

# EFX: A Simpler Approach and an (Almost) Optimal Guarantee via Rainbow Cycle Number

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The existence of EFX allocations is a fundamental open problem in discrete fair division. Since the general problem has been elusive, progress is made on two fronts: (i) proving existence when the number of agents is small, and (ii) proving the existence of relaxations of EFX. In this paper, we improve and simplify the state-of-the-art results on both fronts with new techniques.

For the case of three agents, the existence of EFX was first shown with additive valuations and then extended to a more general class of nice-cancellable valuations. Both results are obtained through an algorithm that moves in the space of partial EFX allocations, improving a certain potential as long as there are unallocated goods. However, the update rules to move from one partial EFX allocation to another are very involved, cumbersome, and fail if any one agent has a general monotone valuation function. We simplify and improve this result by showing the existence of EFX allocations when two of the agents have general monotone valuations, and one has MMS-feasible valuation (a strict generalization of nice-cancelable valuation functions). In contrast to the previous approaches, our algorithm moves in the space of complete allocations, improving a potential, as long as the allocation is not EFX.

Secondly, we consider relaxations of EFX allocations, namely, approximate-EFX allocations and EFX allocations with few unallocated goods (charity). Through a promising new method using a problem in extremal combinatorics called Rainbow Cycle Number (RCN), [Chaudhury et al. 2021] managed to show the existence of  $(1 - \epsilon)$ -EFX allocation with sub-linear charity, namely  $O((n/\epsilon)^{4/5})$  charity, where  $n$  is the number of agents. This is done by upper bounding the RCN by  $O(d^4)$  in  $d$ -dimension. They conjectured this number to be  $O(d)$  and gave a matching lower bound. We improve the upper bound to  $O(d \log d)$  and thereby get (almost) optimal charity of  $\tilde{O}((n/\epsilon)^{1/2})$  that is possible through this method. Our technique is based on the *probabilistic method*. We also derandomize the approach to construct such an allocation in polynomial time.

A full version of this paper can be found at <https://arxiv.org/abs/2205.07638>.

## ACKNOWLEDGEMENTS

Jugal Garg was supported by NSF Grant CCF-1942321. Ruta Mehta was supported by NSF Grant CCF-1750436. Bhaskar Ray Chaudhury was partially supported by NSF Grants CCF-1942321 and CCF-1750436.

## ACM Reference Format:

Hannaneh Akrami, Noga Alon, Bhaskar Ray Chaudhury, Jugal Garg, Kurt Mehlhorn, and Ruta Mehta. 2023. EFX: A Simpler Approach and an (Almost) Optimal Guarantee via Rainbow Cycle Number. In *Proceedings of the 24th ACM Conference on Economics and Computation (EC '23)*, July 9–12, 2023, London, United Kingdom. ACM, New York, NY, USA, 1 page. <https://doi.org/10.1145/3580507.3597799>

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EC '23, July 9–12, 2023, London, United Kingdom

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ACM ISBN 979-8-4007-0104-7/23/07.

<https://doi.org/10.1145/3580507.3597799>