

Sequential sum-of-squares programming for analysis of nonlinear systems^{*}

Torbjørn Cunis¹ and Benoît Legat²

Abstract—Numerous interesting properties in nonlinear systems analysis can be written as polynomial optimization problems with nonconvex sum-of-squares problems. To solve those problems efficiently, we propose a sequential approach of local linearizations leading to tractable, convex sum-of-squares problems. Local convergence is proven under the assumption of strong regularity and the new approach is applied to estimate the region of attraction of a polynomial aircraft model.

I. INTRODUCTION

Polynomials that can be written as a sum of squares are a strict subset of the nonnegative polynomials. While determining whether a given polynomial does not assume negative values is computationally hard, Parillo [1] showed in his seminal paper that convex optimization over sum-of-squares polynomials can be reduced to semidefinite programming. His works, as well as the development of the dual approach via moments by Lasserre [2], and the advent of efficient algorithms for semidefinite problems laid the foundation for numerical analysis of nonlinear systems with polynomial dynamics that is today known as sum-of-squares programming.

Applications of convex sum-of-squares programming include stability verification for hybrid systems [3–7], optimization algorithms and optimization-based control [8–10], control synthesis [11–14], and many more. As these approaches often make use of Lyapunov-type functions and dissipativity inequalities, many sum-of-squares constraints for polynomial dynamics can be viewed as the natural extension of linear matrix inequalities for linear systems [15]. However, unlike in the linear case, most properties of nonlinear systems such as asymptotic stability, invariance, or controllability often are valid on a region of the system’s state-space only. The problem of determining the region of attraction [16], for example, thus consists of finding a Lyapunov candidate V and a region X (often a sublevel set) as well as certifying that V decays strictly on X . If V and the describing function of X are polynomial decision variables, estimating the region of attraction is a nonconvex, nonlinear sum-of-squares problem by the Positivstellensatz of the reals [17].

Despite nonlinear sum-of-squares problems being computationally hard, local analysis of stability and other properties of polynomial dynamics with sum-of-squares programming has been extensively studied [18–28]. Here, the (mostly

bilinear) nonconvex constraints have been mitigated by bisections [29], coordinate descent [30], and combinations of both. Yet, except for quasiconvex problems, convergence is not guaranteed (see remarks in [31]). Given that the underlying semidefinite problems scale notoriously with the polynomial degree, it is desirable to limit the number of convex evaluations.

In this paper, we take inspiration from sequential convex programming [32–36] and study a sequential approach for nonlinear conic problems which we combine with a line search using a merit function from Powell [37]. The nonlinear problem is linearized around a solution candidate in order to obtain an affine conic problem. For the sum-of-squares cone, sum-of-squares toolboxes such as SPOT, SOSOPT, or SOSTOOLS are readily available to solve the local problems via reduction to a semidefinite program; and more recently, direct implementations of the sum-of-squares cone have been proposed [38, 39]. Similar to the affine case, nonlinear sum-of-squares problems could directly be reduced to a nonlinear semidefinite program; yet the authors are only aware of the toolbox SUMOFSQUARES.JL [40] for that purpose, which is limited to quadratic expressions. Moreover, the semidefinite representation of a sum-of-squares polynomial is nonunique (see comments in [1, Section 3.2]); yet uniqueness of the solution is usually assumed for convergence.

We prove local convergence of the sequence of convex problems using a result from variational analysis [41] that builds upon the implicit function theorem for strongly regular generalized equations by Robinson [42]. As this result is stated for (possibly infinite dimensional) Banach spaces, our analysis works in the general setting of nonlinear conic programs with convex cones embedded in Banach spaces. Since the vector space of polynomials is not complete, we limit ourselves to optimization problems with fixed polynomial degree but our sequential algorithm can be applied to other cones as well. We further investigate the line search based on the dual theory of affine sum-of-squares optimization. Numerical results for practical engineering problems demonstrate that sequential sum-of-squares programming significantly reduces the number of convex problems to be solved and thus the computation time compared to previous, iterative approaches.

Our proof generalizes [35] in two aspects. First, we consider convex cones in arbitrary Banach spaces rather than embedded in $\mathbb{R}^n$. Second, we show that the convergence still holds if a line search is used to improve convergence speed. Moreover, by use of variational analysis, our paper provides a simpler proof while obtaining a tighter convergence rate.

^{*}Supported by the authors’ institutions.

¹T. Cunis is with the Institute of Flight Mechanics and Control, University of Stuttgart, 70569 Stuttgart, Germany. tcunis@ifr.uni-stuttgart.de

²B. Legat is with the ESAT Department, KULeuven, Leuven, Belgium. benoit.legat@esat.kuleuven.be

The remainder of the paper is organized as follows: Section II introduces the tools from variational analysis and Section III motivates and states the problem of nonlinear sum-of-squares optimization. The sequential programming approach is detailed in Section IV and local convergence is proven in Section V. In Section VI, the sequential approach is applied to problems from nonlinear system analysis.

Notation: $\mathbb{N}$ (resp., $\mathbb{N}_0$) and $\mathbb{R}$ denote the natural numbers excluding (resp., including) zero and the reals, respectively. For some $m \in \mathbb{N}$, the set of symmetric (resp., positive semidefinite) matrices in $\mathbb{R}^{m \times m}$ is $\mathbb{S}_m$ (resp., $\mathbb{S}_m^+$).

II. PRELIMINARIES

Let X , Y , and P be Banach spaces. The dual space X^* is set of linear operators $l : X \rightarrow \mathbb{R}$ is X^* with evaluation $\langle \cdot, \cdot \rangle : X^* \times X \rightarrow \mathbb{R}$. Moreover, the adjunct of a linear mapping $A : X \rightarrow Y$ is the linear mapping $A^* : Y^* \rightarrow X^*$ satisfying $\langle l, A(\xi) \rangle = \langle A^*(l), \xi \rangle$ for all $\xi \in X$ and $l \in Y^*$.

A. Normals & Gradients

A convex cone is a set $C \subset X$ satisfying $r_1\xi_1 + r_2\xi_2 \in C$ for all $\xi_1, \xi_2 \in C$ and $r_1, r_2 \in \mathbb{R}_{\geq 0}$. The dual cone of C is defined as

$$C^* = \{v \in X^* \mid \langle v, \xi \rangle \geq 0 \text{ for all } \xi \in C\}$$

and the dual of C^* is isometric to C . Moreover, for a convex set $\Omega \subset X$, the normal cone mapping $N_\Omega : X \rightrightarrows X^*$ is given by

$$N_\Omega(\xi_0) = \{w \in X^* \mid \langle w, \xi - \xi_0 \rangle \leq 0 \text{ for all } \xi \in \Omega\}$$

if $\xi_0 \in \Omega$, and $N_\Omega(\xi_0) = \emptyset$ otherwise.

Definition 1: The (Fréchet) derivative of a nonlinear function $g : X \rightarrow Y$ at $\xi_0 \in X$ is a linear mapping $\nabla g(\xi_0) : X \rightarrow Y$ satisfying

$$\lim_{\xi \rightarrow \xi_0} \frac{g(\xi_0) + \nabla g(\xi_0)(\xi - \xi_0) - g(\xi)}{\|\xi_0 - \xi\|} = 0$$

and g is (Fréchet) differentiable if and only if $\nabla g(\xi_0)$ exists for all $\xi_0 \in X$.

B. Continuity & Regularity

Let $h : X \times P \rightarrow Y$ be a function and $H : X \rightrightarrows Y$ be a set-valued mapping. h is said to be *Lipschitz continuous* with respect to ξ around $(\xi_0, \pi_0) \in \text{int dom } h$ if and only if

$$\limsup_{\substack{\xi, \xi' \rightarrow \xi_0, \xi \neq \xi' \\ \pi \rightarrow \pi_0}} \frac{\|h(\xi', \pi) - \psi(\xi, \pi)\|}{\|\xi' - \xi\|} = \kappa$$

with constant $\kappa < \infty$. Moreover, H is said to have a *single-valued localization* $h : X' \rightarrow Y'$ around $\xi_0 \in X$ for $v_0 \in Y$ if and only if $X' \subset X$ and $X' \subset X$ are neighbourhoods of ξ_0 and v_0 , respectively, $h(\xi_0) = v_0$, and $H(\xi) \cap Y' = \{h(\xi)\}$ for all $\xi \in X'$. If the inverse $H^{-1} : v \mapsto \{\xi \in X \mid v \in H(\xi)\}$ has a single-valued localization around v_0 for ξ_0 that is Lipschitz continuous around v_0 with constant γ , then H is called *strongly regular* at ξ_0 for v_0 with constant γ .

Theorem 1 (Theorem 8.8 of [41]): Take $\psi : X \times P \rightarrow Y$ and $N : X \rightrightarrows Y$; suppose ψ is Lipschitz continuous

with respect to π around (ξ_0, π_0) with constant κ and $0 \in \psi(\xi_0, \pi_0) + N(\xi_0)$; if there exists $\psi_0 : X \rightarrow Y$ such that $\psi_0(\xi) = \psi(\xi, \pi)$ around ξ_0 if $\pi \rightarrow \pi_0$ and $\psi_0 + N$ is strongly regular at ξ_0 for 0 with constant γ , then the mapping

$$H : \pi \mapsto \{\xi \in X \mid \psi(\xi, \pi) + N(\xi) \ni 0\}$$

has a Lipschitz continuous, single-valued localization around π_0 for ξ_0 with constant $\gamma\kappa$. ◀

III. SUM-OF-SQUARES OPTIMIZATION

Let $x = (x_1, \dots, x_n)$ be a tuple of free variables and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ a multi-index; a polynomial π in x up to degree d is a linear combination

$$\pi = \sum_{\|\alpha\|_1 \leq d} c_\alpha x^\alpha$$

where $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\|\alpha\|_1 = \sum_i \alpha_i$. The set $\mathbb{R}_d[x]$ of polynomials in x with real coefficients $c_\alpha \in \mathbb{R}$ up to degree d forms a vector space with norm $\|\cdot\|$.

Definition 2: A polynomial $\pi \in \mathbb{R}_d[x]$ is a *sum-of-squares polynomial* ($\pi \in \Sigma_d[x]$) if and only if there exist $m \in \mathbb{N}$ and $\pi_1, \dots, \pi_m \in \mathbb{R}_d[x]$ such that $\pi = \sum_{i=1}^m (\pi_i)^2$.

It is easy to see that $\Sigma_d[x]$ forms a convex cone in $\mathbb{R}_d[x]$. Moreover, its dual cone $\Sigma_d[x]^*$ is isometric to the cone of sum-of-squares polynomials [2].

To avoid confusion with the Fréchet derivative ∇ (with respect to the space of polynomials), we are going to use $\partial_x : \mathbb{R}_d[x] \rightarrow \mathbb{R}_d[x]^{1 \times n}$ for the differentiation operator with respect to the free variables x .

A convex sum-of-squares optimization problem can be reduced to a semidefinite program. This is the fundamental result of [1, Theorem 3.3], which reads as follows; denote by $s_d \in \mathbb{N}_0$ the number of monomials up to degree d of a polynomial in x . A polynomial $\pi \in \mathbb{R}_{2d}[x]$ is sum-of-squares if and only if there exists a matrix $Q \in \mathbb{S}_{s_d}^+$ satisfying $\pi = \zeta^\top Q \zeta$, where $\zeta \in \mathbb{R}_d[x]^{s_d}$ is the vector of monomials up to degree d . The solution for Q is usually not unique, leading to the *implicit* (or kernel) and *explicit* (or image) relaxations of an affine sum-of-squares problem.

A. Motivation

Consider a continuous-time dynamic system defined by the differential equation

$$\dot{x} = \phi(x) \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state vector and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a polynomial function satisfying $\phi(0) = 0$. Many system-theoretic properties on a domain $\mathcal{D} \subset \mathbb{R}^n$ can be written as polynomial dissipativity inequality of the form

$$\forall x \in \mathbb{R}^n, x \in \mathcal{D} \implies \partial_x V(x) \phi(x) \leq S(x) \tag{2}$$

where $V \in \mathbb{R}[x]$ and $S \in \mathbb{R}[x]$ are called storage function and supply rate, respectively. The set $\mathcal{D}$ usually depends on V ; for example, if $\mathcal{D}$ is a sublevel set of V and V and S are positive definite polynomials, then Eq. (2) is LaSalle's condition for asymptotic stability [43, Theorem 2].

In the sum-of-squares literature, the dissipativity condition is rewritten using the so-called generalized S-procedure [29]. Suppose $\mathcal{D} = \{x \in \mathbb{R}^n \mid \ell_V(x) \geq 0\}$ for some $\ell_V \in \mathbb{R}_d[x]$, then if there exists $\varsigma \in \Sigma_d[x]$ such that

$$(S - \partial_x V \phi) - \varsigma \ell_V \in \Sigma_{d'}[x] \quad (3)$$

then V and S satisfy (2). However, since ℓ_V depends on V , the sum-of-squares constraint (3) is nonlinear. Previous approaches to optimize over nonlinear sum-of-squares constraint relied on solving for one variable at a time while keeping the remaining variables fixed (see, e.g., [29–31]).

B. Nonlinear optimization problem

In general, a polynomial optimization problem with non-convex sum-of-squares constraints takes the form of a non-linear optimization

$$\min_{\xi \in X} \langle f, \xi \rangle \quad \text{s.t. } g(\xi) \in D \text{ and } \xi \in C \quad (4)$$

where X and Y are Banach spaces, $f \in X^*$ is a linear cost, $g : X \rightarrow Y$ is a differentiable constraint mapping, and $C \subset X$ and $D \subset Y$ are convex cones. In the case of sum-of-squares, X and Y correspond to spaces of polynomial up to a finite degree with sum-of-squares cones C and D and g takes polynomial values.

Define the Lagrangian as $L(\xi, l) = \langle f, \xi \rangle - \langle l, g(\xi) \rangle$, where $l \in Y^*$ is a Lagrange multiplier; the Karush-Kuhn-Tucker (KKT) conditions for (4) at $\xi_0 \in X$ are

$$\begin{aligned} f - \nabla g(\xi_0)^* l - s &= 0 \\ \langle s, \xi_0 \rangle &= 0, \quad \langle l, g(\xi_0) \rangle = 0 \\ \xi_0 &\in C, \quad g(\xi_0) \in D \end{aligned}$$

where $s \in C^*$ and $l \in D^*$ are dual variables associated with the cone constraints.

Define $\vartheta = (\xi, l)$ and $\mathcal{T} = X \times Y^*$. With a small abuse of notation, we identify $g(\xi)$ as an element of D^{**} using the canonical isomorphism between D and D^{**} . Then the KKT conditions are equivalent to $(-s, -g(\xi))$ belonging to the normal cone of $C \times D^*$ at $(\xi_0, l_0) \in \mathcal{T}$. The KKT conditions can thus be written as generalized equation

$$\varphi(\vartheta) + N(\vartheta) \ni 0 \quad (5)$$

where $\varphi : \mathcal{T} \rightarrow \mathcal{T}^*$ and $N : \mathcal{T} \rightrightarrows \mathcal{T}^*$ are defined as

$$\varphi : \vartheta \mapsto (f - \nabla g(\xi)^* l, g(\xi))$$

and

$$N : \vartheta \mapsto \{(v, \zeta) \in \mathcal{T}^* \mid v \in N_C(\xi), \zeta \in N_{D^*}(l)\}$$

respectively. We denote the solutions to (5) by $\Theta \subset \mathcal{T}$.

Assumption 1: The set Θ is nonempty.

Under a suitable constraint qualification, existence of a KKT point is necessary for an optimal solution of (4).

IV. SEQUENTIAL PROGRAMMING

We propose to approach the nonlinear problem (4) with a sequence of local, convex problems. To that extent, let $\xi^k \in X$ with $k \in \mathbb{N}_0$ be the solution of the k -th iteration and $l^k \in Y^*$ be the associated Lagrange multiplier. Pick tolerances $\epsilon_k, \epsilon_k^* > 0$ for the primal and dual solutions as well as a small weight $\eta > 0$. Our next instance (ξ^{k+1}, l^{k+1}) is subject to the steps:

1) Solve the convex problem at ξ^k ,

$$\min_{\xi \in X, \varsigma \in Y} \langle f, \xi \rangle \quad (6a)$$

$$\text{s.t. } g(\xi^k) + \nabla g(\xi^k)(\xi - \xi^k) = \varsigma \quad (6b)$$

$$\text{and } \xi \in C, \quad \varsigma \in D \quad (6c)$$

and denote the optimal solution as ξ_+ and the associated Lagrange multiplier as l_+ .

2) Solve the line search

$$\min_{r \in \mathbb{R}} \psi(r) - \eta r \quad \text{s.t. } 0 < r \leq 1 \quad (7)$$

where $\psi : r \mapsto L(r\xi_+ + (1-r)\xi^k, l_+)$, and denote the optimal solution as $\hat{r}$.

3) Set $\xi^{k+1} = \hat{r}\xi_+ + (1-\hat{r})\xi^k$ and $l^{k+1} = \hat{r}l_+ + (1-\hat{r})l^k$. We terminate the iteration if both $\|\xi^{k+1} - \xi^k\| \leq \epsilon_k$ and $\|l^{k+1} - l^k\|_* \leq \epsilon_k^*$, where $\|\ell\|_*$ denotes the operator norm of $\ell : Y \rightarrow \mathbb{R}$. Otherwise, we repeat the steps for $k+1$.

As linear problem, (6) has a dual problem at ξ^k , viz.

$$\max_{l \in Y^*, s \in X^*} \langle l, \gamma_k \rangle \quad (8a)$$

$$\text{s.t. } f - \nabla g(\xi^k)^* l - s = 0 \quad (8b)$$

$$\text{and } l \in D^*, \quad s \in C^* \quad (8c)$$

where $\gamma_k \stackrel{\text{def}}{=} \nabla g(\xi^k)\xi^k - g(\xi^k)$. In the following analysis, we will assume that the Lagrange multiplier l_+ is the optimal solution of (8), provided it exists. If $(\xi_0, l_0) \in \mathcal{T}$ satisfy the KKT conditions

$$\begin{aligned} f - \nabla g(\xi^k)^* l_0 - s &= 0 \\ \langle s, \xi_0 \rangle &= 0, \quad \langle l_0, \nabla g(\xi^k)\xi_0 - \gamma_k \rangle = 0 \\ (\xi_0, l_0) &\in C \times D^*, \quad \nabla g(\xi^k)\xi_0 - \gamma_k \in D \end{aligned}$$

for some $s \in C^*$, then ξ_0 and l_0 are optimal solutions for (6) and (8), respectively, and satisfy $\langle f, \xi_0 \rangle = \langle l_0, \gamma_k \rangle$.

V. THEORETICAL ANALYSIS

We are going to prove local convergence of the sequential algorithm using a parametrized version of the generalized equation (5); define

$$\hat{\varphi} : (\vartheta, \xi^k) \mapsto (f - \nabla g(\xi^k)^* l, \nabla g(\xi^k)\xi - \gamma_k)$$

then $\vartheta_0 = (\xi_0, l_0) \in \Theta$ if and only if it solves

$$\hat{L}(\vartheta, \xi^k) \stackrel{\text{def}}{=} \hat{\varphi}(\vartheta, \xi^k) + N(\vartheta) \ni 0 \quad (9)$$

at $\xi^k = \xi_0$. In other words, the generalized equation (9) can be understood as linearization of (5) around ξ^k . The set of KKT points of (6) at $\xi^k \in X$ is given by

$$H(\xi^k) = \{\vartheta \in \mathcal{T} \mid \hat{L}(\vartheta, \xi^k) \ni 0\}$$

the *solution map* of (9).

The following result is a special case of [44, Theorem 2F.1] for solution mappings of monotone variational inequalities and proved here for completeness.

Lemma 1: Let $\xi^k \in X$; if $H(\xi^k)$ is nonempty, then $H(\xi^k)$ is a convex set containing (ξ_+, l_+) .

Proof: Denote the set of optimal solutions to (6) and (8) by $S \subset X \times Y^*$. Assume that $H(\xi^k)$ is nonempty, take $(\xi_0, l_0) \in H(\xi^k)$ and $(\xi_+, l_+) \in S$. By sufficiency of the KKT conditions, $(\xi_0, l_0) \in S$ and $\langle f, \xi_0 \rangle - \langle l_0, \gamma_k \rangle = 0$. Since $\langle f, \xi_+ \rangle \leq \langle f, \xi_0 \rangle$ and $\langle l_+, \gamma_k \rangle \geq \langle l_0, \gamma_k \rangle$ by primal and dual optimality,

$$\begin{aligned} 0 &\geq \langle f, \xi_+ \rangle - \langle l_+, \gamma_k \rangle \\ &= \langle \nabla g(\xi^k)^* l_+ + s, \xi_+ \rangle - \langle l_+, \gamma_k \rangle \\ &= \langle l_+, \nabla g(\xi^k) \xi_+ - \gamma_k \rangle + \langle s, \xi_+ \rangle \end{aligned}$$

with $s \in C^*$. Since $\langle l_+, \nabla g(\xi^k) \xi_+ - \gamma_k \rangle \geq 0$ and $\langle s, \xi_+ \rangle \geq 0$, the inequalities are tight and $(\xi_+, l_+) \in H(\xi^k)$. Hence, $H(\xi^k) = S$, a convex set. ■

Combining Assumption 1 with Lemma 1, a KKT point (ξ_0, l_0) of the nonlinear problem (4) is a candidate stationary condition of the sequential algorithm. However, we have yet to prove that (ξ_+, l_+) is the unique solution of the parametrized variational inequality (9) around ξ_0 . To that extent, we make the following, standing assumptions.

Assumption 2: For all $(\xi_0, l_0) \in \Theta$, the mapping $L_0 = \hat{L}(\cdot, \xi_0)$ is strongly regular at (ξ_0, l_0) for 0 with constant γ .

By definition, strong regularity of L_0 requires that $L_0^{-1}(\delta)$ has a single-valued localization around $\delta = 0$ for ϑ_0 , which is equivalent to a perturbed convex problem having unique solutions for small perturbations $\delta \in \mathcal{T}^*$ [35].

Assumption 3: For all $(\xi_0, l_0) \in \Theta$, the gradient $\nabla g(\xi)$ is Lipschitz continuous around ξ_0 and the mapping $\nabla g(\xi)^* l$ has the Lipschitz constant κ with respect to ξ around (ξ_0, l_0) .

If g is twice differentiable at ξ_0 , then $\nabla g(\xi)$ is Lipschitz continuous at ξ_0 and the constant κ is determined by the norm of its second derivative. We note the following implication of Assumption 3.

Lemma 2: Let $\vartheta_0 = (\xi_0, l_0) \in \Theta$; the mapping $\hat{\varphi}(\vartheta, \xi^k)$ is Lipschitz continuous with respect to ξ^k around (ϑ_0, ξ_0) with constant κ .

Proof: By Assumption 3, the first component of $\hat{\varphi}(\vartheta, \xi^k)$ satisfies

$$\limsup_{\substack{\xi^k, \xi^{k'} \rightarrow \xi_0, \xi^k \neq \xi^{k'} \\ l \rightarrow l_0}} \frac{\|\nabla g(\xi^{k'})^* l - \nabla g(\xi^k)^* l\|}{\|\xi^{k'} - \xi^k\|} \leq \kappa$$

and a difference in the second component can be written as

$$\begin{aligned} g(\xi^k) + \nabla g(\xi^k)(\xi - \xi^k) - g(\xi^{k'}) - \nabla g(\xi^{k'})(\xi - \xi^{k'}) \\ = [\nabla g(\xi^{k'}) - \nabla g(\xi^k)](\xi - \xi^k) - e(\xi^k, \xi^{k'}) \end{aligned}$$

where $e : (\xi^k, \xi^{k'}) \mapsto g(\xi^{k'}) + \nabla g(\xi^{k'})(\xi^k - \xi^{k'}) - g(\xi^k)$. By definition of the Fréchet derivative and Lipschitz continuity of $\nabla g(\xi)$ around ξ_0 , we conclude that

$$\limsup_{\xi^k, \xi^{k'} \rightarrow \xi_0, \xi^k \neq \xi^{k'}} \frac{e(\xi^k, \xi^{k'})}{\|\xi^{k'} - \xi^k\|} = 0$$

and

$$\limsup_{\substack{\xi^k, \xi^{k'} \rightarrow \xi_0, \xi^k \neq \xi^{k'} \\ \xi \rightarrow \xi_0}} \frac{\|\nabla g(\xi^{k'}) - \nabla g(\xi^k)\|}{\|\xi^{k'} - \xi^k\|} (\xi - \xi^k) = 0.$$

Combining these results we obtain

$$\limsup_{\substack{\xi^k, \xi^{k'} \rightarrow \xi_0, \xi^k \neq \xi^{k'} \\ \vartheta \rightarrow \vartheta_0}} \frac{\|\hat{\varphi}(\vartheta, \xi^{k'}) - \hat{\varphi}(\vartheta, \xi^k)\|}{\|\xi^{k'} - \xi^k\|} \leq \kappa + 0$$

which is the desired result. ■

We continue our theoretical analysis by proving that, by strong regularity of $\hat{L}(\cdot, \xi^k)$ and Lipschitz continuity of $\hat{\varphi}(\vartheta_0, \cdot)$, the KKT conditions of (6) have a locally unique solution at ξ^k if $\vartheta^k = (\xi^k, l^k)$ is sufficiently close to Θ .

Proposition 1: Let $\vartheta_0 = (\xi_0, l_0) \in \Theta$; the solution map H of (9) has a single-valued localization $\bar{h} : X \rightarrow \mathcal{T}$ around ξ_0 for ϑ_0 ; and $\bar{h}$ is Lipschitz continuous around ξ_0 with constant $\gamma\kappa$.

Proof: Define $\varphi_0 : \vartheta \mapsto \hat{\varphi}(\vartheta, \xi_0)$; then $\varphi_0(\vartheta)$ equals $\hat{\varphi}(\vartheta, \xi^k)$ around ϑ_0 if $\xi^k \rightarrow \xi_0$ by continuity and $\varphi_0 + N = L_0$ is strongly regular with constant γ by Assumption 2. Moreover, $\hat{\varphi}(\vartheta, \xi^k)$ is Lipschitz continuous with respect to ξ^k around (ϑ_0, ξ_0) with constant κ by Lemma 2. By virtue of Theorem 1, there exists a Lipschitz continuous, single-valued localization $\bar{h}$ of H around ξ_0 for ϑ_0 with constant $\gamma\kappa$, the desired result. ■

In consequence, the convex problems (6) and (8) at ξ^k are not only feasible but have unique solutions around ξ_0 .

Lemma 3: Let $\vartheta_0 = (\xi_0, l_0) \in \Theta$; then $\bar{h}(\xi^k) = (\xi_+, l_+)$ around ξ_0 .

Proof: By Proposition 1, there exists a single-valued localization $\bar{h}(\xi^k)$ of H around ξ_0 ; that is, $H(\xi^k)$ is nonempty and, by Lemma 1, a convex set that contains (ξ_+, l_+) around ξ_0 . On the other hand, $H(\xi^k)$ contains the isolated point $\bar{h}(\xi^k)$ and thus is a singleton. Hence, $\bar{h}(\xi^k) = (\xi_+, l_+)$ around ξ_0 . ■

It rests to prove that the next iterate, subject to the line search, converges towards a KKT point as well.

Proposition 2: Let $(\xi_0, l_0) \in \Theta$; if ξ^k, ξ_+ , and l_+ are sufficiently close to (ξ_0, l_0) , then the solution of (7) satisfies $\hat{r} \geq \eta/\kappa$.

Proof: If $\hat{r} < 1$, then $\psi'(\hat{r}) - \eta = 0$. The derivative is

$$\psi'(r) = \langle f - \nabla g(\xi(r))^* l_+, \xi_+ - \xi^k \rangle$$

where $\xi(r) =_{\text{def}} r\xi_+ + (1-r)\xi^k$. Since $(\xi_+, l_+) \in H(\xi^k)$,

$$\begin{aligned} \psi'(0) &= \langle f, \xi_+ \rangle - \langle f - \nabla g(\xi^k)^* l_+, \xi^k \rangle - \langle \nabla g(\xi^k)^* l_+, \xi_+ \rangle \\ &= -\langle \nabla g(\xi^k)^* l_+, \xi_+ - \xi^k \rangle - \langle l_+, g(\xi^k) \rangle - \langle s, \xi^k \rangle \\ &= -\langle s, \xi^k \rangle - \langle l_+, \varsigma \rangle \leq 0 \end{aligned}$$

where the equalities follow from (6) and (8) as well as $\langle f, \xi_+ \rangle = \langle l_+, \gamma_k \rangle$, and the inequality follows from the definition of the dual cone. Since $\nabla g(\xi)^* l$ is Lipschitz continuous around (ξ_0, l_0) by Assumption 2, we have that

$$|\psi'(r) - \psi'(0)| \leq \kappa \|\xi_+ - \xi^k\|^2 r.$$

With $\psi'(0) \leq 0$ and $\psi'(\hat{r}) = \eta > 0$ as well as $\|\xi_+ - \xi^k\| \leq \|\xi_+ - \xi_0\| + \|\xi_0 - \xi^k\| < 1$, if $\hat{r} < 1$ and ξ_+ and ξ^k are sufficiently close to ξ_0 , we conclude that $\kappa\hat{r} \geq \eta$. ■

We combine our results into a local convergence property of the sequential approach.

Theorem 2: Let $\vartheta_0 = (\xi_0, l_0) \in \Theta$ and suppose that $\gamma\kappa < 1$; there exists a constant $\alpha \in (0, 1)$ such that

$$\|\vartheta^{k+1} - \vartheta_0\| \leq \alpha \|\vartheta^k - \vartheta_0\| \quad (10)$$

if $\vartheta^k \in \mathcal{T}$ is sufficiently close to Θ .

Proof: Let $\vartheta^k = (\xi^k, l^k)$; by Proposition 1, there exists a single-valued localization $\bar{h}$ of the solution mapping H around ξ_0 for ϑ_0 satisfying

$$\|\bar{h}(\xi^k) - \bar{h}(\xi^{k'})\| \leq \gamma\kappa \|\xi^k - \xi^{k'}\|$$

for any $\xi^k, \xi^{k'} \in X$ in a neighbourhood of ξ_0 . Then $\vartheta_0 = \bar{h}(\xi_0)$ as well as $\vartheta_+ = (\xi_+, l_+) = \bar{h}(\xi^k)$ by Lemma 3 and

$$\begin{aligned} \|\vartheta^{k+1} - \vartheta_0\| &= \|\hat{r}(\vartheta_+ - \vartheta_0) + (1 - \hat{r})(\vartheta^k - \vartheta_0)\| \\ &\leq \hat{r}\gamma\kappa \|\xi^k - \xi_0\| + (1 - \hat{r})\|\vartheta^k - \vartheta_0\| \\ &\leq (1 - \hat{r} + \hat{r}\gamma\kappa)\|\vartheta^k - \vartheta_0\| \end{aligned}$$

as $\|\xi^k - \xi_0\| \leq \|\vartheta^k - \vartheta_0\|$. Since $\hat{r} \geq \omega > 0$ by Proposition 2, setting $\alpha = 1 - \omega(1 - \gamma\kappa) \in [\gamma\kappa, 1)$ is the desired result. ■

Our assumptions in the proof of Theorem 2 are similar to the assumptions in [35], which proved local convergence of sequential convex programming in the Euclidean space and without a line search. However, if we omit the line search ($\hat{r} \equiv 1$) the rate of convergence we obtain in the proof, namely $\alpha = \gamma\kappa$, is better than this previous result.

VI. NUMERICAL EXAMPLES

The region of attraction of a nonlinear dynamic system $\dot{x} = \phi(x)$ is defined as the set of initial conditions for which the system trajectories converge to an equilibrium point, here the origin. Estimating the region of attraction is a classical problem in nonlinear systems analysis and a recurrent application of sum-of-squares methods, provided that the dynamics are represented by polynomial equations of motion. One aims to find a polynomial Lyapunov candidate function $v \in \mathbb{R}_d[x]$ that decays along trajectories starting in a sublevel set of v , that is, there exists $\iota > 0$ such that

$$\dot{v}(x) = \partial_x v(x)\phi(x) < 0 \quad (11)$$

for all $x \neq 0$ satisfying $v(x) \leq \iota$. If (11) is satisfied, then $\{x \in \mathbb{R}^n \mid v(x) \leq \iota\}$ is an invariant subset of the region of attraction [43, Theorem 2].

TABLE I
COMPUTATION DETAILS OF SEQUENTIAL SUM-OF-SQUARES
PROGRAMMING FOR REGION-OF-ATTRACTION ESTIMATION.

Dynamics	d_0	d_1	d_2	Final value	Iterations	Time
Short-period	2	0	2	-1.515	8	2.77 s
	4	2	4	-1.772	9	3.70 s
Longitudinal	2	0	4	-0.354	20	9.28 s
	4	2	4	-2.788	16 ¹	25.24 s

¹Terminated early due to numerical issues of MOSEK; the nonlinear sum-of-squares constraint was satisfied with a tolerance of 1.52×10^{-5} .

In order to lower bound the volume of the region of attraction estimate, a polynomial shape $p \in \mathbb{R}[x]$ is introduced. For any $\iota > 0$, the nonlinear sum-of-squares problem is given as

$$\begin{aligned} \min_{\substack{v \in \mathbb{R}_{d_0}[x], b \in \mathbb{R} \\ s_1 \in \mathbb{R}_{d_1}[x], s_2 \in \mathbb{R}_{d_2}[x]}} & -b \\ \text{s.t.} & \begin{cases} s_2(v - \iota) - \partial_x v \phi - \varrho \in \Sigma_{d'}[x] \\ s_1(p - b) - v + \iota \in \Sigma_{d''}[x] \\ v - \varrho \in \Sigma_{d_0}[x] \\ s_1 \in \Sigma_{d_1}[x], s_2 \in \Sigma_{d_2}[x] \end{cases} \end{aligned} \quad (12)$$

where ϱ is small, positive-definite polynomial; we choose $\varrho = 10^{-6} \sum_{i=1}^n x_i^2$. Eq. (12) has been considered for region of attraction estimation [20, 23, 31, 45, 46] as well as, with minor modifications, for reachability [11], control synthesis [11, 20], or robust stability [22, 26]. In these works, the bilinearities are split into an iteration of convex and quasiconvex subproblems via coordinate descent.

We have applied sequential sum-of-squares programming to estimate the region of attraction of the short-period (two states) and longitudinal (four states) dynamics of an airplane.² The equations of motion ϕ are cubic polynomials in the two-state case and quintic polynomials in the four-state case; we fix $\iota \equiv 1$; and solve for quadratic and quartic Lyapunov functions in both cases. The degrees of the decision variables s_1 and s_2 as well as number of iterations, computation time, and final value of the objective are detailed in Tab. I. The initial guess for v has been the quadratic Lyapunov function of the linearized dynamics; $b = 1$; and the variables s_1 and s_2 have been initialized to homogeneous polynomials given in the appendix. The computations have been terminated once the change in the primal variables was below a absolute tolerance $\epsilon_k \equiv 10^{-6}$ and the change of the dual variables was below a relative tolerance $\epsilon_k^* = 10^{-6} \|l^k\|_*$.

TABLE II
COMPUTATION DETAILS OF THE ITERATIVE APPROACH OF [31] FOR
REGION-OF-ATTRACTION ESTIMATION.

Dynamics	d_0	d_1	d_2	Final value	Iterations	Time
Short-period	2	0	2	-1.514	20	10.94 s
	4	2	4	-1.760	40	48.06 s
Longitudinal	2	0	4	-0.353	22	113.21 s
	4	2	4	-2.749	66	465.50 s

For comparison, running the iterative approach of [31] took considerably longer without reaching the same final values (Tab. II); the differences are particularly noticeable for quartic Lyapunov functions. The computationally most expensive part of both algorithms are the semidefinite relaxations of the convex sum-of-squares subproblems. In the sequential approach, the semi-definite problem is larger (both with respect to the number of matrix variables N and the number of constraints M) since all polynomial decision variables are solved for at the same time; yet in the

²See [31] and its appendix for details.

iterative approaches, numerous convex problems are solved in each iteration. We compare the per-iteration effort for both approaches to solve the region of attraction estimation problems. Based on the assumption that the computational cost for a semidefinite problem is roughly of order N^3M [47], Fig. 1 details how the order of the computational cost in each iteration grows with the number of states and degree of polynomials for the region of attraction estimation. Overall, the cost of in each iteration of the sequential approach is about ten times lower than the cost of the iterative approach.

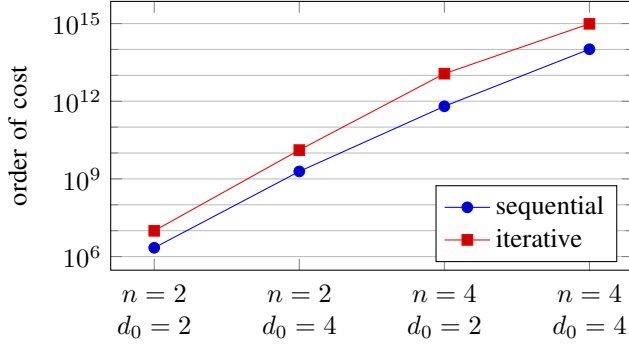


Fig. 1. Comparison of computational cost in each iteration of the sequential sum-of-squares and the iterative approach of [31].

VII. CONCLUSIONS

This paper studies the solution of nonlinear convex programs by a sequential algorithm with the addition of a line search. Theorem 2 shows that the algorithm still converges with the line search. As expected, it does not improve the local rate of convergence α . However, we observe in numerical experiments an improvement in the both radius and rate of global convergence and leave proofs of these behaviors as future work.

While the paper analyses the sequential algorithms on arbitrary (possibly infinite dimensional) Banach spaces, we only apply it to the finite-degree sum-of-squares cone of polynomials with finite degree. In [39], the authors shows how to leverage special properties of the sum-of-squares cone for solving the nonconvex Burer-Monteiro formulation. We are currently investigating whether such refined analysis could allow the sequential algorithm to exploit the structure of the sum-of-squares cone as well.

APPENDIX

For the computations detailed in Tab. I, the following initializations of the multipliers s_1 and s_2 were used:

$$\begin{aligned}
 n = 2, \quad d_0 = 2, \quad s_1 &= 1, & s_2 &= x_1^2 + x_2^2 \\
 n = 2, \quad d_0 = 4, \quad s_1 &= x_1^2 + x_2^2, & s_2 &= x_1^2 + x_2^2 \\
 n = 4, \quad d_0 = 2, \quad s_1 &= 1, & s_2 &= \sum_{i=1}^n x_i^2 \\
 n = 4, \quad d_0 = 4, \quad s_1 &= \sum_{i=1}^n x_i^2, & s_2 &= (\sum_{i=1}^n x_i^2)^2
 \end{aligned}$$

where n is the number of states and d_0 the degree of the Lyapunov function.

All numerical examples were performed on a 2.8GHz quad-core Intel Core i7 processor with 16GB of memory.

We use SOSOPT for the convex sum-of-squares problem and MOSEK for semidefinite programming. Source code is available at <https://github.com/tcunis/bisosprob>.

ACKNOWLEDGMENT

The authors wish to thank Frank Permenter for initiating their collaboration.

REFERENCES

- [1] P. A. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Mathematical Programming, Series B*, vol. 96, no. 2, pp. 293–320, 2003.
- [2] J. B. Lasserre, "Global Optimization with Polynomials and the Problem of Moments," *SIAM Journal on Optimization*, vol. 11, no. 3, pp. 796–817, 2001.
- [3] S. Prajna and a. Papachristodoulou, "Analysis of switched and hybrid systems - beyond piecewise quadratic methods," *Proceedings of the 2003 American Control Conference*, 2003., vol. 4, no. Xi, pp. 2779–2784, 2003.
- [4] A. A. Ahmadi and P. A. Parrilo, "Non-monotonic Lyapunov functions for stability of discrete time nonlinear and switched systems," in *47th IEEE Conference on Decision and Control*, 2008, pp. 614–621.
- [5] A. Papachristodoulou and S. Prajna, "Robust Stability Analysis of Nonlinear Hybrid Systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 5, pp. 1035–1041, 2009.
- [6] H. Ichihara, "Sum of Squares Based Input-to-State Stability Analysis of Polynomial Nonlinear Systems," *SICE Journal of Control, Measurement, and System Integration*, vol. 5, no. 4, pp. 218–225, 2012.
- [7] T. Holicki and C. W. Scherer, "Stability analysis and output-feedback synthesis of hybrid systems affected by piecewise constant parameters via dynamic resetting scalings," *Nonlinear Analysis: Hybrid Systems*, vol. 34, pp. 179–208, 2019. [Online]. Available: <https://doi.org/10.1016/j.nahs.2019.06.003>
- [8] T. H. Summers, K. Kunz, N. Kariotoglou, M. Kamgarpour, S. Summers, and J. Lygeros, "Approximate dynamic programming via sum of squares programming," in *2013 European Control Conference*. Zürich, CH: EUCA, 2013, pp. 191–197.
- [9] S. S. Tan, A. Varvatsiotis, and V. Y. Tan, "Analysis of Optimization Algorithms via Sum-of-Squares," *Journal of Optimization Theory and Applications*, vol. 190, no. 1, pp. 56–81, 2021. [Online]. Available: <https://doi.org/10.1007/s10957-021-01869-0>
- [10] M. Korda and C. N. Jones, "Stability and performance verification of optimization-based controllers," *Automatica*, vol. 78, pp. 34–45, 2017. [Online]. Available: <http://dx.doi.org/10.1016/j.automatica.2016.12.008>
- [11] Z. Jarvis-Wloszek, R. Feeley, W. Tan, K. Sun, and A. Packard, "Some Controls Applications of Sum of Squares Programming," in *Proceedings of the IEEE Conference on Decision and Control*, vol. 5, Maui, US-HI, 2003, pp. 4676–4681.
- [12] C. Ebenbauer and F. Allgöwer, "Analysis and design of polynomial control systems using dissipation inequalities and sum of squares," *Computers and Chemical Engineering*, vol. 30, pp. 1590–1602, 2006.
- [13] Y. Oishi, "Simplified approaches to polynomial design of model predictive controllers," *Proceedings of the IEEE International Conference on Control Applications*, pp. 960–965, 2013.
- [14] M. Vatani and M. Hovd, "Control of Bilinear Power Converters using Sum of Squares Programming," *2015 European Control Conference (ECC)*, pp. 654–659, 2015.
- [15] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. SIAM Studies in Applied and Numerical Mathematics. Philadelphia, PA: Society for Industrial and Applied Mathematics, 1994, no. 15.
- [16] R. Genesio and A. Tesi, "Stability Analysis of Quadratic Systems," *IFAC Proceedings Volumes*, vol. 22, no. 3, pp. 195–199, 1989. [Online]. Available: [http://dx.doi.org/10.1016/S1474-6670\(17\)53633-2](http://dx.doi.org/10.1016/S1474-6670(17)53633-2)
- [17] G. Stengle, "A Nullstellensatz and a Positivstellensatz in Semialgebraic Geometry," *Mathematische Annalen*, vol. 207, no. 2, pp. 87–97, 1974.
- [18] A. Cotraruolo, M. Hosseinzadeh, D. R. Ramirez, D. Limon, and E. Garone, "Reference Dependent Invariant Sets: Sum of Squares Based Computation and Applications in Constrained Control," no. June, 2020. [Online]. Available: <http://arxiv.org/abs/2006.15886>

- [19] L. Khodadadi, B. Samadi, and H. Khaloozadeh, "Estimation of region of attraction for polynomial nonlinear systems: A numerical method," *ISA Transactions*, vol. 53, no. 1, pp. 25–32, 2014.
- [20] T. Cunis, J. P. Condomines, and L. Burlion, "Sum-of-squares flight control synthesis for deep-stall recovery," *Journal of Guidance, Control, and Dynamics*, vol. 43, no. 8, pp. 1498–1511, 2020.
- [21] X. Zheng, Z. She, J. Lu, and M. Li, "Computing multiple Lyapunov-like functions for inner estimates of domains of attraction of switched hybrid systems," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 17, pp. 5191–5212, 2018.
- [22] U. Topcu, A. K. Packard, P. Seiler, and G. J. Balas, "Robust Region-of-Attraction Estimation," *IEEE Transactions on Automatic Control*, vol. 55, no. 1, pp. 137–142, jan 2010.
- [23] U. Topcu, A. Packard, and P. Seiler, "Local stability analysis using simulations and sum-of-squares programming," *Automatica*, vol. 44, no. 10, pp. 2669–2675, 2008.
- [24] M. Newton and A. Papachristodoulou, "Stability of Non-linear Neural Feedback Loops using Sum of Squares," 2022. [Online]. Available: <http://arxiv.org/abs/2204.03913>
- [25] A. Cotorruelo, M. Hosseinzadeh, D. R. Ramirez, D. Limon, and E. Garone, "Reference dependent invariant sets: Sum of squares based computation and applications in constrained control," *Automatica*, vol. 129, p. 109614, 2021. [Online]. Available: <https://doi.org/10.1016/j.automatica.2021.109614>
- [26] A. Iannelli, P. Seiler, and A. Marcos, "Region of attraction analysis with Integral Quadratic Constraints," *Automatica*, vol. 109, p. 108543, 2019.
- [27] H. Yin, M. Arca, A. K. Packard, and P. Seiler, "Backward Reachability for Polynomial Systems on A Finite Horizon," *IEEE Transactions on Automatic Control*, vol. 9286, no. c, pp. 1–8, 2021.
- [28] T. Cunis, J.-P. Condomines, and L. Burlion, "Local stability analysis for large polynomial spline systems," *Automatica*, vol. 113, p. 108773, 2020.
- [29] P. Seiler and G. J. Balas, "Quasiconvex sum-of-squares programming," in *49th IEEE Conference on Decision and Control*, Atlanta, US-GA, 2010, pp. 3337–3342.
- [30] A. Majumdar, A. A. Ahmadi, and R. Tedrake, "Control Design Along Trajectories via Sum of Squares Optimization," in *2013 IEEE International Conference on Robotics and Automation*, Karlsruhe, DE, may 2013, pp. 4039–4046.
- [31] A. Chakraborty, P. Seiler, and G. J. Balas, "Nonlinear region of attraction analysis for flight control verification and validation," *Control Engineering Practice*, vol. 19, no. 4, pp. 335–345, 2011.
- [32] R. W. Freund, F. Jarre, and C. H. Vogelbusch, "Nonlinear semidefinite programming: Sensitivity, convergence, and an application in passive reduced-order modeling," *Mathematical Programming*, vol. 109, no. 2-3, pp. 581–611, 2007.
- [33] Y. Mao, M. Szmuk, X. Xu, and B. Acikmese, "Successive Convexification: A Superlinearly Convergent Algorithm for Non-convex Optimal Control Problems," pp. 1–35, 2018. [Online]. Available: <http://arxiv.org/abs/1804.06539>
- [34] J. L. Li and H. Zhang, "A superlinearly convergent SSDP algorithm for nonlinear semidefinite programming," *Journal of Inequalities and Applications*, vol. 2019, no. 1, 2019. [Online]. Available: <http://dx.doi.org/10.1186/s13660-019-2171-y>
- [35] Q. T. Dinh and M. Diehl, "Local Convergence of Sequential Convex Programming for Nonconvex Optimization," 2010.
- [36] R. Doelman and M. Verhaegen, "Sequential convex relaxation for convex optimization with bilinear matrix equalities," *2016 European Control Conference, ECC 2016*, pp. 1946–1951, 2017.
- [37] M. J. Powell, "Algorithms for nonlinear constraints that use lagrangian functions," *Mathematical Programming*, vol. 14, no. 1, pp. 224–248, 1978.
- [38] D. Papp and S. Yildiz, "Sum-of-Squares Optimization without Semidefinite Programming," *SIAM Journal on Optimization*, vol. 29, no. 1, pp. 822–851, jan 2019. [Online]. Available: <https://epubs.siam.org/doi/10.1137/17M1160124>
- [39] B. Legat, C. Yuan, and P. A. Parrilo, "Low-rank univariate sum of squares has no spurious local minima," *arXiv preprint arXiv:2205.11466*, 2022.
- [40] T. Weisser, B. Legat, C. Coey, L. Kapelevich, and J. P. Vielma, "Polynomial and moment optimization in julia and jump," in *JuliaCon*, 2019. [Online]. Available: <https://pretalx.com/juliacon2019/talk/QZBKAU/>
- [41] A. L. Dontchev, *Lectures on Variational Analysis*, ser. Applied Mathematical Sciences. Cham: Springer, 2021, no. 205.
- [42] S. M. Robinson, "Strongly Regular Generalized Equations," *Mathematics of Operations Research*, vol. 5, no. 1, pp. 43–62, 1980. [Online]. Available: <http://www.jstor.com/stable/3689393>
- [43] J. P. La Salle, "Some Extensions of Liapunov's Second Method," *IRE Transactions on Circuit Theory*, vol. CT-7, no. 4, pp. 520–527, 1960.
- [44] A. L. Dontchev and R. T. Rockafellar, *Implicit Functions and Solution Mappings*, 2nd ed. Springer, 2011.
- [45] A. Chakraborty, P. Seiler, and G. J. Balas, "Susceptibility of F/A-18 Flight Controllers to the Falling-leaf Mode: Nonlinear Analysis," *Journal of Guidance, Control, and Dynamics*, vol. 34, no. 2, pp. 73–85, 2011.
- [46] W. Tan and A. Packard, "Stability region analysis using polynomial and composite polynomial Lyapunov functions and sum-of-squares programming," *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 565–571, 2008.
- [47] D. Peaucelle, D. Henrion, Y. Labit, and K. Taitz, "User's Guide for SEDUMI INTERFACE 1.04," LAAS-CNRS, Toulouse, Tech. Rep., 2002. [Online]. Available: <http://homepages.laas.fr/peaucell/software/sdmguide.pdf>