{A}_\tau := [T_0, T]\mathbf{1}_n \cup (\{T_0, T\}^n)$$

represent the set of points in the set $[T_0, T]^n$ where the timer variables are synchronized with a common value in (T_0, T), and, where the value of the timers can only differ from each other during jumps by taking values in the set $\{T_0, T\}$. Then, $\mathcal{H}_1$ renders the set

$$\mathcal{A}_{\text{sync}} := \ker(\mathcal{L})^{\perp} \times \ker(\mathcal{L})^{\perp} \times A_\tau,$$

UGFxS with convergence bound $T^* := 2(T - T_0) + \eta$. □

By leveraging Lemma (4.1), the following theorem, corresponding to the main result of this paper, characterizes stability certificates for the the hybrid dynamics $\mathcal{H}_1$ with respect to the set $\mathcal{A}_q$ defined in (10).

Theorem 4.2: Suppose that Assumption 3.1 is satisfied, and assume that the tunable parameters (T_0, T) satisfy:

$$T^2 - T_0^2 > \frac{1}{2\sigma_2(\mathcal{L})\gamma\delta}, \tag{22}$$

where $\sigma_2(\mathcal{L})$ is the minimum non-zero singular value of $\mathcal{L}$. Then, the HDS $\mathcal{H}_1$ renders UGES the set

$$\mathcal{A} := \left\{(q,p,\tau) \in \mathbb{R}^{3n} \ : \ p = q, \ q \in \mathcal{A}_q, \ \tau \in \mathcal{A}_\tau\right\}.$$

Additionally, for every $i \in \mathcal{V}$, and for all solutions x, the following bound holds during flows

$$|q(t,j)|^2_{\mathcal{A}_q} \leq \frac{T}{T_0}\sqrt{\frac{\sigma_n(\mathcal{L})}{\sigma_2(\mathcal{L})}}\,(1-\eta)^{\frac{\tilde{j}}{2}}\,M_0,$$

where M_0 is a constant that depends on the initial conditions, $\tilde{j} := \max\{0, \lfloor\frac{j-n}{n}\rfloor\}$, and

$$\eta := 1 - \frac{T_0^2}{T^2} - \frac{1}{2T^2\gamma\delta\sigma_2(\mathcal{L})}.$$

$\square$

Remark 4.3: Theorem 4.2 guarantees exponential stability to $\mathcal{A}_q$ with convergence rate conditioned by $1-\eta$. By following similar ideas to the literature on centralized accelerated optimization [25], [34]–[36], we can find a "quasi-optimal" restarting parameter $T = e\sqrt{\frac{1}{2\gamma\delta\sigma_2(\mathcal{L})} + T_0^2}$, which guarantees exponential convergence of order $\mathcal{O}\left(e^{-\sqrt{\sigma_2(\mathcal{L})}}\right)$ as $T_0 \to 0^+$. $\square$

B. On the Role of Resets and Coordination

The key technical elements of the HOPD that allow us to achieve the result of Theorem 4.2 are the resets and the coordination of the resetting timers τ_i. Indeed, as shown in [25] for centralized optimization problems, when high-order dynamics implement vanishing damping, the resulting stability properties can be lost under arbitrarily small disturbances. On the other hand, the incorporation of resets induces suitable uniformity properties in the convergence, which in turn, guarantees a minimum margin of robustness against additive disturbances [28, Ch.7]. Moreover, the reset condition (22) permits to leverage the decrease of a suitable Lyapunov function during resets in order to achieve accelerated convergence of order $\mathcal{O}(\sigma_2(\mathcal{L}))$, which is particularly advantageous when $\sigma_2(\mathcal{L}) \ll 1$.

Finally, we note that this result relies on achieving fixed-time synchronization of the resetting timers τ_i via the coordination map (16). Without such a coordination mechanism, the performance of the HOPD can be substantially inferior when compared to traditional gradient-based algorithms. We illustrate this phenomenon in Section (V) via numerical examples.

V. NUMERICAL EXAMPLE

In this section, we illustrate our results with a simple numerical example. We consider a total resource of 1 that needs to be allocated among 4 different nodes, i.e., $\mathcal{V} := \{1,2,3,4\}$. The matrices and parameters that describe the payoffs of the underlying game are:

$$A = 2I_4, \quad b = \mathbf{1}_4, \quad \gamma = 0.1, \quad \delta = 1.$$

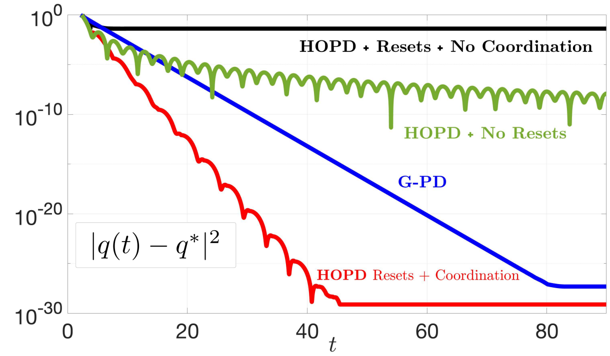

Fig. 1: Evolution of the incentives over time using a ring communication graph.

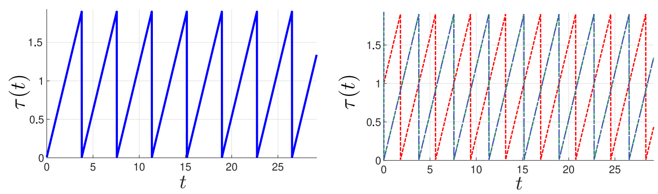

Fig. 2: Coordinated (left) vs Uncoordinated (right) Resetting Timers in the HOPD.

We first simulate the G-PD of [23] and the HOPD using a communication graph characterized by a ring. Next we consider three different scenarios for the HOPD: (a) First, we consider the situation where the HOPD are implemented without resets. (b) Second, we implement the HOPD with uncoordinated resets; (c) Finally, we implement the complete HOPD with coordinated resets. All the results are presented in Figure 1, which shows the evolution in time of the squared error of the incentive q. As observed, the HOPD with coordinated resets generate substantially better performance compared to the standard G-PD of [23], achieving the same "steady state" error in half of the time. In this case, the restarting frequency of the HOPD was selected to be 4s. Another important observation from our numerical experiments is that using HOPD with *uncoordinated* resets generates substantially worse performance compared to the standard first-order G-PD. This poor performance is shown in the black curve of Figure 1. This observation highlights the role of coordination whenever dynamics with momentum and resets are implemented in multi-agent systems.

Finally, we repeat our numerical example in a system with a communication graph characterized by a path. The results are presented in Figure 3. The restarting frequency was selected as 8s. As it can be observed, the graph's structure affects the transient performance of the HOPD. In particular, in this case the difference between the performance of the first-order G-PD dynamics and the HOPD algorithm is more pronounced.

VI. CONCLUSIONS

In this paper, we introduced a class of high-order pricing dynamics for the solution of dynamic incentive problems

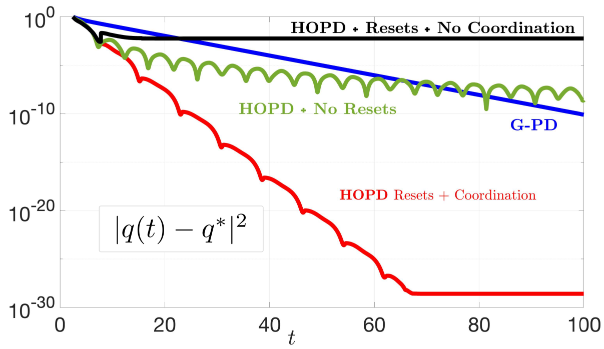

Fig. 3: Evolution of the incentives over time using a line communication graph.

in congestion games under a full utilization assumption. The dynamics incorporate momentum and, when combined with coordinated resets, can achieve better transient performance compared to first-order Welfare gradient dynamics. The stability and convergence properties of the algorithms were studied using tools from hybrid dynamical systems theory. Numerical examples were presented to illustrate the performance of the algorithms. Future research directions will focus on the interconnection of the proposed dynamics with dynamical systems in the loop modeling a class of social dynamics.

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